1.

For each of the four graphs on the attached pages plot sketches of the function which is their derivative at every point. (Please sketch on top of the original graph and hand these in as your solution to this question.)

2.

The decay of a radio-active material can be modelled by the following differential equation, $q'(t) = -k \times q(t)$,

where $q(t)$ is the quantity of the material (in grams, say) that is present at time t. Assuming that $k = 2.0$ and that at $t = t_0 = 0$ we know that $q(t_0) = 10.0$, take **four** steps of Euler's method (with dt = 0.25) to estimate $q(1)$ (i.e. q(t) when $t = 1.0$).

Answer: $q(t_1) =$, $t_1 =$

$q(t_2) =$, $t_2 =$

$q(t_3) =$, $t_3 =$

$q(t_4) =$, $t_4 =$

3.

Let $y(t)$ satisfy the following differential equation and initial condition:

$$y'(t) = y^3 + t^2; \qquad y(1) = -1.$$

Take **three** steps of Euler's method (with d$t = \frac{1}{3}$) to estimate $y(2)$ (i.e. the solution when $t = 2.0$).

Answer: $y(t_1) =$, $t_1 =$

$y(t_2) =$, $t_2 =$

$y(t_3) =$, $t_3 =$

4.

Use two steps of the midpoint rule, with dt = 0.5, to estimate the solution of the problem in the previous question (i.e. the same equation and initial condition) at $t = 2.0$.

Answer: k = , *temp* = in step 1

$y(t_1) =$, $t_1 =$

k = , *temp* = in step 2

$y(t_2) =$, $t_2 =$

5. As shown in lectures, the Euler algorithm applies even in the case of vector equations (as does the midpoint rule but this is not considered in this question):

$$\underline{y}'(t) = \underline{f}(t, \underline{y}).$$

Suppose $\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ then, to solve the system

$$\underline{y}'(t) = \mathbf{A}\underline{y}, \qquad \text{subject to } \underline{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where $A = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$, we can apply Euler's method with:

$$f(t,y) = \mathbf{A}y\,.$$

This gives the following implementation of Euler's method at each step:

$$y^{k+1} = y^k + \mathrm{d}t \times \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix} y^k$$

$$t_{k+1} = t_k + \mathrm{d}t\,.$$

Writing this in component form, and using Python notation, this gives the following at each step:

$$\begin{aligned} y1[k+1] &= y1[k] + \mathrm{d}t \times (-2 \times y1[k] + y2[k]) \\ y2[k+1] &= y2[k] + \mathrm{d}t \times (y1[k] - 4 \times y2[k]) \\ t[k+1] &= t[k] + \mathrm{d}t\,. \end{aligned}$$

Hence take **two** steps of Euler's method, with $\mathrm{d}t = 0.5$, to approximate $y(1)$ (i.e. the solution at $t = 1.0$).

Answer: $y_1(t_1) =$, and
$y_2(t_1) =$, $t_1 =$
$y_1(t_2) =$, and
$y_2(t_2) =$, $t_2 =$

6.

For this question, you may wish to submit figures with your graphical analysis in addition to your written answers to the questions.

Let $y(t) = 4e^{t-1} - t^2 - 2t - 2$ (where e, the base of the natural logarithm, is a constant equal to 2.7182818284...). The function $y(t)$ satisfies the differential equation:

$$y'(t) = y + t^2\,.$$

It is easy to see that for $t = 1$, $y(1) = -1$. We will use this as our initial condition.

(a) Write a Python script, using the Euler method function provided to you for numerical integration, to numerically integrate this differential equation over 10 seconds, to a final time of $t = 11$. Extend your code to include an error analysis. Note that you will need to make some judgements in writing the code (in particular, choosing a time step and which errors might be expedient to compute.)

Hints:

(i) Use the formula given for $y(t)$ to calculate errors.

(ii) Write your code to conveniently compare different choices of $\mathrm{d}t$.

(iii) You might wish to include graphical analysis of the numerical errors, plotting the relevant errors as a function of time (note for Python, you would need to import matplotlib to generate the figures. Alternatively, you might prefer to export the errors and plot them with your favourite plotting tool). This is not required, but will help you to answer the following questions.

(b) Based on your analysis, describe the behaviour of the errors after a long integration time ($t \gg \mathrm{d}t$ and in this case $t \gg 2$). How do the errors obtained with different choices of $\mathrm{d}t$ compare?

(c) Specifying $\mathrm{d}t$ to one significant figure only, determine the most efficient choice of $\mathrm{d}t$ that should provide an error of at most 1% at $t = 11$.

The following optional questions are for extension only, and carry no marks.

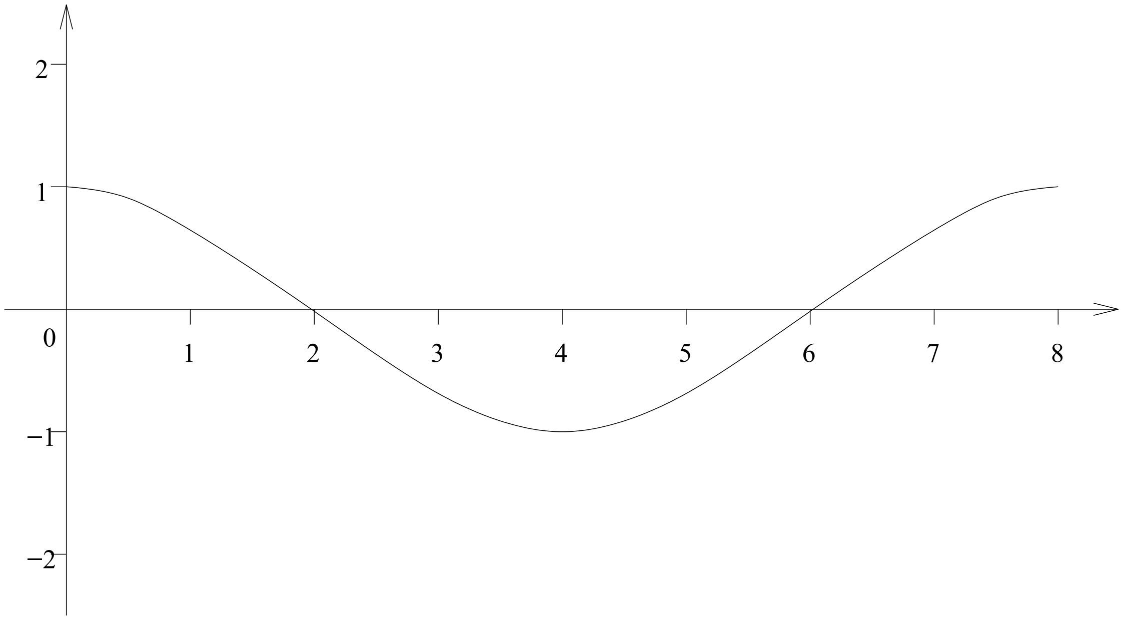
(d) **Optional:** How does the behaviour of the absolute and relative errors change for $t < 2$, $t = 2$ and $t > 2$ and why?

(e) **Optional:** Discuss the implicatons of the choice of $\mathrm{d}t$ you made in item (c). Consider the following points: (i) The criterion used: this was an accuracy requirement on the solution at $t = 11$; (ii) The computational cost. How would you expect this cost to change if you halved the time step? What if you doubled the time step? How does the error scale at times $t < 2$, $t = 11$?

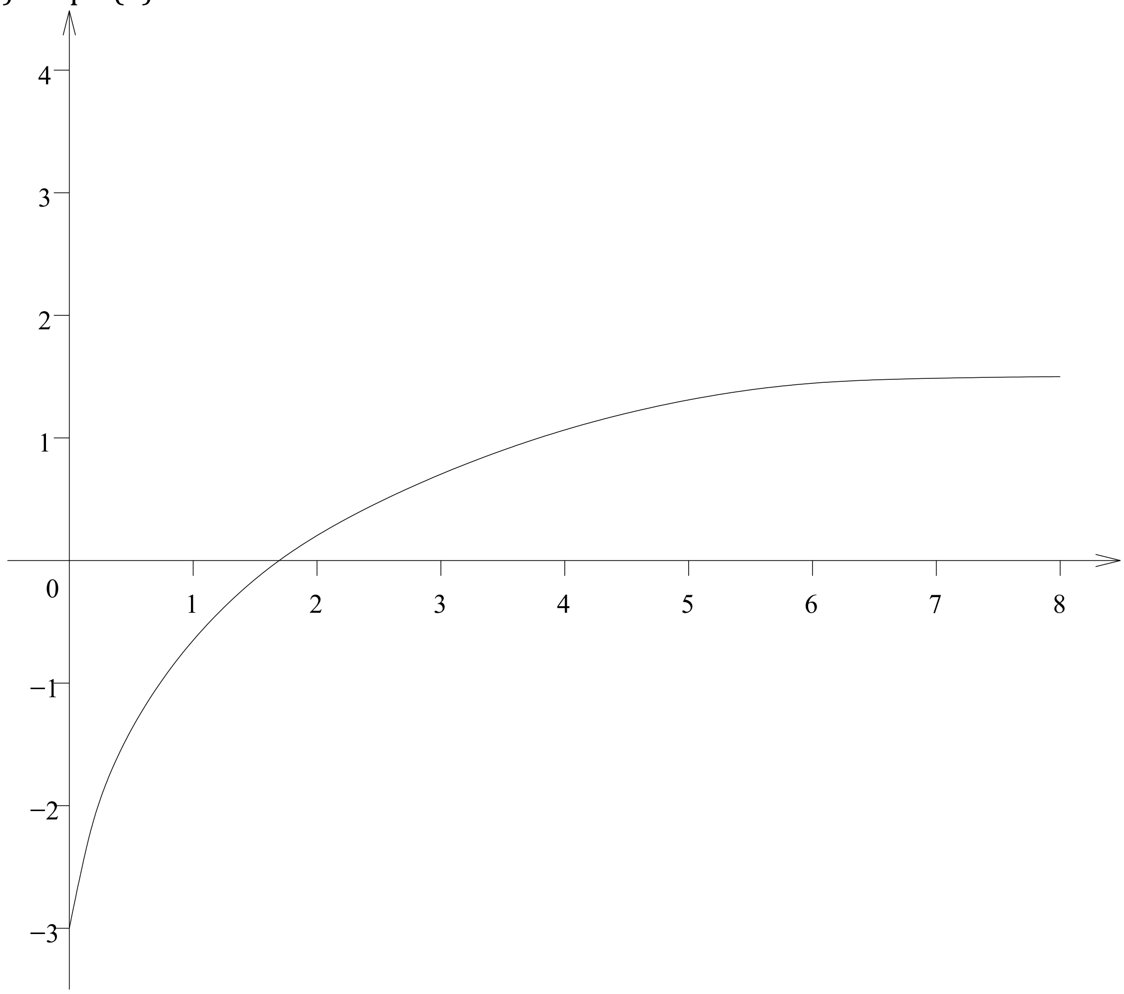
(f) **Optional:** Based on your experience above, provide advice for choosing an appropriate value of $\mathrm{d}t$ (for a given differential equation and integration scheme).

(g) **Optional:** Repeat your analysis with a different integration scheme (midpoint scheme or the Runge Kutta, RK4 scheme) and compare the results. How do the choices of $\mathrm{d}t$ change? How does the error behave after a long time with this new choice of integration scheme?

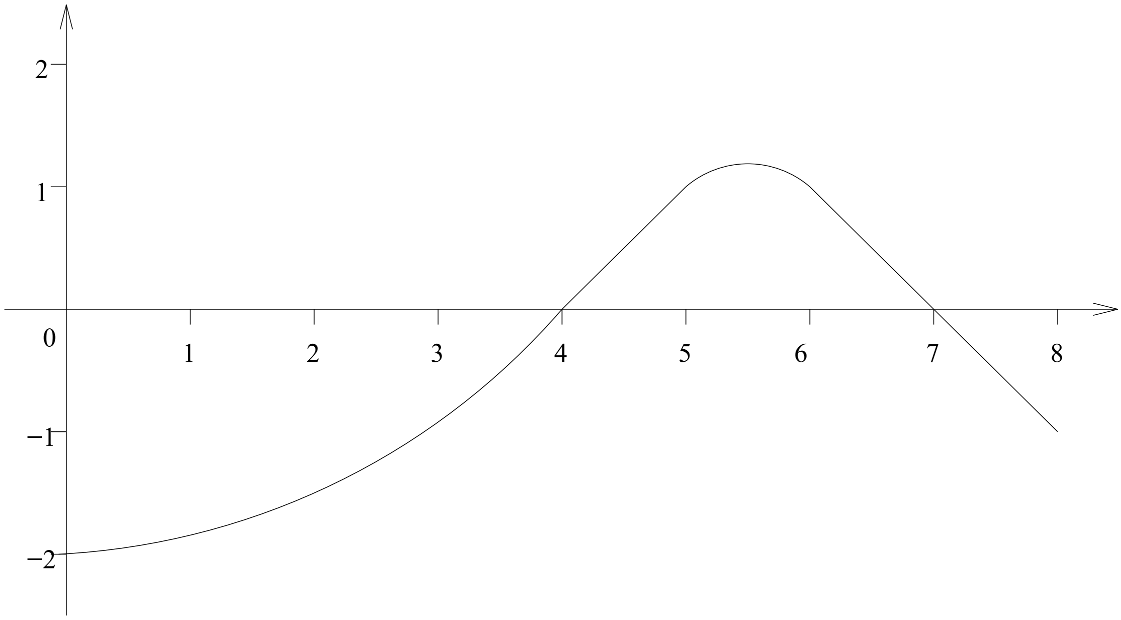
1(a) Graph (a):



1(b) Graph (b):



1(c) Graph (c):



1(d) Graph (d):