REPRESENTATIONS OF SYMMETRIC GROUPS

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Weiyang Fang, November 2021.

Abstract

The representation theory of the symmetric groups is well-studied classically. It was first developed by Frobenius. He first used Young tableaux to describe all the irreducible representations of symmetric groups [Jam78].
In this thesis, we will see each irreducible representation of symmetric group $\mathfrak{S}_d$ corresponds to a Specht module V_λ and some theorems associated with Specht modules such as the Hook length formula and Frobenius's formula.

Contents

Introduction

This thesis aims to provide my understanding of the representation theory of symmetric groups and some of its applications.

Representation theory is a significant research area in pure mathematics nowadays. It is a beautiful mathematical subject closely related to many other branches of mathematics such as combinatorics, probability theory, and number theory. The representation theory of finite groups started with the letter commutation between F. G. Frobenius and R. Dedekind to discuss new ideas on factoring a certain homogeneous polynomial associated with a finite group which is called the "group determinant" in 1896 [Lam98]. Symmetric groups play a central role in the study of group theory. Therefore, the representation theory of symmetric groups is an interesting topic in representation theory.

Let $\mathfrak{S}_d$ be a field of characteristic zero. Let $\mathfrak{S}_d$ be a symmetric group of order d. By Maschke's theorem, $\mathbb{F}\mathfrak{S}_d$ is semisimple algebra. Therefore, representations of $\mathfrak{S}_d$ over $\mathbb{F}$ is semisimple. A representation of group G over field k is called absolutely irreducible if it remains irreducible after any extension of k. A field K is called a splitting field for a group G if every irreducible representation of G over K is absolutely irreducible. It was proved that each field is a splitting field of $\mathfrak{S}_d$ [JK81, Theorem 2.1.12]. In this thesis, we will assume $\mathbb{F}$ is $\mathbb{C}$. It will be no differences if we change the field to $\mathbb{Q}$ or any other fields of characteristic zero.

Not all representations are semisimple. Suppose $\mathbf{k}$ is a field with prime characteristic such that $\mathrm{char}(\mathbf{k})\big||\mathfrak{S}_d|$. Now, the representations of $\mathfrak{S}_d$ over $\mathbf{k}$ is not semisimple anymore, which is called modular representations of symmetric group. One strategy is to study such non semi-simple structure is via Jacobson radical. There is a famous open problem on modular representations of symmetric groups.

Open Problem 1. *Let $A = \mathbf{k}\mathfrak{S}_d$. Determine the Wedderburn components of the semisimple $\mathbf{k}$-algebra $A/\operatorname{rad} A$.*

I acknowledge that this thesis is based on William Fulton and Joe Harris's book [HF91, Chapter 4] and the proof of the Hook length formula in 2.5 is based on the paper [GNW79]. Many of the propositions, theorems, corollaries and lemmas in my thesis can be found in this book, but most of the proofs are written in my own words. For example, I rewrite the proofs of Lemma 2.4.2 and Theorem 2.4.4. Some other propositions, theorems, corollaries and lemmas in my thesis come from exercises of the book. For example, Corollary 4.3.9 and Lemma 2.3.10 are based on Exercise 4.40 and Exercise 4.24 in this book.

Chapter 1

Group Representations

In this chapter, we are going to define the basic concepts in representation theory such as representations of groups, characters and induced representations. We will introduce the notations we are using and some basic propositions in group representations. This chapter is a revision of course Math 5735. All the proofs which we omitted in this chapter can be found in the Math5735 lecture notes [Du21].

1.1 Basic Definitions

Let V be a finite-dimensional vector space over the complex number field $\mathbb{C}$.

Definition 1.1.1. *A representation of a finite group G on V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is a group automorphism of V.*

Definition 1.1.2. *(i) The representation $\rho : G \to \mathbb{C}, g \mapsto 1$ is called trivial representation of G.*
(ii) Let $\mathfrak{S}_d$ be symmetric group of degree d. The representation $\rho : \mathfrak{S}_d \to \mathbb{C}, \delta \mapsto \mathrm{sgn}(\delta)$ is called the sign representation.
(iii) The representation corresponding to $\mathbb{C}G$ module ${}_{\mathbb{C}G}\mathbb{C}G$ is called the regular representation of G.

Definition 1.1.3 (Direct sum of representations)**.** *Let V, W be representations of G. Then $V \oplus W$ also is a representation of G which is defined by*

$$g(v, w) = (gv, gw).$$

Definition 1.1.4 (Tensor product of representations)**.** *Let V, W be representations of G. Then $V \otimes W$ also is a representation of G which is defined by*

$$g(v \otimes w) = gv \otimes gw.$$

Definition 1.1.5 (External tensor product)**.** *Suppose G and H are groups. V_1 and V_2 are representations of G and H. Then the tensor product $V_1 \otimes V_2$ is a group representation of $G \times H$. An element (g, h) of $G \times H$ acts on a basis element $v_1 \otimes v_2$ by*

$$(g, h) \cdot (v_1 \otimes v_2) = gv_1 \otimes hv_2$$

Definition 1.1.6 (Kronecker product [Hor91]). *The Kronecker product of $A = [a_{ij}] \in M_{m,n}(k)$ and $B = [b_{ij}] \in M_{p,q}(k)$ is denoted by $A \otimes B$ and is defined to be the block matrix*

$$A \otimes B \equiv \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(k)$$

1.2 Characters

Proposition 1.2.1. *Let V be a finite dimensional k-space, and $\varphi : V \to V$ a linear map. Then the trace of $\varphi, \operatorname{tr}(\varphi) := \operatorname{tr}(T)$ is independent of the matrix T representing it (with respect to any basis for V in both the domain and codomain).*

Definition 1.2.2 (Character). *Let $\rho : G \to GL(V)$ be a representation of G over V. The character of ρ is the function*

$$\chi_V : G \longrightarrow k, \quad g \longmapsto \operatorname{tr}(\rho(g)).$$

Definition 1.2.3 (Class function). *If $f : G \to \mathbb{C}$ is a mapping satisfying $f(h^{-1}gh) = f(h)$ for all $g, h \in G$, then we call f is a class function on G. The set of all class functions on G is denoted by $\mathrm{CF}(G)$.*

Remark 1.2.4. *The sum and scalar multiples of class function is still class function. Therefore, $\mathrm{CF}(G)$ is a $\mathbb{C}-$vector space. Let $\mathfrak{C}$ be the set of all conjugacy classes of G. Then basis of $\mathrm{CF}(G)$ is given by $\{f_C\}_{C\in\mathfrak{C}}$ such that*

$$f_C(g) = \begin{cases} 1, & \text{if } g \in C; \\ 0, & \text{otherwise.} \end{cases}$$

We can define an inner product on $\mathrm{CF}(G)$ by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g\in G} \alpha(g)\overline{\beta(g)}.$$

Proposition 1.2.5. *Characters are class functions.*

Proposition 1.2.6. *V is a simple representation, iff $\langle \chi_V, \chi_V \rangle = 1$.*

Proposition 1.2.7. *The irreducible characters of a group G form an orthonormal basis for the space of class functions $\mathrm{CF}(G)$.*

Example 1.2.8. *Let G be a finite group. The character of regular representation χ_{reg} is that*

$$\chi_{reg}(g) = \begin{cases} |G|, & \text{if } g = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.2.9. *Let V and W be representations of G, $g, h \in G$. Then*

1. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$,
2. $\chi_{V\oplus W} = \chi_V + \chi_W$,
3. $\chi_V(h^{-1}gh) = \chi_V(g)$.

Theorem 1.2.10. *Let V_1, V_2 be the group representation of G, H. Let χ_i be the character of V_i, then character χ of $V_1 \otimes V_2$ is given by*

$$\chi(g, h) = \chi_1(g)\chi_2(h)$$

Proof. If we express $V_1(g)$ and $V_2(h)$ into matrix forms, then the exterior tensor product of representations becomes Kronecker product of matrices. [DF04, Chapter 11.2 Proposition 17] Therefore, we have that

$$\begin{aligned}\chi(g, h) = \operatorname{tr}(V_1(g) \otimes V_2(h)) = \sum_i \Big(V_1(g)_{i,i} \operatorname{tr}(V_2(h)) \Big) &= \operatorname{tr}(V_1(g)) \operatorname{tr}(V_2(h)) \\ &= \chi_1(g)\chi_2(h).\end{aligned}$$

□

Proposition 1.2.11. *If G is a finite group, then $\chi_{reg} = \sum_{i=1}^{r} \chi_i(1)\chi_i$, where $\chi_1, \dots, \chi_r$ form the set of all irreducible character of G.*

1.3 Restricted and Induced Representations

Let G be a group and H be a subgroup of G. Let W be a representation of H.

Definition 1.3.1 (Induced representation). *The induced representation of G from H, $\operatorname{Ind}_H^G(W)$ is given by*

$$\operatorname{Ind}_H^G(W) = \mathbb{C}G \otimes_{\mathbb{C}H} W \quad \text{with the action } \alpha(\beta \otimes w) = \alpha\beta \otimes w,$$

for all $\alpha, \beta \in \mathbb{C}G$ and $w \in W$. We also use notation $W|^G$ to denote the induced representation of G from H, $\operatorname{Ind}_H^G(W)$.

Definition 1.3.2 (Permutation representation). *A permutation representation of a group G on a set S is a homomorphism from G to the symmetric group of S :*

$$\rho : G \to \operatorname{Sym}(X).$$

Remark 1.3.3. *If W is the 1-dimensional trivial representation of H, then its induced representation is the permutation representation spanned by the coset-space G/H:*

$$\mathbb{C}[G/H] = \operatorname{Ind}_H^G(W).$$

Definition 1.3.4 (Restricted representation). *Given a $\mathbb{C}G$-module M, restriction to H defines a $\mathbb{C}H$-module. We denote this module by $\operatorname{Res}_H^G(M)$ or $M|_H$*

Definition 1.3.5. *Let f be a class function on H. Define the class function f' on G by*

$$f'(s) = \frac{1}{|H|} \sum_{t \in G, t^{-1}st \in H} f(t^{-1}st).$$

Then we say f' is induced by f and denote it by $\operatorname{Ind}(f)$ or $\operatorname{Ind}_H^G(f)$.

In order to compute character $\chi_{\operatorname{Ind} W}$ of $\operatorname{Ind}_H^G W$. We have the following formula.

Theorem 1.3.6. *For each $g \in G$, we have*

$$\chi_{\operatorname{Ind} W(g)} = \sum_{s \in G, s^{-1}gs \in H} \chi_W(s^{-1}gs). \tag{1.3.1}$$

Suppose C is a conjugate class of G and $C \cap H$ decomposes into conjugate classes $D_1, \ldots, D_r$ in H, then we can rewrite (1.3.1) as

$$\chi_{\operatorname{Ind} W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i).$$

In particular, if W is the trivial representation, then

$$\chi_{\operatorname{Ind} W}(C) = \frac{[G : H]}{|C|} |C \cap H|. \tag{1.3.2}$$

Theorem 1.3.7 (Frobenius Reciprocity)**.** *If L is a kH-module and N is a kG-module, then there is an abelian group isomorphism*

$$\operatorname{Hom}_{kG}\left(\operatorname{Ind}_H^G(L), N\right) \cong \operatorname{Hom}_{kH}\left(L, \operatorname{Res}_H^G(N)\right).$$

Definition 1.3.8. *We say that a nonzero R-module is decomposable if it is a direct sum of two nonzero submodules. Otherwise it is indecomposable.*

Definition 1.3.9. *For a ring R, R^* denotes the group of units of R. A (nonzero) ring (or algebra) R is called a local ring (or algebra) if $R - R^*$ is an ideal of R.*

Proposition 1.3.10. *For a finite dimension A-module M, M is indecomposable if and only if $\operatorname{End}_A(M)$ is a local algebra.*

Chapter 2

Irreducible Representations of Symmetric Groups

In this chapter we will work out all irreducible representations of the symmetric group $\mathfrak{S}_d$ and compute their dimensions.

2.1 The Conjugacy Classes of $\mathfrak{S}_d$

Definition 2.1.1 (Partition)**.** *Let $d \in \mathbb{N}$. A partition of d is a sequence of non-negative integers $(\lambda_1, ..., \lambda_r)$ satisfying*

$$n = \lambda_1 + ... + \lambda_r, \quad \lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_r \geqslant 0.$$

If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of n, then we write $\lambda \vdash n$.
We denote $\mathcal{P}(n)$ for the set of all partitions of n.
The number r of nonzero λ_i's is called the length of λ, and is denoted by $l(\lambda)$.

Remark 2.1.2. *If two partitions only differ by the string of zeros at the end, then we regard them as the same partition.*

Example 2.1.3. *Let*

$$\lambda = (3, 2, 1, 0, 0) = (3, 2, 1, 0) = (3, 2, 1) \vdash 6.$$

And $l(\lambda) = 3$.

Example 2.1.4. *If $n = 0$, then only partition of n is that $\lambda = (0)$.*

Definition 2.1.5 (k-transitive)**.** *Let G be a group, Ω is a set. G acts on Ω. Let k is a positive integer such that $k \leqslant |\Omega|$. We say that G is k-transitive on Ω if G is transitive on the set $\Omega^{(k)}$ of k-tuples of distinct elements of Ω, that is, for all $(\alpha_1, \alpha_2, \ldots, \alpha_k), (\beta_1, \beta_2, \ldots, \beta_k) \in \Omega^{(k)}$ there exists $g \in G$ such that*

$$g(\alpha_1) = \beta_1, \quad g(\alpha_2) = \beta_2, \quad ... \quad , g(\alpha_k) = \beta_k$$

$\mathfrak{S}_d$ is d-transitive on $\{1, ..., d\}$.

Proposition 2.1.6. *Two elements in $\mathfrak{S}_d$ are conjugate if and only if they have the same cyclic type.*

Proof. Let $g, h \in \mathfrak{S}_d$. Then we can decompose g into product of disjoint cycles

$$g = C_1 \cdot ... \cdot C_r, \quad \text{each integer in } \{1, ..., d\} \text{ appears once.}$$

By cycle decomposition of the conjugate, $h^{-1}gh = h(C_1)...h(C_2)$. Conversely, suppose g, g' have the same cycle type and g' can be decomposed into product of disjoint cycles such that

$$g' = C_1' \cdot ... \cdot C_r', \text{ each integer in } \{1, ..., d\} \text{ appears once.}$$

$\mathfrak{S}_d$ is d-transitive on $\{1, ..., d\}$. Therefore, there exists $\tau \in \mathfrak{S}_d$ such that $\tau(C_i) = C_i'$ for all $i \in \{1, ..., r\}$. □

Proposition 2.1.7. *There exists a bijection map between the set of all conjugacy classes in $\mathfrak{S}_d$ and $\mathcal{P}(d)$.*

$$\phi : \{\textit{conjugacy class in } \mathfrak{S}_d\} \leftrightarrow \mathcal{P}(d)$$

Proof. Let $\sigma \in \mathfrak{S}_d$. Then we can decompose σ such that $\sigma = C_1 \dots C_r$ where C_i are disjoint cycles with $\#C_1 \geqslant \cdots \geqslant \#C_r$ and each number in $\{1, \cdots, d\}$ appear once. We denote $\#C_i$ to be the length of circle C_i. Then $\phi(\sigma) = (\#C_1, \dots, \#C_r)$ which is a partition of d. By Proposition 2.1.6, different conjugacy class has different cyclic type. Therefore, ϕ is injective. Let $(\lambda_1, \lambda_2, \dots, \lambda_k) \vdash d$. Then we can construct a product of k disjoint cycles such that $\tau = (1, \dots, \lambda_1)(\lambda_1 + 1, \dots, \lambda_1 + \lambda_2) \cdots (1 + \sum_{i=1}^{k-1} \lambda_i, \dots, d)$. Clearly, $\tau \in \mathfrak{S}_d$ and $\phi(d) = (\lambda_1, \lambda_2, \dots, \lambda_k)$. Therefore, ϕ is surjective. □

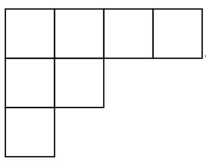
Proposition 2.1.8. *The number of irreducible representations of a finite group G (up to isomorphism) equals the number of conjugacy classes of G.*

Proof. See Math5735 lecture notes [Du21]. □

Corollary 2.1.9. *The number of irreducible representation of $\mathfrak{S}_d$ equals $|\mathcal{P}(d)|$.*

Definition 2.1.10 (Young diagram). *Suppose $\lambda = (\lambda_1, \dots, \lambda_r) \vdash d$. Then a Young diagram of shape λ is a finite collection of boxes arranged in left-justified rows. The number of boxes in row i is λ_i.*

Example 2.1.11. *The integer partition $(4, 2, 1)$ is drawn as the Young diagram*

.

Each partition of d associates with a type of Young diagram. Therefore a Young diagram stands for a conjugacy class of the symmetric group.

Definition 2.1.12 (Young tableau)**.** *Suppose $\lambda \vdash d$. A Young λ-tableau is a Young diagram of shape λ with filling integers $1, \ldots, d$ in each box and each integer appears once.*

Example 2.1.13. *Let $\lambda = (3, 2) \vdash 5$. Then a Young tableau T_λ can be drawn as*

$$T_\lambda = \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 & 5 & \\ \cline{1-2}\end{array}.$$

Definition 2.1.14 (Standard Young tableau)**.** *If a Young tableau satisfies the condition the numbers in each box form a strictly increasing sequence along each row and along each column, then we say it is a standard Young tableau.*

Example 2.1.15. *For the partition $(3, 2)$ of 5,*

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 & 5 & \\ \cline{1-2}\end{array}$$

is not a standard Young tableau.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$$

is a standard Young tableau.

2.2 The Robinson-Schensted-Knuth Correspondence

This section gives a very brief introduction to Robinson-Schensted-Knuth correspondence. The book [Ful96], by William Fulton, is a good reference book to this area.

Let ω be an element of $\mathfrak{S}_d$ such that $\omega = \begin{pmatrix} 1 & 2 & \ldots & d \\ v_1 & v_2 & \ldots & v_d \end{pmatrix}$. Then we can get a pair of standard Young tableaux (P, Q) with the same shape through ω by the following algorithm:

Initial step: We put v_1 in the tableau P and 1 in the tableau Q.

$$P = \boxed{v_1}, \qquad Q = \boxed{1}$$

Iteration steps: If v_i is the largest the entries of the first row of P, then we append v_i to the end of the row of P and append i to the end of the row of Q. Otherwise, find the smallest entry v_k which is strictly larger than v_i. Replace the entry v_k in this box with v_i. (We call v_k is bumped out). We call it "bumping step". We repeat the bumping step for v_k in the next row. We keep repeating bumping step in the following rows until no entry is bumped out. Then we compare Q with P and add addition box to Q and put i in it to make Q and P same shape. Therefore, Q is called recording tableau.

End.
The above algorithm is called Robinson-Schensted-Knuth algorithm (RSK algorithm). We write $\omega \xrightarrow{RSK} (P, Q)$.

Example 2.2.1. *Let* $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$. *Then we do the following steps to generate* (P, Q).
Step 1:

$$P = \begin{array}{|c|}\hline 5 \\ \hline\end{array}, \qquad Q = \begin{array}{|c|}\hline 1 \\ \hline\end{array}.$$

Step 2:

$$P = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}, \qquad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}.$$

Step 3:

$$P = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}, \qquad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}.$$

Step 4:

$$P = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \qquad Q = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Step 5:

$$P = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Theorem 2.2.2 (RSK Theorem)**.** *The Robinson-Schensted-Knuth algorithm set up a one-to-one correspondence between* $\mathfrak{S}_d$ *and pairs of standard Young tableaux* (P, Q) *with the same shape i.e.*

$$\phi : \omega \xrightarrow{RSK} (P, Q),$$

ϕ *is a bijection.*

Proof. We just need to find a inverse algorithm $(P, Q) \to \omega$. Consider the Example 2.2.1, if the largest elements lie in the left corner of the first row, then there is no problem to find the last element added into P.
Suppose the largest element of Q occur at position (i, j) and $i > 1$. Let $v_{(i,j)}$ be element which is at position (i, j) of P. The element bumped $v_{(i,j)}$ is the largest element in the $i-1$ which is strictly less than $v_{(i,j)}$. We call it $v_{(i-1,j')}$. The element bumped $v_{(i-1,j')}$ is the largest element in the $i-2$ which is strictly less than $v_{(i-1,j')}$. We can keep repeating this to find the last element inserted into P.
Clearly, we can repeat the above procedure to take out all elements of P. $\square$

Let $f^\lambda = \#\{\text{standard Young tableau of shape } \lambda\}$.

Corollary 2.2.3.

$$d! = \sum_{\lambda \vdash d} (f^\lambda)^2.$$

Proof. The Robinson-Schensted-Knuth correspondence gives a bijection between $\mathfrak{S}_d$ and pairs of standard Young tableaux of the same shape. $\square$

Example 2.2.4. *Let us consider (P, Q) in the Example 2.2.1. We want to get ω from (P, Q).*
Step 1:

$$P = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}, \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & ? & ? & ? & ? \end{pmatrix}.$$

Step 2:

$$P = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}, \qquad Q = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & ? & ? & ? & 3 \end{pmatrix}.$$

Step 3:

$$P = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}, \qquad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}, \quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & ? & ? & 2 & 3 \end{pmatrix}.$$

Step 4:
The position of largest element in Q is $(1, 3)$. The element in P at position $(1, 3)$ is $\boxed{5}$. $\boxed{4}$ is the only element smaller than $\boxed{5}$. Therefore, $\boxed{4}$ bumped $\boxed{5}$. Similarly, we can see that $\boxed{1}$ bumped $\boxed{4}$. Therefore, we take out $\boxed{1}$.

$$P = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}, \qquad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & ? & 1 & 2 & 3 \end{pmatrix}.$$

Step 5:

$$P = \boxed{5}, \qquad Q = \boxed{1}, \quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ ? & 4 & 1 & 2 & 3 \end{pmatrix}.$$

Step 6:

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

2.3 Young Projectors

Definition 2.3.1 ($\lambda-$tabloid)**.** *A $\lambda-$tabloid $\{T_\lambda\}$ is a tableau T_λ, but the order of numbers in each rows is irrelevant. We also call a $\lambda-$tabloid a tabloid of shape λ.*
Definition 2.3.2 (Row standard Young tableau)**.** *A row standard Young tableau is a Young tableau in which the numbers form a strictly increasing sequence along each row.*
Remark 2.3.3. *For each λ-tabloid $\{T_\lambda\}$, we can rearrange the row elements in increasing order. Then, each λ-tabloid corresponds to a row standard $\lambda-$tableau. Therefore, there exists a bijection such that*

$$\{\lambda - \textit{tableau}\} \to \{\textit{row standard } \lambda\textit{−tableau}\}.$$

Example 2.3.4. *Suppose Young tableau T_λ is* $\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 & 5 \\ \cline{1-2} \end{array}$. *Then the corresponding tabloid $\{T_\lambda\}$ is*

$$\begin{array}{ccc} \hline 4 & 3 & 2 \\ \hline 1 & 5 & \\ \cline{1-2} \end{array} = \begin{array}{ccc} \hline 2 & 3 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} \end{array}.$$

Example 2.3.5. *Let $\lambda = (2,1) \vdash 3$. Then* $\begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ *and* $\begin{array}{cc} \hline 2 & 1 \\ \hline 3 & \\ \cline{1-1} \end{array}$ *are the same λ-tabloids. But* $\begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ *and* $\begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$ *are different λ-tabloids.*

Let $\lambda \vdash d$, T_λ be a Young tableau of shape λ. Define two subgroup of $\mathfrak{S}_d$ on T_λ.

$$P_{T_\lambda} = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each row of } T_\lambda \},$$

$$Q_{T_\lambda} = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each column of } T_\lambda \}.$$

If the corresponding Young tableau is clear, we can write P_λ and Q_λ instead of P_{T_λ} and Q_{T_λ} for short.
Definition 2.3.6 (Young projector)**.** *Let $\lambda \vdash d$. Let T_λ be a Young $\lambda-$tableau. We define a_λ and b_λ be two elements in group algebra $\mathbb{C}\mathfrak{S}_d$ by*

$$a_\lambda = \sum_{g \in P_\lambda} e_g \quad \textit{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) e_g$$

where $\{e_g\}$ is the basis of $\mathbb{C}\mathfrak{S}_d$. Then, we define

$$c_\lambda = a_\lambda \cdot b_\lambda = \sum_{g \in P_\lambda, h \in Q_\lambda} \operatorname{sgn}(h) e_{gh} \in \mathbb{C}\mathfrak{S}_d.$$

The c_λ is called Young projector.
Remark 2.3.7. *Clearly, $P_\lambda \cap Q_\lambda = \{1\}$. Each element $\sigma \in \mathfrak{S}_d$ can be expressed at most one way as $\sigma = pq$ where $p \in P_\lambda$ and $q \in Q_\lambda$. Therefore, $\sum_{g \in P_\lambda, h \in Q_\lambda} \operatorname{sgn}(h) e_{gh} \neq 0$ and $c_\lambda \neq 0$.*

Example 2.3.8. *Let*

$$T_\lambda = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}.$$

Then $P_{T_\lambda} = \langle (2,3) \rangle$, $Q_{T_\lambda} = \langle (1,3) \rangle$. *Therefore,*

$$a_\lambda = e_1 + e_{(2,3)}, \quad b_\lambda = e_1 - e_{(1,3)}.$$

$$c_\lambda = e_1 - e_{(1,3)} + e_{(2,3)} + e_{(2,3)(1,3)}.$$

Let $A = \mathbb{C}\mathfrak{S}_d$ be the group ring of $\mathbb{C}\mathfrak{S}_d$. Let $\lambda \vdash d$. Let T be the corresponding Young tableau of λ.

Lemma 2.3.9. *Let* $g \in \mathfrak{S}_d$. *Then*

$$a_\lambda e_g b_\lambda = \begin{cases} 0, & g \notin P_\lambda Q_\lambda, \\ \operatorname{sgn}(q) c_\lambda, & g = pq \in P_\lambda Q_\lambda, p \in P_\lambda, q \in Q_\lambda. \end{cases}$$

.

Proof. (i) Suppose $g \in P_\lambda Q_\lambda$. Then there exists a unique decomposition of g such that $g = pq, p \in P_\lambda, q \in Q_\lambda$. Therefore,

$$a_\lambda e_g b_\lambda = (a_\lambda e_p)(e_q b_\lambda) = \operatorname{sgn}(q) c_\lambda.$$

(ii) Suppose $g \notin P_\lambda Q_\lambda$. To show $a_\lambda e_g b_\lambda = 0$. All we need to show is that there exists a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\lambda$. Then

$$a_\lambda e_g b_\lambda = a_\lambda e_{tg} b_\lambda = a_\lambda e_g (e_{g^{-1}tg} b_\lambda) = -a_\lambda e_g b_\lambda.$$

Therefore, $a_\lambda e_g b_\lambda = 0$. Let $T' = gT$ be the tableau obtained by replacing the number i in each box of T by $g(i)$. We claim that there are two different integer in the same row of the tableau T and in the same column of T' and t is the transposition of these two integers. Define the column equivalence relation $\sim$ on Young diagram such that $T \sim T^*$ if T and T^* contain the same entries in the same columns. Since $tgT \sim T'$. It follows that $g^{-1}tgT \sim T$. Therefore, $g^{-1}tg \in Q_\lambda$.
Suppose such pair of numbers do not exist, then any two numbers in the first row of T are in different columns of T'. Therefore, there exist $q_1' \in Q_{T_\lambda'}, p_1 \in P_\lambda$ such that $p_1 T$ and $q_1' T'$ have the same first row. Clearly, we can keep doing this on the rest rows of tableau. Finally, we get $p_\alpha T = q_\beta' T'$ for some $p_\alpha \in P_\lambda, q_\beta' \in Q_{T_\lambda'}$. Then $p_\alpha T = q_\beta' g T$, $p_\alpha = q_\beta' g$. Therefore, $g = p_\alpha q_\alpha \in P_\lambda Q_\lambda$ where $q_\alpha = g^{-1}(q_\beta')^{-1} g \in Q_\lambda$. This is contract to $g \notin P_\lambda Q_\lambda$. □

We define the lexicographic ordering on the partitions $\mu, \lambda \vdash d$:

$$\lambda > \mu, \quad \text{if the first non-vanishing } \lambda_i - \mu_i > 0 \ .$$

Lemma 2.3.10. *If* $x \in A$, $\lambda > \mu$, *then* $a_\lambda x b_\mu = 0$. *In particular,* $c_\lambda c_\mu = 0$.

Proof. Let $g \in \mathfrak{S}_n$. Let T_μ be the corresponding Young tableau of μ. If there exists a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\mu$, then by the same argument in the proof of Lemma 2.3.9, $a_\lambda x b_\mu = 0$. Hence, we are done. We claim that there exist a pair of two numbers in the same row of T and in the same column of T_μ and t is the transposition of these two integers.
Because $\lambda > \mu$. Then there exists a the smallest positive m integer such that $\lambda_m - \mu_m > 0$ and $\lambda_j = \mu_j$ for $j \in \{1, 2, ..., m-1\}$. Then $\sum_{i=1}^m \lambda_i > \sum_{i=1}^m \mu_i$. In order to avoid such pair of integers existing, we must put elements of the same row of T_λ into different columns of T_μ. But $\lambda_m > \mu_m$. By pigeonhole principle, it is impossible to spread all the elements of the same row of T_λ into different columns of T_μ. Therefore, such pair of integers must exist. E.g.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \; & \; & \; & \; \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \; & \; \\ \hline \; & \; \\ \cline{1-2} \end{array}.$$

□

2.4 Specht Modules

Definition 2.4.1 (Specht module)**.** *Let $V_\lambda = Ac_\lambda$ which is a subspace of A. The module V_λ is called Specht module.*

Lemma 2.4.2. *We have $c_\lambda^2 = n_\lambda c_\lambda$, where $n_\lambda = \frac{d!}{\dim V_\lambda}$. Hence, c_λ is a scalar multiple of idempotent.*

Proof. Let $l : A \to \mathbb{C}$ be a $\mathbb{C}$-linear function such that

$$l(e_\sigma) = \begin{cases} 0, & \sigma \notin P_\lambda Q_\lambda \\ \operatorname{sgn}(q), & \sigma = pq \in P_\lambda Q_\lambda, \text{ for some } p \in P_\lambda, q \in Q_\lambda. \end{cases}$$

Then

$$\begin{aligned} l(b_\lambda a_\lambda) &= l\Big((\sum_{g \in Q_\lambda} \operatorname{sgn}(g) e_g)(\sum_{g' \in P_\lambda} e_{g'}) \Big) \\ &= \sum_{g \in Q_\lambda} \sum_{g' \in P_\lambda} l(\operatorname{sgn}(g) e_{gg'}) \\ &= \sum_{g \in Q_\lambda} \sum_{g' \in P_\lambda} \operatorname{sgn}(g)^2 \\ &= n_\lambda \quad \text{(for some positive integer } n_\lambda). \end{aligned}$$

By Lemma 2.3.9, $c_\lambda^2 = a_\lambda (b_\lambda a_\lambda) b_\lambda = l(b_\lambda a_\lambda) c_\lambda = n_\lambda c_\lambda$.
Let F be the $\mathbb{C}-$linear endomorphism given by the right multiplication by c_λ on A. From the above calculation we can see that F is multiplication by n_λ on V_λ and

zero on $\ker(c_\lambda)$. Therefore, all the eigenvalues η_i of F are either n_λ or 0. F can be viewed as a scalar multiple of "projection matrix". Then

$$\operatorname{tr}(F) = \sum_i \eta_i = n_\lambda \operatorname{rank}(F) = n_\lambda \dim(V_\lambda).$$

Only the identity of $\mathfrak{S}_d$ can preserve each row and each column of T_λ at same time. Therefore, $P_\lambda \cap Q_\lambda = \{1\}$, where 1 is the identity of $\mathfrak{S}_d$. Then

$$\operatorname{tr}(c_\lambda) = \operatorname{tr}\left(\sum_{g\in P_\lambda, h\in Q_\lambda} \operatorname{sgn}(h)e_{gh}\right) = \operatorname{tr}(e_1) + \operatorname{tr}\left(\sum_{\substack{g\in P_\lambda\setminus\{1\}\\ h\in Q_\lambda\setminus\{1\}}} \operatorname{sgn}(h)e_{gh}\right),$$

The character of identity in the regular representation is $|\mathfrak{S}_d|$. In the regular representation,

$$\operatorname{tr}\left(\sum_{\substack{g\in P_\lambda\setminus\{1\}\\ h\in Q_\lambda\setminus\{1\}}} \operatorname{sgn}(h)e_{gh}\right) = 0.$$

Therefore, the trace of c_λ in regular representation is $|\mathfrak{S}_d| = d!$. In conclusion, $n_\lambda = d!/\dim(V_\lambda)$. □

Proposition 2.4.3. *Let A be an algebra with a non-zero idempotent e and M be an arbitrary left A-module. Then* $\operatorname{Hom}_A(Ae, M) \cong eM$*.*

Proof. Let $f_1 : \operatorname{Hom}_A(Ae, M) \to eM$ such that $f_1(\alpha) = \alpha(e)$. Let $f_2 : eM \to \operatorname{Hom}_A(Ae, M)$ such that $f_2(em) : ae \mapsto aem$. f_1 and f_2 are mutually inverse. □

Theorem 2.4.4. *For each partition λ of d, V_λ is an irreducible representation of $\mathfrak{S}_d$. If $\mu \vdash d$ and $\lambda \neq \mu$, then V_λ and V_μ are not isomorphic.*

Proof. Without loss of generality, we may assume $\lambda \geqslant \mu$. By Proposition 2.4.3 we have that

$$\operatorname{Hom}_A(V_\lambda, V_\mu) \cong c_\lambda A c_\mu.$$

Suppose $\lambda > \mu$. By Lemma 2.3.10, $\operatorname{Hom}_A(V_\lambda, V_\mu) = 0$.
Suppose $\lambda = \mu$. By Lemma 2.4.2, $\operatorname{Hom}_A(V_\lambda, V_\mu) \cong \mathbb{C}c_\lambda$. A division algebra is a local algebra. By Proposition 1.3.10, it is indecomposable. V_λ is semisimple. Hence, it is irreducible.
In conclusion, V_λ is irreducible and $V_\lambda \cong V_\mu$ iff $\lambda = \mu$. By Corollary 2.1.9, the Specht modules V_λ exhaust all the irreducible representations of $\mathfrak{S}_d$. □

Definition 2.4.5 (Polytabloid)**.** *Let T_λ be a Young tableau. Then the corresponding polytabloid e_{T_λ} is*

$$e_{T_\lambda} = b_\lambda\{T_\lambda\}.$$

Example 2.4.6. *Suppose* T_λ *is* $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$. *Then*

$$Q_\lambda = \langle (1,4),(2,5)\rangle,$$

$$\{T_\lambda\} = \begin{array}{ccc}\hline 1&2&3\\\hline 4&5&\\\cline{1-2}\end{array},$$

$$b_\lambda = \sum_{g\in Q_\lambda} \operatorname{sgn}(g)e_g = e_1 - e_{(1,4)} - e_{(2,5)} + e_{(1,4)(2,5)}.$$

Therefore,

$$e_{T_\lambda} = \begin{array}{ccc}\hline 1&2&3\\\hline 4&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 4&2&3\\\hline 1&5&\\\cline{1-2}\end{array} - \begin{array}{ccc}\hline 1&5&3\\\hline 4&2&\\\cline{1-2}\end{array} + \begin{array}{ccc}\hline 4&5&3\\\hline 1&2&\\\cline{1-2}\end{array}.$$

Let $r_i(\{T_\lambda\})$ be the row index in which i appears in $\{T_\lambda\}$. We define a linear ordering on Young tabloids by saying that,
(i) if $\lambda > \mu$, then $\{T_\lambda\} > \{T_\mu\}$,
(ii) if

$$(r_d(\{T_\lambda\}), r_{d-1}(\{T_\lambda\}), \ldots) > \left(r_d\left(\{T'_\lambda\}\right), r_{d-1}\left(\{T'_\lambda\}\right), \ldots\right)$$

in the lexicographical order, then $\{T_\lambda\} > \{T'_\lambda\}$.

Example 2.4.7. *The linear ordering we defined above puts the* $(3,1)$*-shape Young tabloids in the following order:*

$$\begin{array}{ccc}\hline 1&2&3\\\hline 4&&\\\cline{1-1}\end{array} > \begin{array}{ccc}\hline 1&2&4\\\hline 3&&\\\cline{1-1}\end{array} > \begin{array}{ccc}\hline 1&3&4\\\hline 2&&\\\cline{1-1}\end{array} > \begin{array}{ccc}\hline 2&3&4\\\hline 1&&\\\cline{1-1}\end{array},$$

because

$$(2,1,1,1) > (1,2,1,1) > (1,1,2,1) > (1,1,1,2).$$

Proposition 2.4.8. *Suppose* T_λ *is a standard Young tableau, then for any non-identity* $q \in Q_\lambda$*, we have that*

$$\{q \cdot T_\lambda\} < \{T_\lambda\}.$$

Proof. The entries along each column of standard Young tableau are strictly increasing. Therefore, if an entry of $\{T\}$ is moved up by q, then a relatively smaller entry will be moved down by q. □

Proposition 2.4.9. *Let* G *be a finite group. The sum of squares of degrees of all distinct irreducible representations of* G *equals the order of* G*:*

$$\sum_i \dim^2(V_i) = |G|.$$

Proof. It was proved in Math5735 lecture. See [Du21]. □

Theorem 2.4.10. *The set $E_{T_\lambda} = \{e_{T_\lambda} : T_\lambda$ is a standard λ-tableau$\}$ forms a basis of V_λ.*

Proof. (i) To show the elements e_{T_λ} are linearly independent.
By the Example 2.4.6 and Proposition 2.4.8, we can easily see that each e_{T_λ} has form such that

$$e_{T_\lambda} = \{T_\lambda\} + \sum_{\substack{\text{some } \{T\},\\ \{T\}<\{T_\lambda\}}} \pm\{T\}.$$

Let $\{T'\}$ be the maximal Young tabloid. Consider the linear equation $\sum \beta_{T_\lambda} e_{T_\lambda}=0$ where each β_{T_λ} is the coefficient of this linear equation. There are no term in this linear equation that can cancel $e_{T'}$. Therefore, the coefficient of $e_{T'}$ must be 0. We can repeat this step on the second largest Young tabloid. Then all the coefficients are 0 inductively. This also shows that $\dim(V_\lambda) \geqslant f^\lambda$.
(ii) To show the elements in E_{T_λ} span V_λ.
By Corollary 2.2.3 and Proposition 2.4.9,

$$|\mathfrak{S}_d| = d! = \sum_\lambda \dim^2(V_\lambda) \geqslant \sum_\lambda (f^\lambda)^2 = d!.$$

Therefore, $\dim(V_\lambda)$ must equal to f^λ. The elements in E_{T_λ} span V_λ. □

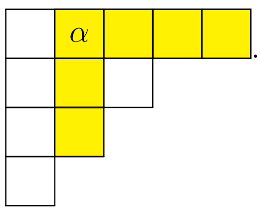
Proposition 2.4.11. *Every irreducible representation of $\mathfrak{S}_d$ is realizable over $\mathbb{Q}$.*

Proof. By Theorem 2.4.10, all the Specht modules can be constructed over $\mathbb{Q}$. □

Corollary 2.4.12. *$\mathbb{Q}$ is a splitting field of $\mathfrak{S}_d$.*

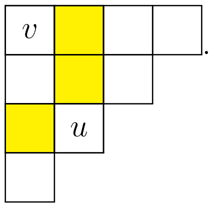
2.5 The Hook Length Formula

Now, we are going to compute the dimension of the Specht module V_λ.
Definition 2.5.1 (Hook)**.** *Let (i,j) be a box of Young diagram of shape λ. The (i,j) Hook is (i,j) and all boxes directly below or directly right of it. Hook length at (i,j) is the number of boxes in (i,j) Hook. We denote Hook length at (i,j) by $H_{(i,j)}$.*
Definition 2.5.2 (co-Hook)**.** *Let u be a box of Young diagram of shape λ. The co-Hook of u is a set of all boxes directly above and all boxes directly left of it. We denote co-Hook of u by* $\mathrm{coHook}(u)$. *Note that $u \notin \mathrm{coHook}(u)$.*
Example 2.5.3. *Consider the following Young tableau. Let $\alpha = (2,1)$. The Hook of α is marked with yellow,*



The Hook length of α $H_\alpha = 6$.

Example 2.5.4. *Consider the following Young tableau:*



The elements in coHook(u) *are marked with yellow and* coHook$(v) = \varnothing$*.*

There are many ways to prove the Hook length formula. The following probabilistic proof of the Hook length formula we gave is built on the paper [GNW79]. It is a very interesting proof.

Theorem 2.5.5 (Hook length formula)**.** *For a Specht module V_λ, the dimension of V_λ is given by*

$$\dim(V_\lambda) = \frac{d!}{\prod H_{(i,j)}},$$

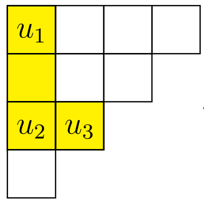
where the product is over all boxes (i, j) of Young tableau.

Proof. By Theorem 2.4.10, $\dim(V_\lambda) = f^\lambda$. Therefore, all we need is to show $f^\lambda = d!/\prod h_\lambda(i, j)$.
Fix a standard Young λ-tableau T_λ. The probability of picking an arbitrary Young tableau T from the set of all standard Young λ-tableaux which equals to T_λ is $P(T = T_\lambda) = (f^\lambda)^{-1}$. We assume every standard tableau has a same probability to be chosen. If we can prove $P(T = T_\lambda) = \prod H_{(i,j)}/d!$, then $\prod H_{(i,j)}/d! = (f^\lambda)^{-1}$ and hence we are done. Consider the following tableau

$$T' = \begin{array}{|c|c|c|} \hline a & d & f \\ \hline b & e & g \\ \hline c & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Suppose T' is standard, then $a < d < f, a < b < c$. For the same reason, we also have $d < f, d < e$. Then it is easy to see that a λ-tableau T_λ is standard iff each integer in the box (i, j) is the smallest integer in its Hook, and the largest number d can only appear in a corner box. If a box (i, j) has a Hook length 1, then we say it is a corner box. For example, the corner boxes of T' are boxes that are labelled with either g or c. Consider we generate a random standard Young tableau with uniform probability $(f^\lambda)^{-1}$ in the following way. We choose randomly a corner box (i, j) to put d, then we block this box and keep repeating this step (randomly choose a corner box (i', j') from the remaining to put $(d-1)$ then block it). The remaining question is how to choose the corner box.

Consider the following experiment: a box u in T_λ is random chosen with uniform probability $1/d$. Then another box $u' \neq u$ is chosen from the Hook of u with uniform probability $1/(H_u - 1)$. We repeat these steps until reaching a corner box v. The path u to v is called a Hook walk. E.g.,



.

The above picture shows a Hook walk from u_1 to u_3, $(u_1 \to u_2 \to u_3)$. Note that we can either go down or go right in each move.

There are 2 ways to block the corner box of T'. Therefore, $f^{(3,3,1)} = f^{(3,3)} + f^{(3,2,1)}$. In general, $f^\lambda = \sum_{\mu\uparrow\lambda} f^\mu$ where $\mu \uparrow \lambda$ means $\mu \vdash (d-1)$ and μ obtains by deleting a corner box of the Young λ-tableau. Define

$$F_\lambda = \frac{d!}{\prod_{(i,j)\in T_\lambda} H_{(i,j)}}.$$

We claim that the probability of a corner box v to be chosen in the above experiment is F_θ/F_λ where θ is the partition obtained by deleting corner box v from T_λ. Every trail stops at a corner box. Then, $1 = \sum_{\mu\uparrow\lambda} F_\mu/F_\lambda$. Therefore, $F_\lambda = \sum_{\mu\uparrow\lambda} F_\mu$. Then, $f^\lambda = F_\lambda$ follows inductively.

Now, we are going to prove the above claim.
Consider a Hook walk $P : u = (a,b) = (a_1,b_1) \to (a_2,b_2) \to \ldots \to (a_m,b_m) = (\alpha,\beta) = v$. Let $A = \{a_1,a_2,\ldots,a_m\}$ and $B = \{b_1,b_2,\ldots,b_m\}$ be the sets of vertical and horizontal projection of P.

Suppose T_λ contains only one row. Then it is easy to see that

$$\frac{F_\theta}{F_\lambda} = \frac{(d-1)!}{d!} \prod_{1\leqslant j<\beta} \frac{H_{(i,\beta)}}{H_{(i,\beta)}-1} = \frac{1}{d} \prod_{1\leqslant j<\beta} \frac{H_{(i,\beta)}}{H_{(i,\beta)}-1}.$$

Therefore, in general,

$$\begin{aligned}\frac{F_\theta}{F_\lambda} &= \frac{1}{d} \prod_{1\leqslant i<\alpha} \frac{H_{(i,\beta)}}{H_{(i,\beta)}-1} \prod_{1\leqslant j<\beta} \frac{H_{(\alpha,j)}}{H_{(\alpha,j)}-1} \\ &= \frac{1}{d} \prod_{1\leqslant i<\alpha} \left(1 + \frac{1}{H_{(i,\beta)}-1}\right) \prod_{1\leqslant j<\beta} \left(1 + \frac{1}{H_{(\alpha,j)}-1}\right) \qquad (2.5.1)\end{aligned}$$

We claim that the conditional probability of random trail starting at (a,b) has a vertical and horizontal projections A and B is that

$$P(A,B|a,b) = \prod_{\substack{i\in A\\ i\neq\alpha}} \frac{1}{H_{(i,\beta)}-1} \prod_{\substack{j\in B\\ j\neq\beta}} \frac{1}{H_{(j,\alpha)}-1}. \tag{2.5.2}$$

Denote the right hand side of (2.5.2) by $\prod(A,B)$. By the law of total probability,

$$P(A,B|a,b) = \frac{1}{H_{(a,b)}-1}\Big(P(A-a_1,B|a_2,b_1) + (P(A,B-b_1|a_1,b_2)\Big) \tag{2.5.3}$$

where $P(A-a_1,B|a_2,b_1)$ is the conditional probability given the first move which goes right and $(P(A,B-b_1|a_1,b_2)$ is the conditional probability given the first move which goes down.
Let prove equation (2.5.2) by induction on m.

Suppose $m=2$: Then $A=\{a_1,a_2\}$ and $B=\{b_1,b_2\}$. The Hook walk consists of only one move. Clearly, $P(A,B|a,b)=\frac{1}{H_{(a,b)}}$. Therefore, (2.5.2) holds when $m=2$.

Assume (2.5.2) holds for $m=k$ for any integer $k>2$. We want to show that it still holds when $m=k+1$. We may assume that,

$$P(A-a_1,B|a_2,b_1) = \prod_{\substack{i\in A-a_1\\ i\neq\alpha}} \frac{1}{H_{(i,\beta)}-1} \prod_{\substack{j\in B\\ j\neq\beta}} \frac{1}{H_{(j,\alpha)}-1} \quad = (H_{(a_1,\beta)}-1)\cdot\prod(A,B),$$

$$P(A,B-b_1|a_1,b_2) = (H_{(\alpha,b_1)}-1)\cdot\prod(A,B).$$

Note that

$$(H_{(a_1,\beta)}-1)+(H_{(\alpha,b_1)}-1) = (H_{(a,b)}-1).$$

Therefore,

$$\begin{aligned}
P(A,B|a,b) &= \frac{1}{H_{(a,b)}-1}\Big(P(A-a_1,B|a_2,b_1)+(P(A,B-b_1|a_1,b_2)\Big)\\
&= \frac{1}{H_{(a,b)}-1}\Big((H_{(\alpha,b_1)}-1)\cdot\prod(A,B)+(H_{(a_1,\beta)}-1)\cdot\prod(A,B)\Big)\\
&= \frac{H_{(a_1,\beta)}+H_{(\alpha,b_1)}-2}{H_{(a,b)}-1}\prod(A,B)\\
&= \prod(A,B).
\end{aligned}$$

$P(\alpha,\beta)$ can be computed by summing over all condition probability with respect to the first box given then summing over all vertical and horizontal projection of possible Hook walk. Hence,

$$P(\alpha,\beta) = \frac{1}{d}\sum P(A,B|a,b),$$

where the sum runs over all possible $A = \{a_1, a_2, .., \alpha\}$ and $B = \{b_1, b_2, .., \beta\}$ and $(a, b) = (a_1, b_1)$ is a random start. Recall $v = (\alpha, \beta)$. If we expand the right hand side of (2.5.1), then we have that

$$\begin{aligned}
&\frac{1}{d}\prod_{1\leqslant i<\alpha}\left(1+\frac{1}{H_{(i,\beta)}-1}\right)\prod_{1\leqslant j<\beta}\left(1+\frac{1}{H_{(\alpha,j)}-1}\right)\\
=&\frac{1}{d}\prod_{v'\in\mathrm{coHook}(v)}\left(1+\frac{1}{H_{v'}-1}\right)\\
=&\frac{1}{d}\left(1+\sum_{\mathcal{V}\subseteq\mathrm{coHook}(v)}\prod_{u\in\mathcal{V}}\frac{1}{H_u-1}\right)
\end{aligned}$$

Note that the equation (2.5.2) can be written as

$$P(A, B|a, b) = \prod_{(i,j)\in\mathcal{U}}\frac{1}{H_{(i,j)}-1},$$

for some $\mathcal{U} \subseteq \mathrm{coHook}(v)$. Therefore, $\frac{F_\theta}{F_\lambda} = P(\alpha, \beta)$. □

Example 2.5.6 (Sign representation of $\mathfrak{S}_d$)**.** *Let $d \geqslant 2$. Recall a sign representation of $\mathfrak{S}_d$ is that*

$$\rho_{\mathrm{sgn}}(g) = \mathrm{sgn}(g), \quad g \in \mathfrak{S}_d.$$

Clearly, the sign representation of $\mathfrak{S}_d$ is 1 dimensional. Let $\lambda = (1, 1, ..., 1, 1) \vdash d$. Then the corresponding Young tableau can be drawn as

$$T_\lambda = \begin{array}{|c|}\hline v_1\\\hline v_2\\\hline \end{array} \\ \vdots \\ \begin{array}{|c|}\hline v_{d-1}\\\hline v_d\\\hline\end{array},$$

where each $v_i \in \{1, 2, .., d\}$. If we label each box with its Hook length, then we can get the following picture

$$\begin{array}{|c|}\hline d\\\hline d-1\\\hline \end{array} \\ \vdots \\ \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}.$$

By Hook length formula, $\dim(V_\lambda) = \frac{d!}{d!} = 1$*. It is easy to see* $P_\lambda = \{1\}$ *and* $Q_\lambda = \mathfrak{S}_d$*. Then*

$$a_\lambda = e_1 \quad \textit{and} \quad b_\lambda = \sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g.$$

Therefore,

$$c_\lambda = \sum_{h \in P_\lambda, g \in Q_\lambda} \mathrm{sgn}(g) e_g = \sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g.$$

Let $h_1, h_2 \in \mathfrak{S}_d$ *such that* $\mathrm{sgn}(h_1) = 1$ *and* $\mathrm{sgn}(h_2) = -1$*. Then*

$$e_{h_1}\Big(\sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g\Big) = \sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_{h_1 g} = \sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g$$

and

$$e_{h_2}\Big(\sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g\Big) = \sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_{h_2 g} = -\Big(\sum_{g \in \mathfrak{S}_d} \mathrm{sgn}(g) e_g\Big).$$

Therefore, V_λ *is the sign representation of* $\mathfrak{S}_d$*.*

Chapter 3

Symmetric Functions

In order to compute the characters of the Specht modules, we need background on symmetric functions. In this chapter, we are going to introduce some basic knowledge of symmetric functions. In the next chapter, we will see how the Frobenius characteristic map builds a magic link between representation theory and symmetric functions.
For the further study of symmetric functions, the books [Mac95] and [Sta99] are good introductions to this area.

3.1 Basic Symmetric Polynomials

Let $x = (x_1, x_2, .., x_k)$. Consider the $\mathbb{C}[x]$ which is the polynomial ring in the variable x over $\mathbb{C}$. Define an action $\mathfrak{S}_d$ on $\mathbb{C}[x]$ such that

$$\sigma.f(x_1, x_2, ..., x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}),$$

where $f(x_1, x_2, ..., x_k) \in \mathbb{C}[x]$ and $\sigma \in \mathfrak{S}_d$.
Definition 3.1.1 (Symmetric polynomial). *A polynomial $p(x_1, x_2, \dots, x_k)$ is called a symmetric polynomial in k variables if it satisfies the condition*

$$p(x_1, x_2, \dots, x_k) = p(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}), \quad \textit{for all } \sigma \in \mathfrak{S}_k.$$

Recall the degree of a monomial $m = x_1^{a_1} x_2^{a_2} \cdot ... \cdot x_k^{a_k}$ is

$$\deg(m) = \sum_{i=1}^{k} a_i.$$

The set of all homogeneous symmetric functions in the variable x of degree n over $\mathbb{C}$ is denoted by $\Lambda^n_{\mathbb{C}}(x)$. We write $\Lambda^n_{\mathbb{C}}$ instead of $\Lambda^n_{\mathbb{C}}(x)$ for short. For $p(x) \in \Lambda_{\mathbb{C}}(x)$, we can write p instead of $p(x)$ for short. Suppose $f, g \in \Lambda^n_{\mathbb{C}}$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g \in \Lambda^n_{\mathbb{C}}$. Therefore, $\Lambda^n_{\mathbb{C}}$ is an $\mathbb{C}-$vector space. Define

$$\Lambda_{\mathbb{C}}(x) = \Lambda^0_{\mathbb{C}}(x) \oplus \Lambda^1_{\mathbb{C}}(x) \oplus \Lambda^2_{\mathbb{C}}(x) \oplus .... \quad .$$

If $f, g \in \Lambda_{\mathbb{C}}$,then $fg \in \Lambda_{\mathbb{C}}$. Therefore, $\Lambda_{\mathbb{C}}$ is a graded $\mathbb{C}-$algebra.

Definition 3.1.2 (Complete homogeneous symmetric polynomial)**.** *The p-th complete symmetric polynomial of k variables $x_1, x_2, \ldots, x_k$ is*

$$h_p(x_1, x_2, \ldots, x_k) = \sum_{\substack{l_1+l_2+\cdots+l_k=p,\\ l_i \geqslant 0}} x_1^{l_1} x_2^{l_2} \cdot \ldots \cdot x_k^{l_k}.$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of d. Then

$$h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdot \ldots \cdot h_{\lambda_k},$$

where h_{λ_i} is λ_i-th complete symmetric polynomial.

Example 3.1.3. *Let $x = (x_1, x_2)$ and $\lambda = (2, 1)$.*

$$\begin{aligned} &h_1(x) = x_1 + x_2, \\ &h_2(x) = x_1^2 + x_2^2 + x_1 x_2, \\ &h_\lambda(x) = h_1(x) \cdot h_2(x) = (x_1 + x_2)(x_1^2 + x_2^2 + x_1 x_2). \end{aligned}$$

Definition 3.1.4 (Monomial symmetric polynomial)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of d such that $k \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0$ and $n \leqslant k$. Then the monomial symmetric polynomial of k variables $x = (x_1, x_2, \ldots, x_k)$ corresponding to λ is*

$$m_\lambda(x) = \sum_{\{\alpha_1,\ldots,\alpha_n\} \subset \{1,\ldots,k\}} x_{\alpha_1}^{\lambda_1} \cdot x_{\alpha_2}^{\lambda_2} \cdot \ldots \cdot x_{\alpha_n}^{\lambda_n},$$

the sum is over all distinct monomials that can be obtained from $x_{\alpha_1}^{\lambda_1} \cdot x_{\alpha_2}^{\lambda_2} \cdot \ldots \cdot x_{\alpha_n}^{\lambda_n}$ by permuting the variables and $\alpha_i \neq \alpha_j$, if $i \neq j$.

Example 3.1.5.

$$\begin{aligned} &m_{(1,1)}(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3, \\ &m_{(k)}(x_1, x_2, \cdots, x_n) = x_1^k + x_2^k + \ldots + x_n^k. \end{aligned}$$

Clearly, if $\lambda \vdash d$, then $m_\lambda(x)$ is a homogeneous polynomial of degree d.

Definition 3.1.6 (Elementary symmetric polynomial)**.** *Let p be a integer such that $1 \leqslant p \leqslant n$. The elementary symmetric polynomials in k variables $x_1, \ldots, x_k$ are* *defined by*

$$\begin{aligned}
&e_0(x) = 1,\\
&e_1(x) = \sum_{i=1}^{n} x_i,\\
&e_2(x) = \sum_{1\leqslant i_1<i_2\leqslant n} x_{i_1}x_{i_2},\\
&\cdots\\
&e_p(x) = \sum_{1\leqslant i_1<\cdots<i_j\leqslant n} x_{i_1}\cdot \ldots \cdot x_{i_j},\\
&\cdots\\
&e_n(x) = x_1x_2\cdots x_k
\end{aligned}$$

$e_p(x)$ is called the p-th symmetric polynomial in $x_1,\ldots,x_k$.
Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of d. Then

$$e_\lambda(x) = e_{\lambda_1}(x)e_{\lambda_2}(x)\cdots e_{\lambda_k}(x).$$

Example 3.1.7. *Let $x = (x_1, x_2, x_3)$. Then*

$$\begin{aligned}
&e_{(1,1)}(x) = (x_1 + x_2 + x_3)^2,\\
&e_{(2,1)}(x) = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3).
\end{aligned}$$

Definition 3.1.8 (Classical definition of Schur polynomials)**.** *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $x = (x_1, \ldots, x_k)$. Then the Schur polynomial corresponding to λ is*

$$s_\lambda(x) = \frac{|x_j^{\lambda_i+k-i}|}{|x_j^{k-i}|}, \tag{3.1.1}$$

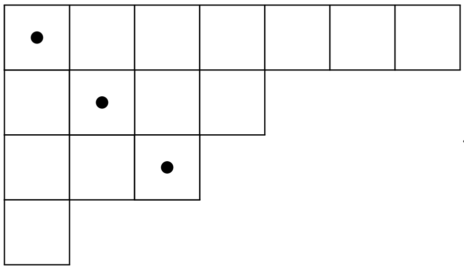
where

$$|x_j^{\lambda_i+k-i}| = \det\begin{pmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & \cdots & x_n^{\lambda_1+k-1}\\ x_1^{\lambda_2+k-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+k-2}\\ \vdots & & \ddots & \vdots\\ x_1^{\lambda_k} & x_2^{\lambda_k} & \cdots & x_k^{\lambda_k}\end{pmatrix},$$

$$|x_j^{k-i}| = \det\begin{pmatrix} x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1}\\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2}\\ \vdots & & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{pmatrix} = \prod_{1\leqslant j<i\leqslant k}(x_j - x_i).$$

Schur polynomials might be the most important among these symmetric polynomials. It is not quite obvious to see Schur polynomials are symmetric from the definition. We will give evidence it is actually symmetric below.
If we flip a Young diagram of shape λ along its main diagonal, we obtain another Young diagram with shape μ. We call μ is the conjugate partition of λ.

Example 3.1.9. *Let* $\lambda = (7, 4, 3, 1)$ *which is a partition of* 15*. Then the corresponding Young diagram* D_λ *is*



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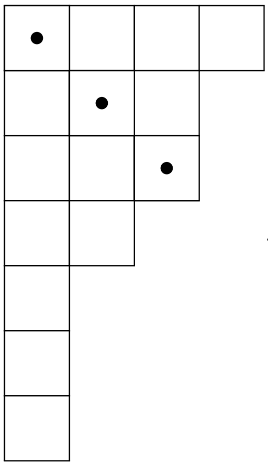
We flip D_λ *along its main diagonal. Then we get Young diagram* D_μ *that is*



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Therefore, $\mu = (4, 3, 3, 2, 1, 1, 1)$ *is the conjugate partition of* λ.

Theorem 3.1.10 (Jacobi-Trudy identity)**.** *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *be a partition of* d *and* $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ *be the conjugate partition to* λ*. Then*

$$s_\lambda = |h_{\lambda_i+j-i}| = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_1+k-2} \\ \vdots & & \ddots & \vdots \\ h_{\lambda-k+1} & \cdots & \cdots & h_{\lambda_k} \end{pmatrix},$$

$$s_\lambda = |e_{\mu_i+j-i}| = \begin{pmatrix} e_{\mu_1} & e_{\mu_1+1} & \cdots & e_{\mu_1+l-1} \\ e_{\mu_2-1} & e_{\mu_2} & \cdots & e_{\mu_1+l-2} \\ \vdots & & \ddots & \vdots \\ e_{\mu-l+1} & \cdots & \cdots & e_{\mu_l} \end{pmatrix}.$$

Proof. See [Sag01, Chapter 4.5]. □

Corollary 3.1.11. *Scuhr polynomials are symmetric and homogeneous.*

Proof. By the Jacobi-Trudy identity, a Schur polynomial can be expressed as a determinant in terms of the complete homogeneous symmetric polynomials. Determinant is a multi-linear function. Therefore, they are symmetric and homogeneous. □

Definition 3.1.12 (Weak composition)**.** *A weak composition of d is a sequence of non-negative integers such that*

$$\alpha = (\alpha_1, \ldots, \alpha_l)\,, \text{ with } \sum_{i=1}^{l} \alpha_i = d.$$

Let $A = (a_{i,j})_{i,j \geqslant 1}$ be an integer matrix with finite non-zero entries. We define the row sum vector $\text{row}(A)$ and column sum vector $\text{col}(A)$ by

$$\begin{aligned} \text{row}(A) &= (r_1, r_2, r_3, ...), \\ \text{col}(A) &= (c_1, c_2, c_3, ...), \end{aligned}$$

where $r_i = \sum_j a_{i,j}$ and $c_j = \sum_i a_{i,j}$. Let $\alpha = (\alpha_1, \alpha_2, ...)$ be a weak composition of d. We define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}....$

Proposition 3.1.13. *Let $\lambda \vdash d$, then*

$$h_\lambda = \sum_{\mu \vdash d} N_{\lambda\mu} m_\mu,$$

where $N_{\lambda\mu}$ is the number of $\mathbb{N}$-matrix A which satisfying $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$.

Proof. Let x^α be a term of $h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdot ....$ Then x^α is obtained by choosing a term $x_1^{\alpha_{i1}} x_2^{\alpha_{i2}}...$ form each h_{λ_i} such that

$$\prod_i (x_1^{\alpha_{i1}} x_2^{\alpha_{i2}}...) = x^\alpha,$$

which is exactly the same as choosing $(a_{i,j})$ be $\mathbb{N}$-matrix A which satisfying $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$. □

Proposition 3.1.14. *Let $\lambda, \mu \vdash d$, then $N_{\lambda\mu} = N_{\mu\lambda}$.*

Proof. A $\mathbb{N}$-matrix A satisfies $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$, iff transpose matrix A^T satisfies $\text{row}(A^T) = \mu$ and $\text{col}(A^T) = \lambda$. □

3.2 The Combinatorial Definition of Schur Polynomials

Let $X = (x_1, x_2, \ldots, x_k)$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash d$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. Therefore, $l(\lambda) \leqslant k$. Let F be an arbitrary symmetric polynomial of degree d in X. Let

$$\psi_\lambda(F) = [F]_\lambda, \tag{3.2.1}$$

where $[F]_\lambda$ denotes the coefficients of $X^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in P and

$$\omega_\lambda(F) = [\Delta \cdot F]_l \tag{3.2.2}$$

where $\Delta = \Delta(x) = \prod_{i<j}(x_i - x_j)$ and $l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k)$. F is a linear combination of symmetric monomials. Therefore, we have the following expression of F,

$$F = \sum_{\lambda \vdash d} \psi_\lambda(F) m_\lambda. \tag{3.2.3}$$

From the definition of Schur polynomials, it is easy to see that the coefficient of X^l in Δs_λ is 1. Therefore,

$$F = \sum_{\lambda \vdash d} \omega_\lambda(F) s_\lambda. \tag{3.2.4}$$

According to [Sag01], we define a semistandard Young tableau as following.

Definition 3.2.1 (semistandard Young tableau)**.** *A semistandard Young tableau (SSYT) is defined as a filling of a Young diagram such that the rows are weakly increasing and the columns are strictly increasing. Let*

$$\mathrm{SSYT}(\lambda) = \{T : T \textit{ is a semi-standard Young Tableau of shape } \lambda\}.$$

Definition 3.2.2 (Content)**.** *The content of an semistandard Young tableau T is the weak composition α where α_i is the number of occurrences of i in T. We denote the content of T by $\operatorname{co} T$.*

Example 3.2.3. *Let*

$$T_\lambda = \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

T_λ is a semi-standard Young tableau and $\operatorname{co} T_\lambda = (1, 1, 2, 1)$.

Definition 3.2.4 (Kostka numbers)**.** *Any two partitions μ, λ of k, the Kostka numbers $K_{\lambda\mu}$ is the number ways to fill the box of Young diagram of shape λ with μ_1 1's, μ_2 $2's$, $\ldots$ to get a semistandard Young tableau. That is*

$$K_{\lambda\mu} = \#\{T \in \mathrm{SSYT}(\lambda) : \operatorname{co} T = \mu\}.$$

Example 3.2.5. *Suppose $\lambda = (3,2)$, $\mu = (1,1,2,1)$. Then we need to use $1,2,3,3,4$ to fill the Young diagram of shape λ. Then*

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\ , \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$$

are the three semistandard Young tableaux. Therefore, $K_{\lambda\mu} = 3$.

Remark 3.2.6.

$$K_{\lambda\mu} = \begin{cases} 0, & \textit{if } \lambda < \mu; \\ 1, & \textit{if } \lambda = \mu. \end{cases}$$

If $\operatorname{co} T = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, then we define X^T by $X^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$.

In [Mac95, Chapter I, §5], it was shown that (3.1.1) could be expressed as a sum of monomials. Therefore, we have the following combinatorial definition of Schur polynomials.

Definition 3.2.7 (Combinatorial definition of Schur polynomials)**.** *The Schur polynomial corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is defined by*

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} X^T.$$

Proposition 3.2.8.

$$s_\lambda = \sum_{\mu \vdash d, \mu \leqslant \lambda} K_{\lambda\mu} m_\mu.$$

Proof. This proposition directly comes from the combinatorial definition of Schur polynomials. The coefficient of X^T in s_λ is the number of semistandard Young tableau which content is $\operatorname{co} T$. That is $K_{\lambda\mu}$ where $\mu = \operatorname{co} T$. □

Corollary 3.2.9.

$$K_{\lambda\mu} = \psi_\mu(S_\lambda).$$

Proposition 3.2.10. *Suppose $\lambda = (d)$, then*

$$s_{(d)} = h_d.$$

Proof. The corresponding T_λ just contains only one row and the integers in boxes of T_λ form a weakly increasing sequence. □

Corollary 3.2.11. *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash d$. Then*

$$h_\lambda = s_{(\lambda_1)} \cdot s_{(\lambda_2)} \cdot \ldots \cdot s_{(\lambda_k)}.$$

Proposition 3.2.12. *Suppose $\lambda = (1,1,1,\ldots,1) = (1^d) \vdash d$, then*

$$s_{(1^d)} = e_d.$$

Proof. The corresponding T_λ contains only one column, and the integers in boxes of T_λ form a strictly increasing sequence. $\square$

Theorem 3.2.13 (Cauchy identity)**.** *Let* $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k)$*. Then*

$$\prod_{i,j=1}^{k} \frac{1}{(1 - x_i y_j)} = \sum_{l(\lambda) \leqslant k} s_\lambda(x) s_\lambda(y),$$

where the sum runs over all partitions λ *of any non-negative integers such that* $l(\lambda) \leqslant k$*.*

Proof. See [Sag20, Chapter 7.5]. $\square$

Lemma 3.2.14. *Let* F *be an arbitrary symmetric polynomial of degree* d *in* X*. Then,*

$$\psi_\lambda(F) = \sum_{\mu \vdash d} K_{\mu\lambda} \omega_\mu(F).$$

Proof. By (3.2.3) and (3.2.4),

$$\begin{aligned} \sum_{\lambda \vdash d} \psi_\lambda(P) m_\lambda = P = \sum_{\mu \vdash d} \omega_\mu(P) s_\mu &= \sum_{\mu, \lambda \vdash d} \omega_\mu(P) K_{\mu\lambda} m_\lambda \\ &= \sum_{\lambda \vdash d} \left(\sum_{\mu \vdash d} K_{\mu\lambda} \omega_\mu(P) \right) m_\lambda. \end{aligned}$$

$\square$

3.3 Newton Polynomials

Let $x = (x_1, x_2, x_3, \ldots, x_k)$ and $y = (y_1, y_2, y_3, \ldots, y_k)$ be two sets of independent variables.

Definition 3.3.1 (Newton polynomial)**.** *For* $r \geqslant 1$*, the* r*-th Newton polynomial in* X *is*

$$P_r(x_1, \ldots, x_k) = x_1^r + \ldots + x_k^r.$$

Let $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ *be a* d*-tuple of non-negative integers with* $\sum_{\alpha=1}^{d} \alpha i_\alpha = d$*. Set*

$$P^{(\mathbf{i})} = P_1^{i_1} \cdot P_2^{i_2} \cdot \ldots \cdot P_d^{i_d},$$

where $P_j^{i_j} = (P_j)^{i_j}$*.*
Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *and* $n \leqslant k$*. Define* P_λ *by*

$$P_\lambda = P_{\lambda_1} P_{\lambda_2} \ldots P_{\lambda_n}.$$

Definition 3.3.2. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ *be a* d*-tuple of non-negative integers with* $\sum_{\alpha \geqslant 1} \alpha i_\alpha = d$*. Define*

$$z(\mathbf{i}) = i_1! 1^{i_1} i_2! 2^{i_2} \ldots i_d! d^{i_d},$$

$$\lambda_{\mathbf{i}} = (d^{i_d}, \ldots, 1^{i_1}).$$

Example 3.3.3. *Let* $\mathbf{i} = (d, 0, ..., 0)$*. Then* $\lambda_{\mathbf{i}} = (1, 1, ..., 1) \vdash d$*.*

Define $\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})})$. By (3.2.4),

$$P^{(\mathbf{i})} = \sum_{\lambda \vdash d} \omega_\lambda(\mathbf{i}) s_\lambda. \tag{3.3.1}$$

Lemma 3.3.4.

$$\prod_{i,j=1}^{k} \frac{1}{1 - x_i y_j} = \exp\left(\frac{1}{q} \sum_{q \geqslant 1} P_q(x) P_q(y)\right).$$

Proof. Note that $\ln\left(\frac{1}{1-z}\right) = \sum_{q \geqslant 1} \frac{z^q}{q}$. Then,

$$\begin{aligned}
\ln\left(\prod_{i,j=1}^{k} \frac{1}{1 - x_i y_i}\right) &= \sum_{i,j=1}^{k} \ln\left(\frac{1}{1 - x_i y_j}\right) \\
&= \sum_{i,j=1}^{k} (\sum_{q \geqslant 1} \frac{1}{q} x_i^q y_j^q) \\
&= \sum_{q \geqslant 1} \frac{1}{q} (\sum_{i \geqslant 1} x_i^q)(\sum_{j \geqslant 1} y_j^q) \\
&= \sum_{q \geqslant 1} \frac{1}{q} P_q(x) P_q(y).
\end{aligned}$$

□

Define

$$\delta_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda = \mu; \\ 0, & \text{otherwise.} \end{cases}$$

For any partition $\lambda = (d^{a_d}, (d-1)^{a_{d-1}}, \ldots, 1^{a_1}) \vdash d$, define

$$z_\lambda = d^{a_d} a_d! (d-1)^{a_{d-1}} a_{d-1}! \ldots \tag{3.3.2}$$

Example 3.3.5. *Suppose* $\lambda = (4, 4, 4, 2, 2)$*, then* $z_\lambda = 4^3 3! 2^2 2!$.

Lemma 3.3.6. *Let*

$$\mathcal{I}_d = \{\mathbf{i} = (i_1, i_2, ..., i_d) : \sum_{\alpha \geqslant 1} \alpha i_\alpha = d, l(\lambda_{\mathbf{i}}) \leqslant k\}.$$

Let $\lambda', \mu' \vdash d$ *and* $l(\lambda') \leqslant k, l(\mu') \leqslant k$*. Then*

$$\sum_{\mathbf{i} \in \mathcal{I}} \frac{1}{z(\mathbf{i})} \omega_{\lambda'}(\mathbf{i}) \omega_{\mu'}(\mathbf{i}) = \delta_{\lambda',\mu'}.$$

Proof. Define $\mathcal{I}_0 = \{\mathbf{i} = (0)\}$. Let $\mathcal{J} = \cup_{d\geqslant 0}\mathcal{I}_d$. Note that $e^x = \sum_{k\geqslant 0}\frac{x^k}{k!}$. By the Cauchy identity and Lemma 3.3.4, we have that

$$\begin{aligned}
\sum_{l(\lambda)\leqslant k} s_\lambda(x)s_\lambda(y) &= \prod_{i,j=1}^{k}\frac{1}{1-x_iy_i} \\
&= \prod_{j\geqslant 1}\exp\left(\frac{1}{j}P_j(x)P_j(y)\right) \\
&= \prod_{j\geqslant 1}\sum_{q\geqslant 0}\frac{P_j^q(x)P_j^q(y)}{q!j^q} \\
&= \sum_{\mathbf{i}\in\mathcal{J}}\frac{1}{z(\mathbf{i})}P^{(\mathbf{i})}(x)P^{(\mathbf{i})}(y) \\
&= \sum_{\mathbf{i}\in\mathcal{J}}\frac{1}{z(\mathbf{i})}\big(\sum_{l(\lambda)\leqslant k}\omega_\lambda(\mathbf{i})s_\lambda(x)\big)\big(\sum_{l(\mu)\leqslant k}\omega_\mu(\mathbf{i})s_\mu(y)\big) \qquad \text{by (3.3.1).}
\end{aligned}$$

□

Proposition 3.3.7.

$$h_d(x) = \sum_{\lambda\vdash d, l(\lambda)\leqslant k} z_\lambda^{-1}P_\lambda(x).$$

Proof. Define $H(t) = \sum_{d=0}^{\infty} h_d(x)t^d$. Clearly,

$$H(t) = \sum_{d_1,\ldots,d_k\geqslant 0} x_1^{d_1}\cdot\ldots\cdot x_k^{d_k}t^{d_1+\ldots+d_k} = \prod_{i=1}^{k}\frac{1}{1-x_it}.$$

Let $y = (t,0,0,\ldots,0)$. Note that

$$\prod_{j\geqslant 1}\exp\left(\frac{1}{j}P_j(x)P_j(y)\right) = \sum_{l(\lambda)\leqslant k}\frac{1}{z_\lambda}P_\lambda(x)P_\lambda(y).$$

By the proof of Lemma 3.3.6,

$$\begin{aligned}
\sum_{l(\lambda)\leqslant k} s_\lambda(x)s_\lambda(t) &= H(t) \\
&= \sum_{d\geqslant 0} h_d(x)t^d \\
&= \sum_{\lambda\vdash d, d\geqslant 0, l(\lambda)\leqslant k}\frac{1}{z_\lambda}P_\lambda(x)t^d.
\end{aligned}$$

Therefore,

$$h_d(x) = \sum_{\lambda\vdash d, l(\lambda)\leqslant k} z_\lambda^{-1}P_\lambda(x).$$

□

Chapter 4

The Frobenius's Formula and the Frobenius Characteristic Map

In this chapter, we will introduce Frobenius's formula and the Frobenius characteristic map. The famous Frobenius's formula which is introduced by G. Frobenius is used to compute the characters of irreducible representations of the symmetric groups. The Frobenius characteristic map gives a bijection between $\mathfrak{S}_d$ and the space of homogeneous symmetric functions of degree d.

4.1 The Frobenius's Formula

Now we are going to calculate the character of the Specht module V_λ. Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash d$ with $k \leqslant d$. Let $x = (x_1, \dots, x_k)$ where $x_1, \dots, x_k$ are independent variables. Define the power sum $P_j(x)$ and the discriminant $\Delta(x)$ by

$$P_j(x) = \sum_{i=1}^{k} x_i^j, \quad 1 \leqslant j \leqslant d,$$

$$\Delta(x) = \prod_{i<j} (x_i - x_j).$$

The $\Delta(x)$ is also called the Vandermonde determinant.

Definition 4.1.1. *Define*

$$[f(x)]_{(l_1,\dots,l_k)} = \textit{coefficient of } x_1^{l_1} \cdots x_k^{l_k} \textit{ in } f.$$

where $f(x) = f(x_1, ..., x_k)$ which is a polynomial in x and $(l_1, ..., l_k)$ is a $k-$tuple of non-negative integers.

Let $\mathbf{i}$ be a sequence of non-negative integers such that $\mathbf{i} = (i_1, i_2, \dots, i_d), \sum_{\alpha=1}^{d} \alpha i_\alpha = d$. Let $C_\mathbf{i}$ be the conjugacy class in $\mathfrak{S}_d$ determined by $\mathbf{i}$. $C_\mathbf{i}$ consists of all the permutation in $\mathfrak{S}_d$ that have i_1 1-cycles, i_2 2-cycles,..., i_d d-cycles.

Theorem 4.1.2 (Frobenius's formula)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash d$ and $g \in \mathfrak{S}_d$. Set $l_1 = \lambda_1 + k - 1, \quad l_2 = \lambda_2 + k - 2, \dots, l_k = \lambda_k$. The the character of V_λ evaluated on*

$g \in C_{\mathbf{i}}$ *is given by the formula*

$$\chi_\lambda(g) = \chi_\lambda(C_{\mathbf{i}}) = [\Delta(x) \prod_{j=1}^{d} P_j(x)^{i_j}]_{(l_1,\ldots,l_k)}.$$

This section aims to give a proof of Frobenius's formula.

Definition 4.1.3 (Young subgroup). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash d$. Then the corresponding Young subgroup of $\mathfrak{S}_d$ is*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{d-\lambda_k+1,\ldots,d\}}.$$

Lemma 4.1.4.

$$|C_{\mathbf{i}}| = \frac{d!}{\prod_{j\geqslant 1} j^{i_j} i_j!}.$$

Proof. Note that $C_{\mathbf{i}} = \{g \in \mathfrak{S}_d : \forall j \in \{1,\ldots,d\},\ g \text{ has } i_j\ j\text{-cycles}\}$. There are $d!$ ways to arrange $1,\ldots,d$. For each $j-$cycles, there are j ways to express the same permutation. (E.g., $(1,2,3) = (2,3,1) = (3,1,2)$). There are i_j $j-$cycles. The order we listed the cycles doesn't matter. (e.g (1,2)(3,4)(5)(6)=(3,4)(1,2)(6)(5)). Therefore,

$$|C_{\mathbf{i}}| = \frac{d!}{\prod_{j\geqslant 1} j^{i_j} i_j!}.$$

□

Lemma 4.1.5.

$$|C_{\mathbf{i}} \cap \mathfrak{S}_\lambda| = \sum \prod_{p=1}^{k} \frac{\lambda_p!}{\prod_{j=1}^{d} j^{r_{(p,j)}} r_{(p,j)}!},$$

where the sum runs over all collections of non-negative integer $(c_{(p,j)})_{\substack{1\leqslant p\leqslant k\\ 1\leqslant j\leqslant d}}$ satisfying

$$\begin{aligned} &r_{(p,1)} + 2r_{(p,2)} + \cdots + dr_{(p,d)} = \lambda_p, \\ &r_{(1,q)} + r_{(2,q)} + \cdots + r_{(k,q)} = i_q, \quad q \in \{1,2,\ldots,d\}. \end{aligned}$$

Proof. Note that $\mathfrak{S}_\lambda \cong \oplus_{p=1}^{k} \mathfrak{S}_{\lambda_p}$. Therefore, the cardinality of $\mathfrak{S}_\lambda$ is $\prod_{p=1}^{k} \lambda_p!$. Let T_λ be the Yong tableau with shape λ. We fill the box of T_λ with integers 1 to d in ascending order from left to right and top to bottom. Then $\mathfrak{S}_\lambda = \{g \in \mathfrak{S}_d : g \text{ fixes tabloid}\{T_\lambda\}\}$. Let $r_{(p,j)}$ be the number of cycles of length j whose elements lie in the p row of T_λ. Therefore, the number of elements in $C_{\mathbf{i}}$ which fix $\{T_\lambda\}$ is

$$\sum \prod_{p=1}^{k} \frac{\lambda_p!}{\prod_{j=1}^{d} j^{r_{(p,j)}} r_{(p,j)}!}.$$

□

Let $U_\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_d} \mathbb{1}$ where $\mathbb{1}$ is the trivial representation of $\mathfrak{S}_\lambda$. By Remark 1.3.3, $U_\lambda = \mathbb{C}[\mathfrak{S}_d/\mathfrak{S}_\lambda]$.
Let $\psi_\lambda = \chi_{U_\lambda}$ be the character of U_λ.

Theorem 4.1.6.

$$\psi_\lambda(C_{\mathbf{i}}) = [\prod_{j=1}^{d} P(x)^{i_j}]_\lambda. \tag{4.1.1}$$

Proof. By Lemma 4.1.4 and Lemma 4.1.5, we have

$$|C_{\mathbf{i}}| = \frac{d!}{\prod_{j\geqslant 1} j^{i_j} i_j!},$$

$$|C_{\mathbf{i}} \cap \mathfrak{S}_d| = \sum \prod_{p=1}^{k} \frac{\lambda_p!}{\prod_{j=1}^{d} j^{r_{(p,j)}} r_{(p,j)}!},$$

where the sum runs over all collections of non-negative integer $\{r_{(p,j)}\}_{\substack{1\leqslant p\leqslant k\\ 1\leqslant j\leqslant d}}$ satisfying

$$\begin{aligned} &r_{(p,1)} + 2r_{(p,2)} + \cdots + dr_{(p,d)} = \lambda_p, \\ &r_{(1,q)} + r_{(2,q)} + \cdots + r_{(k,q)} = i_q, \quad q \in \{1, 2, \ldots, d\}. \end{aligned}$$

By (1.3.2), we have

$$\psi(C_{\mathbf{i}}) = \frac{[\mathfrak{S}_d : \mathfrak{S}_\lambda]}{|C_{\mathbf{i}}|} |C_{\mathbf{i}} \cap \mathfrak{S}_\lambda| = \frac{\prod_{j=1}^{d} j^{i_j} i_j!}{d!} \cdot \frac{d!}{\prod_{p=1}^{k} \lambda_p} \cdot \sum \prod_{p=1}^{k} \frac{\lambda_p!}{\prod_{j=1}^{d} j^{r_{(p,j)}} r_{(p,j)}!}.$$

Simplifying, then we get

$$\psi_\lambda(C_{\mathbf{i}}) = \sum_{r_{(p,j)}} \prod_{j=1}^{d} \frac{i_j!}{\prod_{p=1}^{k} r_{(p,j)}!}. \tag{4.1.2}$$

$$\prod_{j=1}^{d} P(x)^{i_j} = (x_1 + ... + x_k)^{i_1} \cdot (x_1^2 + ... + x_k^2)^{i_2} \cdot ... \cdot (x_1^d + ... + x_k^d)^{i_d}$$

Fix $q \in \{1, \ldots, d\}$, we have the multinomial expansion

$$(x_1^q + ... + x_k^q)^{i_q} = \sum_{(r_{(p,q)})_{1\leqslant p\leqslant d}} \frac{i_q!}{r_{(1,q)}! \cdot ... \cdot r_{(k,q)}!} x_1^{jr_{(1,q)}} \cdots x_k^{jr_{(k,q)}},$$

where $(r_{(p,q)})$ is collection of non-negative integers satisfying

$$r_{(1,q)} + r_{(2,q)} + ... + r_{(k,q)} = i_q, \quad q \in \{1, 2, ..., d\}.$$

Then, it is easy to see (5.0.2) is exactly the coefficient of $x^\lambda = x_1^{\lambda_1} \cdot ... \cdot x_k^{\lambda_k}$ in $\prod_{j=1}^{d} P(x)^{i_j}$. □

Let $\mu \vdash d$.

Definition 4.1.7. *Let $C_{\mathbf{i}}$ be the conjugacy class in $\mathfrak{S}_d$ with cycle type $\mathbf{i}$. Then we define $\omega_\lambda(\mathbf{i})$ by*

$$\omega_\lambda(\mathbf{i}) = [\Delta \cdot P^{(\mathbf{i})}]_l, \quad l = (\lambda_1 + k - 1, \lambda, \lambda_2 + k - 2, ..., \lambda_k).$$

Let P be an arbitrary symmetric polynomial of degree d in x. By Lemma 3.2.14, we have that

$$[P]_\lambda = \sum_{\mu \vdash d} K_{\mu\lambda}[\Delta \cdot P]_{(\mu_1+k-1,\mu_2+k-2,...,\mu_k)}.$$

Recall $K_{\lambda\lambda} = 1$. Therefore,

$$\psi_\lambda(C_{\mathbf{i}}) = \sum_{\mu \vdash d} K_{\mu\lambda}\omega_\mu(\mathbf{i}) = \omega_\lambda(\mathbf{i}) + \sum_{\mu \vdash d, \mu > \lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}). \tag{4.1.3}$$

By Lemma 3.3.6 and Lemma 4.1.4, we have

$$\sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot ... \cdot d^{i_d} i_d!} \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \delta_{\lambda,\mu}$$

Then

$$\frac{1}{d!} \sum_{\mathbf{i}} |C_{\mathbf{i}}| \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i}) = \delta_{\lambda,\mu}. \tag{4.1.4}$$

This result shows ω_λ which is a function on the conjugacy class of $\mathfrak{S}_d$ satisfy orthonormal relations. Therefore, ω_λ must be an irreducible characters of $\mathfrak{S}_d$.

Proposition 4.1.8. *$U_\lambda \cong Aa_\lambda$ as $\mathbb{C}\mathfrak{S}_d$-module.*

Proof. Define $\phi : U_\lambda = \mathbb{C}[\mathfrak{S}_d/\mathfrak{S}_\lambda] \to Aa_\lambda, \sigma\mathfrak{S}_\lambda \mapsto \sigma a_\lambda$ for all $\sigma \in \mathfrak{S}_d$. We can easily check that ϕ actually is a module isomorphism. $\square$

Theorem 4.1.9. *Let χ_λ be the character of the Specht module V_λ. Then for all conjugacy class $C_{\mathbf{i}}$ of $\mathfrak{S}_d$,*

$$\chi_\lambda(C_{\mathbf{i}}) = \omega_\lambda(\mathbf{i}).$$

Proof. By the previous proposition, $U_\lambda \cong Aa_\lambda$. There is a surjection $\phi : Aa_\lambda \to V_\lambda, x \mapsto xb_\lambda$. Therefore, U_λ whose character is ψ_λ contains V_λ. It implies

$$\psi_\lambda = \sum_\mu n_{\lambda,\mu}\chi_\mu, \qquad n_{\lambda,\lambda} \geqslant 1, \text{ all } n_{\lambda,\mu} \geqslant 0. \tag{4.1.5}$$

Recall (4.1.3),

$$\psi_\lambda(C_{\mathbf{i}})) = \sum_{\mu \vdash d, \mu \geqslant \lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}).$$

Consider (4.1.5) and (4.1.3), each ω_λ must be a virtual character. Therefore, we can write

$$\omega_\lambda = \sum_\mu m_{\lambda,\mu}\chi_\mu,$$

for some $m_{\lambda,\mu} \in \mathbb{Z}$. We have already shown that ω_λ are orthonormal. Therefore, $\langle \omega_\lambda, \omega_\lambda \rangle = 1 = \sum_\mu m_{\lambda,\mu}^2$. Therefore, ω_λ is $\pm\chi$ for some irreducible character χ. Fix λ. Since $K_{\lambda\lambda} = 1$ and

$$\sum_{\mu \geqslant \lambda} K_{\mu\lambda}\omega_\mu(\mathbf{i}) = \sum_{\mu \geqslant \lambda} n_{\mu\lambda}\chi_\mu.$$

By the linear independence of characters, we have that $\chi_\lambda = \omega_\lambda$ and $\sum_{\mu>\lambda}\chi_\mu = \sum_{\mu>\lambda}\omega_\mu$. Therefore,

$$\psi_\lambda = \omega_\lambda + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu.$$

Compare the above equation with (4.1.5) and all characters are linear independence. Then χ_μ must equal to ω_μ for all $\mu > \lambda$. □

Corollary 4.1.10 (Young's rule)**.** *The Kostka number $K_{\mu\lambda}$ is the multiplicity of irreducible representation V_μ in the induced representation U_λ:*

$$U_\lambda \cong \bigoplus_{\mu \geqslant \lambda} V_\mu^{\oplus K_{\mu\lambda}}, \qquad \psi_\lambda = \sum_{\mu \geqslant \lambda} K_{\mu\lambda}\chi_\mu.$$

Corollary 4.1.11.

$$\dim(V_\lambda) = \frac{d!}{l_1! \cdot \ldots \cdot l_k!}\prod_{i<j}(l_i - l_j)$$

Proof. The identity permutation 1 in $\mathfrak{S}_d$ has form $1 = (1)(2)\cdots(d)$. There are d 1-cycles in 1. Therefore, $\mathbf{i} = (d, 0, ..., 0)$ and $C_\mathbf{i}$ is the conjugacy class of identity in $\mathfrak{S}_d$. By Frobenius's formula,

$$\dim(V_\lambda) = \chi_\lambda(C_\mathbf{i}) = [\Delta(x)(x_1 + \cdots + x_k)^d]_{(l_1,\ldots,l_k)},$$

where $l_i = \lambda_1 + k - i$ for $i = 1, ..., k$ and $\Delta(x)$ is the Vandermonde determinant:

$$\Delta(x) = \begin{vmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) x_k^{\sigma(1)-1} \cdot \ldots \cdot x_1^{\sigma(k)-1}.$$

By multinomial expansion,

$$(x_1 + \ldots + x_k)^d = \sum \frac{d!}{r_1! \cdot \ldots \cdot r_k!} x_1^{r_1} \cdot \ldots \cdot x_k^{r_k},$$

where the sum is over all k-tuples of non-negative integers $(r_1, \ldots, r_k)$ such that $\sum_{i=1}^{k} r_i = d$. Therefore,

$$[\Delta(x)(x_1 + \cdots + x_k)^d]_{(l_1,\ldots,l_k)} = \sum \operatorname{sgn}(\sigma) \frac{d!}{(l_1 - \sigma(k) + 1)! \cdot \ldots \cdot (l_k - \sigma(1) + 1)!}$$

where the sum goes over $\sigma \in \mathfrak{S}_k$ such that $l_{k-i+1} - \sigma(i) + 1 > 0$ for all $1 \leqslant i \leqslant k$. Thus,

$$\begin{aligned}
&[\Delta(x)(x_1 + \cdots + x_k)^d]_{(l_1,\ldots,l_k)} \\
=&\sum \operatorname{sgn}(\sigma) \frac{d!}{(l_1 - \sigma(k) + 1)! \cdot \ldots \cdot (l_k - \sigma(1) + 1)!} \\
=&\frac{d!}{\prod_{j=1}^{k} l_j} \sum_{\sigma \in \mathfrak{S}_k} [l_1(l_1 - 1) \cdot \ldots \cdot (l_1 - \sigma(k) + 2)] \cdots [l_k \cdot (l_k - 1) \cdot \ldots \cdot (l_k - \sigma(1) + 2)] \\
=&\frac{d!}{l_1! \cdot \ldots \cdot l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k - 1) & \cdots & \prod_{i=0}^{k-2}(l_k - i) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & l_1 & l_1(l_1 - 1) & \cdots & \prod_{i=0}^{k-2}(l_1 - i) \end{vmatrix} \\
=&\frac{d!}{l_1! \cdot \ldots \cdot l_k!} \begin{vmatrix} 1 & l_k & l_k^2 & \cdots & l_k^{k-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & l_1 & l_1^2 & \cdots & l_1^{k-2} \end{vmatrix} \qquad \text{(column reduction)} \\
=&\frac{d!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j).
\end{aligned}$$

□

Corollary 4.1.11 suggests another method to prove the Hook length formula. If Young tableau T_λ only contain one column, then the hook lengths of the boxes of T_λ are $l_1, ..., l_d$. Therefore, we can prove that

$$\frac{d!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j) = \frac{d!}{\prod_{(i,j) \in T_\lambda} H_{(i,j)}}$$

by induction on the number of columns that T_λ contains.

Example 4.1.12. *Consider $\mathfrak{S}_4$. Let $\lambda = (2, 2) \vdash 4$. Let $\mathbf{i} = (4, 0, 0, 0)$. Then*

$$\begin{aligned}
\chi_\lambda(C_{\mathbf{i}}) &= [(x_1 - x_2)(x_1 + x_2)^4]_{(3,2)} \\
&= [x_1^5 + 3x_1^4 x_2 + 2x_1^3 x_2^2 - 2x_1^2 x_2^3 - 3x_1 x_2^4 - x_2^5]_{(3,2)} \\
&= 2.
\end{aligned}$$

Corollary 4.1.13. *Each character of $\mathfrak{S}_d$ has values in $\mathbb{Z}$.*

Proof. This is an immediate conclusion of the Frobenius's formula. □

4.2 A Scalar Product

Let $\mathbf{x} = \{x_1, x_2, ...\}$ be a set of infinitely independent variables. Let $\Lambda^n = \Lambda^n_{\mathbb{C}}(\mathbf{x})$ and $\Lambda = \bigoplus_{i \geqslant 0} \Lambda^i$. Let $\mathrm{par}(n) = \#\{\lambda : \lambda \vdash n\}$.

Proposition 4.2.1. *The space Λ^n has basis $\{m_\lambda : \lambda \vdash n\}$, where $m_\lambda = m_\lambda(\mathbf{x})$. And so $\dim(\Lambda^n) = \mathrm{par}(n)$.*

Proof. Clearly, m_λ are linearly independent. All we need to show is that m_λ spans Λ^n.

Let $f \in \Lambda^n$ and $\lambda = (\lambda_1, ..., \lambda_k) \vdash n$. Suppose $x_1^{\lambda_1} x_2^{\lambda_2} ... x_k^{\lambda_k}$ appears in f with coefficient c. Then $f - cm_{(\lambda_1, \lambda_2, .., \lambda_k)}$ is still symmetric. We can keep doing this until reaching zero. □

Proposition 4.2.2. *The set $\{s_\lambda : \lambda \vdash n\}$ forms a basis of Λ^n.*

Proof. The cardinality of this set is $\mathrm{par}(n)$. Therefore, all we need to show is that s_λ are linearly independent. Let $C = (c_{\lambda,\mu})$ be the matrix expressing s_μ in terms of m_λ. Suppose we can find a matrix C with an ordering partition such that C is upper triangular and all the diagonal elements are non-zero then C^{-1} exists and s_λ are also basis. By Proposition 3.2.8, $s_\lambda = \sum_{\mu \vdash n, \mu \leqslant \lambda} K_{\lambda\mu} m_\mu$ and $K_{\lambda\lambda} = 1$. Therefore, the set $\{s_\lambda : \lambda \vdash n\}$ forms a basis of Λ^n. □

Proposition 4.2.3. *The set $\{h_\lambda : \lambda \vdash n\}$ forms a basis of Λ^n.*

Proof. The cardinality of this set is $\mathrm{par}(n)$. By the Jacobi-Trudy identity, the set $\{h_\lambda : \lambda \vdash n\}$ can generate the set $\{s_\lambda : \lambda \vdash n\}$. □

Proposition 4.2.4. *The set $\{P_\lambda : \lambda \vdash n\}$ forms a basis of Λ^n.*

Proof. The cardinality of this set is $\mathrm{par}(n)$. By Proposition 3.3.7, the set $\{P_\lambda : \lambda \vdash n\}$ can generate the set $\{h_\lambda : \lambda \vdash n\}$. □

Definition 4.2.5 (Scalar product)**.** *Let V be a complex vector space. A scalar product is a sesquilinear form $V \times V \to \mathbb{C}$, which is denoted by $\langle , \rangle$. Let $\{u_j\}$ and $\{v_j\}$ be the bases of V. Then a scalar product on V is uniquely determined by the values of $\langle u_i, v_i \rangle$. In particular, if*

$$\langle u_j, v_i \rangle = \delta_{i,j}, \text{ for all } i, j,$$

($\delta_{i,j}$ is Kronecker delta), then we say $\{u_j\}$ and $\{v_j\}$ are the dual bases of V.

We now define a scalar product on Λ such that $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual bases on Λ by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}.$$

Proposition 4.2.6. *The Scalar product on Λ is Hermitian symmetry. i.e., $\langle f, g \rangle = \overline{\langle g, f \rangle}$, where $f, g \in \Lambda$.*

Proof. All we need to show is that $\langle h_\lambda, h_\mu \rangle = \langle h_\mu, h_\lambda \rangle$. By Proposition 3.1.13,

$$\langle h_\lambda, h_\mu \rangle = \langle \sum_{v \vdash d} N_{\lambda v} m_v, h_\mu \rangle = \langle N_{\lambda\mu} m_\mu, h_\mu \rangle = N_{\lambda\mu} = N_{\mu\lambda} = \langle h_\mu, h_\lambda \rangle.$$

□

Proposition 4.2.7. *The Scalar product on Λ is positive definite, i.e., $\langle f, g \rangle > 0$ for all non-zero $f \in \Lambda$.*

Proof. See [Sta99, Corollary 7.9.4]. □

Therefore, the scalar product $\langle , \rangle$ on Λ we defined actually is an inner product.

Proposition 4.2.8. *The set $\{s_\lambda : \lambda \vdash n\}$ is an orthonormal basis for Λ^n, i.e. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.*

Proof. See [Sta99, Corollary 7.12.2]. □

Proposition 4.2.9. *Let $\mu \vdash n$. Then*

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda,$$

Proof. $\{s_\lambda : \lambda \vdash n\}$ is the basis of Λ^n. Therefore, $h_\mu = \sum_\lambda a_{\lambda\mu} s_\lambda$ where $a_{\lambda\mu}$ is unknown coefficient. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Then, $\langle h_\lambda, s_\mu \rangle = a_{\lambda\mu}$. By the definition of scalar product, $\langle h_\mu, m_v \rangle = \delta_{\mu,v}$. By Proposition 3.2.8, $s_\lambda = \sum K_{\lambda\mu} m_\mu$. Therefore, $\langle h_\lambda, s_\mu \rangle = K_{\lambda\mu} = a_{\lambda\mu}$. □

4.3 The Frobenius Characteristic Map

Let $\mathrm{CF}^d = \mathrm{CF}(\mathfrak{S}_d)$ be the space of class function on $\mathfrak{S}_d$. Let $\lambda = (\lambda_1, ..., \lambda_k) \vdash d$. Define

$$P_\lambda = \prod_{i=1}^{k} P_{\lambda_i},$$

where P_{λ_i} is Newton polynomial of degree λ_i in $\mathbf{x}$.

Example 4.3.1.

$$P_{(2,1)} = (x_1^2 + x_2^2 + ...)(x_1 + x_2 + ...).$$

Definition 4.3.2 (Frobenius characteristic map)**.** *The Frobenius characteristic map* $\mathrm{ch} : \mathrm{CF}^d \to \Lambda^d$ *is defined by*

$$\mathrm{ch}(\chi) = \sum_{\mu \vdash d} z_\mu^{-1} \chi_\mu P_\mu,$$

where χ_μ is the value of χ on the class μ and z_μ is defined in (3.3.2).

Let G be any finite group and $\mathcal{A}$ be any algebra over $\mathbb{C}$. Consider the bilinear form $G \times G \to \mathcal{A}$ such that
$$\langle \psi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\phi(g^{-1}),$$
where $\psi, \phi : G \to \mathcal{A}$. If $\mathcal{A} = \Lambda^d$, then we have the following. Note that g and g^{-1} are in the same conjugacy class in $\mathfrak{S}_d$. Therefore,
$$\mathrm{ch}(\chi) = \sum_{\mu \vdash d} z_\mu^{-1} \chi_\mu P_\mu = \frac{1}{n!} \sum_{g \in \mathfrak{S}_d} \chi(g) P_{c(g)} = \langle \chi, \Psi \rangle,$$
where c maps g to its cycle type and $c(g) = c(g^{-1})$. Define $\Psi(g) = P_{c(g)}$. Clearly, the Frobenius characteristic map ch is a linear map.
Let $\chi_{\mathbb{1}_{\mathfrak{S}_d}}$ be the character of trivial representation of $\mathfrak{S}_d$.
Proposition 4.3.3.
$$\mathrm{ch}(\chi_{\mathbb{1}_{\mathfrak{S}_d}}) = h_d$$

Proof. By Proposition 3.3.7, $\mathrm{ch}(\chi_{\mathbb{1}_{\mathfrak{S}_d}}) = \sum_{\mu \vdash d} z_\mu^{-1} P_\mu = h_d$. □

Let $f \in \mathrm{CF}(\mathfrak{S}_d) = \mathrm{CF}^d$ and $g \in \mathrm{CF}(\mathfrak{S}_m) = \mathrm{CF}^m$. Define the pointwise product $f \times g \in \mathrm{CF}(\mathfrak{S}_d \times \mathfrak{S}_m)$ by
$$(f \times g)(u, v) = f(u)g(v).$$
Suppose V_f and V_g are the representations of $\mathfrak{S}_d$ and $\mathfrak{S}_m$ and f and g are the character of V_f and V_g, then $f \times g$ is the character of the tensor representation $V_f \otimes V_g$. Define induction product
$$V_f \circ V_g = \mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} (V_f \otimes V_g), \quad f \circ g = \mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} (f \times g).$$
Let $\mathrm{CF} = \bigoplus_{i \geqslant 0} \mathrm{CF}^i$. Extend scalar product on CF^d to CF by setting $\langle f, g \rangle = 0$, if $f \in \mathrm{CF}^d, g \in \mathrm{CF}^m, m \neq d$.
Proposition 4.3.4. *$\{P_\lambda\}$ is an orthogonal basis of Λ, i.e.*
$$\langle P_\lambda, P_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

Proof. The proof can be found in [Sta99, Proposition 7.9.3]. □

Proposition 4.3.5. *The linear map* ch *is an isometry, i.e.*
$$\langle f', g' \rangle_{\mathrm{CF}^d} = \langle \mathrm{ch}\, f', \mathrm{ch}\, g' \rangle_{\Lambda^d} \quad .$$

Proof.

$$\begin{aligned}
\langle \operatorname{ch} f', \operatorname{ch} g' \rangle &= \left\langle \sum_{\lambda \vdash d} z_\lambda^{-1} f'(\lambda) P_\lambda, \sum_{\mu \vdash d} z_\mu^{-1} g'(\mu) P_\mu \right\rangle \\
&= \sum_{\lambda \vdash d} z_\lambda^{-1} f'(\lambda) g'(\lambda) \qquad \text{(Proposition 4.3.4)} \\
&= \langle f', g' \rangle.
\end{aligned}$$

□

Proposition 4.3.6. *The Frobenius characteristic map* ch : CF → Λ *is a ring isomorphism and satisfies.*

$$\operatorname{ch}(f \circ g) = (\operatorname{ch} f)(\operatorname{ch} g)$$

Proof.

$$\begin{aligned}
\operatorname{ch}(f+g) &= \langle f+g, \Psi \rangle = \langle f, \Psi \rangle + \langle g, \Psi \rangle = \operatorname{ch}(f) + \operatorname{ch}(g). \\
\operatorname{ch}(f \circ g) &= \operatorname{ch}(\operatorname{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} (f \times g)) \\
&= \langle \operatorname{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} (f \times g), \Psi \rangle \\
&= \langle f \times g, \operatorname{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{d+m}} \Psi \rangle_{\mathfrak{S}_d \times \mathfrak{S}_m} \quad \text{(Frobenius Reciprocity)} \\
&= \frac{1}{d!m!} \sum_{v \in \mathfrak{S}_d} \sum_{u \in \mathfrak{S}_m} f(v) g(u) \Psi(vu) \\
&= \frac{1}{d!m!} \sum_{v \in \mathfrak{S}_d} \sum_{u \in \mathfrak{S}_m} f(v) g(u) \Psi(v) \Psi(u) \\
&= \left(\frac{1}{d!} \sum_{v \in \mathfrak{S}_d} f(v) \Psi(v) \right) \left(\frac{1}{m!} \sum_{u \in \mathfrak{S}_m} f(u) \Psi(u) \right) \\
&= \langle f, \Psi \rangle_{\mathfrak{S}_d} \langle g, \Psi \rangle_{\mathfrak{S}_m} \\
&= (\operatorname{ch} f)(\operatorname{ch} g).
\end{aligned}$$

The linear map ch is an isometry. Therefore, it is injective. The collection of all Newton polynomials forms a basis of Λ. Therefore, it is bijective. □

Recall ψ_λ is the character of U_λ.

Proposition 4.3.7.

$$\operatorname{ch}(\psi_\lambda) = h_\lambda$$

Proof.

$$\begin{aligned}
\operatorname{ch}(\psi_\lambda) &= \operatorname{ch}(\psi_{\lambda_1} \circ \psi_{\lambda_2} \circ \ldots \circ \psi_{\lambda_k}) \\
&= \operatorname{ch}(\psi_{\lambda_1}) \operatorname{ch}(\psi_{\lambda_2}) \ldots \operatorname{ch}(\psi_{\lambda_k}) \\
&= h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_k} \qquad \text{(Proposition 4.3.3)} \\
&= h_\lambda.
\end{aligned}$$

□

Proposition 4.3.8.

$$\mathrm{ch}(\chi_\lambda) = s_\lambda$$

Proof. By Young's rule, $\psi_\lambda = \sum_\mu K_{\mu\lambda}\chi_\mu$. Apply the Frobenius characteristic map to the both side of the equation, $h_\lambda = \sum_\mu K_{\mu\lambda}\,\mathrm{ch}(\chi_\mu)$. $\{\chi_\mu\}$ is linearly independent. The Frobenius characteristic map is isomorphic. Therefore, $\{\mathrm{ch}(\chi_\mu)\}$ is linearly independent. Compare the previous equation with Proposition 4.2.9, $\mathrm{ch}(\chi_\mu)$ must equal to s_μ. □

Corollary 4.3.9. *Suppose χ_λ is the character of irreducible representation V_λ. Then*

$$\chi_\lambda = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma)\psi_{(\lambda_1+\sigma(1)-1,\lambda_2+\sigma(2)-2,\ldots,\lambda_k+\sigma(k)-k)},$$

where $\psi_{(\lambda_1+\sigma(1)-1,\lambda_2+\sigma(2)-2,\ldots,\lambda_k+\sigma(k)-k)} = 0$, if any $\lambda_i + \sigma(i) - i < 0$.

Proof. By the Jacobi-Trudy identity, we have that $s_\lambda = |h_{\lambda_i+j-i}|$. Apply the inverse map of the Frobenius characteristic map to the both side of the equation, $\chi_\lambda = |\psi_{\lambda_i+j-i}|$. By Leibniz formula of determinant,

$$\chi_\lambda = |\psi_{\lambda_i+j-i}| = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma)\psi_{(\lambda_1+\sigma(1)-1,\lambda_2+\sigma(2)-2,\ldots,\lambda_k+\sigma(k)-k)}.$$

□

4.4 The Littlewood-Richardson Rule

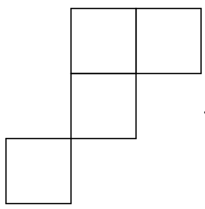
Quoting from [Ale], we define skew Young tableaux, standard skew Young tableaux and semi-standard Young tableaux as follows.

Definition 4.4.1 (Skew Young tableau). *Consider two partitions, $\lambda \vdash (d+m)$ and $\mu \vdash d$. With those partitions one can associate Young tableau denoted by T_λ and T_μ. Suppose that each box of T_μ is also a box of T_λ. The set-difference $T_{\lambda/\mu}$ contains exactly m cells. It is called a skew Young tableau.*

Definition 4.4.2 (Standard skew Young tableau). *If a skew Young tableau is filled with integers from 1 to d in increasing order in each row and in each column, then it is called a standard skew Young tableau.*

Definition 4.4.3 (Semi-standard skew Young tableau). *If a skew Young tableau is filled with integers from 1 to m, the rows are only non-decreasing, and the columns are increasing, then it is called a semi-standard skew Young tableau.*

Example 4.4.4. *If $\lambda = (3, 2, 1)$ and $\mu = (1, 1)$, then the corresponding Skew Young tableau $T_{\lambda/\mu}$ can be drawn as*



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Definition 4.4.5 (Lattice word)**.** *A lattice word is a sequence composed of positive integers, in which every prefix contains at least as many positive integers i as integers $i+1$.*

Example 4.4.6. *The sequence $(1,1,1,2,2)$ is a lattice word, but $(1,2,1,2,2,1,1)$ is not a lattice word, since the sub-sequence $(1,2,1,2,2)$ contains more two's than one's.*

Example 4.4.7. *Let $\lambda = (4,3,2)$ and $\mu = (2,1)$.*

$$T_{\lambda/\mu} = \begin{array}{cccc} & & \boxed{1} & \boxed{2} \\ & \boxed{3} & \boxed{4} & \\ \boxed{5} & \boxed{6} & & \end{array}, \qquad T'_{\lambda/\mu} = \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{2} & \boxed{2} & \\ \boxed{3} & \boxed{3} & & \end{array}.$$

$T_{\lambda/\mu}$ is a standard skew Young tableau and $T'_{\lambda/\mu}$ is a semi-standard skew Young tableau.

Definition 4.4.8 (Reading word)**.** *Giving a Young tableau T_λ, read the rows from bottom to top and left to right within a row to form the reading word.*

According to [Ale], we can define a Littlewood–Richardson skew tableaux as follows.

Definition 4.4.9 (Littlewood–Richardson (skew) tableaux)**.** *A Littlewood–Richardson (skew) tableau is a skew semistandard tableau with the additional property that the reading word is a lattice word.*

Example 4.4.10. *Let*

$$T = \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{2} & \boxed{2} & \\ \boxed{1} & \boxed{3} & & \end{array}.$$

The reading word of T is $(1,3,2,2,1,1)$ which is a lattice word. Therefore, T is a Littlewood–Richardson tableau.

Definition 4.4.11 (Littlewood-Richardson number)**.** *The Littlewood-Richardson number $c^{\nu}_{\lambda\mu}$ count the number of Littlewood-Richardson skew tableaux of shape ν/λ with content μ.*

Example 4.4.12. *If $\nu = (5,4,2)$, $\lambda = (3,2)$ and $\mu = (3,2,1)$, then the Littlewood-Richardson number $c^{\nu}_{\lambda\mu} = 2$. The two Littlewood–Richardson tableaux are drawn as*

$$\begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & \boxed{2} & \boxed{2} & \\ \boxed{1} & \boxed{3} & & & \end{array}, \qquad \begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & \boxed{1} & \boxed{2} & \\ \boxed{2} & \boxed{3} & & & \end{array}.$$

Our combinatorial definition of Schur polynomials also makes sense in defining skew Schur polynomials.

Definition 4.4.13 (skew Schur polynomial)**.** *The skew Schur polynomial $s_{\nu/\lambda}$ is defined as*

$$s_{\nu/\lambda} = \sum_{T \in \mathrm{SSYT}(\nu/\lambda)} X^T.$$

Theorem 4.4.14 (Littlewood-Richardson Rule)**.** *Let $\lambda \vdash d$, $\mu \vdash m$. Then*

$$s_\lambda s_\mu = \sum_{\nu \vdash (d+m)} c^\nu_{\lambda\mu} s_\nu$$
$$s_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} s_\mu.$$

Proof. See [Sag01, Chapter 4.9]. □

4.5 Applications of the Littlewood–Richardson Rule

Theorem 4.5.1. *If $\mu \vdash m$, then*

$$V_\lambda \circ V_\mu = \mathrm{Ind}^{\mathfrak{S}_{d+m}}_{\mathfrak{S}_d \times \mathfrak{S}_m} \left(V_\lambda \otimes V_\mu\right) \cong \bigoplus_{\nu \vdash (d+m)} V_\nu^{\oplus c^\nu_{\lambda\mu}}.$$

In particular, taking $m = 1$ and V_μ is trivial, gives $\mathrm{Ind}^{\mathfrak{S}_{d+1}}_{\mathfrak{S}_d} V_\lambda = \oplus_\nu V_\nu$, where the sum the sum over all ν whose Young diagram is obtained from that of Young diagram of shape λ by adding one box.

Proof. By Theorem 4.4.14,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$
$$\mathrm{ch}^{-1}(s_\lambda s_\mu) = \mathrm{ch}^{-1}(\sum_\nu c^\nu_{\lambda\mu} V_\nu)$$
$$\chi_\lambda \circ \chi_\mu = \sum_\nu c^\nu_{\lambda\mu} \chi_\nu$$
$$V_\lambda \circ V_\mu \cong \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}.$$

□

Corollary 4.5.2. *If $\nu \vdash (d+m)$, then*

$$\mathrm{Res}^{\mathfrak{S}_{d+m}}_{\mathfrak{S}_d \times \mathfrak{S}_m} V_\nu \cong \bigoplus_{\lambda \vdash d, \mu \vdash m} (V_\lambda \otimes V_\mu)^{\oplus c^\nu_{\lambda\mu}},$$

In particular, taking $m = 1$ and V_μ is trivial, gives $\operatorname{Ind}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+1}} V_\lambda = \oplus_\nu V_\nu$, where the sum goes over all ν whose Young diagram is obtained from that of λ by removing one box.

Proof. By Frobenius reciprocity,

$$\begin{aligned}\dim\left(\operatorname{Hom}_{\mathbb{C}[\mathfrak{S}_{d+m}]}(V_\lambda \circ V_\mu, V_\nu)\right) &= \dim\left(\operatorname{Hom}_{\mathbb{C}[\mathfrak{S}_d \times \mathfrak{S}_m]}(V_\lambda \otimes V_\mu, \operatorname{Res}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} V_\nu)\right) \\ &= c_{\lambda\mu}^\nu.\end{aligned}$$

By definition, external tensor product is bilinear. Both V_λ and V_μ are irreducible. Then $V_\lambda \otimes V_\mu$ is indecomposable. Hence, $V_\lambda \otimes V_\mu$ is irreducible. Therefore, $c_{\lambda\mu}^\nu$ is the multiplicity of $V_\lambda \otimes V_\mu$ in $\operatorname{Res}_{\mathfrak{S}_d \times \mathfrak{S}_m}^{\mathfrak{S}_{d+m}} V_\nu$. □

If we define $\operatorname{ch}(s_{\nu/\lambda}) = \chi_{\nu/\lambda}$, then the corresponding representation of $\chi_{\nu/\lambda}$ is $V_{\nu/\lambda}$. The skew Specht module $V_{\nu/\lambda}$ is called the skew representation of $\mathfrak{S}_m$, where $m = \sum_i \lambda_i - \sum_j \mu_j$.

By Theorem 4.4.14, we may have the following result.

Proposition 4.5.3.

$$V_{\nu/\lambda} = \bigoplus_\mu V_\mu^{\oplus c_{\lambda\mu}^\nu}.$$

Therefore, we can decompose a skew representation $V_{\nu/\lambda}$ into direct sum through irreducible representations V_μ through the Littlewood-Richardson rule.

Chapter 5

Conclusion

5.1 Summary

Now, we should have a good understanding of the irreducible representations of $\mathfrak{S}_d$ over the field with characteristic 0.
In Chapter 2, we have seen that all irreducible representations of $\mathfrak{S}_d$ can be constructed by Specht modules V_λ and we can use the Hook length formula to calculate the $\dim(V_\lambda)$.
In chapter 3, we gave a brief introduction to symmetric functions.
In chapter 4, we proved Frobenius's formula which is the formula to calculate the characters of V_λ and we introduced the Frobenius characteristic map, which is a bridge between representation theory and symmetric functions.

5.2 Future Work

The representation theory of symmetric groups is a very big area. There is a lot of future work I would like to do. Define an infinite symmetric group $\mathfrak{S}_\infty$ by

$$\mathfrak{S}_\infty = \bigcup_{d \geqslant 1} \mathfrak{S}_d.$$

The representation theory of $\mathfrak{S}_\infty$ is an interesting topic to study.
I also would like to study the modular representation theory of $\mathfrak{S}_d$.

Bibliography

[Ale] P. Alexandersson. *The symmetric functions catalog*. Online. URL: `https://www.math.upenn.edu/~peal/polynomials/`.

[DF04] D.S. Dummit and R.M. Foote. *Abstract algebra*. eng. Hoboken, NJ, 2004.

[Du21] J. Du. *MATH5735 Modules and Representation Theory Lecture Notes*. UNSW, 2021.

[Ful96] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996. DOI: `10.1017/CBO9780511626241`.

[GNW79] C. Greene, A. Nijenhuis, and H.S. Wilf. "A probabilistic proof of a formula for the number of Young tableaux of a given shape". eng. In: *Advances in mathematics (New York. 1965)* 31.1 (1979), pp. 104–109. ISSN: 0001-8708.

[HF91] J. Harris and W. Fulton. *Representation theory : a first course*. eng. Graduate texts in mathematics ; 129. New York: Springer-Verlag, 1991. ISBN: 0387974954.

[Hor91] R.A. Horn. *Topics in matrix analysis*. eng. Cambridge ; New York, 1991.

[Jam78] G.D. James. *The Representation Theory of the Symmetric Groups*. eng. Lecture Notes in Mathematics, 682. Berlin, Heidelberg: Springer Berlin Heidelberg : Imprint: Springer, 1978. ISBN: 3540357114.

[JK81] G.D. James and A. Kerber. *The Representation Theory of the Symmetric Group*. 1st. Encyclopedia of mathematics and its applications 16. Section, Algebra. Addison-Wesley Pub. Co., Advanced Book Program, 1981. ISBN: 9780201135152.

[Lam98] T.Y. Lam. "Representations of finite groups : a hundred years, part I". In: *Notices of the American Mathematical Society* 45 (1998), pp. 361–372.

[Mac95] I.G. Macdonald. *Symmetric functions and Hall polynomials*. eng. 2nd ed. Oxford mathematical monographs. Oxford : New York: Clarendon Press ; Oxford University Press, 1995. ISBN: 0198534892.

[Sag01] B.E. Sagan. *The symmetric group : representations, combinatorial algorithms, and symmetric functions*. eng. New York, 2001.

[Sag20] B.E. Sagan. *Combinatorics: The Art of Counting*. Graduate Studies in Mathematics. American Mathematical Society, 2020. ISBN: 9781470460327. URL: `https://books.google.com.au/books?id=DYgEEAAAQBAJ`.

[Sta99] R.P. Stanley. *Enumerative combinatorics. Volume 2*. eng. Cambridge studies in advanced mathematics ; 62. Cambridge: Cambridge University Press, 1999. ISBN: 0511609582.

Frequently Used Notation

$\mathfrak{S}_d$ symmetric group of order d. 1
CF(G) The set of all class functions on G. 4
$\vdash$ is a partition of for integers. 7
$\mathcal{P}(n)$ the set of all partitions of n. 7
$l(\lambda)$ the length of λ. 7
f^λ the number of standard Young tableau of shape λ. 11
$\{T_\lambda\}$ tabloid of T_λ. 12
V_λ Specht module corresponding to λ. 14
e_{T_λ} polytabloid corresponding to T_λ. 15
$\Lambda^n_{\mathbb{C}}(x)$ the set of all homogeneous symmetric function in the variable x of degree n over $\mathbb{C}$. 23
h_λ complete homogeneous symmetric polynomial associated with λ. 24
m_λ monomial symmetric polynomial associated with λ. 24
e_λ elementary symmetric polynomial associated with λ. 25
s_λ Schur polynomial associated with λ. 25, 29
$\Delta(x)$ Vandermonde determinant. 28, 33
$K_{\lambda\mu}$ Kostka number associated with λ, μ. 28
P_λ Newton polynomial associated with λ. 30
$\delta_{\lambda,\mu}$ Kronecker delta. 31
$\mathfrak{S}_\lambda$ Young subgroup associated with λ. 34
χ_λ the character of V_λ. 36
ch Frobenius characteristic map. 41
$\circ$ induction product. 41
$c^\nu_{\lambda\mu}$ Littlewood-Richardson number. 44

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