Twisted q-Yangians and Sklyanin determinants

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ABSTRACT. q-Yangians can be viewed as quantum deformations of the upper triangular loop Lie algebras, and also be viewed as deformation of the Yangian algebra. In this paper, we study the twisted q-Yangians as coideal subalgebras of the quantum affine algebra introduced by Molev, Ragoucy and Sorba. We investigate the invariant theory of the quantum symmetric spaces in affine types AI, AII and use the Sklyanin determinants to study the invariant theory and show that they also obey classical type identities similar to the quantum coordinate algebras of finite types.

1. Introduction

Let \mathfrak{g} be the complex simple Lie algebra, the Yangian $Y(\mathfrak{g})$ is defined by Drinfeld [3] as the algebraic structure to solve the rational Yang-Baxter equation. As an algebra, $Y(\mathfrak{gl}_N)$ deforms $\mathfrak{gl}_N[t]$ and there are two important homomorphisms:

$$(1.1) U(\mathfrak{gl}_N) \hookrightarrow Y(\mathfrak{gl}_N), Y(\mathfrak{gl}_N) \longrightarrow U(\mathfrak{gl}_N)$$

thus many representational problems of $U(\mathfrak{gl}_N)$ can be better understood over the Yangian algebra $Y(\mathfrak{gl}_N)$, see [18] for examples.

However for other classical types $\mathfrak{g}_N = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$, there are no such evaluation homomorphisms $Y(\mathfrak{g}_N) \longrightarrow U(\mathfrak{g}_N)$. Olshanski [25] introduced the twisted Yangian $Y(\mathfrak{g}_N)$ for $\mathfrak{g}_N = \mathfrak{o}_N, \mathfrak{sp}_N$ as coideal subalgebras of $Y(\mathfrak{gl}_N)$ corresponding to the orthogonal and symplectic types. It is known that $Y(\mathfrak{g}_N)$ are also deformations of the enveloping algebras $U(\mathfrak{g}_N)$ and more importantly there are canonical homomorphism maps

$$U(\mathfrak{g}_N) \hookrightarrow Y(\mathfrak{g}_N), \qquad Y(\mathfrak{g}_N) \longrightarrow U(\mathfrak{g}_N)$$

which provide the natural lifting and have important applications in studying $U(\mathfrak{g}_N)$.

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On the other hand, the enveloping algebra $U(\mathfrak{g})$ has another quantum deformation $U_q(\mathfrak{g})$ corresponding to the trigonometric Yang-Baxter R-matrix introduced by Drinfeld and Jimbo [3, 10].

For a classical symmetric pair $(\mathfrak{g},\mathfrak{k})$, the twisted quantum enveloping algebra $U_q^{\mathrm{tw}}(\mathfrak{k})$ was introduced by Noumi [21] and Dijkhuizen-Noumi-Sugitani [22, 23]. According to [21], the algebra $U_q^{\mathrm{tw}}(\mathfrak{so}_N)$ is isomorphic to the one introduced by [7]. In the first part of the paper, we consider the twisted quantum enveloping algebras $U_q^{\mathrm{tw}}(\mathfrak{o}_N)$ and $U_q^{\mathrm{tw}}(\mathfrak{sp}_N)$ corresponding to the symmetric pairs.

(1.2)
$$\text{AI}: \quad (\mathfrak{gl}_N, \mathfrak{o}_N), \\ \text{AII}: \quad (\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n}).$$

In 2003, Molev, Ragoucy, and Sorba used the R-matrix method to formulate the q-Yangian $Y_q(\mathfrak{gl}_N)$ as a q-deformation of $Y(\mathfrak{gl}_N)$ by replacing the rational R-matrix by the spectral trigonometric R-matrix [20]. They introduced twisted q-Yangians $Y_q^{\mathrm{tw}}(\mathfrak{g}_N)$ as coideal subalgebras of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ [2] corresponding to the symmetric spaces of orthogonal and symplectic types. Note that unlike the Yangian case, they are generated by quadratic elements from both the upper and lower triangular subalgebras $U_q^{\pm}(\widehat{\mathfrak{gl}}_N)$ according to the imaginary triangular decompositions

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^+ \oplus \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^-$$

where $\widehat{\mathfrak{g}}^{\pm} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]t^{\pm 1}$, Lu established the isomorphism between the twisted q-Yangians and affine iquantum groups associated to type AI [17]. It turns out that there are naturally homomorphisms:

$$U_q^{\mathrm{tw}}(\mathfrak{g}_N) \hookrightarrow Y_q^{\mathrm{tw}}(\mathfrak{g}_N),$$

$$Y_q^{\mathrm{tw}}(\mathfrak{g}_N) \longrightarrow U_q^{\mathrm{tw}}(\mathfrak{g}_N).$$

A different approach to quantum symmetric spaces was described for arbitrary symmetric pair $(\mathfrak{g},\mathfrak{k})$ by Letzter [16], which shows the importance of the coideal construction. In type A case, the quantum minors and quantum determinants satisfy nice algebraic equations that generalize the classical identities [8, 5, 14, 15]. Recently we have studied the dual quantum symmetric spaces of finite types AI and AII for the quantum coordinate algebra and determined their centers using the Sklyanin determinants and quantum Pfaffians [12], and we have showed that the Sklyanin determinants satisfy similar identities as the quantum determinants [13]. It is natural to ask for a similar construction for the quantum symmetric spaces associated with the quantum loop algebras.

The aim of this paper is to study the quantum affine symmetric spaces of orthogonal and symplectic types and also partly generalize the invariant theory of the Yangians [19] to the quantum affine algebras. After reviewing the quantum symmetric spaces of type AI, AII as in [20], we formulate the quantum affine symmetric spaces using the R-matrix method [6, 26] subject

to certain reflective RTT equation and introduce the Sklyanin determinant in both cases.

In the third part of the paper we give several identities for the quantum Sklyanin determinant. The key identities can be expressed as minor identities for the quantum Sklyanin determinant, which correspond to the classical identities [1] for the quantum determinant over the quantum general linear group. Similar to the Yangian and twisted Yangians (cf. [19]), we will generalize the q-Jacobi identities, q-Cayley's complementary identities, the q-Sylvester identities and Muir's theorem to Sklyanin determinants both in the quantum orthogonal and quantum symplectic situations. In a sense, we have generalized several key identities for the general linear, orthogonal and symplectic groups to their counterparts to the quantum affine symmetric spaces.

2. Coideal subalgebras of $U_q(\mathfrak{gl}_N)$

In this section we briefly recall the orthogoanl and symplectic coideals $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{so}_N)$ and $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_N)$ of the quantum algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ [20] to prepare our further study of the quantum affine algebras. Let R be the matrix in $\operatorname{End}(\mathbb{C}^N\otimes\mathbb{C}^N)\simeq (\operatorname{End}\mathbb{C}^N)^{\otimes 2}$ defined by:

$$R = q \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ij} \otimes E_{ji},$$

where E_{ij} are the unit matrices in End \mathbb{C}^N , and q is a complex number often assumed to be not a root of unity. It is known that R satisfies the well-known Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} \in \operatorname{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$ acts on the *i*th and *j*th copies of \mathbb{C}^N as R does on $\mathbb{C}^N \otimes \mathbb{C}^N$.

The quantum enveloping algebra $U_q(\mathfrak{gl}_N)$ is generated by the elements l_{ij}^{\pm} with $1 \leq i, j \leq N$ subject to the relations

(2.1)
$$l_{ij}^{-} = l_{ji}^{+} = 0, \ 1 \le i < j \le N,$$

$$(2.2) l_{ii}^+ l_{ii}^- = l_{ii}^- l_{ii}^+ = 1, \ 1 \le i \le N,$$

$$(2.3) RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R, RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R,$$

Here L^{\pm} are the matrices

(2.4)
$$L^{\pm} = \sum_{i,j} l_{ij}^{\pm} \otimes E_{ij},$$

which are regarded as elements in the algebra $U_q(\mathfrak{gl}_N) \otimes \operatorname{End}(\mathbb{C}^N)$. The elements in the R-matrix relation (2.3) are elements in $U_q(\mathfrak{gl}_N) \otimes \operatorname{End}(\mathbb{C}^N) \otimes$ $\operatorname{End}(\mathbb{C}^N)$ and the subindices of L^{\pm} indicate the copies of $\operatorname{End}(\mathbb{C}^N)$ where L^{\pm} act; e.g. $L_1^{\pm}=L^{\pm}\otimes 1$. In terms of the generators the defining relations between the l_{ij}^{\pm} can be written as

$$(2.5) q^{\delta_{ij}} l_{ia}^{\pm} l_{jb}^{\pm} - q^{\delta_{ab}} l_{jb}^{\pm} l_{ia}^{\pm} = (q - q^{-1})(\delta_{b < a} - \delta_{i < j}) l_{ja}^{\pm} l_{ib}^{\pm}$$

where $\delta_{i < j}$ equals 1 if i < j and 0 otherwise.

The relations involving both l_{ij}^+ and l_{ij}^- have the following form

$$(2.6) q^{\delta_{ij}} l_{ia}^{+} l_{jb}^{-} - q^{\delta_{ab}} l_{jb}^{-} l_{ia}^{+} = (q - q^{-1}) (\delta_{b < a} l_{ja}^{-} l_{ib}^{+} - \delta_{i < j} l_{ja}^{+} l_{ib}^{-})$$

2.1. Orthogonal case. The twisted quantum enveloping algebra $U_q^{\text{tw}}(\mathfrak{o}_N)$ is an unital algebra generated by $s_{ij}, 1 \leq i < j \leq N$ subject to the reflection relations

(2.7)
$$s_{ij} = 0, \ 1 \le i < j \le N,$$
$$s_{ii} = 1, \ 1 \le i \le N,$$
$$RS_1 R^t S_2 = S_2 R^t S_1 R$$

where $S = (s_{ij})_{N \times N}$ and $R^t = R^{t_1}$ denotes the partial transpose in the first tensor factor:

$$(2.8) R^{t_1} = q \sum_{1 \le i \le N} e_{ii} \otimes e_{ii} + \sum_{i \ne j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ji} \otimes e_{ji}.$$

The reflection relations can be written as

$$q^{\delta_{aj}+\delta_{ij}}s_{ia}s_{jb} - q^{\delta_{ab}+\delta_{ib}}s_{jb}s_{ia} = (q-q^{-1})q^{\delta_{ai}}(\delta_{b

$$+ (q-q^{-1})(q^{\delta_{ab}}\delta_{b

$$+ (q-q^{-1})^{2}(\delta_{b$$$$$$

where $\delta_{i < j}$ or $\delta_{i < j < k}$ equals 1 if the subindex inequality is satisfied and 0 otherwise.

THEOREM 2.1. [20] The map $S = L^-(L^+)^t$ defines an algebra embedding of $U_q^{tw}(\mathfrak{o}_N) \longrightarrow U_q(\mathfrak{gl}_N)$. The monomials

$$(2.10) s_{21}^{k_{21}} s_{31}^{k_{31}} s_{32}^{k_{32}} \cdots s_{N1}^{k_{N1}} s_{N2}^{k_{N2}} \cdots s_{N,N-1}^{k_{N,N-1}}$$

form a basis of the algebra $U_q^{tw}(\mathfrak{o}_N)$, where k_{ij} are nonnegative integers.

Regarding $U_q^{tw}(\mathfrak{o}_N)$ as a subalgebra of $U_q(\mathfrak{gl}_N)$, we introduce another matrix \overline{S} by

$$(2.11) \overline{S} = L^+ L^{-t},$$

then

$$(2.12) \overline{S} = 1 - q + qS^t.$$

In terms of the matrix elements, $\overline{s}_{ij} = qs_{ji}$ for i < j. So, the elements \overline{s}_{ij} belong to the subalgebra $U_q^{\text{tw}}(\mathfrak{o}_N)$.

The subalgebra $U_q^{tw}(\mathfrak{o}_N)$ is a coideal of $U_q(\mathfrak{gl}_N)$, as the image of the generator s_{ij} under the coproduct is given by

(2.13)
$$\Delta(s_{ij}) = \sum_{k,l=1}^{N} l_{ik}^{-} l_{jl}^{+} \otimes s_{kl}.$$

2.2. Symplectic case. Let G be the $2n \times 2n$ matrix G defined by

(2.14)
$$G = q \sum_{k=1}^{n} E_{2k-1,2k} - \sum_{k=1}^{n} E_{2k,2k-1}.$$

For $1 \le k \le n$, we denote

$$(2.15) (2k-1)' = 2k, (2k)' = 2k-1.$$

The twisted quantum enveloping algebra $U_q^{\text{tw}}(\mathfrak{sp}_N)$ is an unital algebra generated by $s_{ij}, 1 \leq i, j \leq N$ and the elements $s_{ii'}^{-1}, i = 1, 3, \dots, 2n - 1$, subject to the reflection relation

$$(2.16) RS_1 R^t S_2 = S_2 R^t S_1 R,$$

and in addition the following relations

$$(2.17) s_{ij} = 0 ext{ for } i < j ext{ with } j \neq i',$$

$$(2.18) s_{ii'} s_{ii'}^{-1} = s_{ii'}^{-1} s_{ii'} = 1, i = 1, 3, \dots, 2n - 1,$$

$$(2.19) s_{i'i'}s_{ii} - q^2 s_{i'i}s_{ii'} = q^3, i = 1, 3, \dots, 2n - 1.$$

We still use the symbol s_{ij} for the generators of $U_q^{tw}(\mathfrak{sp}_N)$ since they also satisfy the reflection equation (2.16) in both cases. It will be clear from the context whether they are the generators of the orthogonal algebra or the symplectic algebra.

Theorem 2.2. [20] The map $S \to L^-G(L^+)^t$ defines an algebra embedding of $U_q^{tw}(\mathfrak{g}_N) \longrightarrow U_q(\mathfrak{gl}_N)$. The monomials

(2.20)
$$\prod_{i=1,3,\dots,2n-1}^{\vec{r}} s_{i1}^{k_{i1}} s_{i2}^{k_{i2}} \cdots s_{ii'}^{k_{ii'}}, s_{i'i'}^{k_{i'i'}}, s_{i'1}^{k_{i'1}} \cdots s_{i,i-2}^{k_{i',i'-2}}$$

form a basis of the algebra $U_q^{tw}(\mathfrak{o}_N)$, where $k_{ii'}$ with $i=1,3,\ldots,2n-1$ are arbitrary integers and the remaining exponents k_{ij} are any nonnegative integers.

Regarding $U_q^{\mathrm{tw}}(\mathfrak{sp}_N)$ as subalgebra of $U_q(\mathfrak{gl}_N)$, we introduce another matrix \overline{S} by

$$(2.21) \overline{S} = L^+ G L^{-t}.$$

Then it is easy to see the following relations between the matrix elements of S and \overline{S} : for any i = 1, 3, ..., 2n - 1

$$(2.22) \overline{s}_{ii} = -q^{-2} s_{ii}, \ \overline{s}_{i'i'} = -q^{-2} s_{i'i'},$$

(2.23)
$$\overline{s}_{i'i} = -q^{-1}s_{ii'}, \ \overline{s}_{ii'} = -q^{-1}s_{i'i} + (1 - q^{-2})s_{ii'},$$

and for any i < j, $j \neq i'$,

$$\overline{s}_{ij} = -q^{-1}s_{ji}.$$

Thus, the elements \overline{s}_{ij} belong to the subalgebra $U_q^{tw}(\mathfrak{sp}_N)$.

The algebra $U_q^{tw}(\mathfrak{sp}_N)$ is another coideal of $U_q(\mathfrak{gl}_N)$, as the image of the generators under the coproduct are given by

(2.25)
$$\Delta(s_{ij}) = \sum_{k,l=1}^{N} l_{ik}^{-} l_{jl}^{+} \otimes s_{kl},$$

(2.26)
$$\Delta(s_{ii'}^{-1}) = l_{i'i'}^{-1} l_{ii}^{+} \otimes s_{ii'}^{-1}.$$

3. Quntum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$

The R-matrix R is invertible, let $P = \sum_{ij} E_{ij} \otimes E_{ji}$ and $PR^{-1}P$ becomes another R-matrix, explicitly

$$(3.1) PR^{-1}P = q^{-1}\sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} - (q - q^{-1})\sum_{i > j} E_{ij} \otimes E_{ji}.$$

For any two variables u, v, we introduce the R-matrix $R(u, v) = uPR^{-1}P - vR$:

(3.2)
$$R(u,v) = (u-v) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1}u - qv) \sum_{i} E_{ii} \otimes E_{ii} + (q^{-1} - q)u \sum_{i > j} E_{ij} \otimes E_{ji} + (q^{-1} - q)v \sum_{i < j} E_{ij} \otimes E_{ji},$$

which satisfies the spectral parameter dependent Yang-Baxter equation

$$(3.3) R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$

where both sides take values in $\operatorname{End}(\mathbb{C}^N) \otimes \operatorname{End}(\mathbb{C}^N) \otimes \operatorname{End}(\mathbb{C}^N)$ and the subindices indicate the copies of $\operatorname{End} \mathbb{C}^N$, e.g. $R_{12}(u,v) = R(u,v) \otimes 1$ etc.

The matrix R(u, v) is invertible and

(3.4)
$$R(u,v)R'(u,v) = (qu - q^{-1}v)(q^{-1}u - qv)1 \otimes 1.$$

where R'(u, v) is obtained from R(u, v) by replacing q with q^{-1} .

We also need the normalized R matrix

(3.5)
$$\widetilde{R}(x) = \frac{R(x,1)}{q^{-1}x - q}, \qquad R(x) = f(x)\widetilde{R}(x)$$

while the formal power series

(3.6)
$$f(x) = 1 + \sum_{k=1}^{\infty} f_k x^k, \qquad f_k = f_k(q),$$

is uniquely determined by the relation

(3.7)
$$f(xq^{2N}) = f(x)\frac{(1-xq^2)(1-xq^{2N-2})}{(1-x)(1-xq^{2N})}.$$

The R-matrix R(x) satisfies the following crossing symmetry relations [4]:

(3.8)
$$R_{12}^{-1}(x)^{t_2}D_2R_{12}^{t_2}(xq^{2N}) = D_2, \qquad R_{12}^{t_1}(xq^{2N})D_1R_{12}^{-1}(x)^{t_1} = D_1,$$
 where D is the diagonal matrix $\operatorname{diag}(q^{N-1}, q^{N-3}, \dots, q^{1-N}).$

The quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ is generated by $l_{ij}^{\pm(r)}$ where $1 \leq i, j \leq N$ and r runs over nonnegative integers. Let $L^{\pm}(u) = (l_{ij}^{\pm}(u))$ be the matrix

(3.9)
$$L^{\pm}(u) = \sum_{i,j=1}^{N} l_{ij}^{\pm}(u) \otimes E_{ij},$$

where $l_{ij}^{\pm}(u)$ are formal series in $u^{\pm 1}$ respectively,

(3.10)
$$l_{ij}^{\pm}(u) = \sum_{r=0}^{\infty} l_{ij}^{\pm(r)} u^{\pm r}.$$

The defining relations are

$$l_{ij}^{-(0)} = l_{ji}^{+(0)} = 0, \ 1 \le i < j \le N,$$

$$l_{ii}^{-(0)} l_{ii}^{+(0)} = l_{ii}^{+(0)} l_{ii}^{-(0)} = 1, \ 1 \le i \le N,$$

$$R(u/v) L_1^{\pm}(u) L_2^{\pm}(v) = L_2^{\pm}(v) L_1^{\pm}(u) R(u/v),$$

$$R(uq^{-c}/v) L_1^{+}(u) L_2^{-}(v) = L_2^{-}(v) L_1^{+}(u) R(uq^{c}/v).$$

The quantum enveloping algebra $U_q(\mathfrak{gl}_N)$ is a natural subalgebra of $U_q(\widehat{\mathfrak{gl}}_N)$ defined by the embedding

(3.12)
$$l_{ij}^{\pm} \mapsto l_{ij}^{\pm(0)}$$
.

Moreover, there is an algebra homomorphism $U_q(\widehat{\mathfrak{gl}}_N) \to U_q(\mathfrak{gl}_N)$ called the evaluation homomorphism defined by

(3.13)
$$L^{+}(u) \mapsto L^{+} - uL^{-}, L^{-}(u) \mapsto L^{-} - u^{-1}L^{+}, q^{c} \mapsto 1.$$

3.1. Quantum determinants. Consider the tensor product $U_q(\mathfrak{gl}_N) \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m}$, we have the following relation:

(3.14)
$$R(u_1, \dots, u_m) L_1^{\pm}(u_1) \cdots L_m^{\pm}(u_m) = L_m^{\pm}(u_m) \cdots L_1^{\pm}(u_1) R(u_1, \dots, u_m)$$

where

(3.15)
$$R(u_1, \ldots, u_m) = \prod_{1 \le i \le j \le m}^{\rightarrow} R_{ij}(u_i, u_j),$$

and the product is taken in the lexicographical order on the pairs (i, j). Consider the q-permutation operator $P^q \in \operatorname{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ defined by

(3.16)
$$P^{q} = \sum_{i} E_{ii} \otimes E_{ii} + q \sum_{i>j} E_{ij} \otimes E_{ji} + q^{-1} \sum_{i$$

The symmetric group \mathfrak{S}_m acts on the space $(\mathbb{C}^N)^{\otimes m}$ via $s_i \mapsto P^q_{s_i} := P^q_{i,i+1}$ far $i=1,\ldots,m-1$, where s_i denotes the transporition $(i,\ i+1)$. If $\sigma=s_{i_1}\cdots s_{i_l}$ is a reduced decomposition of an element $\sigma\in\mathfrak{S}_m$ we set $P^q_{\sigma}=P^q_{s_{i_1}}\cdots P^q_{s_{i_l}}$. The q-antisymmetrizer is then defined by

(3.17)
$$A_m^q = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}_{\sigma} \cdot P_{\sigma}^q$$

It is known that the q-antisymmetrizer can be rewritten in $(\operatorname{End}\mathbb{C}^N)^{\otimes r}$

(3.18)
$$R(1, q^{-2}, \dots, q^{-2m+2}) = m! \prod_{0 \le i < j \le m-1} (q^{-2i} - q^{-2j}) A_m^q.$$

Using the commutation relation (3.11) and the Yang-Baxter equation (3.3) we have that

$$(3.19) A_m^q L_1^{\pm}(u) \cdots L_m^{\pm}(q^{-2m+2}u) = L_m^{\pm}(q^{-2m+2}u) \cdots L_1^{\pm}(u) A_m^q.$$

which can be rewritten as

(3.20)
$$\frac{1}{m!} \sum_{i_k, j_k} l^{\pm i_1 \cdots i_m}_{j_1 \cdots j_m}(u) \otimes E_{i_1 j_1} \otimes \cdots \otimes E_{i_m j_m}$$

where the coefficients $l^{\pm i_1 \cdots i_m}_{j_1 \cdots j_m}(u) \in \mathrm{U}_q(\mathfrak{gl}_N)[[u^{-1}]]$ are called the *quantum* minors. Due to the q-antisymmetrizer, the quantum minor are given by the following formula. If $i_1 < \cdots < i_m$, then

$$(3.21) l_{j_1\cdots j_m}^{\pm i_1\cdots i_m}(u) = \sum_{\sigma\in\mathfrak{S}_m} (-q)^{-l(\sigma)} \cdot l_{i_{\sigma(1)}j_1}^{\pm}(u) \cdots l_{i_{\sigma(m)}j_m}^{\pm}(q^{-2m+2}u).$$

Subsequently for any $\tau \in \mathfrak{S}_m$ we have

(3.22)
$$l^{\pm i_{\tau(1)} \cdots i_{\tau(m)}}_{j_1 \cdots j_m}(u) = (-q)^{l(\tau)} l^{\pm i_1 \cdots i_m}_{j_1 \cdots j_m}(u),$$

where $l(\sigma)$ denotes the length of the permutation σ . If $j_1 < \cdots < j_m$ (and the i_r are arbitrary) then

$$(3.23) l_{j_1\cdots j_m}^{\pm i_1\cdots i_m}(u) = \sum_{\sigma\in\mathfrak{S}_m} (-q)^{l(\sigma)} \cdot l_{i_m j_{\sigma(m)}}^{\pm}(q^{-2m+2}u) \cdots l_{i_1 j_{\sigma(1)}}^{\pm}(u)$$

and for any $\tau \in \mathfrak{S}_m$ we have

(3.24)
$$l^{\pm i_1 \cdots i_m}_{j_{\tau(1)} \cdots j_{\tau(m)}}(u) = (-q)^{-l(\tau)} l^{\pm i_1 \cdots i_m}_{j_1 \cdots j_m}(u)$$

The quantum determinants of the matrices $L^{\pm}(u)$ are defined as N-minors:

(3.25)
$$\det_{q} L^{\pm}(u) = l^{\pm 1 \cdots N}_{1 \cdots N}(u),$$

which satisfies the following relation:

(3.26)
$$A_N^q L_1^{\pm}(u) \cdots L_N^{\pm}(q^{2-2N}u) = L_N^{\pm}(q^{2-2N}u) \cdots L_1^{\pm}(u) A_N^q$$
$$= \det_q L^{\pm}(u) A_N^q.$$

Suppose that $I = \{i_1 < i_2 < \dots < i_m\}$, $J = \{j_1 < j_2 < \dots < j_m\}$, we denote $l_{j_1 \dots j_m}^{\pm i_1 \dots i_m}(u)$ by $\det_q (L^{\pm}(u)_J^I)$. If I = J, we denote $\det_q (L^{\pm}(u)_I) = \det_q (L^{\pm}(u)_I^I)$.

We define the comatrix $\widehat{L}^{\pm}(u)$ by

(3.27)
$$\widehat{L}^{\pm}(u)L^{\pm}(q^{2-2N}u) = \det_q(L^{\pm}(u))I.$$

Proposition 3.1. The matrix elements of $\hat{L}_{ij}^{\pm}(u)$ are given by

(3.28)
$$\widehat{L}_{ij}^{\pm}(u) = (-q)^{j-i} l^{\pm 1, \dots, \hat{j} \dots, N}_{1, \hat{i}, \dots, N}(u).$$

Moreover, we have the relations

(3.29)
$$L^{\pm}(u)^{t}D\widehat{L}^{\pm}(q^{-2}u)^{t}D^{-1} = \det_{q}(L^{\pm}(u))I.$$

PROOF. Multiplying $(L_N^{\pm}(q^{2-2N}u))^{-1}$ from the right of the formulas

$$(3.30) A_N^q L_1^{\pm}(u) L_2^{\pm}(q^{-2}u) \dots L_N^{\pm}(q^{2-2N}u) = A_N^q \det_q(L^{\pm}(u)).$$

we get that

$$(3.31) A_N^q L_1^{\pm}(u) L_2^{\pm}(q^{-2}u) \dots L_{N-1}^{\pm}(q^{4-2N}u) = A_N \widehat{L}_N^{\pm}(u).$$

Applying both sides to the vector

$$(3.32) e_1 \otimes \cdots \hat{e}_i \otimes e_N \otimes e_i$$

and comparing the coefficients of $e_1 \otimes \cdots \otimes e_N$ we get the equation (3.28). Denote by $A^q_{\{2,\ldots,N\}}$ the q-antisymmetrizer over the copies of $End(\mathbb{C}^N)$ labeled by $\{2,\ldots,N\}$. Then $A^q_N = A^q_N A^q_{\{2,\ldots,N\}}$ and

$$(3.33) A_N^q L_1^{\pm}(u) A_{\{2,\dots,N\}}^q L_2^{\pm}(q^{-2}u) \dots L_N^{\pm}(q^{2-2N}u) = A_N^q \det_q(L^{\pm}(u)).$$

Applying both sides to the vector

$$(3.34) e_i \otimes e_1 \otimes \cdots \hat{e_j} \otimes e_N,$$

we get that

(3.35)
$$\sum_{k=1}^{N} (-q)^{j-k} l_{ki}^{\pm}(u) l_{1,\hat{j},\dots,N}^{\pm 1,\dots,\hat{k}\dots,N}(q^{-2}u) = \delta_{ij} \det_{q}(L^{\pm}(u))$$

It can be written as

(3.36)
$$L^{\pm}(u)^{t} D\left(\widehat{L}^{\pm}(q^{-2}u)\right)^{t} D^{-1} = \det_{q}(L^{\pm}(u)).$$

3.2. Minor identities for quantum determinants. Multiplying the RTT relation from both sides consecutively by the inverses to R(u,v), $L_1^{\pm}(u)$ and $L_2^{\pm}(v)$, then using the relation (3.4) we have that

(3.37)
$$R'(u/v)L_{1}^{\pm}(u)^{-1}L_{2}^{\pm}(v)^{-1} = L_{2}^{\pm}(v)^{-1}L_{1}^{\pm}(u)^{-1}R'(u/v),$$
$$R'(u/v)L_{1}^{+}(u)^{-1}L_{2}^{-}(v)^{-1} = L_{2}^{-}(v)^{-1}L_{1}^{+}(u)^{-1}R'(u/v),$$

The following theorem is an analog of Jacobi's ratio theorem for quantum determinants.

THEOREM 3.2. Let $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_k)$ be two subsets of [1, N] of the same cardinality, and $I^c = \{i_{k+1} < \cdots < i_N\}$ and $J^c = \{j_{k+1} < \cdots < j_N\}$ be their complements. Then

$$(-q)^{-l(I,I^c)} \det_q(L^{\pm}(u)_J^I)$$

$$= (-q)^{-l(J,J^c)} \det_q(L^{\pm}(u)) \det_{q^{-1}} \left((L^{\pm}(q^{2-2N}u)^{-1})_{I^c}^{J^c} \right).$$

PROOF. Multiplying $(L_N^{\pm}(q^{2-2N}u))^{-1}, \cdots, (L_{k+1}^{\pm}(q^{-2k}u))^{-1}$ from the right of the formula

(3.39)
$$A_N^q L_1^{\pm}(u) L_2^{\pm}(q^{-2}u) \dots L_N^{\pm}(q^{2-2N}u) = A_N^q \det_q(L^{\pm}(u)),$$

we get that

(3.40)
$$A_N^q L_1^{\pm}(u) L_2^{\pm}(q^{-2}u) \dots L_k^{\pm}(q^{2-2k}u) \\ = \det_q(L^{\pm}(u)) A_N^q L_N^{\pm}(q^{2-2N}u)^{-1} \dots L_{k+1}^{\pm}(q^{-2k}u)^{-1}.$$

Applying both sides to the vector $e_{j_1} \otimes \dots e_{j_k} \otimes e_{i_N} \otimes \dots \otimes e_{i_{k+1}}$ and comparing the coefficient of $e_1 \otimes e_2 \otimes \dots \otimes e_N$, we obtain that

$$(-q)^{-l(I,I^c)} \det_q(L^{\pm}(u)_J^I)$$

$$= (-q)^{-l(J,J^c)} \det_q(L^{\pm}(u)) \det_{q^{-1}} \left((L^{\pm}(q^{2-2N}u)^{-1})_{I^c}^{J^c} \right).$$

As a special case with $I = J = \emptyset$, we have the following corollary.

Corollary 3.3.

(3.42)
$$\det_{q}(L^{\pm}(u))\det_{q^{-1}}(L^{\pm}(q^{2-2N}u)^{-1}) = 1.$$

The following is an analogue of Schur's complement theorem.

Theorem 3.4. Let

(3.43)
$$L^{\pm}(u) = \begin{pmatrix} L_{11}^{\pm}(u) & L_{12}^{\pm}(u) \\ L_{21}^{\pm}(u) & L_{22}^{\pm}(u) \end{pmatrix},$$

be the block matrix such that $L_{11}^{\pm}(u)$ and $L_{22}^{\pm}(u)$ are submatrices of size $k \times k$ and $(N-k) \times (N-k)$ respectively. Then

$$\det_q(L^{\pm}(u))$$

$$= \det_{q}(L_{11}^{\pm}(u)) \det_{q}\left(L_{22}^{\pm}(q^{-2k}u) - L_{21}^{\pm}(q^{-2k}u)L_{11}^{\pm}(q^{-2k}u)^{-1}L_{12}^{\pm}(q^{-2k}u)\right)$$

$$= \det_{q}(L_{22}^{\pm}(u)) \det_{q}(L_{22}^{\pm}(q^{2(k-N)}u) - L_{21}^{\pm}(q^{2(k-N)}u)L_{11}^{\pm}(q^{2(k-N)}u)^{-1}L_{12}^{\pm}(q^{2(k-N)}u))$$

PROOF. We Denote by $X^{\pm}(u)$ the inverse of $L^{\pm}(u)$ and write it as

(3.44)
$$X^{\pm}(u) = \begin{pmatrix} X_{11}^{\pm}(u) & X_{12}^{\pm}(u) \\ X_{21}^{\pm}(u) & X_{22}^{\pm}(u) \end{pmatrix},$$

then

(3.45)
$$X_{11}^{\pm}(u) = \left(L_{11}^{\pm}(u) - L_{12}^{\pm}(u)L_{22}^{\pm}(u)^{-1}L_{21}^{\pm}(u)\right)^{-1}, X_{22}^{\pm}(u) = \left(L_{22}^{\pm}(u) - L_{21}^{\pm}(u)L_{11}^{\pm}(u)^{-1}L_{12}^{\pm}(u)\right)^{-1}.$$

It follows from Theorem 3.2 that

(3.46)
$$\det_q(L_{11}^{\pm}(u)) = \det_q(L^{\pm}(u))\det_{q^{-1}}\left(X_{22}^{\pm}(q^{2-2N}u)\right).$$

Since the matrix $L_{22}^{\pm}(u)-L_{21}^{\pm}(u)L_{11}^{\pm}(u)^{-1}L_{12}^{\pm}(u)$ is the inverse of $X_{22}^{\pm}(u)$, it satisfies the RTT relations. By Corollary 3.3,

(3.47)
$$\det_{q^{-1}}(X_{22}^{\pm}(u))^{-1} = \det_{q}\left(L_{22}^{\pm}(q^{-2k}u) - L_{21}^{\pm}(q^{-2k}u)L_{11}^{\pm}(q^{-2k}u)^{-1}L_{12}^{\pm}(q^{-2k}u)\right)$$

Combing (3.46) and (3.47), we obtain the first equation. The second equation can be proved similarly.

Using Jacobi's theorem we obtain the following analog of Cayley's complementary identity for quantum determinants.

Theorem 3.5. Suppose a minor identity for quantum determinants is given:

(3.48)
$$\sum_{i=1}^{k} b_i \prod_{i=1}^{m_i} \det_q \left((L^{\epsilon_{ij}}(u))_{J_{ij}}^{I_{ij}} \right) = 0,$$

where I_{ij} and J_{ij} are subsets of [1,N], $\epsilon_{ij} = +$ or - and $b_i \in \mathbb{C}(q)$. Then the following identity holds

$$(3.49) \quad \sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} (-q)^{l(I_{ij}^c, I_{ij}) - l(J_{ij}^c, J_{ij})} \det_q(L^{\epsilon_{ij}}(u))^{-1} \det_q(L^{\epsilon_{ij}}(u))^{J_{ij}^c} = 0,$$

where b'_i is obtained from b_i by changing q by q^{-1} .

PROOF. The matrix $L^{\pm}(u)^{-1}$ satisfies the q^{-1} -RTT relations. Applying the minor identity to $L^{\pm}(u)^{-1}$ we get that

(3.50)
$$\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \det_{q^{-1}} \left((L^{\epsilon_{ij}}(u)^{-1})_{J_{ij}}^{I_{ij}} \right) = 0,$$

It follows from Theorem 3.2 that

$$\begin{array}{ll} \det_{q^{-1}} \left((L^{\epsilon_{ij}}(u)^{-1})_{J_{ij}}^{I_{ij}} \right) \\ & = (-q)^{l(I_{ij}^c,I_{ij}) - l(J_{ij}^c,J_{ij})} \det_q (L^{\epsilon_{ij}}(q^{2N-2}u))^{-1} \det_q (L^{\epsilon_{ij}}(q^{2N-2}u)_{I_{ij}^c}^{C}) \end{array}$$

The proof is completed by replacing u with $q^{2-2N}u$.

The following theorem is an analog of Muir's law for quantum determinants.

Theorem 3.6. Suppose a quantum minor determinant identity is given:

(3.52)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \det_q \left((L^{\epsilon_{ij}}(u))_{J_{ij}}^{I_{ij}} \right) = 0,$$

where $I'_{ij}s$ are subsets of $T = \{1, 2, ..., N\}$, $\epsilon_{ij} = +$ or - and $b_i \in \mathbb{C}(q)$. Let K be the set $\{N, ..., N + M\}$. Then the following identity holds

(3.53)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \det_q((L^{\epsilon_{ij}}(u))_K)^{-1} \det_q((L^{\epsilon_{ij}}(u))_{J_{ij} \cup K}^{I_{ij} \cup K}) = 0.$$

PROOF. Applying Cayley's complementary identity for the set T, we get that

(3.54)
$$\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} (-q)^{l(T \setminus I_{ij}, I_{ij}) - l(T \setminus J_{ij}, J_{ij})} \det_q (L^{\epsilon_{ij}}(u)_T)^{-1} \cdot \det_q (L^{\epsilon_{ij}}(u)_{T \setminus I_{ij}}^{T \setminus J_{ij}}) = 0.$$

The equation (3.53) is similarly shown by using Cayley's complementary identity for the set $T \cup K$.

The following is an analog of Sylvester's theorem for the quantum determinants given by Hopkins and Molev [9]. We will give a proof using Muir's Law.

Theorem 3.7. Let $T=\{1,\cdots,N\}$, $K=\{N+1,\cdots,N+M\}$. Then the mapping $l_{ij}^{\pm}(u)\mapsto l_{j,N+1,\cdots,N+M}^{\pm i,N+1,\cdots,N+M}(u)$ defines an algebra morphism $U_q(\mathfrak{gl}_N)\to U_q(\mathfrak{gl}_{M+N})$. Denote $\tilde{l}_{ij}^{\pm}(u)$ by the image of $l_{ij}^{\pm}(u)$ and $\tilde{L}^{\pm}(u)=$

$$(\tilde{l}_{ij}^{\pm}(u))$$
. Then

(3.55)
$$\det_{q}(\tilde{L}^{\pm}(u)) = \det_{q}(L^{\pm}(u)) \prod_{i=1}^{N-1} \det_{q}(L^{\pm}(q^{-2i}u)_{K})$$

PROOF. It follows from Muir's Law (Theorem 3.6) and centralility of quantum minors that the map defines an algebra morphism. Applying Muir's law to the equation

(3.56)
$$\det_{q}(L^{\pm}(u)_{T}) = \sum_{\sigma \in \mathfrak{S}_{N}} (-q)^{-l(\sigma)} \cdot l^{\pm}_{\sigma(1)1}(u) \cdots l^{\pm}_{\sigma(N)N}(q^{2-2N}u)$$

we get that

(3.57)
$$\det_{q}(L^{\pm}(u)_{K})^{-1}\det_{q}(L^{\pm}(u))$$
$$= \det_{q}(\tilde{L}^{\pm}(u)) \prod_{i=0}^{N-1} \det_{q}(L^{\pm}(q^{-2i}u)_{K})^{-1}$$

Consequently we see that

(3.58)
$$\det_q(\tilde{L}^{\pm}(u)) = \det_q(L^{\pm}(u)) \prod_{i=1}^{N-1} \det_q(L^{\pm}(q^{-2i}u)_K).$$

Let A be a $N \times N$ matrix. For any subset I of [1, N], we denote by A_I the principal submatrix of A with rows and columns indexed by the elements of I. The following relation involving the determinants and the permanents were first established by Muir:

(3.59)
$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{I \subset [1,N] \\ |I|=k}} \det(A_I) \operatorname{per}(A_{[1,N]\setminus I}) = 0.$$

In the following, we give an analog of Muir's identity for quantum affine algebra. Denote by H^q_m the q-symmetrizer

$$(3.60) H_m^q = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} P_\sigma^q.$$

LEMMA 3.8. In the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$

$$(3.61) L_1^{\pm}(u) \cdots L_m^{\pm}(q^{-2m+2}u)H_m^q = H_m^q L_1^{\pm}(u) \cdots L_m^{\pm}(q^{-2m+2}u)H_m^q.$$

PROOF. Since $H_2^q + A_2^q = 1$, using the RTT relations we have that

$$(3.62) A_2^q L_1^{\pm}(u) L_2^{\pm}(q^{-2}u) H_2^q = A_2^q L_2^{\pm}(q^{-2}u) L_1^{\pm}(u) H_2^q = 0.$$

Then

(3.63)
$$P^{q}L_{1}^{\pm}(u)L_{2}^{\pm}(q^{-2}u)H_{2}^{q} = L_{1}^{\pm}(u)L_{2}^{\pm}(q^{-2}u)H_{2}^{q}$$
$$P^{q}L_{2}^{\pm}(q^{-2}u)L_{1}^{\pm}(u)H_{2}^{q} = L_{2}^{\pm}(q^{-2}u)L_{1}^{\pm}(u)H_{2}^{q}.$$

The lemma follows from the first equation.

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THEOREM 3.9. The following are Muir's identities for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$.

(3.64)
$$\sum_{r=0}^{k} (-1)^r t r_{1,\dots,k} H_r^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u) = 0,$$

(3.65)
$$\sum_{r=0}^{k} (-1)^r tr_{1,\dots,k} A_r^q H_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u) = 0,$$

where $H^q_{\{r+1,\ldots,k\}}$ and $A^q_{\{r+1,\ldots,k\}}$ denote the symmetrizer and antisymmetrizer over the copies of $End(\mathbb{C}^N)$ labeled by $\{r+1,\ldots,k\}$ respectively.

PROOF. We show the following identity:

$$(3.66) tr_{1,\dots,k}H_r^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u)$$

$$= tr_{1,\dots,k} \frac{r(k-r+1)}{k} H_r^q A_{\{r,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u)$$

$$+ tr_{1,\dots,k} \frac{(r+1)(k-r)}{k} H_{r+1}^q A_{\{r,\dots,k\}}^q L_1^{\pm}(u) \cdots L_m^{\pm}(q^{-2m+2}u).$$

In fact, using the group algebra of \mathfrak{S}_k , we have that

$$(k-r+1)A_{\{r,\dots,k\}}^q = A_{\{r+1,\dots,k\}}^q - (k-r)A_{\{r+1,\dots,k\}}^q P_{r,r+1}^q A_{\{r+1,\dots,k\}}^q,$$

$$(r+1)H_{r+1}^q = H_r^q + rH_r^q P_{r,r+1}^q H_r^q.$$

Therefore

$$(3.67) tr_{1,\dots,k}H_r^q A_{\{r+1,\dots,k\}}^q P_{(r,r+1)}^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u)$$

$$= tr_{1,\dots,k}H_r^q P_{(r,r+1)}^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u) A_{\{r+1,\dots,k\}}^q$$

$$= tr_{1,\dots,k}H_r^q P_{(r,r+1)}^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u).$$

Similarly,

$$(3.68) tr_{1,\dots,k}H_r^q P_{r,r+1}^q H_r^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u)$$

$$= tr_{1,\dots,k}H_r^q P_{(r,r+1)}^q A_{\{r+1,\dots,k\}}^q L_1^{\pm}(u) \cdots L_k^{\pm}(q^{-2k+2}u).$$

These imply equation (3.66). Therefore the telescoping sum equals to zero. The second equation can be proved by the same arguments.

3.3. Liouville formula. Now we introduce a family of central elements for the quantum affine algebra [11]. Let

(3.69)
$$z^{\pm}(u)^{-1} = \frac{1}{N} tr \left(L^{\pm}(u) D^{-1} L^{\pm} (q^{-2N} u)^{-1} D \right).$$

Theorem 3.10. [11] On the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$, we have that

(3.70)
$$z^{\pm}(u)^{-1} = \frac{\det_q(L^{\pm}(u))}{\det_q(L^{\pm}(q^{-2}u))}$$

Moreover, we also have

(3.71)
$$z^{\pm}(u)^{-1} = \frac{1}{N} tr \left(D^{-1} L^{\pm} (q^{-2N} u)^{-1} D L^{\pm}(u) \right)$$

(3.72)
$$z^{\pm}(u) = \frac{1}{N} tr \left(D^{-1} (L^{\pm}(u)^{t})^{-1} D L^{\pm} (q^{-2N} u)^{t} \right)$$
$$= \frac{1}{N} tr \left(L^{\pm} (q^{-2N} u)^{t} D^{-1} (L^{\pm}(u)^{t})^{-1} D \right)$$

PROOF. We compute that

$$z^{\pm}(u)^{-1} = \frac{1}{N} tr \left(L^{\pm} T(u) D^{-1} L^{\pm} (q^{-2N} u)^{-1} D \right)$$

$$= \frac{1}{N} tr \left(L^{\pm}(u) D^{-1} \hat{L}^{\pm} (q^{-2} u) D \right) \det_{q} (L^{\pm}(q^{-2} u))^{-1}$$

$$= \frac{\det_{q} (L^{\pm}(u))}{\det_{q} (L^{\pm}(q^{-2} u))}$$

By the same argument,

(3.73)
$$\frac{1}{N} tr \left(D^{-1} L^{\pm} (q^{-2N} u)^{-1} D L^{\pm} (u) \right) = \frac{\det_q(L^{\pm}(u))}{\det_q(L^{\pm}(q^{-2u}))}$$

It follows from Proposition 4.9 that

$$(3.74) (L^{\pm}(u)^{t})^{-1} = D\widehat{L}^{\pm}(q^{-2}u)^{t}D^{-1}\det_{q}(L^{\pm}(u))^{-1}$$

then

$$(3.75) \quad tr\left(D^{-1}(L^{\pm}(u)^{t})^{-1}DL^{\pm}(q^{-2N}u)^{t}\right) = tr\left(\widehat{L}^{\pm}(q^{-2}u)^{t}L^{\pm}(q^{-2N}u)^{t}\right)$$

Using the comtrix, we immediately have that

$$(3.76) \quad \frac{1}{N} tr \left(D^{-1} (L^{\pm}(u)^t)^{-1} D L^{\pm} (q^{-2N} u)^t \right) = \frac{\det_q (L^{\pm}(q^{-2} u))}{\det_q (L^{\pm}(u))} = z^{\pm}(u)$$

The other equation can be proved similarly.

Proposition 3.11. The following relations hold

$$QL_{1}^{\pm}(u)D_{2}(L_{2}^{\pm}(q^{-2N}u)^{-1})^{t}D_{2}^{-1}$$

$$=D_{2}(L_{2}^{\pm}(q^{-2N}u)^{-1})^{t}D_{2}^{-1}L_{1}^{\pm}(u)Q = Qz^{\pm}(u)^{-1}$$

$$QD_{1}^{-1}(L_{1}^{\pm}(u)^{t})^{-1}D_{1}L_{2}^{\pm}(q^{-2N}u)$$

$$=L_{2}^{\pm}(q^{-2N}u)D_{1}^{-1}(L_{1}^{\pm}(u)^{t})^{-1}D_{1}Q = Qz^{\pm}(u).$$
(3.78)

Proof. Multiplying both sides of the relation

(3.79)
$$R(u,v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u,v)$$

by $L_2^{\pm}(v)^{-1}$ and taking partial transpose with respect to the second copy, we get that

$$(3.80) R(u,v)^{t_2} (L_2^{\pm}(v)^{-1})^t L_1^{\pm}(u) = L_1^{\pm}(u) (L_2^{\pm}(v)^{-1})^t R(u,v)^{t_2}.$$

Multiply the inverse of $R(u, v)^{t_2}$, the above can be rewritten as

$$(3.81) (L_2^{\pm}(v)^{-1})^t L_1^{\pm}(u) (R(u,v)^{t_2})^{-1} = (R(u,v)^{t_2})^{-1} L_1^{\pm}(u) (L_2^{\pm}(v)^{-1})^t.$$

The take $v = q^{-2N}u$,

(3.82)
$$D_{2}(L_{2}^{\pm}(q^{-2N}u)^{-1})^{t}L_{1}^{\pm}(u)D_{2}^{-1}Q$$
$$=QD_{2}L_{1}^{\pm}(u)(L_{2}^{\pm}(q^{-2N}u)^{-1})^{t}D_{2}^{-1}$$

Note that for any X, $QX_1 = QX_2^t$, and $X_1Q = X_2^tQ$. Therefore

$$(3.83) QL_1^{\pm}(u)D_2(L_2^{\pm}(q^{-2N}u)^{-1})^tD_2^{-1} = QL_2^{\pm}(u)^tD_2(L_2^{\pm}(q^{-2N}u)^{-1})^tD_2^{-1}$$
$$= QL_2^{\pm}(u)^tD_2\widehat{L}_2^{\pm}(q^{-2}u)^tD_2^{-1}(\det_q L^{\pm}(q^{-2}u))^{-1}$$

which is $z^{\pm}(u)^{-1}Q$ by Proposition 3.1.

Taking transposition of first copy in the equation (3.79) we get that

(3.84)
$$L_1^{\pm}(u)^t R^t(u,v) L_2^{\pm}(v) = L_2^{\pm}(v) R^t(u,v) L_1^{\pm}(u)^t,$$

where we abbreviate $R^{t_1}(u,v)$ as $R^t(u,v)$. Taking inverses of $L_1^{\pm}(u)^t$ and $R^t(u,v)$, the relation becomes

$$(3.85) L_2^{\pm}(v)(L_1^{\pm}(u)^t)^{-1}R^t(u,v)^{-1} = R^t(u,v)^{-1}(L_1^{\pm}(u)^t)^{-1}L_2^{\pm}(v).$$

Using the crossing symmetry, we have

(3.86)
$$L_{2}^{\pm}(v)(L_{1}^{\pm}(u)^{t})^{-1}D_{1}R^{-1}(q^{-2N}u,v)^{t}D_{1}^{-1}$$
$$= D_{1}R^{-1}(q^{-2N}u,v)^{t}D_{1}^{-1}(L_{1}^{\pm}(u)^{t})^{-1}L_{2}^{\pm}(v).$$

Letting $v = q^{-2N}u$, we rewrite it as

(3.87)
$$L_{2}^{\pm}(q^{-2N}u)D_{1}^{-1}(L_{1}^{\pm}(u)^{t})^{-1}D_{1}Q$$
$$=QD_{1}^{-1}(L_{1}^{\pm}(u)^{t})^{-1}D_{1}L_{2}^{\pm}(q^{-2N}u)$$

$$= QL_1^{\pm}(q^{-2N}u)^t D_1^{-1}(L_1^{\pm}(u)^t)^{-1} D_1$$

which is $z^{\pm}(u)Q$ by Theorem 3.10.

4. Coideal subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$

4.1. Orthogonal twisted q-Yangians. Consider the element $R^t(u, v) := R^{t_1}(u, v)$ obtained from R(u, v):

(4.1)
$$R^{t}(u,v) = (u-v) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1}u - qv) \sum_{i} E_{ii} \otimes E_{ii} + (q^{-1} - q)u \sum_{i > j} E_{ji} \otimes E_{ji} + (q^{-1} - q)v \sum_{i < j} E_{ji} \otimes E_{ji}$$

We define the twisted q-Yangian $Y_q^{tw}(\mathfrak{o}_N)$ as the associative algebra generated by $s_{ij}^{(r)}$, $1 \leq i, j \leq N$ and $r \in \mathbb{Z}_+$ subject to the following relations.

(4.2)
$$s_{ij}^{(0)} = 0, \ 1 \le i < j \le N, \qquad s_{ii}^{(0)} = 1, \ 1 \le i \le N,$$
$$R(u/v)S_1(u)R^t(1/uv)S_2(v) = S_2(v)R^t(1/uv)S_1(u)R(u/v),$$

where

(4.3)
$$s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)} u^{-r}.$$

Theorem 4.1. [20] The map $S(u) \mapsto L^-(uq^{-c})L^+(u^{-1})^t$ defines an algebra embedding of $Y_q^{tw}(\mathfrak{o}_N) \longrightarrow U_q(\widehat{\mathfrak{gl}}_N)$. The ordered monomials in the generators constitute a basis of $Y_q^{tw}(\mathfrak{o}_N)$.

As a subalgebra, $Y_q^{tw}(\mathfrak{o}_N)$ becomes a coideal of $U_q(\widehat{\mathfrak{gl}}_N)$, as

(4.4)
$$\Delta(s_{ij}(u)) = \sum_{k,l=1}^{N} l_{ik}^{-}(u) l_{jl}^{+}(u^{-1}) \otimes s_{kl}(u).$$

Clearly the map $s_{ij} \mapsto s_{ij}^{(0)}$ defines an embedding $U_q^{tw}(\mathfrak{o}_N) \to Y_q^{tw}(\mathfrak{o}_N)$. On the other hand, the mapping

$$(4.5) S(u) \mapsto S + q^{-1}u^{-1}\overline{S}$$

defines an algebra homomorphism $Y_q^{tw}(\mathfrak{o}_N) \to U_q^{tw}(\mathfrak{o}_N)$.

We introduce the matrices $\overline{S}(u) = (\overline{s}_{ij}(u))$ by

(4.6)
$$\overline{S}(u) = L^{+}(uq^{-c})L^{-}(u^{-1})^{t},$$

then one has the following relations.

(4.7)
$$(uq - u^{-1}q^{-1})\overline{s}_{ij}(u) = (uq^{\delta_{ij}} - u^{-1}q^{-\delta_{ij}})s_{ji}(u^{-1}) + (q - q^{-1})(u\delta_{j < i} + u^{-1}\delta_{i < j})s_{ij}(u^{-1}),$$

this implies that the coefficients of the series $\overline{s}_{ij}(u)$ generate $Y_q^{\text{tw}}(\mathfrak{o}_N)$.

4.2. Symplectic twisted -Yangians. The Symplectic twisted q-Yangians $Y_q^{tw}(\mathfrak{sp}_N)$ is generated by $s_{ij}^{(r)}$, with $1 \leq i, j \leq N$ and r runs over nonnegative integers, and $(s_{ii'}^{(0)})^{-1}$ with $i = 1, 3, \ldots, 2n - 1$. The defining relations are

$$s_{ij}^{(0)} = 0 \text{ for } i < j \text{ with } j \neq i',$$

$$s_{i'i'}^{(0)} - q^2 s_{i'i}^{(0)} s_{ii'}^{(0)} = q^3, \text{ for } i = 1, 3, \dots, 2n - 1,$$

$$s_{ii'}^{(0)} (s_{ii'}^{(0)})^{-1} = (s_{ii'}^{(0)})^{-1} s_{ii'}^{(0)} = 1,$$

$$R(u/v) S_1(u) R^t (1/uv) S_2(v) = S_2(v) R^t (1/uv) S_1(u) R(u/v),$$

where

(4.9)
$$s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)} u^{-r}.$$

Theorem 4.2. [20] The map $S(u) \mapsto L^-(uq^{-c})GL^+(u^{-1})^t$ defines an algebra embedding of $Y_q^{tw}(\mathfrak{sp}_N) \longrightarrow U_q(\widehat{\mathfrak{gl}}_N)$. The ordered monomials in the generators constitute a basis of $Y_q^{tw}(\mathfrak{sp}_N)$.

Regarding $Y_q^{tw}(\mathfrak{sp}_N)$ as subalgebra of $U_q(\widehat{\mathfrak{gl}}_N)$, we also have

(4.10)
$$\Delta(s_{ij}(u)) = \sum_{k,l=1}^{N} t_{ik}(u)\overline{t}_{jl}(u^{-1}) \otimes s_{kl}(u).$$

The map $s_{ij} \mapsto s_{ij}^{(0)}, s_{ii'}^{-1} \mapsto (s_{ii}^{(0)})^{-1}$ defines an embedding $U_q^{tw}(\mathfrak{sp}_N) \to Y_q^{tw}(\mathfrak{sp}_N)$. On the other hand, the mapping

(4.11)
$$S(u) \mapsto S + qu^{-1}\overline{S}, \qquad (s_{ii}^{(0)})^{-1} \mapsto s_{ii'}^{-1}$$

defines an algebra homomorphism $\mathbf{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_N) \to \mathbf{U}^{\mathrm{tw}}_q(\mathfrak{sp}_N)$.

We introduce the matrices $\overline{S}(u) = (\overline{s}_{ij}(u))$ by

(4.12)
$$\overline{S}(u) = L^{+}(uq^{-c})GL^{-}(u^{-1})^{t},$$

then one has the following relations.

(4.13)
$$(u^{-1}q - uq^{-1})\overline{s}_{ij}(u) = (uq^{\delta_{ij}} - u^{-1}q^{-\delta_{ij}})s_{ji}(u^{-1})$$
$$+ (q - q^{-1})(u\delta_{i < j} + u^{-1}\delta_{j < i})s_{ij}(u^{-1}),$$

which implies that the coefficients of the series $\overline{s}_{ij}(u)$ also generate $Y_q^{tw}(\mathfrak{sp}_N)$.

4.3. Sklyanin determinant. From now one, the symbol $Y_q^{tw}(\mathfrak{g}_N)$ will denote either $Y_q^{tw}(\mathfrak{o}_N)$ or $Y_q(\mathfrak{sp}_N)$ (with N=2n in the latter). For convenience, we introduce the following R-matrix:

(4.14)
$$\overline{R}(x) = \frac{R(x,1)}{x-1}.$$

For any permutation i_1, i_2, \ldots, i_m of $1, 2, \ldots, m$ we denote

$$\langle S_{i_1}, S_{i_2}, \cdots, S_{i_m} \rangle$$

$$(4.15) = S_{i_1}(u_{i_1})(\overline{R}_{i_1i_2}^t \cdots \overline{R}_{i_1i_m}^t)S_{i_2}(u_{i_2})(\overline{R}_{i_2i_3}^t \cdots \overline{R}_{i_2i_m}^t)\cdots S_{i_m}(u_{i_m}),$$

where $\overline{R}_{i_a i_b}^t = \overline{R}_{i_a i_b}^t (u_{i_a}^{-1} u_{i_b}^{-1})$. The following relation in the algebra $Y_q^{tw}(\mathfrak{g}_N) \otimes (\text{End}\mathbb{C}^N)^{\otimes m}$ follows from the reflection relation

$$(4.16) R(u_1,\ldots,u_m)\langle S_1,S_2,\cdots,S_m\rangle = \langle S_m,S_{m-1},\cdots,S_1\rangle R(u_1,\ldots,u_m).$$

Specializing $u_i = uq^{-2i+2}$, we have that (cf. (3.18))

$$(4.17) A_m^q \langle S_1, S_2, \cdots, S_m \rangle = \langle S_m, S_{m-1}, \cdots, S_1 \rangle A_m^q.$$

This element can be written as

$$(4.18) \frac{1}{m!} \sum_{j_1 \cdots j_m} s_{j_1 \cdots j_m}^{i_1 \cdots i_m}(u) \otimes e_{i_1 j_1} \otimes \cdots \otimes e_{i_m j_m},$$

where the sum is taken over all $i_k, j_k \in \{1, 2, \dots, N\}$. We call $s_{j_1 \dots j_m}^{i_1 \dots i_m}(u)$ the Sklyanin minor associated to the rows $i_1 \dots i_m$ and the columns $j_1 \dots j_m$. Clearly if $i_k = i_l$ or $j_k = j_l$ for $k \neq l$, then $s_{j_1 \dots j_m}^{i_1 \dots i_m}(u) = 0$. Suppose that $i_1 < i_2 < \dots < i_m$ and $j_1 < j_2 < \dots < j_m$. Then the Sklyanin minors satisfy the relations:

$$(4.19) s_{j_{\tau(1)}\cdots j_{\tau(m)}}^{i_{\sigma(1)}\cdots i_{\sigma(m)}}(u) = (-q)^{-l(\sigma)-l(\tau)} s_{j_{1}\cdots j_{m}}^{i_{1}\cdots i_{m}}(u).$$

Note that $s_j^i = s_{ij}(u)$. In particular, $s_{1\cdots N}^{1\cdots N}(u)$ is called the Sklyanin determinant and will be denoted by $\operatorname{sdet}_{q}S(u)$.

The following theorem provides an expression of sdet S(u) in terms of the quantum determinants.

Theorem 4.3. [20] We have

(4.20)
$$\operatorname{sdet} S(u) = \gamma_N(u) \operatorname{det}_q L^-(uq^{-c}) \operatorname{det}_q L^+(q^{2N-2}u^{-1}),$$

where

(4.21)
$$\gamma_N(u) = \begin{cases} 1, & Case \ (\mathfrak{o}_N), \\ \frac{q^{n-2} - q^n u^2}{q^{2n-2} - q^{-2n} u^2}, & Case \ (\mathfrak{sp}_N). \end{cases}$$

We introduce the elements $c_k \in Y_q^{tw}(\mathfrak{g}_N)$ as the coefficients of the following series c(u):

(4.22)
$$c(u) = \gamma_N(u)^{-1} \operatorname{sdet} S(u) = 1 + \sum_{k=1}^{\infty} c_k u^{-k}.$$

COROLLARY 4.4. [20] The coefficients $c_k, k \geq 1$ belong to the center of the algebra $Y_q^{tw}(\mathfrak{g}_N)$.

Proposition 4.5. The coefficients of the Sklyanin determinants sdet(S+ $q^{-1}u^{-1}\overline{S}$) and $\operatorname{sdet}(S+qu^{-1}\overline{S})$ are central elements in the algebras $\operatorname{U}_q^{\operatorname{tw}}(\mathfrak{o}_N)$ and $U_a^{tw}(\mathfrak{sp}_N)$ respectively.

We define the auxiliary minor $\check{s}_{i_1,\dots,i_m}^{i_1,\dots,i_m}$ (u) by

(4.23)
$$m! A_m^q \langle S_1, \dots, S_{m-1} \rangle \overline{R}_{1m}^t (u_1^{-1} u_m^{-1}) \cdots \overline{R}_{m-1,m}^t (u_{m-1}^{-1} u_m^{-1})$$

$$= \sum_{j_1, \dots, j_{m-1}, c} \check{s}_{j_1, \dots, j_{m-1}, c}^{i_1, \dots, i_m} (u) \otimes e_{i_1 j_1} \otimes \dots \otimes e_{i_m c},$$

where the sum is taken over all $i_k, j_k, c \in \{1, 2, \dots, N\}$. If $i_k = i_l$ or $j_k = j_l$ for $k \neq l$, then $\check{s}^{i_1 \cdots i_m}_{j_1 \cdots j_m - 1, c}(u) = 0$. Suppose $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_{m-1}$, then

(4.24)
$$\check{s}_{j_{\tau(1)}\cdots j_{\tau(m-1)},c}^{i_{\sigma(1)}\cdots i_{\sigma(m)}}(u) = (-q)^{l(\sigma)-l(\tau)}\check{s}_{j_{1}\cdots j_{m-1},c}^{i_{1}\cdots i_{m}}(u).$$

$$\sigma \in S_{m}, \tau \in S_{m-1}.$$

LEMMA 4.6. For $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_{m-1}$,

(4.25)
$$s_{j_1,\dots,j_m}^{i_1,\dots,i_m}(u) = \sum_{c=1}^N \check{s}_{j_1,\dots,j_{m-1},c}^{i_1,\dots,i_m}(u) s_{cj_m}(uq^{2-2m})$$

PROOF. The identity follows from the formula

(4.26)
$$A_m^q \langle S_1, \dots, S_m \rangle = A_m^q \langle S_1, \dots, S_{m-1} \rangle \overline{R}_{1m}^t (u_1^{-1} u_m^{-1}) \cdots \overline{R}_{m-1,m}^t (u_{m-1}^{-1} u_m^{-1}) S_m (uq^{2-2m}).$$

For any $1 \leq i, j \leq N$ we define the elements $s_{ij}^{\sharp}(u)$ as follows

$$(4.27) s_{ij}^{\sharp}(u) = \begin{cases} \frac{u^{-1} - u}{qu^{-1} - q^{-1}u} s_{ij}(u) + \frac{(q - q^{-1})u^{-1}}{qu^{-1} - q^{-1}u} s_{ji}(u), i < j, \\ \frac{u^{-1} - u}{qu^{-1} - q^{-1}u} s_{ij}(u) + \frac{(q - q^{-1})u}{qu^{-1} - q^{-1}u} s_{ji}(u), i > j \\ s_{ii}(u), i = j. \end{cases}$$

Note that in the orthogonal case, $s_{ij}^{\sharp}(u) = \overline{s}_{ji}(u^{-1})$.

Proposition 4.7. Suppose $i_1 < i_2 < \cdots < i_{m-1}, j_2 < \cdots, j_{m-1}, j_1 \in$ $\{i_1, \ldots, i_m\}$ and $c \notin \{j_2, \ldots, j_{m-1}\}$. Then

$$\check{s}_{j_1\cdots j_{m-1},c}^{i_1\cdots i_m}(u) = 0, \quad if \ c \notin \{i_1,\ldots,i_m\},$$

(4.28)
$$\check{s}_{j_{1}\cdots j_{m-1},c}^{i_{1}\cdots i_{m}}(u) = 0, \quad if \ c \notin \{i_{1},\ldots,i_{m}\}, \\
\check{s}_{j_{1}\cdots j_{m-1},c}^{i_{1}\cdots i_{m}}(u) = \sum_{r=1}^{m-1} (-q)^{1-r} s_{i_{r},j_{1}}^{\sharp}(u) s_{j_{2},\ldots,j_{m-1}}^{i_{1},\ldots,\hat{i_{r}},\ldots,i_{m-1}}(uq^{-2}),$$

if $c = i_m$.

PROOF. For $c \notin \{j_2, \ldots, j_{m-1}\}$, we have that

$$(4.29) \overline{R}_{2m}^t \cdots \overline{R}_{m-1,m}^t e_{j_2} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c = e_{j_2} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c$$
We compute that

$$(4.30) A_m^q \langle S_1, \dots, S_{m-1} \rangle \overline{R}_{1m}^t \cdots \overline{R}_{m-1,m}^t e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c$$

$$= A_m^q S_1 \overline{R}_{12}^t \cdots \overline{R}_{1,m}^t \langle S_2, \dots, S_{m-1} \rangle e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c,$$

where $S_i = S_i(uq^{2-2i})$. Let $A_{\{2,\dots,m\}}^q$ be the q-antisymmetrizer on the indices $\{2,\ldots,m\}$. Then $A_m^q = A_m^q A_{\{2,\ldots,m\}}^q$, and

$$(4.31) A_{\{2,\dots,m\}}^{q} \overline{R}_{12}^{t} \cdots \overline{R}_{1,m}^{t} = \overline{R}_{1m}^{t} \cdots \overline{R}_{12}^{t} A_{\{2,\dots,m\}}^{q}$$

$$= A_{\{2,\dots,m\}}^{q} \overline{R}_{12}^{t} \cdots \overline{R}_{1,m}^{t} A_{\{2,\dots,m\}}^{q} = A_{\{2,\dots,m\}}^{q} \overline{R}_{1m}^{t} \cdots \overline{R}_{12}^{t} A_{\{2,\dots,m\}}^{q}$$

which follows from the Yang-Baxter equation (3.3).

Then we can rewrite (4.30) as

$$(4.32) \begin{array}{l} A_m^q S_1 \overline{R}_{12}^t \cdots \overline{R}_{1,m}^t A_{\{2,\dots,m\}}^q \langle S_2,\dots,S_{m-1} \rangle e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c \\ = A_m^q S_1 \overline{R}_{12}^t \cdots \overline{R}_{1,m}^t \sum_{j_2,\dots,j_{m-1}} s_{j_2,\dots,j_{m-1}}^{k_2,\dots,k_{m-1}} (q^{-2}u) e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c \end{array}$$

where the sum runs through $k_2 < \cdots < k_{m-1}$. Now let's compute the coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in

$$(4.33) A_m^q S_1 \overline{R}_{12}^t \cdots \overline{R}_{1,m}^t e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c.$$

This is done in several cases:

(i) If $c \notin \{i_1, \ldots, i_m\}$, then $c \neq j_1$ and

$$(4.34) \overline{R}_{1m}^t e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c = e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c,$$

So the basis vectors $e_{r_1} \otimes \cdots \otimes e_{r_m}$ in the expansion of (4.32) only contain those with $c \in \{r_1, \ldots, r_m\}$, thus $\check{s}_{j_1 \cdots j_{m-1}, c}^{i_1 \cdots i_m}(u) = 0$.

(ii) If $c = j_1 = i_m$, then

$$(4.35) A_{\{2,\ldots,m\}}^q e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c = 0$$

if $k_r = j_1$ for some $2 \le r \le m-1$. In order to have $e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m}$, $\{k_2, \dots, k_{m-1}\}$ must be $\{i_1, i_2, \dots, \hat{i_r}, \dots, i_{m-1}\}$ for some $1 \le r \le m-1$ and $j_1 \notin \{i_1, i_2, \dots, \hat{i_r}, \dots, i_{m-1}\}$.

Assume that $i_p < j_1 < i_{p+1}$. For r < p, the coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in (4.33) is

$$(4.36) (-q)^{1-r} \frac{q^{-1}x - q}{x - 1} s_{i_r j_1}(u) + (-q)^{-r} \frac{(q^{-1} - q)x}{x - 1} s_{j_1 i_r}(u),$$

where $x = q^2 u^{-2}$. This element can be written as $(-q)^{1-r} s_{i_r,j_1}^{\sharp}(u)$. For r > p, the coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in (4.33) is

$$(4.37) (-q)^{1-r} \frac{q^{-1}x - q}{x - 1} s_{i_r j_1}(u) + (-q)^{2-r} \frac{(q^{-1} - q)}{x - 1} s_{j_1 i_r}(u),$$

which is equal to $(-q)^{1-r}s_{i_r,j_1}^{\sharp}(u)$.

(iii) Suppose $c = i_m$ and $j_1 = i_p$ for some $1 \le p \le m - 1$. If $j_1 \notin \{k_2, \ldots, k_{m-1}\}$, then

$$A_m^q S_1 R_{12}^t \cdots R_{1,m}^t e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c$$

(4.38)
$$= A_m^q \sum_{k_1=1}^N s_{k_1 j_1}(u) e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c$$

If $j_1 = k_s$ for $2 \le s \le m - 1$, then

$$(4.39) A_m^q S_1 R_{12}^t \cdots R_{1,m}^t e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c = (-q)^{s-2} A_m^q S_1 R_{12}^t \cdots R_{1,m}^t e_{j_1} \otimes e_{k_s} \otimes e_{k_2} \dots e_{k_s} \dots e_{k_{m-1}} \otimes e_c$$

In both cases, in order to have $e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m}$, $\{k_2, \cdots, k_{m-1}\}$ must be $\{i_1, i_2, \cdots, \hat{i_r}, \cdots, i_{m-1}\}$ for some $1 \leq r \leq m-1$.

The coefficient of $e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m}$ in (4.33) is

$$\begin{cases}
(-q)^{1-r} \frac{q^{-1}x-q}{x-1} s_{i_r j_1}(u) + (-q)^{-r} \frac{(q^{-1}-q)x}{x-1} s_{j_1 i_r}(u), & r < p, \\
(-q)^{1-r} \frac{q^{-1}x-q}{x-1} s_{i_r j_1}(u) + (-q)^{2-r} \frac{(q^{-1}-q)x}{x-1} s_{j_1 i_r}(u), & r > p \\
(-q)^{1-r} s_{i_r, j_1}(u), & r = p.
\end{cases}$$

where $x = q^2 u^{-2}$. All of these are equal to $(-q)^{1-r} s_{i_r,j_1}^{\sharp}(u)$. Therefore, in cases (ii) and (iii)

$$(4.41) \check{s}_{j_1\cdots j_{m-1},c}^{i_1\cdots i_m}(u) = \sum_{r=1}^{m-1} (-q)^{1-r} s_{i_r,j_1}^{\sharp}(u) s_{j_2,\dots,j_{m-1}}^{i_1,\dots,\hat{i_r},\dots,i_{m-1}}(uq^{-2}).$$

This completes the proof.

In order to produce an explicit formula for the Sklyanin determinant in the orthogonal case we introduce a map

$$\pi_N: S_N \to S_N, \ p \mapsto p'$$

which was used in the formula for the Sklyanin determinant for the twisted Yangians [18, 20]. The map π_N is defined inductively as follows. Given a set of positive integers $\omega_1 < \cdots < \omega_N$, and consider the action of S_N on these indices. If N=2 we define π_2 as the identity map of $S_2 \to S_2$. For N>2 define a map from the set of ordered pairs (ω_k, ω_l) with $k \neq l$ into themselves by the rule

$$(\omega_{k}, \ \omega_{l}) \mapsto (\omega_{l}, \ \omega_{k}) \ , \ k, \ l < N,$$

$$(\omega_{k}, \ \omega_{N}) \mapsto (\omega_{N-1}, \ \omega_{k}) \ , \ k < N-1,$$

$$(\omega_{N}, \ \omega_{k}) \mapsto (\omega_{k}, \ \omega_{N-1}) \ , \ k < N-1,$$

$$(\omega_{N-1}, \ \omega_{N}) \mapsto (\omega_{N-1}, \ \omega_{N-2}) \ ,$$

$$(\omega_{N}, \ \omega_{N-1}) \mapsto (\omega_{N-1}, \ \omega_{N-2}) \ .$$

Let $p = (p_1, \ldots, p_N)$ be a permutation of the indices $\omega_1, \ldots, \omega_N$. Its image under the map π_N is the permutation $p' = (p'_1, \ldots, p'_{N-1}, \omega_N)$, where the pair (p'_1, p'_{N-1}) is the image of the ordered pair (p_1, p_N) under the map (4.42). Then the pair (p'_2, p'_{N-2}) is found as the image of (p_2, p_{N-1}) under the map (4.42) which is defined on the set of ordered pairs of elements obtained from $(\omega_1, \ldots, \omega_N)$ by deleting p_1 and p_N . The procedure is completed in the same manner by determining consecutively the pairs (p'_i, p'_{N-i}) .

In the following theorem, we give explicit formula for Sklyanin determinants. Note that the explicit formula for Sklyanin determinants in orthogonal case was given in [20].

Theorem 4.8. The Sklyanin determinant $\operatorname{sdet}_q(S(u))$ can be written explicitly as

(4.43)
$$\operatorname{sdet}_{q}(S(u)) = \sum_{p \in S_{N}} (-q)^{l(p')-l(p)} s_{p_{1}p'_{1}}^{\sharp}(u) \cdots s_{p_{n}p'_{n}}^{\sharp}(q^{2-2n}u) \times s_{p_{n+1}p'_{n+1}}(q^{-2n}u) \cdots s_{p_{N}p'_{N}}(q^{2-2N}u).$$

PROOF. For $i_1 < i_2 \cdots < i_m$, we can write

$$(4.44) s_{i_1,\dots,i_{m-1},j_m}^{i_1,\dots,i_m}(u) = \sum_{k=1}^m \check{s}_{i_1,\dots,i_{m-1},i_k}^{i_1,\dots,i_m}(u) s_{i_k,j_m}(q^{2-2m}u).$$

It follows from Proposition 4.7 that

$$\begin{split} s_{i_1,\cdots,i_m}^{i_1,\cdots,i_m} &= s_{i_{m-1},i_{m-1}}^\sharp(u) s_{i_1,\cdots,i_{m-2}}^{i_1,\cdots,i_{m-2}}(q^{-2}u) s_{i_m,j_m}(q^{2-2m}u) \\ &+ (-q)^{2m-2l+3} \sum_{l=1}^{m-2} s_{i_l,i_{m-1}}^\sharp(u) s_{i_1,\cdots,\hat{i_l},\cdots,i_{m-1}}^{i_1,\cdots,\hat{i_l},\cdots,i_{m-1}}(q^{-2}u) s_{i_m,j_m}(q^{2-2m}u) \\ &+ (-q)^{2k-2m+1} \sum_{k=1}^{m-1} s_{i_m,i_k}^\sharp(u) s_{i_1,\cdots,\hat{i_k},\cdots,i_{m-1}}^{i_1,\cdots,\hat{i_k},\cdots,i_{m-1}}(q^{-2}u) s_{i_k,j_m}(q^{2-2m}u) \\ &+ (-q)^{2m-2k-2l} \sum_{k=1}^{m-1} \sum_{l=1}^{k-1} s_{i_l,i_k}^\sharp(u) s_{i_1,\cdots,\hat{i_l},\cdots,\hat{i_k},\cdots,i_{m-1},i_l}^{i_1,\cdots,\hat{i_k},\cdots,i_{m-1},i_l}(q^{-2}u) s_{i_k,j_m}(q^{2-2m}u) \end{split}$$

$$+ (-q)^{2m-2k-2l+2} \sum_{k=1}^{m-1} \sum_{l=k+1}^{m-1} s_{i_l,i_k}^{\sharp}(u) s_{i_1,\dots,\hat{i_k},\dots,\hat{i_l},\dots,\hat{i_l},\dots,i_{m-1},i_l}^{i_1,\dots,\hat{i_k},\dots,\hat{i_l},\dots,i_{m-1}}(q^{-2}u) s_{i_k,j_m}(q^{2-2m}u)$$

Starting with $s_{1,\dots,N}^{1,\dots,N}(u)$, we apply the recurrence relation repeatedly to write the Sklyanin determinant $\operatorname{sdet}_q(X)$ in terms of the generator s_{ij} :

4.4. Minor identities for Sklyanin determinants. We define the Sklyanin comatrix $\widehat{S}(u)$ by

(4.46)
$$\widehat{S}(u)S(q^{2-2N}u) = \operatorname{sdet}_q(S(u))I.$$

PROPOSITION 4.9. The matrix elements $\hat{s}_{ij}(u)$ are given by

(4.47)
$$\hat{s}_{ij}(u) = (-q)^{N-i} \check{s}_{1,\dots\hat{i},\dots,N,j}^{1,\dots,N}(u)$$

for $i \neq j$ and

(4.48)
$$\hat{s}_{ii}(u) = s_{1,\dots\hat{i},\dots,N}^{1,\dots\hat{i},\dots,N}(u).$$

PROOF. Multiplying $S_N(q^{2-2N}u)^{-1}$ from the right of the formulas

$$(4.49) A_N^q \langle S_1, \dots, S_N \rangle = A_N^q \operatorname{sdet}_q(S(u)),$$

we get that

$$(4.50) A_N^q \langle S_1, \dots, S_{N-1} \rangle \overline{R}_{1N}^t \cdots \overline{R}_{N-1,N}^t = A_N^q \hat{S}_N(u).$$

Applying both sides to the vector

$$(4.51) v_{ij} = e_1 \otimes \cdots \hat{e_i} \otimes e_N \otimes e_j$$

and comparing the coefficients of $e_1 \otimes \cdots \otimes e_N$ we get the first formula. Using $\overline{R}_{kN}^t v_{ii} = v_{ii}$ for $1 \leq k \leq N-1$, applying the operators to the vector v_{ii} we obtain the second formula.

Let $C = \operatorname{diag}((-q)^{\frac{N-1}{2}}, (-q)^{\frac{N-3}{2}}, \dots, (-q)^{-\frac{N-1}{2}})$ be the $N \times N$ diagonal matrix. We have that

$$(4.52) R(x)C_1C_2 = C_1C_2R(x),$$

and the crossing symmetry relations can be written as

(4.53)
$$R_{12}^{-1}(x)^{t_2}C_1C_2R_{12}^{t_2}(xq^{2N}) = C_1C_2, R_{12}^{t_1}(xq^{2N})C_1C_2R_{12}^{-1}(x)^{t_1} = C_1C_2.$$

We see that

$$(4.54) (\overline{R}(x)^t)^{-1} = C_1 C_2 \overline{R}' (q^{-2N} x)^t C_1^{-1} C_2^{-1},$$

where $\overline{R}'(x)$ is obtained from $\overline{R}(x)$ by replacing q with q^{-1} . Taking the inverse of both sides of the reflection equation

(4.55)
$$R(u/v)S_1(u)R^t(1/uv)S_2(v) = S_2(v)R^t(1/uv)S_1(u)R(u/v),$$

subsequently

(4.56)
$$R'(u/v)S_1(u)^{-1}C_1C_2\left(R'(1/q^{2N}uv)\right)^tC_1^{-1}C_2^{-1}S_2(v)^{-1}$$
$$=S_2(v)^{-1}C_1C_2\left(R'(1/q^{2N}uv)\right)^tC_1^{-1}C_2^{-1}S_1(u)^{-1}R'(u/v).$$

Multiplying $C_1^{-1}C_2^{-1}$ from the left and C_1C_2 from the right of both sides,

(4.57)
$$R'(u/v)C_1^{-1}S_1(u)^{-1}C_1\left(R'(1/q^{2N}uv)\right)^tC_2^{-1}S_2(v)^{-1}C_2$$
$$=C_2^{-1}S_2(v)^{-1}C_2\left(R'(1/q^{2N}uv)\right)^tC_1^{-1}S_1(u)^{-1}C_1R'(u/v).$$

Denote $C^{-1}S(q^{-N}u)^{-1}C$ by X(u), then X(u) satisfies the q^{-1} -reflection relation, i.e.

(4.58)
$$R'(u/v)X_1(u) \left(R'(1/uv)\right)^t X_2(u) = X_2(u) \left(R'(1/uv)\right)^t X_1(u)R'(u/v).$$

We write S(u) as

(4.59)
$$s_{ij}(u) = \sum_{r=0}^{\infty} S^{(r)} u^{-r}.$$

In the orthogonal case, the matrix $S^{(0)}$ is a lower unitriangular matrix, it is clearly invertible and the inverse of $S^{(0)}$ is lower unitriangular. In the symplectic case, $S^{(0)}$ is invertible and (4.60)

$$\begin{pmatrix} (S^{(0)})^{-1}_{ii} & (S^{(0)})^{-1}_{ii'} \\ (S^{(0)})^{-1}_{i'i} & (S^{(0)})^{-1}_{i'i'} \end{pmatrix} = q^{-3} \begin{pmatrix} s_{i'i'} & -q^2 s_{ii'} \\ (q - q^{-1})s_{ii'} - s_{i'i} & s_{ii}, \end{pmatrix}$$

therefore

$$(4.61) (S^{(0)})^{-1}{}_{i'i'}(S^{(0)})^{-1}{}_{ii} - q^{-2}(S^{(0)})^{-1}{}_{i'i}(S^{(0)})^{-1}{}_{ii'} = q^{-6}s_{ii}s_{i'i'} + q^{-6}((q - q^{-1})s_{ii'} - s_{i'i})s_{ii'} = q^{-3}.$$

Also we have that

(4.62)
$$(S^{(0)})^{-1}_{ij} = 0 \text{ for } i < j \text{ with } j \neq i'.$$

Combining with (4.58), we have the following proposition.

PROPOSITION 4.10. The mapping $S(u) \mapsto X(u)$, $q \mapsto q^{-1}$, defines an automorphism of $Y_q^{\text{tw}}(\mathfrak{g}_N)$.

The following result is the Sklyanin determinant analogue of Jacobi's theorem.

Theorem 4.11. Let $I = \{i_1 < i_2 < \dots < i_k\}$ be a subset of [1, N] and $I^c = \{i_{k+1} < \dots < i_N\}$ the complement of I. Then

$$(4.63) \operatorname{sdet}_{q} S_{I}(u) = \operatorname{sdet}_{q} S(u) \operatorname{sdet}_{q^{-1}} X_{I^{c}}(q^{2-N}u),$$

where
$$X(u) = C^{-1}S(q^{-N}u)^{-1}C$$
.

Proof. The q-antisymmetrizer satisfies the relation

(4.64)
$$C_1 \dots C_N A_N^q C_1^{-1} \dots C_N^{-1} = A_N^q.$$

By the relation

(4.65)
$$A_N^q \langle S_1, \dots, S_N \rangle = \operatorname{sdet} S(u) A_N^q$$

and the definition of $\langle S_1, \dots, S_N \rangle$ we have that

(4.66)
$$A_N^q \langle S_1, \dots, S_k \rangle \overrightarrow{\prod}_{1 \le i \le k < j \le N} \overline{R}_{ij}^t \\ = \operatorname{sdet}_q(X) A_N^q S_N^{-1} (\overline{R}_{N-1,N}^t)^{-1} S_{N-1} \cdots (\overline{R}_{k+1,k+2}^t)^{-1} S_{k+1}^{-1}$$

where
$$u_{i} = uq^{-2i+2}$$
, $S_{i} = S(u_{i})$ and $\overline{R}_{ij}^{t} = \overline{R}_{ij}^{t}(1/u_{i}u_{j})$.

$$A_{N}^{q}\langle S_{1}, \dots, S_{k}\rangle \overrightarrow{\prod}_{1 \leq i \leq k < j \leq N} \overline{R}_{ij}^{t}$$

$$= \operatorname{sdet}_{q}(S(u))A_{N}^{q}S_{N}^{-1}(\overline{R}_{N-1,N}^{t})^{-1}S_{N-1}\cdots(\overline{R}_{k+1,k+2}^{t})^{-1}S_{k+1}^{-1}$$

$$= \operatorname{sdet}_{q}S(u)A_{N}^{q}C_{k+1}\cdots C_{N}X_{N}(q^{N}u_{N})(\overline{R}_{N-1,N}^{t}(1/q^{2N}u_{N-1}u_{N}))^{t}$$

$$X_{N-1}(q^{N}u_{N-1})\cdots(\overline{R}_{k+1,k+2}^{t}(1/q^{2N}u_{k+1}u_{k+2}))^{t}X_{k+1}(q^{N}u_{k+1})$$

$$C_{k+1}^{-1}\cdots C_{N}^{-1}.$$

Applying both sides to the vector $v = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes e_{i_N} \otimes \ldots e_{i_{k+1}}$, we see that v's coefficient in $A_N^q \langle S_1, \ldots, S_k \rangle \overrightarrow{\prod}_{1 \leq i \leq k < j \leq N} \overline{R}_{ij}^t v$ (the LHS) is $\operatorname{sdet} S_I(u)$ due to the fact that

$$(4.68) \qquad \prod_{1 \le i \le k \le j \le N} \overline{R}_{ij} (1/u_i u_j)^t v = v.$$

The coefficient of v in the RHS of (4.67) is $\operatorname{sdet}_{q^{-1}} X_{I^c}(q^{2-N}u)$, which has proved (4.63).

The following is the special case of Theorem 4.11 for $I = \{1, 2, \dots, N\}$.

COROLLARY 4.12. In the algebra $Y_q^{tw}(\mathfrak{g}_N)$ we have that

(4.69)
$$\operatorname{sdet}_{q} S(u) \operatorname{sdet}_{q-1} X(q^{2-N}u) = 1.$$

THEOREM 4.13. For any $1 \le a, b \le N$, one has that

$$(4.70) s_{b,N+1,\dots,N+M}^{a,N+1,\dots,N+M}(u) = (-q)^{b-N} \operatorname{sdet}_q(S(u)) \check{X}_{1,\dots,\hat{a},\dots,N,b}^{1,\dots,N}(q^{2-N-M}u),$$

$$where X(u) = C^{-1} S(q^{-N-M}u)^{-1} C.$$

PROOF. The proof is similar to Jacobi's theorem. Using the relation (4.71)

$$A_{N+M}\langle S_1, \dots, S_{N+1}\rangle \overrightarrow{\prod}_{1\leq i\leq N, N+2\leq j\leq N} \overline{R}_{ij}^t$$

$$= \operatorname{sdet}_q(S(u)) A_{N+M} C_{N+1} \dots C_{N+M} X_{N+M} (q^{N+M} u_{N+M}) (\overline{R}'_{N-1,N})^t \dots$$

$$\dots X_{N+2} (q^{N+M} u_{N+2})$$

$$\cdot (\overline{R}'_{N+1,N})^t \dots (\overline{R}'_{N+1,N+2})^t C_{N+1}^{-1} \dots C_{N+M}^{-1}$$

Applying both sides to the vector $e_{N+1} \otimes \dots e_{N+M} \otimes e_b \otimes e_N \otimes \dots \otimes \widehat{e_a} \dots \otimes e_1$ and comparing the coefficient of $e_1 \otimes e_2 \otimes \dots \otimes e_{N+M}$ we obtain that

$$(4.72) \quad s_{b,N+1,\dots,N+M}^{a,N+1,\dots,N+M}(u) = (-q)^{b-N} \operatorname{sdet}_q(S(u)) \check{X}_{1,\dots,\hat{a},\dots,N,b}^{1,\dots,N}(q^{2-N-M}u)$$

The following is an analogue of Schur's complement theorem.

Theorem 4.14. We write

(4.73)
$$S(u) = \begin{pmatrix} S_{11}(u) & S_{12}(u) \\ S_{21}(u) & S_{22}(u) \end{pmatrix},$$

where S_{11} is $N \times N$ matrix and S_{22} is $M \times M$ matrix. In symplectic case, N and M are even. Then

(4.74)

 $sdet_q(S(u))$

$$= \operatorname{sdet}_{q}(S_{11}(u)) \operatorname{sdet}_{q}(S_{22}(q^{-2N}u) - S_{21}(q^{-2N}u)S_{11}(q^{-2N}u)^{-1}S_{12}(q^{-2N}u))$$

$$= \operatorname{sdet}_{q}(S_{22}(u)) \operatorname{sdet}_{q}(S_{22}(q^{-2M}u) - S_{21}(q^{-2M}u)S_{11}(q^{-2M}u)^{-1}S_{12}(q^{-2M}u))$$

PROOF. We subdivide C and $X(u) = C^{-1}S(q^{-N-M}u)^{-1}C$ into blocks

(4.75)
$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}, \qquad X(u) = \begin{pmatrix} X_{11}(u) & X_{12}(u) \\ X_{21}(u) & X_{22}(u) \end{pmatrix},$$

then

$$X_{11}(u) = C_{11}^{-1} \left(S_{11}(q^{-M-N}u) - S_{12}(q^{-M-N}u) S_{22}(q^{-M-N}u)^{-1} \right)$$

$$(4.76) S_{21}(q^{-M-N}u)^{-1} C_{11}$$

$$X_{22}(u) = C_{22}^{-1} \left(S_{22}(q^{-M-N}u) - S_{21}(q^{-M-N}u) S_{11}(q^{-M-N}u)^{-1} \right)$$

$$(4.77) S_{12}(q^{-M-N}u)^{-1} C_{22}$$

It follows form Theorem 4.11 that

(4.78)
$$\operatorname{sdet}_{q}(S_{11}(u)) = \operatorname{sdet}_{q}(S(u))\operatorname{sdet}_{q^{-1}}(X_{22}(q^{2-N-M}u))$$

Since $X_{22}(u)$ satisfies the q^{-1} reflection relation,

(4.79)
$$C_{22}X_{22}(q^{M}u)^{-1}C_{22}^{-1} = S_{22}(q^{-N}u) - S_{21}(q^{-N}u)S_{11}(q^{-N}u)^{-1}S_{12}(q^{-N}u)$$

satisfies the q reflection relation. By Corollary 4.12,

(4.80)
$$\operatorname{sdet}_{q}(S_{22}(q^{M-N-2}u) - S_{21}(q^{M-N-2}u)S_{11}(q^{M-N-2}u)^{-1} S_{12}(q^{M-N-2}u)\operatorname{sdet}_{q^{-1}}X_{22}(u) = 1.$$

Therefore,

$$sdet_q(S(u))$$

(4.81)
$$= \operatorname{sdet}_{q}(S_{11}(u))\operatorname{sdet}_{q}(S_{22}(q^{-2N}u) - S_{21}(q^{-2N}u)S_{11}(q^{-2N}u)^{-1}$$

$$S_{12}(q^{-2N}u))$$

The second equation can be proved similarly.

Using Jacobi's theorem we obtain the following analogue of Cayley's complementary identity for the Sklyanin determinant.

Theorem 4.15. Suppose a minor identity for the Sklyanin determinant is given:

(4.82)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \operatorname{sdet}_q S_{I_{ij}}(u) = 0,$$

where $I'_{ij}s$ are subsets of [1, N] and $b_i \in \mathbb{C}(q)$. Then the following identity holds

(4.83)
$$\sum_{i=1}^{k} b_i' \prod_{i=1}^{m_i} \operatorname{sdet}_q S(u)^{-1} \operatorname{sdet}_q S_{I_{ij}^c}(u) = 0,$$

where b'_i is obtained from b_i by replacing q by q^{-1} .

PROOF. The matrix $X(u) = C^{-1}S(q^{-N}u)^{-1}C$ satisfies the q^{-1} reflection relations. Applying the minor identity to X(u) we get that

(4.84)
$$\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \operatorname{sdet}_{q^{-1}}(X_{I_{ij}}(u)) = 0.$$

It follows from Theorem 4.11 that

(4.85)
$$\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \operatorname{sdet}_q S(q^{N-2}u)^{-1} \operatorname{sdet}_q S_{I_{ij}^c}(q^{N-2}u) = 0,$$

The proof is completed by replacing u with $q^{2-N}u$.

The following theorem is an analogue of Muir's law for the Sklyanin determinant.

Theorem 4.16. Suppose there is a minor Sklyanin determinant identity

(4.86)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \operatorname{sdet}_q S_{I_{ij}}(u) = 0,$$

where $I'_{ij}s$ are subsets of $I = \{1, 2, ..., N\}$ and $b_i \in \mathbb{C}(q)$. Let J be the set $\{N, ..., N + M\}$. Then the following identity holds

(4.87)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \operatorname{sdet}_q(S_J)^{-1} \operatorname{sdet}_q(S_{I_{ij} \cup J}) = 0.$$

Proof. Applying Cayley's complementary identity respect to the set I, we get that

(4.88)
$$\sum_{i=1}^{k} b'_{i} \prod_{j=1}^{m_{i}} \operatorname{sdet}_{q} S_{I}(u)^{-1} \operatorname{sdet}_{q} S_{I \setminus I_{ij}}(u) = 0,$$

Applying Cayley's complementary identity respect to the set $I \cup J$, we obtain that

(4.89)
$$\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \operatorname{sdet}_q S_J(u)^{-1} \operatorname{sdet}_q S_{I_{ij} \cup J}(u) = 0.$$

In the following we give an analogue of Muir's identities for the Sklyanin determinant.

Lemma 4.17. One has that

where $A^q_{\{2,\ldots,m\}}$ is the q-antisymmetrizer numbered by indices $\{2,\ldots,m\}$.

Proof. Taking transposition with respect to the first factor to Yang-Baxter equation we have

$$(4.91) R_{23}(v,w)R_{12}^t(u,v)R_{13}^t(u,w) = R_{13}^t(u,w)R_{12}^t(u,v)R_{23}(v,w).$$

Taking $v = 1, w = q^{-2}$ we have that

$$(4.92) A_{\{2,3\}}^q R_{12}^t(u) R_{13}^t(uq^2) = R_{13}^t(uq^2) R_{12}^t(u) A_{\{2,3\}}^q.$$

Since $(A_2^q)^2 = A_2^q$, we have

$$(4.93) \qquad A_{\{2,3\}}^q R_{12}^t(u) R_{13}^t(uq^2) A_{\{2,3\}}^q = A_{\{2,3\}}^q R_{12}^t(u) R_{13}^t(uq^2), A_{\{2,3\}}^q R_{12}^t(u) R_{13}^t(uq^2) P_{23}^q = -A_{\{2,3\}}^q R_{12}^t(u) R_{13}^t(uq^2).$$

It implies that

$$(4.94) A_{\{2,\dots,m\}}^q R_{12}^t(u) \cdots R_{1,m}^t(uq^{2m-2}) P_{i,i+1}$$

$$= -A_{\{2,\dots,m\}}^q R_{12}^t(u) \cdots R_{1,m}^t(uq^{2m-2})$$

for any $1 \leq i \leq m-1$. Using the formula for A_m^q we obtain the first equation. Taking transpositions with respect to the first and second factors consecutively to the Yang-Baxter equation we have

$$(4.95) R_{13}^t(u,w)R_{23}^t(v,w)R_{12}^{t_1t_2}(u,v) = R_{12}^{t_1t_2}(u,v)R_{23}^t(v,w)R_{13}^t(u,w).$$

Using the relation $R_{12}^{t_1t_2}(u, v) = P_{12}R_{12}(u, v)P_{12}$, we have

$$(4.96) R_{12}(u,v)R_{13}^t(v,w)R_{23}^t(u,w) = R_{23}^t(u,w)R_{13}^t(v,w)R_{12}(u,v).$$

Taking $u = 1, v = q^{-2}$ we have that

$$(4.97) A_2^q R_{13}^t(q^{-2}w^{-1}) R_{23}^t(w^{-1}) = R_{23}^t(w^{-1}) R_{13}^t(q^{-2}w^{-1}) A_2^q.$$

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Replacing w with $q^{-2}u^{-1}$, we get

$$(4.98) A_2^q R_{13}^t(u) R_{23}^t(q^2 u) = R_{23}^t(q^2 u) R_{13}^t(u) A_2^q.$$

Using the relation $A_2^q + H_2^q = 1$ and $(A_2^q)^2 = A_2^q$, we have

$$(4.99) P_{12}^q R_{13}^t(u) R_{23}^t(q^2 u) H_2^q = R_{13}^t(u) R_{23}^t(q^2 u) H_2^q.$$

Then the second equation can be obtained by the same arguments of the first equation. $\hfill\Box$

The following version of McMahon's theorem holds.

Theorem 4.18. One has that

(4.100)
$$\sum_{r=0}^{k} (-1)^r t r_{1,\dots,k} H_r^q A_{\{r+1,\dots,k\}}^q \langle S_1,\dots,S_k \rangle = 0,$$

(4.101)
$$\sum_{r=0}^{k} (-1)^r tr_{1,\dots,k} A_r^q H_{\{r+1,\dots,k\}}^q \langle S_1,\dots,S_k \rangle = 0,$$

where $A^q_{\{r+1,\ldots,k\}}$ and $H^q_{\{r+1,\ldots,k\}}$ denote the antisymmetrizer and symmetrizer over the copies of $End(\mathbb{C}^k)$ labeled by $\{r+1,\ldots,k\}$.

PROOF. In the following we show that

$$tr_{1,\dots,k}H_r^q A_{\{r+1,\dots,k\}}^q \langle S_1,\dots,S_k \rangle$$

$$(4.102) = tr_{1,\dots,k} \frac{r(k-r+1)}{k} H_r^q A_{\{r,\dots,k\}}^q \langle S_1,\dots,S_k \rangle + tr_{1,\dots,k} H_r^q A_{\{r,\dots,k\}}^q \frac{(k-r)(r+1)}{k} H_{r+1}^q A_{\{r+1,\dots,k\}}^q \langle S_1,\dots,S_k \rangle.$$

The element $\langle S_1, \ldots, S_k \rangle$ can be written as

$$\langle S_1, \dots, S_r \rangle \prod_{\substack{1 \le i \le r \\ r+1 \le j \le k}} \overline{R}_{ij}^t \langle S_{r+1}, \dots, S_k \rangle,$$

where the product is taken in the lexicographical order on the pairs (i, j). It follows from Lemma 4.17 that

$$(4.104) A_{\{r+1,\dots,k\}}^q \prod_{\substack{1 \le i \le r \\ r+1 \le j \le k}} \overline{R}_{ij}^t = A_{\{r+1,\dots,k\}}^q \prod_{\substack{1 \le i \le r \\ r+1 \le j \le k}} \overline{R}_{ij}^t A_{\{r+1,\dots,k\}}^q,$$

(4.105)
$$\prod_{\substack{1 \le i \le r \\ r+1 \le j \le k}}^{r+1 \le j \le k} \overline{R}_{ij}^t H_r^q = H_r^q \prod_{\substack{1 \le i \le r \\ r+1 \le j \le k}} \overline{R}_{ij}^t H_r^q.$$

Then

$$(4.106) A_{\{r+1,\ldots,k\}}^q \langle S_1,\ldots,S_k \rangle = A_{\{r+1,\ldots,k\}}^q \langle S_1,\ldots,S_k \rangle A_{\{r+1,\ldots,k\}}^q,$$

$$(4.107) H_r^q \langle S_1, \dots, S_k \rangle = H_r^q \langle S_1, \dots, S_k \rangle H_r^q.$$

By the relation in the group algebra of \mathfrak{S}_k ,

$$(4.108) (k-r+1)A_{\{r,\dots,k\}}^q = A_{\{r+1,\dots,k\}}^q - (k-r)A_{\{r+1,\dots,k\}}^q P_{r,r+1}^q A_{\{r+1,\dots,k\}}^q, (4.109) (r+1)H_{r+1}^q = H_r^q + rH_r^q P_{r,r+1}^q H_r^q.$$

Thus we have that

$$(4.110) tr_{1,\dots,k}H_{r}A_{\{r+1,\dots,k\}}^{q}P_{r,r+1}^{q}A_{\{r+1,\dots,k\}}^{q}\langle S_{1},\dots,S_{k}\rangle$$

$$= tr_{1,\dots,k}H_{r}^{q}P_{r,r+1}^{q}A_{\{r+1,\dots,k\}}^{q}\langle S_{1},\dots,S_{k}\rangle A_{\{r+1,\dots,k\}}^{q}$$

$$= tr_{1,\dots,k}H_{r}^{q}P_{r,r+1}^{q}A_{\{r+1,\dots,k\}}^{q}\langle S_{1},\dots,S_{k}\rangle$$

Similarly,

$$(4.111) tr_{1,...,k}H_{r}^{q}P_{r,r+1}^{q}H_{r}^{q}A_{\{r+1,...,k\}}^{q}\langle S_{1},...,S_{k}\rangle = tr_{1,...,k}P_{r,r+1}^{q}H_{r}^{q}A_{\{r+1,...,k\}}^{q}\langle S_{1},...,S_{k}\rangle H_{r}^{q} = tr_{1,...,k}P_{r,r+1}^{q}A_{\{r+1,...,k\}}^{q}\langle S_{1},...,S_{k}\rangle H_{r}^{q} = tr_{1,...,k}H_{r}^{q}P_{r,r+1}^{q}A_{\{r+1,...,k\}}^{q}\langle S_{1},...,S_{k}\rangle.$$

These imply the equation (4.102). Therefore we have shown the first equation. The second equation can be proved by the same arguments. \Box

4.5. Sylvester's theorem for the Sklyanin determinant. The following is an analog of Sylvester's theorem for the Sklyanin determinant.

Theorem 4.19. Let $I = \{1, \dots, N\}$, $J = \{N+1, \dots, N+M\}$, where N and M are positive integers such that N and M are even in the symplectic case. Then the mapping $s_{ij}(u) \mapsto s_{j,N+1,\dots,N+M}^{i,N+1,\dots,N+M}(q^Mu)$ defines an algebra morphism $Y_q^{\mathrm{tw}}(\mathfrak{g}_N) \to Y_q^{\mathrm{tw}}(\mathfrak{g}_{N+M})$. Denote $\widetilde{s}_{ij}(u)$ by the image of $s_{ij}(u)$. Then

$$(4.112) \operatorname{sdet}_q(\widetilde{S}(u)) = \prod_{i=1}^{N-1} \operatorname{sdet}_q(S_J(q^{M-2i}u)) \operatorname{sdet}_q(S(q^Mu)).$$

PROOF. Let S(u) be the generator matrix for $Y_q^{\text{tw}}(\mathfrak{g}_{N+M})$, we can write $X(u) = C^{-1}S^{-1}(q^{-N-M}u)C$ as a block matrix matrix

(4.113)
$$\begin{pmatrix} X_{11}(u) & X_{12}(u) \\ X_{21}(u) & X_{22}(u) \end{pmatrix},$$

where $X_{11}(u)$ and $X_{22}(u)$ are respectively matrices of size $N \times N$ and $M \times M$. Then $X_{11}(u)$ satisfies the q^{-1} -reflection relation. The inverse of $X_{11}(q^{2N-2}u)$ is $(\text{sdet}_{q^{-1}}X_{11}(u))^{-1}\hat{X}_{11}(u)$. Denote $Z(u) = DX_{11}^{-1}(q^Nu)D^{-1}$, then Z(u) satisfies the q-reflection relation. It follows from Proposition 4.9 that the (i,j)-th entry of $\hat{X}_{11}(u)$ is $(-q)^{i-N}\check{X}_{1,\cdots,\hat{i},\cdots,N,j}^{1,\cdots,N}(u)$. By Theorem 4.13 we have that

(4.114)

$$\tilde{s}_{ij}(u) = s_{j,M+1,\cdots,M+N}^{i,M+1,\cdots,M+N}(q^M u)
= (-q)^{j-N} \operatorname{sdet}_q(S(q^M u)) \check{X}_{1,\cdots,\hat{i},\cdots,N,j}^{1,\cdots,N}(q^{2-N} u)
= (-q)^{j-i} \operatorname{sdet}_q(S(q^M u)) (\hat{X}_{11}(q^{2-N} u))_{ij}
= (-q)^{j-i} \operatorname{sdet}_q(S(q^M u)) \operatorname{sdet}_{q^{-1}}(X_{11}(q^{2-N} u)) (X_{11}(q^N u)^{-1})_{ij}
= \operatorname{sdet}_q(S(q^M u)) \operatorname{sdet}_{q^{-1}}(X_{11}(q^{2-N} u)) z_{ij}(u)$$

Since $\operatorname{sdet}_q(S(q^M u))$ and $\operatorname{sdet}_{q^{-1}}(X_{11}(q^{2-N}u))$ commute with $z_{ij}(u)$ for any $1 \leq i, j \leq N$, $\tilde{S}(u)$ satisfies the q-reflection relation. This proves the first statement.

By Jacobi's theorem,

(4.115)
$$\operatorname{sdet}_{q}(S(q^{M}u))\operatorname{sdet}_{q^{-1}}(X_{11}(q^{2-N}u)) = \operatorname{sdet}_{q}(S_{J}(q^{M}u)),$$

then $\tilde{s}_{ij}(u) = \operatorname{sdet}_q(S_J(q^M u))z_{ij}(u)$. Using the explicit formula for Sklyanin determinants, we have that

(4.116)
$$\operatorname{sdet}_{q}(\tilde{S}(u)) = \prod_{i=0}^{N-1} \operatorname{sdet}_{q}(S_{J}(q^{M-2i}u)) \operatorname{sdet}_{q}(Z(u)).$$

It follows from Jacobi's identity that

(4.117)
$$\operatorname{sdet}_{q^{-1}}(X_{11}(u))\operatorname{sdet}_q(Z(q^{N-2}u)) = 1.$$

This implies that

$$(4.118) \operatorname{sdet}_q(Z(u)) = \operatorname{sdet}_q(S_J(q^M)u)^{-1} \operatorname{sdet}_q(S(q^Mu)).$$

Therefore,

$$(4.119) \operatorname{sdet}_q(\tilde{S}(u)) = \prod_{i=1}^{N-1} \operatorname{sdet}_q(S_J(q^{M-2i}u)) \operatorname{sdet}_q(S(q^Mu)).$$

4.6. Liouville formula. In this section, we regard the twisted q-Yangians as subalgebras of the quantum affine algebra at level c=0. Recall that $S(u) = L^-(u)GL^+(u^{-1})^t$, $\overline{S}(u) = L^+(u)GL^-(u^{-1})^t$. In the orthogonal case, G=I. In the following we derive a common reflection relation involving both S(u) and $\overline{S}(u)$ which will be used in the Liouville formula.

Proposition 4.20. We have that

$$(4.120) R(u,v)\overline{S}_1(u)R^t(u^{-1},v)S_2(v) = S_2(v)R^t(u^{-1},v)\overline{S}_1(u)R(u,v),$$

PROOF. We consider the equation in the quantum affine algebra. The left side can be written as

(4.121)
$$R(u,v)\overline{S}_{1}(u)R^{t}(u^{-1},v)S_{2}(v) = R(u,v)L_{1}^{+}(u)G_{1}L_{1}^{-}(u^{-1})^{t}R^{t}(u^{-1},v)L_{2}^{-}(v)G_{2}L_{2}^{+}(v^{-1})^{t}.$$

Taking transposition t_1 to the RTT equation with respect to $L^-(u)$, we get that

$$(4.122) L_1^-(u)^t R(u,v)^t L_2^-(v) = L_2^-(v) R(u,v)^t L_1^-(u)^t.$$

Thus,

$$R(u,v)\overline{S}_1(u)R^t(u^{-1},v)S_2(v)$$

$$(4.123) = R(u,v)L_1^+(u)G_1L_2^-(v)R^t(u^{-1},v)L_1^-(u^{-1})^tG_2L_2^+(v^{-1})^t$$

= $R(u,v)L_1^+(u)L_2^-(v)G_1R^t(u^{-1},v)G_2L_1^-(u^{-1})^tL_2^+(v^{-1})^t$.

Using the RTT relation, this is equal to

$$(4.124) L_2^-(v)L_1^+(u)R(u,v)G_1R^t(u^{-1},v)G_2L_1^-(u^{-1})^tL_2^+(v^{-1})^t.$$

Since the matrix G satisfies the reflection relation in both cases, we get

(4.125)
$$R(u,v)\overline{S}_{1}(u)R^{t}(u^{-1},v)S_{2}(v) = L_{2}^{-}(v)L_{1}^{+}(u)G_{2}R^{t}(u^{-1},v)G_{1}R(u,v)L_{1}^{-}(u^{-1})^{t}L_{2}^{+}(v^{-1})^{t}.$$

Taking transposition t_1 and t_2 consecutively to the equation

$$(4.126) R(u,v)L_1^+(u)L_2^-(v) = L_2^-(v)L_1^+(u)R(u,v),$$

we get that

$$(4.127) L_1^+(u)^t L_2^-(v)^t R(u,v)^{t_1 t_2} = R(u,v)^{t_1 t_2} L_2^-(v)^t L_1^+(u)^t.$$

Since $R(u, v)^{t_1 t_2} = PR(u, v)P$,

$$(4.128) L_2^+(u)^t L_1^-(v)^t R(u,v) = R(u,v) L_1^-(v)^t L_2^+(u)^t.$$

As R(u,v) is equal to $R(v^{-1},u^{-1})$ up to constant, replacing u and v by v^{-1} and u^{-1} respectively we have that

$$(4.129) L_2^+(v^{-1})^t L_1^-(u^{-1})^t R(u,v) = R(u,v) L_1^-(u^{-1})^t L_2^+(v^{-1})^t.$$

Then

$$R(u,v)\overline{S}_1(u)R^t(u^{-1},v)S_2(v)$$

$$(4.130) = L_{2}^{-}(v)L_{1}^{+}(u)G_{2}R^{t}(u^{-1},v)G_{1}L_{2}^{+}(v^{-1})^{t}L_{1}^{-}(u^{-1})^{t}R(u,v)$$

$$= L_{2}^{-}(v)G_{2}L_{1}^{+}(u)R^{t}(u^{-1},v)L_{2}^{+}(v^{-1})^{t}G_{1}L_{1}^{-}(u^{-1})^{t}R(u,v).$$

The RTT relation with respect to $L^+(u)$ is equivalent to

$$(4.131) PR(u,v)PL_2^+(u)L_1^+(v) = L_1^+(v)L_2^+(u)PR(u,v)P.$$

Taking transposition t_2 we get

$$(4.132) L_2^+(u)^t R^t(u,v) L_1^+(v) = L_1^+(v) R^t(u,v) L_2^+(u)^t.$$

Replacing u and v by v^{-1} and u respectively we have

$$(4.133) L_2^+(v^{-1})^t R^t(v^{-1}, u) L_1^+(u) = L_1^+(u) R^t(v^{-1}, u) L_2^+(v^{-1})^t.$$

Note that $R^t(v^{-1}, u)$ is equal to $R^t(u^{-1}, v)$ up to a constant. Therefore,

$$R(u,v)\overline{S}_1(u)R^t(u^{-1},v)S_2(v)$$

= $L_2^-(v)G_2L_2^+(v^{-1})^tR^t(u^{-1},v)L_1^+(u)G_1L_1^-(u^{-1},v)$

$$(4.134) = L_2^-(v)G_2L_2^+(v^{-1})^tR^t(u^{-1},v)L_1^+(u)G_1L_1^-(u^{-1})^tR(u,v)$$

= $S_2(v)R^t(u^{-1},v)\overline{S}_1(u)R(u,v)$.

Multiplying from both sides of the relation (4.120) by the inverse of R(u, v), $\overline{S}_1(u)$ and $R^t(u^{-1}, v)$, we get that

(4.135)
$$S_2(v)R^{-1}(u,v)\overline{S}_1(u)^{-1}R^t(u^{-1},v)^{-1} = R^t(u^{-1},v)^{-1}\overline{S}_1(u)^{-1}R^{-1}(u,v)S_2(v).$$

Taking $u = q^{-2N}v^{-1}$ and using the relations (3.4) and (3.8), we have

(4.136)
$$D_1^{-1}S_2(v)\widetilde{R}'(q^{-2N}v^{-2})\overline{S}_1(q^{-2N}v^{-1})^{-1}D_1Q$$
$$=QD_1^{-1}\overline{S}_1(q^{-2N}v^{-1})^{-1}\widetilde{R}'(q^{-2N}v^{-2})S_2(v)D_1.$$

where $\widetilde{R}'(x)$ is the inverse of R(x) up to constant, and it is the R matrix obtained from $\widetilde{R}(x)$ by replacing q with q^{-1} Replacing v with uq^{-2N} .

(4.137)
$$D_1^{-1} S_2(q^{-2N}u) \widetilde{R}'(q^{2N}u^{-2}) \overline{S}_1(u^{-1})^{-1} D_1 Q$$
$$= Q D_1^{-1} \overline{S}_1(u^{-1})^{-1} \widetilde{R}'(q^{2N}u^{-2}) S_2(q^{-2N}u) D_1,$$

Since Q is a one-dimensional operator satisfying $Q^2 = NQ$, the equation must be Q times a series in u^{-1} with coefficients in $Y_q^{\text{tw}}(\mathfrak{g}_N)$. We denote the series by $\alpha_N(u)y(u)$, where

(4.138)
$$\alpha_N(u) = \begin{cases} 1, & \text{Case } (\mathfrak{o}_N), \\ \frac{q^2 u^2 - 1}{q^2 - u^2}, & \text{Case } (\mathfrak{sp}_N). \end{cases}$$

We now fix y(u) by Sklyanin determinants.

Theorem 4.21. On the algebra $Y_q^{\mathrm{tw}}(\mathfrak{g}_N)$, we have that

(4.139)
$$y(u) = \frac{\operatorname{sdet}_{q}(S(q^{-2}u))}{\operatorname{sdet}_{q}(S(u))}.$$

PROOF. We write $S(u)=L^-(u)G{\bf L}^+(u^{-1})^t$, $\overline{S}(u)=L^+(u)GL^-(u^{-1})^t$, then

(4.140)
$$\alpha_N(u)y(u)Q = QD_1^{-1}(L_1^-(u)^t)^{-1}G_1^{-1}L_1^+(u^{-1})^{-1} \\ \widetilde{R}'(q^{2N}u^{-2})L_2^-(q^{-2N}u)G_2L_2^+(q^{2N}u^{-1})^tD_1.$$

Multiplying the inverse of R(u, v) and $L_1^+(u)$ to the relation

$$(4.141) R(u,v)L_1^+(u)L_2^-(v) = L_2^-(v)L_1^+(u)R(u,v)$$

we have that

$$(4.142) L_2^-(v)R(u,v)^{-1}L_1^+(u)^{-1} = L_1^+(u)^{-1}R(u,v)^{-1}L_2^-(v).$$

Thus

$$(4.143)$$

$$\alpha_{N}(u)y(u)Q = QD_{1}^{-1}(L_{1}^{-}(u)^{t})^{-1}G_{1}^{-1}L_{2}^{-}(q^{-2N}u)$$

$$\widetilde{R}'(q^{2N}u^{-2})L_{1}^{+}(u^{-1})^{-1}G_{2}L_{2}^{+}(q^{2N}u^{-1})^{t}D_{1}$$

$$= QD_{1}^{-1}(L_{1}^{-}(u)^{t})^{-1}D_{1}L_{2}^{-}(q^{-2N}u)D_{1}^{-1}G_{1}^{-1}$$

$$\widetilde{R}'(q^{2N}u^{-2})G_{2}L_{1}^{+}(u^{-1})^{-1}D_{1}L_{2}^{+}(q^{2N}u^{-1})^{t}$$

It follows from Proposition 3.11 that

$$(4.144) QD_1^{-1}(L_1^-(u)^t)^{-1}D_1L_2^-(q^{-2N}u) = Qz^-(u).$$

Then

$$(4.145) y(u)Q = z^{-}(u)QD_{1}^{-1}G_{1}^{-1}\widetilde{R}'(q^{2N}u^{-2})G_{2}L_{1}^{+}(u^{-1})^{-1}L_{2}^{+}(q^{2N}u^{-1})^{t}D_{1}.$$

By direct computation we have that

(4.146)
$$QD_1^{-1}G_1^{-1}\widetilde{R}'(q^{2N}u^{-2})G_2D_1 = D_1^{-1}G_2\widetilde{R}'(q^{2N}u^{-2})G_1^{-1}D_1Q = Q\beta_N(u)$$

where the constant $\beta_N(u)$ is given by

(4.147)
$$\beta_N(u) = \begin{cases} 1, & \text{Case } (\mathfrak{o}_N), \\ \frac{q^2 u^2 - q^{2N}}{q^{2N+2} - u^2}, & \text{Case } (\mathfrak{sp}_N). \end{cases}$$

By definition of $z^+(u)$, we have

$$QD_1^{-1}L_1^+(u^{-1})^{-1}D_1L_2^+(q^{2N}u^{-1})^t$$

$$= QL_1^+(q^{2N}u^{-1})D_1^{-1}L_1^+(u^{-1})^{-1}D_1$$

$$= z^+(q^{2N}u^{-1})^{-1}.$$

On the other hand,

(4.149)
$$y(u) = \frac{\beta_N(u)}{\alpha_N(u)} z^-(u) z^+(q^{2N} u^{-1})^{-1}$$

$$= \frac{\beta_N(u)}{\alpha_N(u)} \frac{\det_q(L^-(q^{-2}u)) \det_q L^+(q^{2N} u^{-1})}{\det_q(L^-(u)) \det_q L^+(q^{2N-2} u^{-1})}.$$

Now it is easy to see that

(4.150)
$$\frac{\beta_N(u)}{\alpha_N(u)} = \frac{\gamma_N(q^{-2}u)}{\gamma_N(u)}.$$

Thus,

(4.151)
$$y(u) = \frac{\beta_N(u)}{\alpha_N(u)} z^-(u) z^+ (q^{2N} u^{-1})^{-1} = \frac{\operatorname{sdet}_q(S(q^{-2} u))}{\operatorname{sdet}_q(S(u))}.$$

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