The decoding problem of this dynamic bayesian network is written as,

$$
\begin{aligned}
& p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{y}_{1:t}) \\
=& \sum_k p(\mathrm{v}_t, \mathrm{r}_t, k | \mathrm{y}_{1:t}, k) \\
=& \sum_k p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{y}_{1:t}, k) p(k) \\
=& \sum_k p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{y}_{1:t-1}, \mathrm{y}_t, k) p(k) \\
\propto& \sum_k p(\mathrm{v}_t, \mathrm{r}_t, \mathrm{y}_t | \mathrm{y}_{1:t-1}, k) p(k) \\
=& \sum_k p(\mathrm{y}_t | \mathrm{v}_t, \mathrm{r}_t, \mathrm{y}_{1:t-1}, k) p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{y}_{1:t-1}) p(k | \mathrm{y}_{1:t-1}) p(k) \\
=& \sum_k p(\mathrm{y}_t | \mathrm{v}_t, \mathrm{r}_t, \mathrm{y}_{1:t-1}, k) p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{y}_{1:t-1}, k) p(k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t, \mathrm{r}_t, \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j | \mathrm{y}_{1:t-1}, k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j, \mathrm{y}_{1:t-1}, k) \\
& p(\mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j | \mathrm{y}_{1:t-1}, k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j, k) \delta_{t-1}(i, j, k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t, \mathrm{r}_t | \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j, k) \delta_{t-1}(i, j, k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t | \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j, k) \\
& p(\mathrm{r}_t | \mathrm{v}_t, \mathrm{v}_{t-1} = i, \mathrm{r}_{t-1} = j, k) \delta_{t-1}(i, j, k) \\
=& p(\mathrm{y}_t | \mathrm{r}_t) \sum_i \sum_j \sum_k p(k) p(\mathrm{v}_t | \mathrm{v}_{t-1} = i, k) p(\mathrm{r}_t | \mathrm{v}_t, \mathrm{r}_{t-1} = j) \delta_{t-1}(i, j, k)
\end{aligned}
$$

In other words,

$$\delta_t(i,j,k) = p(\mathrm{y}_t|\mathrm{r}_t)\sum_i\sum_j p(\mathrm{v}_t|\mathrm{v}_{t-1}=i,k)p(\mathrm{r}_t|\mathrm{v}_t,\mathrm{r}_{t-1}=j)\delta_{t-1}(i,j,k)$$

The decoding problem can be seen as a function of recursive equation δ.

$$p(\mathrm{v}_t=i,\mathrm{r}_t=j|\mathrm{y}_{1:t}) = \sum_k \delta_t(i,j,k)p(k)$$