This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the WCNC 2009 proceedings.

Improved MDS-Based Multi-Target Tracking Algorithm

Davide Macagnano and Giuseppe Thadeu Freitas de Abreu
Centre for Wireless Communications - University of Oulu
P.O.Box 4500, 90014-Finland
[macagnan,giuseppe]@ee.oulu.fi

***Abstract*— We consider the problem of tracking multiple targets in the presence of imperfect and incomplete ranging information using an MDS-based tracking algorithm. An advantage of this technique is that tracking accuracy is independent on target dynamics. The main feature of the aforementioned algorithm, which we proposed in an earlier work, is that tracking is performed over the eigenspace of a Nyström-Gram kernel matrix constructed with no *a-priori* knowledge of the statistics of target trajectories. Consequently tracking becomes a problem of updating the eigenspace given new input data, which is achieved with an iterative Jacobian eigen-decomposition technique. In this paper it is first shown how to improve the aforementioned eigen-decomposition to fully exploit the structure of the reconstructed Gram kernel matrix, then how to use the similarity existing between subsequent Gram matrices to efficiently track the relative sub-spaces. The performance and computational complexity of two techniques, namely, the Multidimensional Scaling (MDS)-based tracking algorithm and SMACOF are investigated. As a result, the MDS-based tracking algorithm with Jacobian eigen-space updating is shown to achieve the same performance as the SMACOF algorithm, but at a significantly lower complexity.**

I. Introduction

Multidimensional scaling (MDS) is a technique used to map a set of objects into points of an Euclidean space of typically low dimension with basis on their mutual dissimilarities, as measured by a certain metric, often the Euclidean distance, such that the dissimilarities computed from the mapped points match the observed dissimilarities [1].

There are two major approaches to solve the metric MDS problem: the *classic* approach, which is based on an algebraic procedure and involves the eigen-decomposition of a kernel matrix, positive semidefinite transformation of the dissimilarity matrix, $e.g.$ the Euclidean Distance Matrix (EDM) [2]; and the *least-squares* (LS) approach, which involves the optimization of a stress (generally least-squares) function [3].

From the above, it is clear that metric MDS can be used as a distance-based multi-source localization algorithm, as demonstrated $e.g.$ in [4], although this approach has the drawbacks of requiring that all, in the classic case, pairwise distances between sources be collected and processed at a central unit. This may be a problem in at least a portion of the multi-source localization scenarios of interest, in particular those characterized by networks with meshy topologies of relatively large dimensions, where node-to-node distances can only be measured within the vicinity of each source, resulting in an incomplete EDM.

In some of the multi-source localization applications of interest, however, metric MDS can be a suitable distance-based localization technique. A prominent example is localization of cellular or indoor wireless devices with basis on their distances to a number of inter-wired based stations of known locations.

In such cases the typical network topology is a superposition of star topologies (node-to-anchors) and the classic MDS solution is feasible because traffic is not an issue, since each device has a direct communication link to all base stations, and because the lack of direct connectivity amongst the devices to be localized translates to a well-structured incompleteness of the EDM which can be coped with by the Nyström approximation [5]. We have shown that for such *star-like* topologies, the classic MDS-based localization technique can also be modified into a tracking algorithm if the eigen-decomposition step of the algebraic solution is performed iteratively, such that the eigen-pairs corresponding to the current observation can be computed via updating the eigen-pairs of the previous one [6]. It was shown that the classic MDS-based tracking algorithm in fact has very low computational complexity [6]-[7], and its performance is completely independent on the target dynamics, that in the context of multi-target tracking can be variable and unknown [6]. In this paper, we reconsider the problem of tracking multiple dynamic targets subjected to imperfect ranging information, improving the previous result by introducing an approximation of the eigendecomposition used in the algorithm.

The reminder of this article is as follow. In section II the formulation of the MDS-based tracking technique and the Jacobian-eigendecomposition are briefly reviewed. In section III two variations of Jacobian eigendecomposition used to solve the MDS problem are discussed and compared in complexity against the Symmetric QR eigendecomposition, and in performance (precision) versus the standard Singular Value Decomposition (SVD).

In section IV it is explained how to track the subspaces associated to two subsequent Gram matrices and how this affects the sweeping strategy to be used in the Jacobian algorithm. Concluding section V compares the complexity and performance for the proposed solution against a widely know optimization-based solution to the MDS problem, the SMACOF algorithm (Scaling by Majorizing a COmplicated Function). Subsection V-A investigates the scalability of both MDS-based techniques, showing the superior efficiency of the

978-1-4244-2948-6/09/$25.00 ©2009 IEEE

algebraic solution. Subsection V-B compares the performance of the aforementioned techniques. The results indicate that the eigenspace MDS-based tracking algorithm, while being substantially less complex than the optimization-based, achieves the same performance.

II. MDS-Based Tracking Algorithm

As mentioned in section I, MDS is a multivariate method that provides a spatial representation of the data under investigation. Given the matrix containing all the similarities-dissimilarities δ_{ij} between N points, MDS maps their configuration in an embedding space such that the difference between δ_{ij} and the relative distances d_{ij} is minimal. Let $\mathbf{X}$ denote the $[N \times \mu]$ matrix containing the set of coordinates for N objects in a μ dimensional embedding space, and choose $\{\delta_{ij}\} = \{d_{ij}\}$. The set of all the euclidean pairwise distance between two generic objects i and j in the scenario, is

$$d_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^{\mathrm{T}} \cdot (\mathbf{x}_i - \mathbf{x}_j)\,. \tag{1}$$

Then the MDS problem can be solved algebraically [2] as the least square solution on the inner product matrix $\mathbf{B} = \mathbf{X} \cdot \mathbf{X}^{\mathrm{T}}$. The classical scaling solution, provided the EDM $\mathbf{D}$ ($[\mathbf{D}]_{ij} = \mathbf{d}_{ij}$), constructs the following Gram matrix

$$\mathbf{G} = -\frac{1}{2} \cdot \mathbf{J} \cdot \mathbf{D}^{\circ 2} \cdot \mathbf{J}, \tag{2}$$

where $\mathbf{J} = \mathbf{I}_N - \mathbf{1}_N \mathbf{1}_N^{\mathrm{T}}/N$ is the double centering matrix, $\mathbf{1}_N$ the N-by-1 unitary vector, $\mathbf{I}_N$ the N-by-N identity matrix and $\circ$ indicating the point-wise (Hadamart) product.

As proved in [1] and also used in section IV, $\mathbf{G}$ is equivalent to $\mathbf{B}$ (for $\mathbf{X}$ centered at the origin). Therefore at any time, the MDS technique allows to recover the location $\mathbf{Y}$ of all sensors in the network (up to rotation, scaling, translation and reflection) from the complete EDM by

$$\mathbf{Y} = [\mathbf{V}]_{1:\mu} \cdot [\mathbf{\Lambda}]_{1:\mu}^{\frac{1}{2}}, \tag{3}$$

$$\mathbf{G} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{\mathrm{T}}. \tag{4}$$

In [5] it was shown how to address the same problem from an incomplete, nevertheless well structured set of $\{\delta_{ij}\}$, by means of the Nyström approximation of the Gram Kernel associated to $\mathbf{X}$. This same technique can be adapted to solve localization problems in dynamic scenarios over incomplete EDM $\tilde{\mathbf{D}}$ considering only anchor-to-anchor $\mathbf{D}_{\mathrm{A}^2}$ and anchor-to-target distances $\mathbf{D}_{\mathrm{AM}}$ per sampling, as

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D}_{\mathrm{A}^2} & \mathbf{D}_{\mathrm{AM}} \\ \mathbf{D}_{\mathrm{AM}}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}. \tag{5}$$

Provided that the coordinates $\mathbf{X}_{\mathrm{A}}$ of at least $A > \mu$ anchor nodes are known with respect to an absolute system of coordinates, the solution $\mathbf{Y}$ can be re-oriented via Procrustes transformation [8], returning $\hat{\mathbf{X}}$.

A. Jacobian Eigendecomposition

As indicated by equation (3), tracking with the MDS algorithm requires the repetitive computation of the eigen-decomposition of $\mathbf{G}$ at the different time instances. This can be efficiently done using the Jacobian algorithm [9] summarized by

$$\mathbf{A}_{k+1} \longleftarrow \mathbf{R}(i_k, j_k, \theta_k)^{\mathrm{T}} \cdot \mathbf{A}_k \cdot \mathbf{R}(i_k, j_k, \theta_k), \tag{6}$$

where $\mathbf{R}(i_k, j_k, \theta_k)$ are orthogonal similarity transformations corresponding to the Givens rotation matrices [9]. The approximate eigenvector matrix of $\mathbf{A}$ obtained after K_{E} iterations is given by

$$\mathbf{V} = \prod_{k=1}^{K_{\mathrm{E}}} \prod_{(i_k, j_k)} \mathbf{R}(i_k, j_k, \theta_k). \tag{7}$$

A Jacobian-like eigen-decomposition algorithm capable of finding a single matrix $\mathbf{R}$ that jointly (approximately) diagonalizes a set of matrices $\mathcal{A} \triangleq \{\mathbf{A}_1, ..., \mathbf{A}_M\}$ also exists [10]. In this Jacobian-like joint-diagonalization technique, one iterates the expression

$$\mathcal{A}_{k+1} = \mathbf{R}(i_k, j_k, \theta_k)^{\mathrm{T}} \cdot \mathcal{A}_k \cdot \mathbf{R}(i_k, j_k, \theta_k) \tag{8}$$

where $\mathbf{R}(i_k, j_k, \theta_k)$ is the solution of

$$\min_{\theta(i,j)} \sum_{m=1}^{M} \mathrm{off}\left(\mathbf{R}^{\mathrm{T}}(i,j,\theta) \cdot \mathbf{A}_m \cdot \mathbf{R}(i,j,\theta)\right), \tag{9}$$

with

$$\mathrm{off}\,(\mathbf{A}) \triangleq \sum_{i \neq j} \|a_{i,j}\|^2. \tag{10}$$

Convergence and stability of this algorithm are rigorously proved in [10]. Fortunately, closed-form expression for the optimum angles solving the minimization problem defined by equation (9) was discovered in [11] as

$$\theta_{\mathrm{Opt}}(i,j) = \frac{1}{2} \cdot \mathrm{atan}\left(\frac{a_{i,j} + a_{j,i}}{a_{i,i} - a_{j,j}}\right). \tag{11}$$

The solution thereby is extremely simple and is applicable to any set of equi-dimentional matrices $\mathcal{A}$, regardless of symmetry and commutativity properties.

As noted in [6], the Jacobian algorithm with closed-form optimum rotation angles can be used as an eigenspectrum tracking algorithm as follows. Consider the two Gram matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ corresponding to consecutive observations $\tilde{\mathbf{D}}_1$ and $\tilde{\mathbf{D}}_2$, and assume that the eigenvector matrix $\mathbf{V}_1$ of $\mathbf{G}_1$ is known. Then, $\mathbf{V}_2$ can be easily computed by initializing the Jacobian algorithm with $\mathbf{V}_1$. This leads to the following solution for the eigen-decomposition of $\mathbf{G}_2$,

$$\mathbf{V}_2 = \mathbf{V}_1 \cdot \prod_{k=1}^{K_{\mathrm{E}}} \prod_{(i_k, j_k)} \mathbf{R}(i_k, j_k, \theta_k). \tag{12}$$

In addition, in [6] it was discussed how the number of iterations K_{E} required to update $\mathbf{V}_1$ into $\mathbf{V}_2$ is related to the average value for the target dynamic $\bar{\nu}$.

III. Improvement I: Efficient Stop Criterion for Jacobian Eigendecomposition

Denoting with λ_i the ith eigenvalue of a generic matrix $\mathbf{A}$, then its Frobenius norm is

$$\|\mathbf{A}\|_{\mathrm{F}}^2 \triangleq \sum_{i=1}^{N}\sum_{j\neq i}^{N} a_{ij}^2 \equiv \sum_{i=1}^{N} \lambda_i^2. \tag{13}$$

Provided that $\mathbf{A}$ is a Hermitian matrix ($i.e$ $\mathbf{A}$=$\mathbf{A}^*$), and $\mathbf{B} = \mathbf{HAH}^*$, with $\mathbf{H}$ a unitary transformation, then assuming the dimension of the eigen-space equal to μ, since $\|\mathbf{A}\|_{\mathrm{F}}^2 = \mathrm{trace}(\mathbf{A}^*\mathbf{A}) = \sum_{i=1}^{\mu}\lambda_i^2$ and using the fact that unitary transformation do not change the eigenvalues, it follows that $\|\mathbf{A}\|_{\mathrm{F}}^2 = \|\mathbf{B}\|_{\mathrm{F}}^2$.

Consequently, being plane rotations unitary transformation, neither the trace nor the Frobenius norm of the matrix to be eigendecomposed are changed by the Jacobian algorithm.

Since $\|\mathbf{A}\|_{\mathrm{F}}^2$ can be decomposed into the terms $\alpha(\mathbf{A}) = \sum a_{ii}^2$ and its complement $2\beta(\mathbf{A}) = \mathrm{off}(\mathbf{A})$, with the latter equivalent to equation (10), then every time a plane rotation $\mathbf{R}(i_k, j_k, \theta)$ is applied to $\mathbf{A}$ the two terms are modified into:

$$\alpha\left(\mathbf{R}^{\mathrm{T}}\mathbf{AR}\right) = \alpha(\mathbf{A}) + 2a_{ij}^2, \tag{14}$$

$$\beta\left(\mathbf{R}^{\mathrm{T}}\mathbf{AR}\right) = \beta(\mathbf{A}) - a_{ij}^2. \tag{15}$$

Independently on the order, β is known to be monotonic decreasing (consequently α is monotonic increasing) and converging to zero, making $\mathbf{A}$ to tend to a fixed diagonal matrix similar to the initial [12]. Applying the optimal off-diagonal search strategy, namely always the biggest plane rotation possible, then the algorithm is proved to converge quadratically with the number of $sweeps$ K_{E}, $i.e$ the number of full sets of plain rotations applied to the upper triangular part of $\mathbf{A}$ [12]. In subsection II-A it was shown how the Jacobian algorithm performs the eigendecomposition through the optimization of equation (9) until a threshold value ϵ (sufficiently close to zero) is reached, namely

$$2\beta(\mathbf{A}) < \epsilon. \tag{16}$$

This type of eigendecomposition has the advantage that both eigenvalues and eigenvectors are obtained simultaneously, the latter as an orthogonal set, it is extremely accurate and converges with a number of sweeps that is proportional to $\log_2(N)$ [9].

However, in the form presented thereby, it would be too computationally demanding for our needs. Indeed, if compared against the symmetric QR solution, the complexity necessary to compute two sweeps in the Jacobin algorithm would be sufficient to perform a full eigendecomposition using the QR one [9]. As discusses hereafter and in section IV, within the MDS framework it is possible to make use of several modifications to substantially decrease the complexity without affecting the performance of the tracking algorithm.

To begin with, remembering that the Frobenius norm is preserved under the plane rotations applied by the Jacobian algorithm, the stopping criterion can be rewritten using, instead of the squared off-diagonal terms of $\mathbf{A}$ defined by equation (16), the following condition on the squared diagonal terms

$$\alpha(\mathbf{A}) \geq \|\mathbf{A}\|_{\mathrm{F}}^2 \cdot (1-\epsilon). \tag{17}$$

This modification, although simple, already gives several advantages. Figure 1 shows the accuracy of the algorithm to estimate both eigenvalues and eigenvectors (dependent on ϵ), while Figure 2 shows the gain in complexity of this stopping criterion. Moreover an additional small gain is provided by the repetitive computation of $\alpha(\mathbf{A})$ rather than $\beta(\mathbf{A})$. Indeed while the former requires $(2N)$ operations, the latter approximately $\left(N^2+N\right)$. Concerning the precision of the eigendecomposition, Figures 1-3 show that within the context of this paper, small errors on the eigenvalues/eigenvectors don't affect the overall performance of the tracking algorithm.

Although better, as Figure 2 shows, the criterion proposed by equation (17) alone does not beat the Symmetric QR eigendecomposition yet. Nevertheless, using the fact that to complete $\mathbf{G}$ from $\tilde{\mathbf{D}}$, the Nyström approximation becomes necessary [5], then it is possible to improve further the stopping criterion.

Being the Nyström method based on the the approximation of the full eigenspectrum of $\mathbf{G}$ trough its first rows (in our case equivalent to the number of anchor nodes), then it is possible to change the condition imposed by equation (17) further assuming $\sum_{i=\mu+1}^{N}\lambda_i = 0$. This corresponds to the case of perfect observation, under which $\mathbf{G}$ is positive semidefinite with $\mathrm{rank}(\mathbf{G}) = \mu$, meaning $\lambda_i = 0$ for $\{i = \mu+1 \ldots N\}$.

However, with fallible data, the spurious roots don't equal to zero but vary positively and negatively around it [1]. Within the MDS context, some authors already assumed the sum of these spurious eigenvalues equal to zero, like in [13] to solve the MDS additive constant problem and in [14] to define the dimensionality of the problem through the trace criterion. Here the same idea is used in connection with the spectral approximation/troncation operated on $\mathbf{G}$ by Nyström, yielding to the following stopping criteria for the Jacobian algorithm

$$\gamma(\mathbf{A}) = \sum_{i=1}^{\mu} a_{ii} \geq \mathrm{trace}(\mathbf{A}) \cdot (1-\epsilon). \tag{18}$$

As before, Figure 1 shows the precision in computing both the eigenvalues and eigenvectors by means of equation (18). It can be appreciated how, stopping the Jacobian algorithm one iteration before than what defined by the equation, namely $(\gamma - 1)$, the error on the eigenvalues/eigenvectors increases drastically, validating equation (18) as a good candidate to capture the structure of $\mathbf{G}$ approximated by Nyström.

This is further confirmed in Figure 2, where it is shown that the same localization accuracy of the stopping criterions defined by equations (16)-(17) is obtained at expense of a much lower complexity, even less than the one required by the Symmetric QR algorithm. In addition, the computation of this metric only requires μ sums.

As previously discussed, being $\mathbf{G}$ hermitian positive semi-definite (under perfect ranging), from the spectral theorem

follows that its eigendecomposition, given by equation (4), is also equivalent to the singular value decomposition (SVD). Therefore to evaluate the precision of the different stopping criterion defined by equations (16)-(18) to compute the first μ eigenvectors/eigenvalues, namely $\mathbf{V}_{1:\mu}$ and $\mathbf{\Lambda}_{1:\mu}$, the results are compared against the corresponding singular vectors $\mathbf{E}_{1:\mu}$ and values $\mathbf{\Omega}_{1:\mu}$ obtained by SVD of the same Gram matrix $\mathbf{G}$. The subspace difference between $\mathbf{V}_{1:\mu}$ and $\mathbf{E}_{1:\mu}$ is computed accordingly to [9] as

$$\phi\left(\mathbf{E}_{1:\mu}, \mathbf{V}_{1:\mu}\right) = \|\mathbf{P}_{\mathrm{E}} - \mathbf{P}_{\mathrm{V}}\|_2, \tag{19}$$

with $\mathbf{P}_{\mathrm{E}} = \mathbf{E}_{1:n}\mathbf{E}_{1:n}^{\mathrm{T}}$ representing the orthogonal projection matrix for the space associated to $\mathbf{E}_{1:\mu}$ and similarly $\mathbf{P}_{\mathrm{V}}$ [9].

While to evaluate the error to compute the first μ eigenvalues, the following normalized error is considered

$$\eta = \sum_{i=1}^{\mu}\left(\frac{|\omega_i - \lambda_i|}{\omega_i}\right). \tag{20}$$

IV. Improvement II: Efficient Sweeping Strategy for Subspace Tracking

Hereafter we intend to show the relations between two subsequent Gram matrices $\mathbf{G}_1$ and $\mathbf{G}_2$, and how equation (12) allows to track their subspace.

Given $\mathbf{X}_1$ as the set of coordinates for the N points at the time 1, at the next time $\mathbf{X}_2 = \mathbf{X}_1 + \Delta\mathbf{X}$, and consequently the inner product matrix $\mathbf{B}_2 = \mathbf{X}_2 \cdot \mathbf{X}_2^{\mathrm{T}}$ can be written as

$$\mathbf{B}_2 = (\mathbf{X}_1 + \Delta\mathbf{X}) \cdot (\mathbf{X}_1 + \Delta\mathbf{X})^{\mathrm{T}}. \tag{21}$$

This equation can be rewritten for two generic points $\{i, j\}$, with $\{i, j = 1 \ldots N\}$, as $b_{2,ij} = \mathbf{x}_{2,i} \cdot \mathbf{x}_{2,j}^{\mathrm{T}}$, with $\mathbf{x}_{2,i} = (\mathbf{x}_{1,i} + \delta_i)$ and $\mathbf{x}_{2,j} = (\mathbf{x}_{1,j} + \delta_j)$ and $\delta_{\cdot}$ as the coordinate difference between two successive times for the $\{i, j\}$ points. Since classical MDS recovers $\mathbf{X}$ through the eigendecomposition of $\mathbf{G}$, it is useful to see how $\mathbf{B}_2$ and $\mathbf{G}_2$ relates to $(\mathbf{X}_1 + \mathbf{\Delta X})$ through $\mathbf{D}_2$. Substituting in equation (1) the expression for $\mathbf{x}_{2,i}$ and $\mathbf{x}_{2,j}$, under the condition that the centroid of the configuration at both times 1 and 2 are at the origin (to avoid the indeterminacy of the solution due to translation), namely $\sum_{i=1}^{N}\mathbf{x}_{1,i,\mu} = 0$ and $\sum_{i=1}^{N}\delta_{i,\mu} = 0$, then

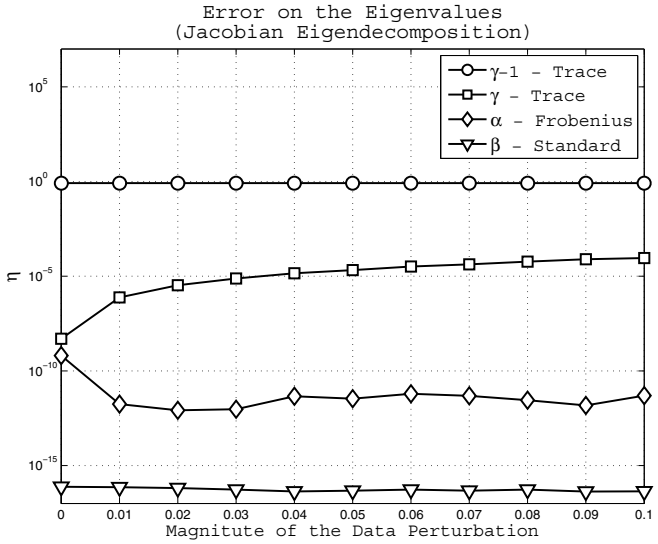
$$\frac{1}{N}\sum_{i=1}^{N} d_{2,ij}^2 = \frac{1}{N}\sum_{i=1}^{N} \|\left(\mathbf{x}_{1,i} + \delta_i\right)\|_2^2 + \|\left(\mathbf{x}_{1,j} + \delta_j\right)\|_2^2, \tag{22}$$

$$\frac{1}{N}\sum_{i=1}^{N} d_{2,ij}^2 = \frac{1}{N}\sum_{j=1}^{N} \|\left(\mathbf{x}_{1,j} + \delta_j\right)\|_2^2 + \|\left(\mathbf{x}_{1,i} + \delta_i\right)\|_2^2, \tag{23}$$

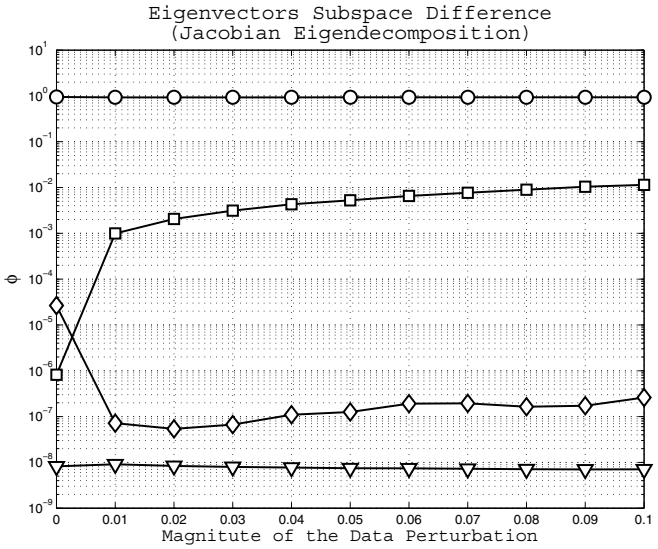
$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} d_{2,ij}^2 = \frac{2}{N}\sum_{i=1}^{N} \|\left(\mathbf{x}_i + \delta_i\right)\|_2^2. \tag{24}$$

Rewriting $b_{2,ij}$ using equation (1) follows

$$b_{2,ij} = -\frac{1}{2}\left(d_{2,ij}^2 - \|\left(\mathbf{x}_{1,i} + \delta_i\right)\|_2^2 - \|\left(\mathbf{x}_{1,j} + \delta_j\right)\|_2^2\right), \tag{25}$$



(a) Error η for the different stopping criterion for the Jacobian algorithm.



(b) Error ϕ for the different stopping criterion for the Jacobian algorithm.

Fig. 1. Error η and ϕ as a function of the perturbation on $\tilde{\mathbf{D}}$ with $\epsilon = 10^{-8}$

and using equation (22)-(24), then $b_{2,ij}$ can be written as a function of $d_{2,ij}^2$ as

$$b_{2,ij} = -\frac{1}{2}\left(d_{2,ij}^2 - \frac{1}{N}\sum_{i=1}^{N} d_{2,ij}^2 - \frac{1}{N}\sum_{i=1}^{N} d_{2,ij}^2 + \frac{1}{N^2}\sum_{i,j=1}^{N} d_{2,ij}^2\right). \tag{26}$$

Since equation (26) corresponds to the double-centered kernel defined by equation (2) and computed for $\mathbf{D}_2$ [1], the equivalence between $\mathbf{G}_2$ and $\mathbf{B}_2$ is proved. Consequently $\mathbf{G}_2$ can be rewritten using equation (21) as

$$\mathbf{G}_2 = \mathbf{X} \cdot \mathbf{X}^{\mathrm{T}} + \Delta\mathbf{X} \cdot \Delta\mathbf{X}^{\mathrm{T}} + \mathbf{X} \cdot \Delta\mathbf{X}^{\mathrm{T}} + \Delta\mathbf{X} \cdot \mathbf{X}^{\mathrm{T}}. \tag{27}$$

Assuming the eigenvector matrix $\mathbf{V}_1$, solution for $\mathbf{G}_1$, as known, then $\mathbf{V}_2$ can be easily computed by initializing the

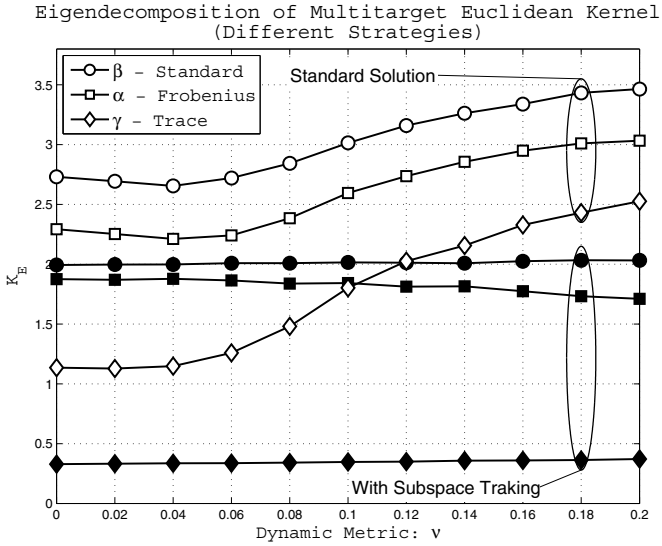


Fig. 2. Number of sweeps K_{E} for the different stopping criterion with and without application of equation (12) as function of ν, $\sigma = 0.01$.

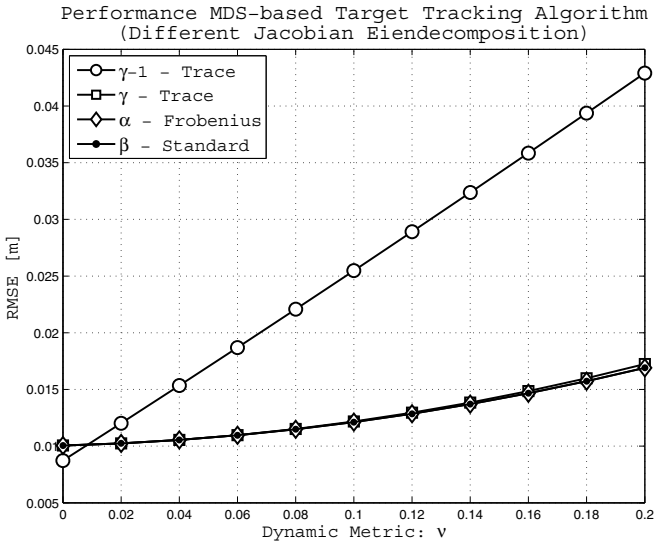


Fig. 3. Accuracy for the MDS-based Tracking algorithm for the different stopping criterions, 2D scenario with 4 anchors placed at the corners of a $[1 \times 1]\,[m]$ space, $\sigma = 0.01$.

Jacobian algorithm with $\mathbf{V}_1$. Indeed, applying $\mathbf{V}_1$ to equation (27) (namely $\mathbf{V}_1^{\mathrm{T}} \cdot \mathbf{G}_2 \cdot \mathbf{V}_1$) yields to

$$\mathbf{T}_1 = \mathbf{V}_1^{\mathrm{T}} \cdot \mathbf{B}_1 \cdot \mathbf{V}_1, \tag{28}$$

$$\mathbf{T}_2 = \mathbf{V}_1^{\mathrm{T}} \cdot \left(\Delta\mathbf{X} \cdot \Delta\mathbf{X}^{\mathrm{T}}\right) \cdot \mathbf{V}_1, \tag{29}$$

$$\mathbf{T}_3 = \mathbf{\Lambda}^{\frac{1}{2}} \cdot \Delta\mathbf{X}^{\mathrm{T}} \cdot \mathbf{V}_1, \tag{30}$$

$$\mathbf{T}_4 = \mathbf{V}_1^{\mathrm{T}} \cdot \Delta\mathbf{X} \cdot \mathbf{\Lambda}^{\frac{1}{2}}. \tag{31}$$

Clearly $\mathbf{T}_1 = \mathbf{\Lambda}$, while the other terms can be seen as perturbations on $\mathbf{G}_1$. Since $\mathrm{rank}\,(\boldsymbol{G}_1) = \mu$, then $\mathbf{\Lambda}$ is diagonal with only the first μ entrances different from zero. Consequently only the corresponding rows-columns of $\mathbf{T}_3$ - $\mathbf{T}_4$ will be different from 0 (note that $\mathbf{T}_3 = \mathbf{T}_4^{\mathrm{T}}$). In addition, since $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} d_{ii}^2$ [1], it follows that the elements in the first μ rows of $\mathbf{T}_3$, namely columns of $\mathbf{T}_4$, on average are multiplied by

$$\pi = \left(\frac{1}{\mu}\sum_{i=1}^{N} d_{ii}^2\right)^{1/2}. \tag{32}$$

Therefore, unless $trace\left(\Delta\mathbf{X} \cdot \Delta\mathbf{X}^{\mathrm{T}}\right)$ is at least of the same order of $trace\,(\mathbf{G})$, then the dominant terms in equation (27) will be give by $\mathbf{T}_3$-$\mathbf{T}_4$. Consequently, for relatively small changes $\Delta\mathbf{X}$, namely low average velocities $\bar{\nu}$, the cyclic row-by-row sweeping strategy to compute the Jacobian eigendecomposition can be considered close to the optimal off-diagonal search strategy [12], with the clear advantage of a simpler implementation and a faster execution.

V. Results

Benefits and eventual weaknesses deriving from the usage of this MDS-based tracking algorithm were investigated and compared against a widely known parametric approach, *e.g* Extended Kalman Filter (EKF), in [6]. Following, the performance of the MDS-based tracking algorithm was improved including a wavelet-based filtering block on the time series measured at the anchors [7]. In the same article it was shown that this additional block, while being computationally efficient, because of its ability to study the local regularity of functions, it allows the detection of the singularities affecting the TOA ranging received at the anchors in correspondence of Line of Sight (LoS) non-Line of Sight (NLoS) transitions.

In [15] the algebraic MDS-based subspace tracking algorithm was compared, in its standard version (equation (16)), against SMACOF, an optimization based solution to the MDS problem [1] [3]. It was shown how the two solutions combined with the wavelet-based pre-filtering suggested in [15], reach approximately the same performance, with SMACOF being less complex. In addition some results from the implementation of the algorithm into real devices were shown and discussed under both LoS and mixed LoS-NLoS real scenarios.

In Figure 1 the gain in complexity achievable by the Jacobian algorithm for the different stop criterions defined by equation (16)-(18) have been compared with and without application of the subspace tracking defined by equation (12). Following the overall complexity for SMACOF is compared against the MDS-based Tracking Algorithm solution with the new stopping criterion. Concluding the performance for the two techniques are compared as a function of the target dynamics ν under different noise conditions, modeled as $\mathcal{N}\left(0, \sigma^2\right)$.

A. Complexity

The computational complexity required by the two algorithms here considered as a function of the number of mobile targets is shown in Figure 4. In the case of SMACOF, due to its fast convergence properties, only 20 iterations are considered. For the algebraic MDS-based tracking algorithm, the stopping criterion defined by equation (18) is considered.

Although the complexity for the proposed technique is slightly dependent on the target dynamics, it always remains

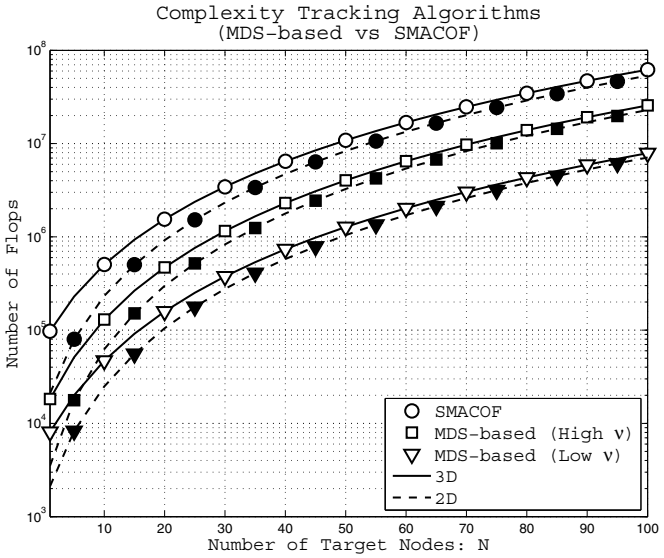


Fig. 4. Computational Complexity.

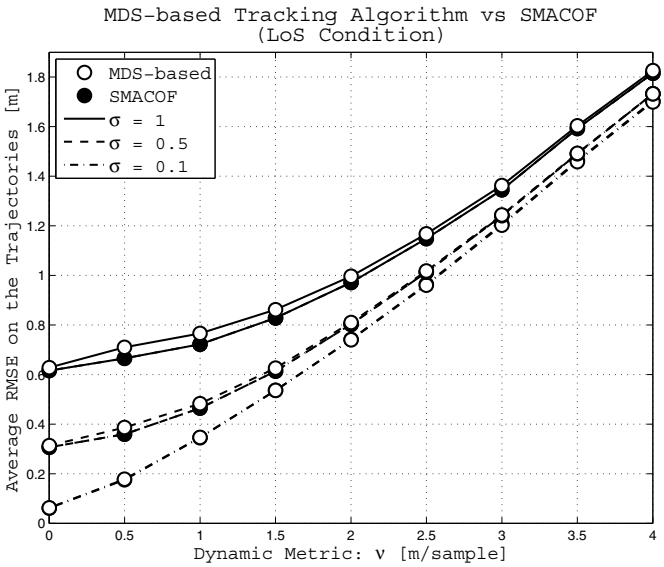


Fig. 5. RMSE for the MDS-based Tracking algorithm vs. SMACOF.

substantially lower that the one necessary to SMACOF. In addition, as shown in Figure 1, the proposed solution perform the eigendecomposition at a significantly lower complexity than the Symmetric QR decomposition, that, as discussed in section III equates to $K_{\mathrm{E}} = 2$ for the Jacobian algorithm.

B. Performance

The algorithms are compared through simulation. A scenario of 4 anchors placed at the corners of a 2D square with edges 10[m] long is considered. Each anchor performs Time of Arrival (TOA) measurements between itself and each one of the 10 mobile targets in the scenario. The observations are subject to an error distributed accordingly to $\mathcal{N}\left(0, \sigma^2\right)$. As done in [7], a wavelet-based pre-filter is used to low-pass filter the TOA data. This allows, up to a certain velocity, to distinguish/remove the fast time-scale process introduced from the noise and affecting the anchor-to-target TOA measurements, with clear benefit in terms of accuracy on the estimated positions. The same pre-filtering block is applied here.

The targets follow independent random trajectories generated accordingly to a first order autoregressive model (AR). Figure 5 shows that the algorithms reach the same performance for different values of noise σ and variable average target velocity ν.

VI. Conclusion

In this work, we have shown that a Jacobian-based eigenspace tracking algorithm can be used to efficiently solve the multi-source tracking problem in a non-parametric fashion. Specifically, the proposed algorithm, exploiting the structure inhered in the Nyström approximation, it is able to track the sub-space associated to subsequent Gram matrices. This yields to a substantial gain in the complexity required to perform the necessary eigendecomposition without compromising the final localization error. For these reasons the proposed algorithm is a suitable candidate to solve the localization problem in large wireless networks in a centralized fashion.

References

[1] T. F. Cox and M. A. A. Cox, *Multidimensional Scaling*, 2nd ed. Chapman & Hall/CRC, 2000.
[2] W. Torgerson, "Multidimensional scaling 1: Theory and method," *Psychometrica*, vol. 17, pp. 401 – 419, 1952.
[3] J. D. Leeuw, "Convergence of the majorization method for multidimensional scaling," *Journal of Classification*, vol. 5, pp. 163–180, 1988.
[4] K. W. Cheung and H. C. So, "A multidimensional scaling framework for mobile location using time-of-arrival measurements," *IEEE Trans. Signal Processing*, vol. 53, no. 2, pp. 460–470, Feb. 2005.
[5] J. C. Platt, "Fastmap, metricmap, and landmark mds are all nyström algorithms," Microsoft Research Technical Report, Tech. Rep. MSR-TR-2004-26, 2004. [Online]. Available: http://citeseer.ist.psu.edu/732858.html
[6] D. Macagnano and G. Abreu, "Tracking multiple dynamic targets with multidimensional scaling," in *Proc. IEEE Personal Indoor Mobile Radio Communication*, Sept.3-7 2007.
[7] D. Macagnano and G. T. F. de Abreu, "Tracking multiple dynamic targets in los-nlos condition with multidimensional scaling," in *IEEE 5th Workshop on Positioning, Navigation and Communication*, 2008.
[8] P. D. Fiore, "Efficient linear solution of exterior orientation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 23, no. 2, pp. 140 – 148, February 2001.
[9] G. H. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed. New York, NY: Johns Hopkins Univ. Press, Nov. 1996.
[10] A. Bunse-Gerstner, R. Byers, and V. Mehrmann, "Numerical methods for simultaneous diagonalization," *SIAM Journal on Matrix Analysis and Applications*, vol. 14, no. 4, pp. 927 – 949, 1993.
[11] J.-F. Cardoso and A. Souloumiac, "Jacobi angles for simultaneous diagonalization," *SIAM Journal on Matrix Analysis and Applications*, vol. 17, no. 1, pp. 161 – 164, 1996.
[12] B. N.Parlett, *The Symmetric Eigenvalue Problem*, ser. Series in Computational Mathematics. Prentice Hall, 1980.
[13] S. J. Messick and R. P. Abelson, "The additive constant problem in multidimensional scaling." *Psychometrika*, vol. 21, no. 1, March 1956.
[14] R. Sibson, "Studies on the robustness of multidimensional scaling: Perturbation analysis of classical scaling," *Journal of the Royal Statistical Society. Series B*, vol. 41, no. 2, pp. 217–229, 1979.
[15] J. S. Davide Macagnano and G. Abreu, "Tracking multiple targets from incomplete ranging with multidimensional scaling," in *WPMC*, September 2008.