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# PROPAGATING TEMPORAL RELATIONS OF INTERVALS BY MATRIX

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*Traditional temporal relations propagating is based on Allen's Interval Algebra. This paper proposes an alternative method to propagate temporal relations among intervals, in which* $5 \times 5$ *matrices are used to represent temporal relations of intervals. Hence, the propagation of temporal relations is transformed into a numerical computation. For efficiency, we use the special values of the thirteen matrices to determine the possible temporal relations between two given intervals by using only the final resultant matrix so as to optimize the propagation. To evaluate the utility of the proposed technique, we have implemented the matrix representation in Java. The experimental results demonstrate that the approach is efficient and promising.*

In the real world, changes cannot be avoided. To conquer and exploit the real world for our life, we must catch the properties of time. So, Allen proposed a time world model of *time interval* (simply called interval) calculus—Interval Algebra (IA) (Allen 1983). The model divides the relationships among intervals into 13 kinds of temporal forms, laying a foundation for dealing with temporal relationships in applications.

The major challenge in IA is the propagation of temporal relations. To confront this problem, traditional models are designed to improve IA. For example, a number of researches have been focused on constraint satisfactory problems (Allen and Koomen 1983; Haddawy 1996; Ladkin and Maddux 1994; Meiri 1996; Mouhoub, Charpillet, and Haton 1998; Nebel 1997; Pirri and Reiter 1995; Schwalb, Kask, and Dechter 1994; Vilain and Kautz 1986) and solving tractable subclasses of Allen's interval algebra (Nebel and Burckert 1995; Schwalb and Dechter 1997; Schwalb 1997; Tolba, Charpillet, and Haton 1991; Vila and Reichgelt 1996). However, propagating temporal relations is still non-tractable in general (Nebel and Buckert 1995; Nebel 1997).

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This paper proposes an alternative method for propagating temporal relations of intervals. This method is to apply thirteen $5 \times 5$ matrices to represent the thirteen temporal relations described in Allen's model (Allen 1983). It transforms the propagation of temporal relations into a numerical computation.

Consider the interval constraint problem on intervals $I_1, I_2, \ldots, I_n$. Let the possible relations of $I$ and $J$ be $R(I, J)$. In our approach, $R(I, J)$ can be described as a $5 \times 5$ matrix. Assume relations $R(I_1, I_2), R(I_2, I_3), \ldots, R(I_{n-1}, I_n)$ be given, and $M(I_1, I_2), M(I_2, I_3), \ldots, M(I_{n-1}, I_n)$ be the corresponding matrices ($M(I_i, I_j)$ describes relational constrains $R(I_i, I_j)$ on the intervals $I_i$ and $I_j, i, j = 1, 2, \ldots, n$).

According to Allen's interval calculus, the propagation of the temporal relations between $I_1$ and $I_n$ is as follows. It must first determine the possible relations between $I_1$ and $I_3$ by $R(I_1, I_2)$ and $R(I_2, I_3)$; secondly, determine the possible relations between $I_1$ and $I_4$ by $R(I_1, I_3)$ and $R(I_3, I_4)$; $\cdots$; finally, determine the possible relations between $I_1$ and $I_n$ by $R(I_1, I_{n-1})$ and $R(I_{n-1}, I_n)$. In particular, it needs multiple accesses to a $13 \times 13$ relational table in each of the above $n-1$ steps. For instance, let $R(I_1, I_2)$ and $R(I_2, I_3)$ contain 5 and 6 elements respectively; the first step needs 30 accesses to the $13 \times 13$ relational table.

In our approach, the temporal relation between $I_1$ and $I_n$ can directly be depended by computing a $5 \times 5$ matrix $M(I_1, I_n)$ as

$$M(I_1, I_n) = M(I_1, I_2) * M(I_2, I_3) * \cdots * M(I_{n-1}, I_n).$$

Hence, propagating temporal relations by matrix is neat. Our experimental results also demonstrate that the approach is efficient and promising.

## RELATED WORK

To capture the properties of time, some temporal logics are built. They fall into abstract model termporal logics and reified temporal logics. For instance, abstract model temporal logics like Templog (Abadi and Manna 1989) is based on linear and branching time temporal logics and temporal logic programming languages (Tang 1989); reified temporal logics like the token reification temporal logic (Halpern and Sholam 1991; Vila and Reichgelt 1996), temporal extensions and implementations (Hrycej 1993; Tang 1989; Yampratoom and Allen 1993), and more general works on temporal logic (Barringer, Kuiper, and Pnueli 1984; Staab and Hahn 1999). Certainly, there are also a number of researches on time, such as adding synchronization constraints between concurrent agents such that their concurrent execution satisfies a given temporal property (Shoham and Tennenholtz 1994; Stuart 1985), and a method for synchronizing multiagent plans from goals described by a temporal logic (Kabanza 1995), natural language understanding (Allen 1984), general planning (Allen and Koomen 1983; Haddawy 1996; Pirri and Reiter 1995), and binary constraint problems (Ladkin and Maddux 1994; Meiri 1996; Mouhoub, Charpillet, and Haton 1998; Nebel and Buckert 1995; Nebel 1997; Schwalb, Kask, and Dechter 1994; Schwalb and Dechter 1997; Schwalb 1997; Tolba, Charpillet, and Haton 1991; Vila and Reichgelt 1996, Vilain and Kautz 1986). Much of the work on temporal reasoning in Artificial Intelligence are summarized in Vila (1944).

In 1983, Allen proposed a time world model of *time interval* (simply called interval) calculus—Interval Algebra (IA) (Allen 1983). The model divides the relationships among intervals into 13 kinds of temporal forms, laying a foundation for dealing with temporal relationships in applications. In fact, Allen's model applies the relations of the endpoints of two intervals to describe the interval temporal relations between them. This model has two clear specialities: simple and intuitive. Presently, it has been taken as a basic framework for temporal reasoning. And this model has also successfully applied to multi-agents (Kraus, Wilkenfeld, and Zlotkin 1995; Shoham and Tennenholtz 1994), to databases for manipulating their time relationships (Snodgrass 1987; Gadia and Yeung 1991; Moeira and Edelweiss 1999), to specifically design for simulations (Tuzhilin 1995), and to specify temporal properties of concurrent systems (Barringer, Kuiper, and Pnueli 1984).

Since Allen's work on binary interval relations, many researchers have further investigated temporal reasoning based on constraint propagation techniques (Drakengren and Jonsson 1997; Gerevini and Schubert 1995; Nebel and Burckert 1995; Schwalb and Dechter 1997). Given a constraint network, the constraint satisfaction problems a detailed introduction can be found in Allen (1983); Ladkin and Maddux (1994) are

(1) to decide whether there exists a solution that satisfies the constraints;
(2) to find the corresponding minimal network where the constraints are as explicit as possible.

The above tasks have been proven to be NP-complete in the general case. So, many researchers attempt to improve them (Drakengren and Jonsson 1997; Gerevini and Schubert 1995; Nebel and Burckert 1995; Schwalb and Dechter 1997). Accordingly, we establish an alternative method to propagate temporal relations by matrix in this paper.

# TEMPORAL MODEL DEFINITION

In this section, we present the temporal model by matrix. There are many time models for temporal logics in the current literature. For example, time can be discrete or dense, bounded or unbounded, and linear or branching. For convenience, we consider the discrete linear temporal domain in this paper. Suppose the universe of time is $U = [0, +\infty)$, and

$$\forall t_1, t_2 \in U(t_1 < t_2 \vee t_1 = t_2 \vee t_1 > t_2).$$

*Definition* 1. Let $I \subseteq U$. If $a \leq b \leq c \wedge a \in I \wedge c \in I \longrightarrow b \in I$, then $I$ is called a convex interval (or simply called as interval) over $U$.

## Relations between Intervals

As you know, the relation between two given intervals $I$ and $J$ can be completely determined by the relations among the end-points ($I^-, I^+, J^-$, and $J^+$) of $I$ and $J$ due to the fact that an interval is uniquely determined by its two end-points. For example, given intervals $I_1 = [2, 7], I_2 = [7, 18]$, $I_3 = [0, 2]$, and $I_4 = [5, 24]$, we can determine the relation between $I_1$ and any one of other intervals as follows, $I_1$ *meets* $I_2, I_1$ *met-by* $I_3$, and $I_1$ *overlaps* $I_4$ according to the relations among the endpoints of these intervals. This model is certainly simple and also directly reflects intuitive relations of intervals. Actually, there is alternative way to describe the temporal relations of intervals. That is to capture the temporal relations in numerical computation by matrix. Because we can define some operators on matrices for some application targets, the propagation of temporal relations can be transformed into a numerical problem. We will see shortly that this method is more convenient to propagate temporal relations than Allen's model does. It can also save much more time than Interval Algebra.

Apparently, an interval partitions the real line (or the universe of time) into three parts: the left of the interval, the interval, and the right of the interval. For example, given an interval [3, 8], it partitions the real line into three parts as: $(-\infty, 3)$, [3,8], and $(8, +\infty)$. To capture the relations *meets* and *met-by* of intervals, we can also divide an interval into three parts: the left end-point, the set of all inner-points, the right end-point. This means interval [3,8] can be divided into three parts: $\{3\}$, (3,8), and $\{8\}$. According to the above partitions, it is easy to find out the fact that an interval can be uniquely determined by its five parts: its left, its left end-point, its inner, its right end-point, and its right. For instance, the five parts of interval [3,8] are: $(-\infty, 3), \{3\}, (3, 8)$, $\{8\}$, and $(8, +\infty)$. Also, [3,8] is uniquely determined by its five parts above. Consequently, we can also determine the temporal relations of intervals using these five parts. This idea is illustrated by the following example:

*Example 1* Consider two given intervals $I_1 = [2, 7]$ and $I_2 = [7, 18]$ with relation $I_1$ meets $I_2$. The five parts of $I_1$ are as follows: $(-\infty, 2), \{2\}$, (2,7), $\{7\}$, and $(7, +\infty)$. And the five parts of $I_2$ are as follows: $(-\infty, 7), \{7\}$, (7,18), $\{18\}$, and $(18, +\infty)$. Certainly, we have $(-\infty, 2) \cap (-\infty, 7) \neq \emptyset, (-\infty, 2) \cap \{7\} = \emptyset, (-\infty, 2) \cap (7, 18) = \emptyset, (-\infty, 2) \cap \{18\} = \emptyset, (-\infty, 2) \cap (18, +\infty) = \emptyset$. The intersections are listed in Table 1 as follows:

**TABLE 1** The Intersections Among the Five Parts of $I_1$ and $I_2$

| $\cap$ | $(-\infty, 2)$ | $\{2\}$ | (2,7) | $\{7\}$ | $(7, +\infty)$ |
|---|---|---|---|---|---|
| $(-\infty, 7)$ | – | – | – | $\emptyset$ | $\emptyset$ |
| $\{7\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | – | $\emptyset$ |
| (7,18) | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | – |
| $\{18\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | – |
| $(18, +\infty)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | – |

where "–" stands for non-empty set.

The above intersections among the five parts of $I_1$ and $I_2$ depict the temporal relation as "$I_1$ meets $I_2$". As we have seen, each intersection has only two values: empty and non-empty. If we substitute 0 and 1 for empty and non-empty respectively, the above intersections can be represented as a matrix as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, as you will see shortly, we can determine the rest of the possible temporal relations between $I$ and $J$ using the relations among the five-parts of $I$ and $J$.

The idea of propagating temporal relations is to calculate the given matrices. We now demonstrate the propagation by an example as follows:

*Example 2* Consider intervals $I_1 = [2, 7], I_2 = [7, 18], I_3 = [11, 24]$; the relations $I$ meets $J$ and $I_2$ overlaps $I_3$; the matrix of $I$ meets $J$ was obtained in the above, and the matrix $I_2$ overlaps $I_3$ is as:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can now obtain the relation between $I_1$ and $I_3$ by the operation on the above two matrices as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where "$\circ$" is a operator for matrices that is indicated as logical "AND". An element of the resultant matrix at *i*th row and *j*th column is the logical "AND" of the ith row of the matrix before "$\circ$" and the *j*th column of the matrix after "$\circ$". For instance, the element of the resultant matrix at 1th row and 4th column is as

$$(1 \circ 0) \uplus (1 \circ 1) \uplus (1 \circ 0) \uplus (0 \circ 0) \uplus (0 \circ 0) = 1$$

where "$\uplus$" stands for logical "OR". "$\circ$" and "$\uplus$" will be interpreted as late.

By the resultant matrix, we can determine the relation "$I_1 < I_2$". It matches the real temporal relation between $I_1$ and $I_2$. In this way, we implement the propagation of the temporal relations of intervals by numerical computation.

Surely, this model takes a simple but non-intuitive path to describe the temporal relations of intervals. However, it can be easy to transform the propagation of the temporal relations of intervals into numerical computation.

## Matrix Representation

As we have seen, the temporal relations of intervals can be digitized by matrix. We now formally define them.

Formally, and interval $I$ divides the universe of time $U$ into five-parts: $I^L, I^-, I^1, I^+, I^R$, where $I^L, I^-, I^1, I^+, I^R$ are the set of the left outer-points of $I$, the set of the left end-point of $I$, the set of the inner-points of $I$, the set of the right end-point of $I$, and the set of the right outer-points of $I, I^R = U$, for

$a, b \in \{I^L, I^-, I^1, I^+, I^R\}$, if $a \neq b$, then $a \cap b = \emptyset$. According to this partition, we can exhibit a new visual angle to the temporal relations of intervals below.

*Definition 2* Let $I, J$ be two intervals, the operator "$\bullet$" is defined as,

$$I^* \bullet J^@ \in \{0, 1\}, \text{ and}$$
$$I^* \bullet J^@ = 1 \text{ iff the intersection of } I^* \text{ and } J^@ \text{ is non-empty.}$$

where $*, @ \in \{L, -1, +, R\}$.

*Lemma 1.* Let $I, J$ and $K$ be three intervals, the operator "$\bullet$" satisfies the following rules,

($R1$) *idempotent law:* $\overbrace{I^* \bullet I^* \bullet \cdots \bullet I^*}^{n} = 1$
($R2$) *commutativity law:* $I^* \bullet J^@ = J^@ \bullet I^*$
($R3$) *associativity law:* $(I^* \bullet J^@) \bullet K^{\%} = I^* \bullet (J^@ \bullet K^{\%})$
($R4$) *absorptive law:* $I^* \bullet (I^* \bullet J^@) = I^* \bullet J^@$
($R5$) $U \bullet I^* = I^* \bullet U = 1$
($R6$) $\emptyset \bullet I^* = I^* \bullet \emptyset = 0$

where $*, @, \% \in \{L, -1, +, R\}$.

*Proof:* They can be proved directly by the properties of sets.

For example, let $I = [0, 5]$ and $J = [5, 12]$, then $I^L = (-\infty, 0), I^- = \{0\}$, $I^1 = (0, 5), I^+ = \{5\}$, and $I^R = (5, +\infty); J^L = (-\infty, 5); J^- = \{5\}, J^1 = (5, 12)$, $J^+ = \{12\}$, and $J^R = (12, +\infty)$. Because $I$ and $J$ are intervals, $I^*$ and $J^@$ both are non-empty sets for $*, @ \in \{L, -, 1, +, R\}$. Hence, $\overbrace{I^* \bullet I^* \bullet \cdots \bullet I^*}^{n} \neq \emptyset = 1$, $U \bullet I^* = I^* \bullet U \neq \emptyset = 1, \emptyset \bullet I^* = I^* \bullet \emptyset = \emptyset = 0$. Again, because $I^1 \bullet J^R = \emptyset = 0$ and $J^R \bullet I^1 = \emptyset = 0, I^1 \bullet J^R = J^R \bullet I^1$. The above R3 and R4 can also be illustrated in the same as R2.

Furthermore, we can use a matrix to define temporal relationships between two given intervals as follows:

*Definition 3* Let $I$ and $J$ be two given intervals, the temporal relations between $I$ and $J$ can be described in the following matrix $M_{I,J}$:

$$M_{I,J} = \begin{bmatrix} I^L \bullet J^L & I^- \bullet J^L & I^1 \bullet J^L & I^+ \bullet J^L & I^R \bullet J^L \\ I^L \bullet J^- & I^- \bullet J^- & I^1 \bullet J^- & I^+ \bullet J^- & I^R \bullet J^- \\ I^L \bullet J^1 & I^- \bullet J^1 & I^1 \bullet J^1 & I^+ \bullet J^1 & I^R \bullet J^1 \\ I^L \bullet J^+ & I^- \bullet J^+ & I^1 \bullet J^+ & I^+ \bullet J^+ & I^R \bullet J^+ \\ I^L \bullet J^R & I^- \bullet J^R & I^1 \bullet J^R & I^+ \bullet J^R & I^R \bullet J^R \end{bmatrix}$$

For example, let $I = [8, 11]$ and $J = [18, 24]$, then $I^L = (-\infty, 8), I^- = \{8\}$, $I^1 = (8, 11), I^+ = \{11\}$, and $I^R = (11, +\infty)$; $J^L = (-\infty, 18)$, $J^- = \{18\}$, $J^1 = (18, 24), J^+ = \{24\}$, and $J^+ = (24, +\infty)$. And $M_{I,J}$ is as follows:

$$\begin{bmatrix} (-\infty,8)\bullet(-\infty,18) & \{8\}\bullet(-\infty,18) & (8,11)\bullet(-\infty,18) & \{11\}\bullet(-\infty,18) & (11,+\infty)\bullet(-\infty,18) \\ (-\infty,8)\bullet\{18\} & \{8\}\bullet\{18\} & (8,11)\bullet\{18\} & \{11\}\bullet\{18\} & (11,+\infty)\bullet\{18\} \\ (-\infty,8)\bullet(18,24) & \{8\}\bullet(18,24) & (8,11)\bullet(18,24) & \{11\}\bullet(18,24) & (11,+\infty)\bullet(18,24) \\ (-\infty,8)\bullet\{24\} & \{8\}\bullet\{24\} & (8,11)\bullet\{24\} & \{11\}\bullet\{24\} & (11,+\infty)\bullet\{24\} \\ (-\infty,8)\bullet(24,+\infty) & \{8\}\bullet(24,+\infty) & (8,11)\bullet(24,+\infty) & \{11\}\bullet(24,+\infty) & (11,+\infty)\bullet(24,+\infty) \end{bmatrix}$$

According to Definition 3, we have

$$M_{I,J} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because $I < J$, the above matrix can be represented as the temporal relation "$<$" according to the discussion in the Section, "Relations between Intervals."

In the same reason, the thirteen different possible relations between two intervals can be represented as the following thirteen matrices. Let $\mu$ and $\gamma$ be the intervals.

(1) $\mu$ BEFORE $\gamma$ (or $\mu < \gamma$). For example, $\mu = [8, 11], \gamma = [18, 24]$. $\mu < \gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_1$.

(2) $\mu$ AFTER $\gamma$ (or $\mu > \gamma$). For example, $\mu = [38, 51], \gamma = [15, 20]$. $\mu > \gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is denoted as $M_2$. And $M_2$ is the converse of $M_1$, or $M_2 = M_1^{-1}$. Note that $M_1^{-1}$ means, for $M_2 = (a_{ij})_{5\times 5}$, $M_1 = (b_{ij})_{5\times 5}$, then $M_2 = M_1^{-1} \longrightarrow a_{ij} = b_{ji}, i, j = 1, 2, 3, 4, 5$.

(3) $\mu$ MEETS $\gamma$ (or $\mu$ m $\gamma$). For example, $\mu = [8, 11], \gamma = [11, 45]$. $\mu$ m $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_3$.

(4) $\mu$ MET BY $\gamma$ (or $\mu$ mi $\gamma$). For example, $\mu = [11, 24], \gamma = [8, 11]$ $\mu$ mi $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is denoted as $M_4$. And $M_4 = M_3^{-1}$.

(5) $\mu$ OVERLAPS $\gamma$ (or $\mu$ o $\gamma$). For example, $\mu = [4, 11], \gamma = [8, 24]$. $\mu$ o $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_5$.

(6) $\mu$ OVERLAPPED BY $\gamma$ (or $\mu$ oi $\gamma$). For example, $\mu = [20, 35], \gamma = [18, 24]$. $\mu$ oi $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is denoted as $M_6$. And $M_6 = M_5^{-1}$.

(7) $\mu$ FINISHES $\gamma$ (or $\mu$ f $\gamma$). For example, $\mu = [39, 45]$, $\gamma = [30, 45]$. $\mu$ f $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_7$.

(8) $\mu$ FINISHED BY $\gamma$ (or $\mu$ fi $\gamma$). For example, $\mu = [8, 45]$, $\gamma = [30, 45]$. $\mu$ fi $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_8$. And $M_8 = M_7^{-1}$.

(9) $\mu$ DURING $\gamma$ (or $\mu$ d $\gamma$). For example, $\mu = [8, 11]$, $\gamma = [1, 24]$. $\mu$ d $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_9$.

(10) $\mu$ CONTAINS $\gamma$ (or $\mu$ di $\gamma$). For example, $\mu = [2, 26]$, $\gamma = [18, 24]$. $\mu$ di $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is denoted as $M_{10}$. And $M_{10} = M_9^{-1}$.

(11) $\mu$ STARTS $\gamma$ (or $\mu$ s $\gamma$). For example, $\mu = [2, 11]$, $\gamma = [2, 45]$. $\mu$ s $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_{11}$.

(12) $\mu$ STARTED BY $\gamma$ (or $\mu$ si $\gamma$). For example, $\mu = [2, 51]$, $\gamma = [2, 45]$. $\mu$ si $\gamma$ if and only if

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix is denoted as $M_{12}$. And $M_{12} = M_{11}^{-1}$.

(13) $\mu$ EQUALS $\gamma$ (or $\mu = \gamma$). For example, $\mu = [31, 51]$, $\gamma = [31, 51]$.

$$M_{\mu,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is denoted as $M_{13}$.

# PROPAGATING TEMPORAL RELATIONS

After defining the thirteen temporal relations of intervals in Allen (1983), Allen advocated the propagation for the given temporal relations of intervals, which is called *interval calculus.* The propagation works based on some axioms. For example, suppose $I_1$, $I_2$, $I_3$ be three given intervals, $I_1 < I_2$, and $I_2$ mi $I_3$. Then by the propagation laws, the possible relations between $I_1$ and $I_3$ is $I_1 < I_3 \vee I_1 \circ I_3 \vee I_1$ m $I_3$. To match this propagation, we now define the temporal relational calculus for the above matrix representation.

## Basic Definitions

To catch Allen's idea on propagation, two operators for this kind of matrices need to be defined. We begin with defining the operators: "$\circ$" and "$\uplus$" mentioned in the Section, "Related Work."

*Definition 4* Let I, J and K be three convex intervals, the operators "$\circ$" and "$\uplus$" are defined as,

$$\begin{aligned}
&(J^! \bullet K^{\%}) \circ (I^* \bullet J^{@}) \in \{0,1\}, \ \textit{and} \\
&(J^! \bullet K^{\%}) \circ (I^* \bullet J^{@}) = 1 \ \textit{iff} \ J^! \bullet K^{\%} = 1 \ \textit{and} \ I^* \bullet J^{@} = 1; \\
&(J^! \bullet K^{\%}) \uplus (I^* \bullet J^{@}) \in \{0,1\}, \ \textit{and} \\
&(J^! \bullet K^{\%}) \uplus (I^* \bullet J^{@}) = 1 \ \textit{iff} \ J^! \bullet K^{\%} = 1 \ \textit{or} \ I^* \bullet J^{@} = 1.
\end{aligned}$$

where $*$, $@$, $!1$, $\%$, $\in \{L, -, 1, +, R\}$. Intuitively, "$\circ$" means logical "AND", "$\uplus$" means logical "OR".

*Theorem 1.* Let I, J and K be three convex intervals, the operators "$\circ$" and "$\uplus$" satisfy

$$\begin{aligned}
&(i)\ (J^! \bullet K^{\%}) \circ (I^* \bullet J^!) = I^* \bullet K^{\%} \\
&(ii)\ (J^! \bullet K^{\%}) \uplus (I^* \bullet J^!) = J^! \bullet I^* \bullet K^{\%}
\end{aligned}$$

*Proof:* They can be proved directly by Lemma 1 and the properties of sets.

*Lemma 2* Let I, J, K, and T, be four convex intervals, the operator "$\circ$" and "$\uplus$" satisfy the following rules,

$(G1)$ idempotent law: $\overbrace{(I^{*} \bullet J^{@}) \circ (I^{*} \bullet J^{@}) \circ \cdots \circ (I^{*} \bullet J^{@})}^{n} = I^{*} \bullet J^{@}$, $\overbrace{(I^{*} \bullet J^{@}) \uplus (I^{*} \bullet J^{@}) \uplus \cdots \uplus (I^{*} \bullet J^{@})}^{n} = I^{*} \bullet J^{@}$.

$(G2)$ commutativity law: $(I^{*} \bullet J^{@}) \circ (J^{!} \bullet K^{\%}) = (J^{!} \bullet K^{\%}) \circ (I^{*} \bullet J^{@})$, $(I^{*} \bullet J^{@}) \uplus (J^{!} \bullet K^{\%}) = (J^{!} \bullet K^{\%}) \uplus (I^{*} \bullet J^{@})$.

$(G3)$ associativity law: $(I^{*} \bullet J^{@}) \circ ((J^{!} \bullet K^{\%}) \circ (K^{\#} \bullet T^{?})) = ((I^{*} \bullet J^{@}) \circ (J^{!} \bullet K^{\%})) \circ (K^{\#} \bullet T^{?})$, $(I^{*} \bullet J^{@}) \uplus ((J^{!} \bullet K^{\%}) \uplus (K^{\#} \bullet T^{?})) = ((I^{*} \bullet J^{@}) \uplus (J^{!} \bullet K^{\%})) \uplus (K^{\#} \bullet T^{?})$.

$(G4)$ assignment law: $(I^{*} \bullet J^{@}) \circ ((J^{!} \bullet K^{\%}) \uplus (K^{\#} \bullet T^{?})) = ((I^{*} \bullet J^{@}) \circ (J^{!} \bullet K^{\%})) \uplus (I^{*} \bullet J^{@}) \circ (K^{\#} \bullet T^{?}))$, $(I^{*} \bullet J^{@}) \uplus ((J^{!} \bullet K^{\%}) \circ (K^{\#} \bullet T^{?}) = ((I^{*} \bullet J^{@})) \uplus (J^{!} \bullet K^{\%})) \circ ((I^{*} \bullet J^{@}) \uplus (K^{\#} \bullet T^{?}))$,

where $*, @, !, \%, ?, \# \in \{L, -, 1, +, R\}$.

*Proof:* They can be proved directly by Lemma 1, Theorem 1 and the properties of sets.

We now define the operator "$\circ$" on this kind of matrices as follows.

*Definition 5* Let $I, J$ and $K$ be three intervals, the temporal relational matrices $M_{I,J}, M_{J,K}$ be

$$M_{I,J} = \begin{bmatrix} I^{L} \bullet J^{L} & I^{-} \bullet J^{L} & I^{1} \bullet J^{L} & I^{+} \bullet J^{L} & I^{R} \bullet J^{L} \\ I^{L} \bullet J^{-} & I^{-} \bullet J^{-} & I^{1} \bullet J^{-} & I^{+} \bullet J^{-} & I^{R} \bullet J^{-} \\ I^{L} \bullet J^{1} & I^{-} \bullet J^{1} & I^{1} \bullet J^{1} & I^{+} \bullet J^{1} & I^{R} \bullet J^{1} \\ I^{L} \bullet J^{+} & I^{-} \bullet J^{+} & I^{1} \bullet J^{+} & I^{+} \bullet J^{+} & I^{R} \bullet J^{+} \\ I^{L} \bullet J^{R} & I^{-} \bullet J^{R} & I^{1} \bullet J^{R} & I^{+} \bullet J^{R} & I^{R} \bullet J^{R} \end{bmatrix}$$

$$M_{J,K} = \begin{bmatrix} J^{L} \bullet K^{L} & J^{-} \bullet K^{L} & J^{1} \bullet K^{L} & J^{+} \bullet K^{L} & J^{R} \bullet K^{L} \\ J^{L} \bullet K^{-} & J^{-} \bullet K^{-} & J^{1} \bullet K^{-} & J^{+} \bullet K^{-} & J^{R} \bullet K^{-} \\ J^{L} \bullet K^{1} & J^{-} \bullet K^{1} & J^{1} \bullet K^{1} & J^{+} \bullet K^{1} & J^{R} \bullet K^{1} \\ J^{L} \bullet K^{+} & J^{-} \bullet K^{+} & J^{1} \bullet K^{+} & J^{+} \bullet K^{+} & J^{R} \bullet K^{+} \\ J^{L} \bullet K^{R} & J^{-} \bullet K^{R} & J^{1} \bullet K^{R} & J^{+} \bullet K^{R} & J^{R} \bullet K^{R} \end{bmatrix},$$

then we define "$\circ$" operator of matrices for propagating temporal relationships as follows:

$$M_{I,K} = M_{J,K} \circ M_{I,J},$$

where,

$$
\begin{aligned}
I^L \bullet K^L = {} & (J^L \bullet K^L) \circ (I^L \bullet J^L) \uplus \quad (J^- \bullet K^L) \circ (I^L \bullet J^-) \uplus \\
& (J^1 \bullet K^L) \circ (I^L \bullet J^1) \uplus \quad (J^+ \bullet K^L) \circ (I^L \bullet J^+) \uplus \\
& (J^R \bullet K^L) \circ (I^L \bullet J^R) \uplus,
\end{aligned}
$$

$$
\begin{aligned}
I^- \bullet K^L = {} & (J^L \bullet K^L) \circ (I^- \bullet J^L) \uplus \quad (J^- \bullet K^L) \circ (I^- \bullet J^-) \uplus \\
& (J^1 \bullet K^L) \circ (I^- \bullet J^1) \uplus \quad (J^+ \bullet K^L) \circ (I^- \bullet J^+) \uplus \\
& (J^L \bullet K^R) \circ (I^- \bullet J^R) \uplus,
\end{aligned}
$$

$$\cdots\cdot$$

In other words, let $M_{I,K} = (a_{ij})_{5\times5}$, $M_{J,K} = (b_{ij})_{5\times5}$, and $M_{I,J} = (c_{ij})_{5\times5}$, then

$$a_{ij} = (b_{i1} \circ c_{1j}) \uplus (b_{i2} \circ c_{2j}) \uplus \cdots \uplus (b_{i5} \circ c_{5j})$$

$i, j = 1, 2, 3, 4, 5.$

*Theorem 2* Definition 5 can match the propagation of temporal relation among intervals I, J and K.

*Proof:* By Lemma 1, Lemma 2, Theorem 1 and the properties of sets, we have

$$
\begin{aligned}
& (J^L \bullet K^L) \circ (I^L \bullet J^L) \uplus (J^- \bullet K^L) \circ (I^L \bullet J^-) \uplus (J^1 \bullet K^L) \circ (I^L \bullet J^1) \uplus \\
& (J^+ \bullet K^L) \circ (I^L \bullet J^+) \uplus (J^R \bullet K^L) \circ (I^L \bullet J^R) \uplus \\
= {} & (I^L \bullet K^L) \uplus (I^L \bullet K^L) \uplus (I^L \bullet K^L) \uplus (I^L \bullet K^L) \uplus (I^L \bullet K^L) = I^L \bullet K^L.
\end{aligned}
$$

In the same reasons, we have

$$
\begin{aligned}
& (J^L \bullet K^L) \circ (I^- \bullet J^L) \uplus (J^- \bullet K^L) \circ (I^- \bullet J^-) \uplus (J^1 \bullet K^L) \circ (I^- \bullet J^1) \uplus \\
& (J^+ \bullet K^L) \circ (I^- \bullet J^+) \uplus (J^R \bullet K^L) \circ (I^- \bullet J^R) \uplus = I^L \bullet K^L,
\end{aligned}
$$

$$\cdots\cdot$$

So, Definition 5 can match the propagation of temporal relation among intervals $I$, $J$ and $K$.

Again, we can define the operator "$\uplus$" for this kind of matrices as follows:

*Definition 6* Let I, J and K be three intervals, $M_{I,J}$, $M_{J,K}$ be the temporal relational matrices. The operator "$\uplus$" of matrices is defined as,

$$N = M_{I,J} \uplus M_{J,K} = (b_{ij})_{5\times 5} \uplus (c_{ij})_{5\times 5} = (b_{ij} \uplus c_{ij})_{5\times 5} = (d_{ij})_{5\times 5}$$

where $N = (d_{ij})_{5\times 5}$, $M_{I,J} = (b_{ij})_{5\times 5}$, and $M_{J,K} = (c_{ij})_{5\times 5}$. Or,

$$N = M_{I,J} \uplus M_{J,K} = \begin{bmatrix} I^L \bullet J^L & I^- \bullet J^L & I^1 \bullet J^L & I^+ \bullet J^L & I^R \bullet J^L \\ I^L \bullet J^- & I^- \bullet J^- & I^1 \bullet J^- & I^+ \bullet J^- & I^R \bullet J^- \\ I^L \bullet J^1 & I^- \bullet J^1 & I^1 \bullet J^1 & I^+ \bullet J^1 & I^R \bullet J^1 \\ I^L \bullet J^+ & I^- \bullet J^+ & I^1 \bullet J^+ & I^+ \bullet J^+ & I^R \bullet J^+ \\ I^L \bullet J^R & I^- \bullet J^R & I^1 \bullet J^R & I^+ \bullet J^R & I^R \bullet J^R \end{bmatrix}$$

$$\uplus \begin{bmatrix} J^L \bullet K^L & J^- \bullet K^L & J^1 \bullet K^L & J^+ \bullet K^L & J^R \bullet K^L \\ J^L \bullet K^- & J^- \bullet K^- & J^1 \bullet K^- & J^+ \bullet K^- & J^R \bullet K^- \\ J^L \bullet K^1 & J^- \bullet K^1 & J^1 \bullet K^1 & J^+ \bullet K^1 & J^R \bullet K^1 \\ J^L \bullet K^+ & J^- \bullet K^+ & J^1 \bullet K^+ & J^+ \bullet K^+ & J^R \bullet K^+ \\ J^L \bullet K^R & J^- \bullet K^R & J^1 \bullet K^R & J^+ \bullet K^R & J^R \bullet K^R \end{bmatrix} = (d_{ij})_{5\times 5}$$

where,

$$\begin{aligned} d_{11} &= (I^L \bullet J^L) \uplus (J^L \bullet K^L), & d_{12} &= (I^- \bullet J^L) \uplus (J^- \bullet K^L), \\ d_{13} &= (I^1 \bullet J^L) \uplus (J^1 \bullet K^L), & d_{14} &= (I^+ \bullet J^L) \uplus (J^+ - \bullet K^L), \\ d_{15} &= (I^R \bullet J^L) \uplus (J^R \bullet K^L), & d_{21} &= (I^L \bullet J^-) \uplus (J^L \bullet K^-), \\ d_{22} &= (I^- \bullet J^-) \uplus (J^- \bullet K^-), & d_{23} &= (I^1 \bullet J^-) \uplus (J^1 \bullet K^-), \\ d_{24} &= (I^+ \bullet J^-) \uplus (J^+ \bullet K^-), & d_{25} &= (I^R \bullet J^-) \uplus (J^R \bullet K^-), \\ d_{31} &= (I^L \bullet J^1) \uplus (J^L \bullet K^1), & d_{32} &= (I^- \bullet J^1) \uplus (J^- \bullet K^1), \\ d_{33} &= (I^1 \bullet J^1) \uplus (J^1 \bullet K^1), & d_{34} &= (I^+ \bullet J^1) \uplus (J^+ \bullet K^1), \\ d_{35} &= (I^R \bullet J^1) \uplus (J^R \bullet K^1), & d_{41} &= (I^L \bullet J^+) \uplus (J^L \bullet K^+), \\ d_{42} &= (I^- \bullet J^+) \uplus (J^- \bullet K^+), & d_{43} &= (I^1 \bullet J^+) \uplus (J^1 \bullet K^+), \\ d_{44} &= (I^+ \bullet J^+) \uplus (J^+ \bullet K^+), & d_{45} &= (I^R \bullet J^+) \uplus (J^R \bullet K^+), \\ d_{51} &= (I^L \bullet J^R) \uplus (J^L \bullet K^R), & d_{52} &= (I^- \bullet J^R) \uplus (J^- \bullet K^R), \\ d_{53} &= (I^1 \bullet J^R) \uplus (J^1 \bullet K^R), & d_{54} &= (I^+ \bullet J^R) \uplus (J^+ \bullet K^R), \end{aligned}$$
$$d_{55} = (I^R \bullet J^R) \uplus (J^R \bullet K^R).$$

*Definition 7* Let $I$, $J$ and $K$ be three intervals, $M_{I,J}$, $M_{J,K}$ be the temporal relational matrices. Suppose $M_{I,K} = M_{I,J} \circ M_{J,K}$, then the possible relations of $I$ and $K$ are determined as follows. For $1 \leqslant i \leqslant 13$, if

$$M_i \uplus M_{I,K} = M_{I,K},$$

then the relation denoted with $M_i$ is a possible relation of $I$ and $K$.

We now illustrate the use of the above definitions by an example as follows:

*Example 3* Let $I$, $J$ and $K$ be three intervals; $I$ *m* $J$ and $J$ oi $K$. Then

$$M_{I,K} = M_{J,K} \circ M_{I,J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Certainly,

$$M_5 \uplus M_{I,K} = M_{I,K}$$
$$M_9 \uplus M_{I,K} = M_{I,K}$$
$$M_{11} \uplus M_{I,K} = M_{I,K}$$

and $((1 \leq i \leq 13) \wedge (i \neq 2))(M_i \uplus M_{I,K} \neq M_{I,K})$. This means, $I \circ K$, or $I$ *d* $K$, or $I$ *s* $K$.

## Algorithm of Propagating Temporal Relations

We can now construct the algorithm for propagating the temporal relations between $I$, $J$ and $K$. And an example is also presented to demonstrate the use of the algorithm.

Algorithm 1 *TemRelPropagation*

**Input**: $M_i$ $(1 \geq i \leq 13)$, $M_{I,J}$, $M_{J,K}$: *matrices;*
**Output**: $\alpha$*: possible temporal relations between I and K*;
(1) **Input** $M_i$ $(1 \geq i \leq 13)$, $M_{I,J}$, $M_{J,K}$;
    $\alpha := \emptyset$;
(2) $M_{I,K} := M_{J,K} \circ M_{I,J}$;
(3) **For** $i := 1$ **To** 13 **do**
    **Begin**
      $N := M_i \uplus M_{I,K}$;
      **If** $N = M_{I,K}$ **Then**
        $\alpha := \alpha \cup \{i\}$;
    **End**;
(4) **Output** the values of "$\alpha$";
(5) **End**.

Algorithm TemRelPropagation is used to compute the possible temporal relations between interval $I$ and interval $K$ when the relations between interval $I$ and interval $J$ and the relations between interval $J$ and interval $K$ are given. Step (1) reads the known matrices. And the resultant matrix is calculated in Step (2). Step (3) determines which relations are possible relations between $I$ and $J$ according to Definition 7. Step (4) outputs the result: all possible relations between $I$ and $J$.

We now illustrate the use of the above propagation by an example as follows:

*Example 4* Let $I$, $J$, $K$, and $L$ be four intervals, $I > J$, $J\ s\ K$, and $K\ \mathrm{mi}\ L$. We now compute the temporal relations between $I$ and $L$. We first get $M_{I,K}$ as follows.

$$M_{I,K} = M_{J,K} \circ M_{I,J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Secondly, we can calculate $M_{I,L}$ by $M_{I,K}$ and $M_{K,L}$ as follows.

$$M_{I,L} = M_{K,L} \circ M_{I,K} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Finally, we can determine the temporal relations between $I$ and $L$ by the final resultant matrix $M_{I,L}$. According to Definition 7, we can find out $M_2 \uplus M_{I,K} = M_{I,K}$, and $M_i \uplus M_{I,K} \neq M_{I,K}$ for $i \neq 2$. Consequently, the possible relations between $I$ and $K$ are $\alpha = \{>\}$.

## OPTIMIZING THE PROPAGATION

As have seen, Allen's idea can be implemented in matrix method and the propagation of temporal relations is certainly transformed into a numerical problem. Because the temporal relations can be described in $5 \times 5$ matrices, the running time of propagating temporal relations are mainly determined by the computation on given $5 \times 5$ matrices.

To present properties of logics and set theory, its efficiency is temporarily neglected in the above description for matrix representation. We now construct a more efficient algorithm for propagating temporal relations in this section.

There are two ways to improve Algorithm 1. The first one is to reduce the time of determining temporal relations after the final resultant matrix is obtained. The second one is to lessen the time on operating matrices. We first discuss the two methods in the following subsections, and than present an efficient algorithm to propagate temporal relations of intervals.

## Properties of the Thirteen Matrices

Recalling Example 4, we can directly determine the temporal relations between $I$ and $L$ according to the **final resultant matrix** $M_{I,L}$. However, for any one of the thirteen possible matrices, we must first logically "OR" the possible matrix and $M_{I,L}$. And then compare the final resultant matrix and $M_{I,L}$. Finally, the relation can be determined whether or not it is a possible relation between $I$ and $L$. It is a time-consuming procedure. Actually, this work can be finished by matching the special values of the thirteen matrices defined in Section 3. For this goal, we now look for the special values in these matrices.

For description, let $M_i = (m_{jk})_{5\times 5}$, where $1 \leq i \geq 13$. For matrix $M_1$, its special value is $m_{15} = 1$. Certainly, any one of other 12 matrices is with $m_{15} = 0$. Presently, we can determine whether or not relation "<" is a possible relation between two given intervals by only the value of the final resultant matrix at 1st row and 5th column. For example, consider the final resultant matrix $M_{I,K}$ in Example 3. Because $m_{15} = 0$ in matrix $M_{I,K}$, ">" is not a possible relation between $I$ and $K$. Hence, we can save much time in this way.

Also, for matrix $M_6$, its special values are $m_{32} = 1$ and $m_{43} = 1$. Certainly, any one of other 12 matrices is with either $m_{32} = 0$ or $m_{43} = 0$. In the same reason, we can list all special values of the 13 matrices in Table 2 where "s-value1" stands for the first special value, "s-value2" stands for the second special value.

Certainly, we can determine whether or not a relation is a possible relation between two given intervals by only the special values listed in Table 3. In other words, for a final resultant matrix $M = (m_{IK})_{5\times 5}$, if $m_{15} = 1$, "<" is a

**TABLE 2** The Special Values of the Thirteen Matrices

| matrix | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | $M_9$ | $M_{10}$ | $M_{11}$ | $M_{12}$ | $M_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s-value1 | $m_{15} = 1$ | $m_{51} = 1$ | $m_{24} = 1$ | $m_{42} = 1$ | $m_{23} = 1$ | $m_{32} = 1$ | $m_{32} = 1$ | $m_{23} = 1$ | $m_{32} = 1$ | $m_{23} = 1$ | $m_{22} = 1$ | $m_{22} = 1$ | $m_{32} = 1$ |
| s-value2 | | | | | $m_{34} = 1$ | $m_{43} = 1$ | $m_{44} = 1$ | $m_{44} = 1$ | $m_{34} = 1$ | $m_{43} = 1$ | $m_{34} = 1$ | $m_{43} = 1$ | $m_{44} = 1$ |

possible relation; if $m_{51} = 1$, "$>$" is a possible relation; if $m_{24} = 1$, "*m*" is a possible relation; if $m_{42} = 1$, "*mi*" is a possible relation; if $m_{23} = 1$ and $m_{34} = 1$, "*o*" is a possible relation; if $m_{32} = 1$ and $m_{43} = 1$, "*oi*" is a possible relation; if $m_{32} = 1$ and $m_{44} = 1$, "*f*" is a possible relation; if $m_{23} = 1$ and $m_{44} = 1$, "*fi*" is a possible relation; if $m_{32} = 1$ and $m_{34} = 1$, "*d*" is a possible relation; if $m_{23} = 1$ and $m_{43} = 1$, "*di*" is a possible relation; if $m_{22} = 1$ and $m_{34} = 1$, "*s*" is a possible relation; if $m_{22} = 1$ and $m_{43} = 1$, "*si*" is a possible relation; if $m_{22} = 1$ and $m_{44} = 1$, " = " is a possible relation. They are listed in Table 3.

We now illustrate the use of the above method by an example as follows.

**TABLE 3** Determining Relations by Special Values

| special value | $m_{15} = 1$ | $m_{51} = 1$ | $m_{24} = 1$ | $m_{42} = 1$ | $m_{23} = 1$ $m_{34} = 1$ | $m_{32} = 1$ $m_{43} = 1$ | $m_{32} = 1$ $m_{44} = 1$ | $m_{23} = 1$ $m_{44} = 1$ | $m_{32} = 1$ $m_{34} = 1$ | $m_{23} = 1$ $m_{43} = 1$ | $m_{22} = 1$ $m_{34} = 1$ | $m_{22} = 1$ $m_{43} = 1$ | $m_{22} = 1$ $m_{44} = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| relation | $<$ | $>$ | *m* | *mi* | *o* | *oi* | *f* | *fi* | *d* | *di* | *s* | *si* | = |

*Example 5* Let *I*, *J*, *K*, and *L* be four intervals, *I m J*, *J* oi *K*, and *K* si *L*. We now compute the temporal relations between *I* and *J*. We first can get $M_{I,L}$ as follows.

$$M_{I,K} = M_{J,K} \circ M_{I,J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$M_{I,L} = M_{K,L} \circ M_{I,K} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then we use the above special values to determine the possible temporal relations between *I* and *L* by the final resultant matrix $M_{I,L}$. We have, $m_{51} = 1$, "$>$" is a possible relation; $m_{42} = 1$, "m" is a possible relation; $m_{23} = 1$ and $m_{34} = 1$, "o" is a possible relation; $m_{32} = 1$ and $m_{43} = 1$, "*oi*" is a possible relation; $m_{32} = 1$ and $m_{44} = 1$, "f" is a possible relation; $m_{23} = 1$ and $m_{44} = 1$, "fi" is a possible relation; $m_{32} = 1$ and $m_{34} = 1$, "d" is a possible relation; $m_{23} = 1$ and $m_{43} = 1$, "di" is a possible relation; $m_{22} = 1$ and $m_{34} = 1$, "s" is a possible relation; $m_{22} = 1$ and $m_{43} = 1$, "si" is a possible relation; $m_{22} = 1$ and $m_{44} = 1$, "$=$" is a possible relation. Consequently, the possible temporal relations between $I$ and $L$ are $\{>, mi, o, oi, f, fi, d, di, s, si, =\}$.

As we have seen, the special values of the thirteen matrices can be used to simplify the procedure of determining the possible temporal relations between two given intervals by the final resultant matrix. And the running time can be apparently reduced. Another way to decrease the running time is presented in next subsection.

## Simplifying Operators on Matrices

For the sake of logical significant level, we use 1 and 0 stand for a nonempty set and an empty set, respectively, and the operators defined for matrices are also limited to match logical operations. In this way, some theorems become clear.

Actually, these operators on the matrices can be replaced by arithmetical operators for implementation. Because logical operations are usually time-consuming more than arithmetical operations, we can save much running time by defining arithmetical operators on the thirteen matrices. In our opinion, we can replace "$\circ$" and "$\uplus$" with "$*$" and "$+$" respectively. In this way, Table 3 would be changed into Table 4 as follows.

**TABLE 4** Determining Relations by New Special Values

| special value | $m_{15} > 0$ | $m_{51} > 1$ | $m_{24} > 0$ | $m_{42} > 0$ | $m_{23} > 0$ $m_{34} > 0$ | $m_{32} > 0$ $m_{43} > 0$ | $m_{32} > 0$ $m_{44} > 0$ | $m_{23} > 0$ $m_{44} > 0$ | $m_{32} > 0$ $m_{34} > 0$ | $m_{23} > 0$ $m_{43} > 0$ | $m_{22} > 0$ $m_{34} > 0$ | $m_{22} > 0$ $m_{43} > 0$ | $m_{22} > 0$ $m_{44} > 0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| relation | $<$ | $>$ | $m$ | $mi$ | $o$ | $oi$ | $f$ | $fi$ | $d$ | $di$ | $s$ | $si$ | $=$ |

The use of this table is the same as Table 3. We now show this procedure by an example as follows.

*Example 6* For Example 5, we used the arithmetical operators: $*$ and $+$ on matrices to compute the temporal relations between $I$ and $J$. We can get $M_{I,L}$ as follows:

$$M_{I,K} = M_{J,K} * M_{I,J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

$$M_{I,L} = M_{K,L} * M_{I,K} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}.$$

We now use the above new special values to determine the possible temporal relations between I and L by the final resultant matrix $M_{I,L}$. According to Table 4, $m_{51} > 0$, "$>$" is a possible relation; $m_{42} > 0$, "*mi*" is a possible relation; $m_{23} > 0$ and $m_{34} > 0$, "*o*" is a possible relation; $m_{32} > 0$ and $m_{43} > 0$, "*oi*" is a possible relation; $m_{32} > 0$ and $m_{44} > 0$, "*f*" is a possible relation; $m_{23} > 0$ and $m_{44} > 0$, "*fi*" is a possible relation; $m_{32} > 0$ and $m_{34} > 0$, "*d*" is a possible relation; $m_{23} > 0$ and $m_{43} > 0$, "*di*" is a possible relation; $m_{22} > 0$ and $m_{34} > 0$, "*s*" is a possible relation; $m_{22} > 0$ and $m_{43} > 0$, "*si*" is a possible relation; $m_{22} > 0$ and $m_{44} > 0$, "$=$" is a possible relation. Consequently, the possible temporal relations between I and L are $\{>, mi, o, oi, f, fi, d, di, s, si, =\}$.

After "$\circ$" and "$\uplus$" are replaced with "$*$" and "$+$" respectively, the running time is certainly cut down.

## The Efficient Algorithm

We now construct the efficient algorithm for propagating the temporal relations between *I*, *J*, and *K*, which the above two methods are used to decrease the running time.

Algorithm 2. *Efficient Propagation*

***Input***: $M_i$ $(1 \geq i \leq 13)$, $M_{I,J}$, $M_{J,K}$ : *matrices;*

***Output:*** $\alpha$: *possible temporal relations between I and K;*

(1) ***Input***: $M_i$ $(1 \geq i \leq 13)$, $M_{I,J}$, $M_{J,K}$;
    $\alpha := \emptyset$
(2) $M_{I,K} := M_{J,K} * M_{I,J}$;
(3) **For** the special values of each matrix of the thirteen matrices **do**
    **If** the conditions hold in $M_{I,K}$ **Then**
        $\alpha := \alpha \cup$ the corresponding relation;
(4) **Output** the values of "$\alpha$";
(5) **End-all**.

where algorithm EfficientPropagation uses arithmetical operators: $*$ and $+$ to compute the possible temporal relations between interval *I* and interval *K* when the relations between interval *I* and interval *J* and the relations between interval *J* and interval *K* are given. Step (1) reads the known matrices. And the resultant matrix is calculated in Step (2). Step (3) determines which relations are the possible relations between *I* and *J* according to Table 4. Step (4) outputs the result: all possible relations between *I* and *J*.

# EXPERIMENTS

As we have seen, the matrix representation provides us with an elegant way to catch temporal relations of intervals, which the propagation of temporal relations can be transformed into a numerical problem in thirteen $5 \times 5$ matrices. And the running time of propagating temporal relations are mainly determined by the computation on given $5 \times 5$ matrices. In particular, we suggested two ways to improve Algorithm 1. The first one is to reduce the time of determining temporal relations after the final resultant matrix is obtained. The second one is to lessen the time on operating matrices. To elucidate the efficiency and effectiveness of our matrix representation, we now display our experimental results in this section.

## Datasets

To study the effectiveness and efficiency, we have performed several experiments for our matrix model. Our algorithms are implemented on DELL-Pentium III using Java + +. The experimental results demonstrate that the approach is efficient and promising.

To evaluate the utility of the performance enhancements described above, we ran our matrix method and Allen's model on four artificial datasets, which each dataset contains four groups of data. In each dataset, the first group has fifteen intervals $\{I_1, I_2, \ldots, I_{15}\}$, the second group has twenty intervals $\{J_1, J_2, \ldots, J_{20}\}$, the third group has twenty five intervals $\{K_1, K_2, \ldots, K_{25}\}$, and the fourth group has thirty intervals $\{L_1, L_2, \ldots, L_{30}\}$. For simplicity, the possible relations set between $I_1$ and $I_2$, the possible relations set between $I_2$ and $I_3$, and the possible relations set between $I_i$ and $I_{i+1}$ are given in the first dataset. And the possible relations set between $I_1$ and $I_{15}$ is what we want to solve. The slight detailed information is listed in Tables 5, 6, 7, and 8.

**TABLE 5** The Properties of the First Dataset

| | 1st group | 2nd group | 3rd group | 4th group |
|---|---|---|---|---|
| intervalno | 15 | 20 | 25 | 30 |
| relationset | $S_{1,2}, S_{2,3}, \ldots, S_{14,15}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,20}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,25}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,30}$ |
| mostRS | 3 | 3 | 3 | 4 |
| smallestRS | 2 | 2 | 1 | 2 |

where "intervalno" stands for the number of intervals in a group, "relationset" the possible temporal relations in a group, $S_{ij}$ the set of the possible temporal relations between $I_i$ and $I_j$, "mostRS" the most number among all $|S_{ij}|$ in a group, "smallestRS" the smallest number among all $|S_{ij}|$ in a group, $|S_{ij}|$ the number of possible temporal relations between $I$ and $J$ (or the elements in $S_{ij}$).

**TABLE 6** The Properties of the Second Dataset

| | 1st group | 2nd group | 3rd group | 4th group |
|---|---|---|---|---|
| intervalno | 15 | 20 | 25 | 30 |
| relationset | $S_{1,2}, S_{2,3}, \ldots, S_{14,15}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,20}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,25}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,30}$ |
| mostRS | 4 | 6 | 7 | 6 |
| smallestRS | 2 | 3 | 3 | 3 |

The possible relations set between $J_1$ and $J_{20}$ is what we want to solve.

**TABLE 7** The Properties of the Third Dataset

| | | | | |
|---|---|---|---|---|
| intervalno | 15 | 20 | 25 | 30 |
| relationset | $S_{1,2}, S_{2,3}, \ldots, S_{14,15}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,20}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,25}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,30}$ |
| mostRS | 5 | 7 | 8 | 9 |
| smallestRS | 3 | 3 | 4 | 5 |

The possible relations set between $K_1$ and $K_{25}$ is what we want to solve.

**TABLE 8** The Properties of the Fourth Dataset

| | 1st group | 2nd group | 3rd group | 4th group |
|---|---|---|---|---|
| intervalno | 15 | 20 | 25 | 30 |
| relationset | $S_{1,2}, S_{2,3}, \ldots, S_{14,15}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,20}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,25}$ | $S_{1,2}, S_{2,3}, \ldots, S_{14,30}$ |
| mostRS | 6 | 9 | 9 | 11 |
| smallestRS | 4 | 5 | 4 | 6 |

The possible relations set between $L_1$ and $L_{30}$ is what we want to solve.

**TABLE 9** The Results of Matrix Method and Allen's Model

| | 1st group | 2nd group | 3rd group | 4th group | dataset |
|---|---|---|---|---|---|
| timeofMM | 1 | 2 | 3 | 4 | $D_1$ |
| timeofAM | 3 | 5 | 9 | 12 | $D_1$ |
| timeofMM | 1 | 2 | 3 | 4 | $D_2$ |
| timeofAM | 5 | 7 | 10 | 13 | $D_2$ |
| timeofMM | 1 | 2 | 3 | 4 | $D_3$ |
| timeofAM | 5 | 7 | 10 | 14 | $D_3$ |
| timeofMM | 1 | 2 | 4 | 4 | $D_4$ |
| timeofAM | 6 | 8 | 11 | 14 | $D_4$ |

## Results and Analyses

Table 9 shows the results of running the matrix method (simply called as MM) and Allen's model (simply called as AM) over the four artificial datasets: $D_1$, $D_2$, $D_3$, and $D_4$, where "*i*th group" stands for the *i*th group

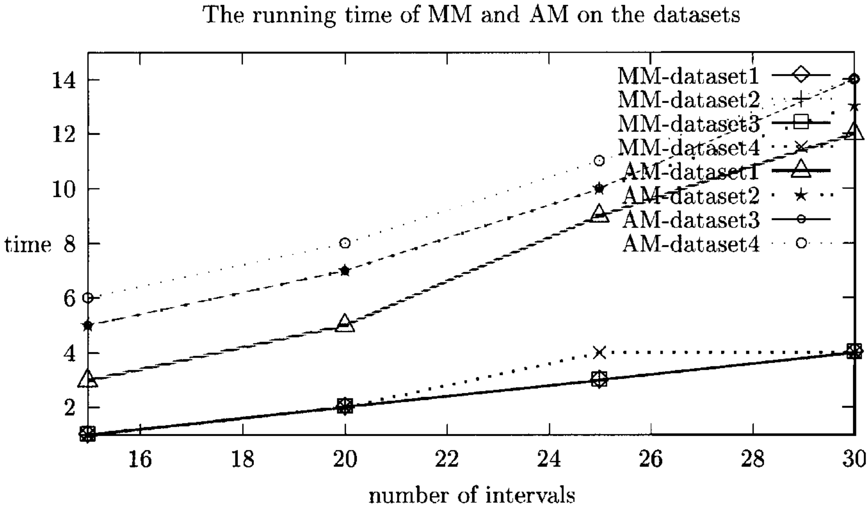


**FIGURE 1.** The comparison of *MM* and *AM* on the four datasets.

data in a dataset, "timeofMM" the running time of the matrix method over a group data, "timeofAM" the running time of the Allen's model over a group data, the unit of time is second. Figure 1 illustrates, the running time results of the proposed algorithm *MM* and the algorithm *AM*.

In the four experiments, the running results (the four sets of possible temporal relations of intervals) in our matrix method (MM) axe the same that in Allen's model (AM). The experiments also show that our method is faster than Allen's model. The main reason of resulting the above weakness is that the temporal reasoning (or propagation) in Allen's model must do multiple accesses to a $13 \times 13$ table in each reasoning step, which the $13 \times 13$ table is the possible temporal relations between any two temporal relations of the thirteen temporal relations. Indeed, our method applies the special values of the thirteen matrices (16 special values listed in Table 4) to determine the possible temporal relations between two given intervals by the final resultant matrix. Also the running time can be apparently reduced.

## CONCLUSION

Interval algebra has successfully applied to multi-agents (Kraus, Wilkenfeld, and Zlotkin 1995), to temporal databases for manipulating their time relationships (Gadia and Yeung 1991; Snodgrass 1987), to specifically design for simulations (Tuzhilin 1995) to specify temporal properties of concurrent systems (Barringer, Kuiper, and Pnueli 1984). The major challenge in IA is the propagation of temporal relations. So, a number of researches have been focused on constraint satisfactory problems (Allen and Koomen 1983; Haddawy 1996; Ladkin and Maddux 1994; Meiri 1996; Mouhoub, Charpillet, and Haton 1998; Nebel 1997; Pirri and Reiter 1995; Schwalb, Kask, and Dechter 1994; Vilain and Kautz 1986). However, the proposed models can only make improvements on IA for some applications. Though there are many excellent contributions (Nebel and Burckert 1995; Schwalb and Dechter 1997; Schwalb 1997; Tolba, Charpillet, and Haton 1991; Vila and Reichgelt 1996) solving tractable subclasses of Allen's interval algebra, it is still an identical limitation in these models that the propagation of temporal relations is non-tractable in general. We advocated an alternative method to represent temporal relations in this paper, in which thirteen $5 \times 5$ matrices were used to describe thirteen temporal relations defined in Allen (1983). In this way, the propagation of temporal relations can be transformed into a computable problem. The keys of this work are as follows:

- Established an alternative method of representing interval calculus with thirteen $5 \times 5$ matrices.
- Advocated an efficient algorithm of the propagation based on two methods.
- Evaluated the effectiveness and efficiency of matrix method by experiments.

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