Sensor Network Localization, Euclidean Distance Matrix Completions, and Graph Realization

[Extended Abstract] *

Yichuan Ding[†]
Stanford University
Management Science and Engineering
Stanford, California, USA
y7ding@stanford.edu

Nathan Krislock[‡]
University of Waterloo
Department of Combinatorics and Optimization
Waterloo, Ontario, Canada
ngbkrislock@uwaterloo.ca

Jiawei Qian[§]
Cornell University
School of Operations Research and Information Engineering
Ithaca, New York, USA
jq35@cornell.edu

Henry Wolkowicz[¶]
University of Waterloo
Department of Combinatorics and Optimization
Waterloo, Ontario, Canada
hwolkowicz@uwaterloo.ca

ABSTRACT

We study Semidefinite Programming, $\boldsymbol{SDP}$, relaxations for Sensor Network Localization, $\boldsymbol{SNL}$, with anchors and with noisy distance information. The main point of the paper is to view $\boldsymbol{SNL}$ as a (nearest) Euclidean Distance Matrix, $\boldsymbol{EDM}$, completion problem and to show the advantages for using this latter, well studied model. We first show that the current popular $\boldsymbol{SDP}$ relaxation is equivalent to known relaxations in the literature for $\boldsymbol{EDM}$ completions. The existence of anchors in the problem is *not* special. The set of anchors simply corresponds to a given fixed clique for the graph of the $\boldsymbol{EDM}$ problem. We next propose a method of projection when a large clique or a dense subgraph is identified in the underlying graph. This projection reduces the size, and improves the stability, of the relaxation. In addition, the projection/reduction procedure can be repeated for other given cliques of sensors or for sets of sensors, where many distances are known. Thus, further size reduction can be obtained.

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1. INTRODUCTION

We study the sensor network localization problem, $\boldsymbol{SNL}$, with anchors. The anchors have fixed known locations and the sensor-sensor and sensor-anchor distances are known (approximately) if they are within a given (radio) range. The problem is to approximate the positions of all the sensors, given that we have only this partial information on the distances. We use semidefinite programming, $\boldsymbol{SDP}$, relaxations to find approximate solutions to this problem.

In the last few years, there has been an increased interest in the $\boldsymbol{SNL}$ problem with anchors. In particular, $\boldsymbol{SDP}$ relaxations have been introduced that are specific to the problem with anchors. In this paper we emphasize that the existence of anchors is not special. The $\boldsymbol{SNL}$ problem with anchors can be modelled as a (nearest) Euclidean Distance Matrix, $\boldsymbol{EDM}$, completion problem, a well studied problem. There is no advantage to considering the anchors separately to other sensors. The only property that distinguishes the anchors is that the corresponding set of nodes yields a clique in the graph. This results in the failure of the Slater constraint qualification for the $\boldsymbol{SDP}$ relaxation. We then show that we can take advantage of this liability. We can find the smallest face of the $\boldsymbol{SDP}$ cone that contains the feasible set and project the problem onto this face.

This projection technique yields an equivalent smaller dimensional problem, where the Slater constraint qualification holds. Thus the problem size is reduced and the problem stability is improved. In addition, by treating the anchors this way, we show that other cliques of sensors or dense parts of the graph can similarly result in a reduction in the size of the problem. In fact, not treating other cliques this way can result in instability, due to loss of the Slater constraint qualification.

The geometry of $\boldsymbol{EDM}$ has been extensively studied in the literature, e.g. [12, 9] and more recently in [2, 1] and the references therein. The latter two references studied algorithms based on $\boldsymbol{SDP}$ formulations of the $\boldsymbol{EDM}$ completion problem.

The formulation of the $\boldsymbol{SNL}$ problem as a least squares approximation is presented in Section 2. We continue in Section 3 with background, notation, including information on the linear transformations used in the model. The $\boldsymbol{SDP}$ relaxations are presented in Section 4. This section contains the details for the two main contributions of the paper: i.e. (i) the connection of $\boldsymbol{SNL}$ with $\boldsymbol{EDM}$, and (ii) the projection technique for cliques and dense sets of sensors.

2. SNL PROBLEM FORMULATION

Let the n unknown (sensor) points be $p^1, p^2, \ldots, p^n \in \mathbb{R}^r$, r the embedding dimension; and let the m known (anchor) points be $a^1, a^2, \ldots, a^m \in \mathbb{R}^r$. Let $X^T = [p^1, p^2, \ldots, p^n]$, and $A^T = [a^1, a^2, \ldots, a^m]$. We identify a^i with p^{n+i}, for $i = 1, \ldots, m$, and sometimes treat these as unknowns. We now define

$$P^T := \left(X^T \; A^T \right). \tag{2.1}$$

Note that we can always translate all the sensors and anchors so that the anchors are centered at the origin, i.e. $A^T \leftarrow A^T - \frac{1}{m} A^T e e^T$ yields $A^T e = 0$. We can then translate them all back at the end. In addition, we assume that there are a sufficient number of anchors so that the problem cannot be realized in a smaller embedding dimension. Therefore, to avoid some special trivial cases, we assume the following.

Assumption 2.1. *The number of sensors and anchors, and the embedding dimension satisfy*

$$n >> m > r, \; A^T e = 0, \; \textit{and } A \textit{ is full column rank.}$$

Now define $(\mathcal{N}_e, \mathcal{N}_u, \mathcal{N}_l)$, respectively, to be the index sets of specified (distance values, upper bounds, lower bounds), respectively, of the distances d_{ij} between pairs of nodes from $\{p^i\}_1^n$ (sensors); and let $(\mathcal{M}_e, \mathcal{M}_u, \mathcal{M}_l)$, denote the same for distances between a node from $\{p^i\}_1^n$ (sensor) and a node from $\{a^k\}_1^m$ (anchor). Define (the partial Euclidean Distance Matrix) E with elements

$$E_{ij} = \begin{cases} d_{ij}^2 & \text{if} \quad ij \in \mathcal{N}_e \cup \mathcal{M}_e \\ \|p^i - p^j\|^2 = \|a^{i-n} - a^{j-n}\|^2 & \text{if} \quad i, j > n \\ 0 & \text{otherwise.} \end{cases}$$

The underlying graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with node set $\mathcal{V} = \{1, \ldots, m+n\}$ and edge set $\mathcal{E} = \mathcal{N}_e \cup \mathcal{M}_e \cup \{ij : i, j > n\}$. Note that the subgraph induced by the anchors (the nodes with $j > n$) is complete, i.e. the set of anchors forms a clique in the graph. Similarly, we define the matrix of (squared) upper distance bounds U^b and the matrix of (squared) lower distance bounds L^b for $ij \in \mathcal{N}_u \cup \mathcal{M}_u$ and $\mathcal{N}_l \cup \mathcal{M}_l$, respectively.

We minimize the weighted least squares error.

$$\begin{array}{ll} \min & f_1(P) := \frac{1}{2} \sum\limits_{(i,j)\in\mathcal{N}_e} (W_p)_{ij} (\|p^i - p^j\|^2 - E_{ij})^2 \\ & \quad + \frac{1}{2} \sum\limits_{(i,k)\in\mathcal{M}_e} (W_{pa})_{ik} (\|p^i - a^k\|^2 - E_{ik})^2 \\ & \quad \left(+ \frac{1}{2} \sum\limits_{i,j>n} (W_a)_{ij} (\|p^i - p^j\|^2 - E_{ij})^2 \right) \\ \text{s.t.} & \|p^i - p^j\|^2 \le U_{ij}^b \quad \forall (i,j) \in \mathcal{N}_u \quad \left(n_u = \frac{|\mathcal{N}_u|}{2} \right) \\ & \|p^i - a^k\|^2 \le U_{ik}^b \quad \forall (i,k) \in \mathcal{M}_u \quad \left(m_u = \frac{|\mathcal{M}_u|}{2} \right) \\ & \|p^i - p^j\|^2 \ge L_{ij}^b \quad \forall (i,j) \in \mathcal{N}_l \quad \left(n_l = \frac{|\mathcal{N}_l|}{2} \right) \\ & \|p^i - a^k\|^2 \ge L_{ik}^b \quad \forall (i,k) \in \mathcal{M}_l \quad \left(m_l = \frac{|\mathcal{M}_l|}{2} \right) \\ & \left(\|p^i - p^j\|^2 = E_{ij} \quad \forall i,j > n \right). \end{array} \tag{2.2}$$

This is a *hard* problem to solve due to the nonconvex objective and constraints. We again included the anchor-anchor distances within brackets both in the objective and constraints. This is to emphasize that we could treat them with large weights in the objective or as holding exactly without error in the constraints.

3. DISTANCE GEOMETRY

The geometry for $\boldsymbol{EDM}$ has been studied in e.g. [18, 13, 14, 20], and more recently, in e.g. [2, 1]. Further theoretical properties can be found in e.g. [3, 11, 13, 14, 16, 17, 18, 17]. Since we emphasize that the $\boldsymbol{EDM}$ theory can be used to solve the $\boldsymbol{SNL}$, we now include an overview of the tools needed for $\boldsymbol{EDM}$. In particular, we show the relationships between $\boldsymbol{EDM}$ and $\boldsymbol{SDP}$.

3.1 Linear Transformations and Adjoints Related to EDM

We work in spaces of real matrices, $\mathcal{M}^{s\times t}$, equipped with the trace inner-product $\langle A, B\rangle = \operatorname{trace} A^T B$ and induced Frobenius norm $\|A\|_F^2 = \operatorname{trace} A^T A$. For a given $B \in \mathcal{S}^n$, the space of $n \times n$ real symmetric matrices, the linear transformation $\operatorname{diag}(B) \in \mathbb{R}^n$ denotes the diagonal of B; for $v \in \mathbb{R}^n$, the adjoint linear transformation is the diagonal matrix $\operatorname{diag}^*(v) = \operatorname{Diag}(v) \in \mathcal{S}^n$. We now define two linear operators on $\mathcal{S}^n$:

$$\begin{array}{c} \mathcal{D}_e(B) := \operatorname{diag}(B)\, e^T + e \operatorname{diag}(B)^T, \\ \mathcal{K}(B) := \mathcal{D}_e(B) - 2B, \end{array} \tag{3.3}$$

where e is the vector of ones. By abuse of notation we allow $\mathcal{D}_e$ to act on $\mathbb{R}^n$:

$$\mathcal{D}_e(v) = ve^T + ev^T, \; v \in \mathbb{R}^n.$$

The linear operator $\mathcal{K}$ maps the cone of positive semidefinite matrices (denoted $\boldsymbol{SDP}$) onto the cone of Euclidean distance matrices (denoted $\boldsymbol{EDM}$), i.e. $\mathcal{K}(\boldsymbol{SDP}) = \boldsymbol{EDM}$.

Let $B = PP^T$. Then

$$\begin{array}{rcl} D_{ij} & = & \|p^i - p^j\|^2 \\ & = & (\mathrm{diag}\,(B)e^T + e\mathrm{diag}\,(B)^T - 2B)_{ij} \\ & = & (\mathcal{K}(B))_{ij}\,, \end{array}$$

i.e. the $\boldsymbol{EDM}$ $D = (D_{ij})$ and the points p_i in P are related by $D = \mathcal{K}(B)$, see (3.3). This allows us to change problem $\boldsymbol{EDMC}$ into a $\boldsymbol{SDP}$ problem.

3.2 Properties of the $\mathcal{K}$ Transformation

LEMMA 3.1. *([1]) Define the linear operator on $\mathcal{S}^n$ by*

$$\mathrm{offDiag}\,(S) = S - \mathrm{Diag}\,(\mathrm{diag}\,(S)).$$

Let $J := I - \frac{1}{n}ee^T$. Then, the following holds.

- *The nullspace $\mathcal{N}(\mathcal{K})$ equals the range $\mathcal{R}(\mathcal{D}_e)$.*
- *The range $\mathcal{R}(\mathcal{K})$ equals the hollow subspace of $\mathcal{S}^n$, denoted $S_H := \{D \in \mathcal{S}^n : \mathrm{diag}\,(D) = 0\}$.*
- *The Moore-Penrose generalized inverse*

$$\mathcal{K}^\dagger(D) = -\frac{1}{2} J\,(\mathrm{offDiag}\,(D))\,J.$$

■

4. SDP RELAXATIONS OF SNL BASED ON EDM MODEL

We first study the $\boldsymbol{SDP}$ relaxation used in the recent series of papers on $\boldsymbol{SNL}$, e.g. [6, 4, 19, 5, 15]. (See (4.5) and Section 4.1.3 below.) This relaxation starts by treating the anchors distinct from the sensors. We use a different derivation and model the problem based on classical $\boldsymbol{EDM}$ theory, and show its equivalence with the current $\boldsymbol{SDP}$ relaxation.

4.1 Connections from Current SDP Relaxation to EDM

Let $Y = XX^T$. Then the current $\boldsymbol{SDP}$ relaxation for the feasibility problem for $\boldsymbol{SNL}$ uses

$$Y \succeq XX^T, \text{ or equivalently, } Z_s = \begin{pmatrix} I_r & X^T \\ X & Y \end{pmatrix} \succeq 0. \quad (4.4)$$

This is in combination with the constraints

$$\begin{array}{c} \mathrm{trace}\begin{pmatrix} 0 \\ e_i - e_j \end{pmatrix}\begin{pmatrix} 0 \\ e_i - e_j \end{pmatrix}^T Z_s = E_{ij}, \quad \forall ij \in \mathcal{N}_e \\ \mathrm{trace}\begin{pmatrix} -a_k \\ e_i \end{pmatrix}\begin{pmatrix} -a_k \\ e_i \end{pmatrix}^T Z_s = E_{ij}, \quad \forall ij \in \mathcal{M}_e, i < j = n+k. \end{array} \quad (4.5)$$

4.1.1 Reformulation using Matrices

We use the matrix lifting or linearization $\bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$ and $Z := [I;P][I;P]^T = \begin{pmatrix} I & P^T \\ P & \bar{Y} \end{pmatrix}$, where P defined in (2.1). The dimensions are:

$$\begin{array}{c} X \in \mathcal{M}^{n\times r}; \quad A \in \mathcal{M}^{m\times r}; \quad P \in \mathcal{M}^{(m+n)\times r}; \\ \bar{Y} \in \mathcal{S}^{m+n}; \quad Z \in \mathcal{S}^{m+n+r}. \end{array}$$

Adding the hard quadratic constraint $\bar{Y} = PP^T$ allows us to replace the quartic objective function in (2.2) with a quadratic function. We can now reformulate $\boldsymbol{SNL}$ using matrix notation to get the equivalent $\boldsymbol{EDM}$ problem

$$\begin{array}{llcl} \min & f_2(\bar{Y}) := \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 & & \\ \text{s.t.} & g_u(\bar{Y}) := H_u \circ (\mathcal{K}(\bar{Y}) - \bar{U}^b) & \leq & 0 \\ & g_l(\bar{Y}) := H_l \circ (\mathcal{K}(\bar{Y}) - \bar{L}^b) & \geq & 0 \\ & \bar{Y} - PP^T & = & 0 \\ & \left(\mathcal{K}(\bar{Y})_{22}\right. & = & \left.\mathcal{K}(AA^T)\right), \end{array} \quad (4.6)$$

where $W \in \mathcal{S}^{n+m}$ is the weight matrix having a positive ij-element if $(i,j) \in \mathcal{N}_e \cup \mathcal{M}_e \cup \{(ij) : i,j > n\}$, 0 otherwise. H_u, H_l are 0–1 matrices where the ij-th element equals 1 if an upper (resp. lower) bound exists; and it is 0 otherwise. We include in brackets the constraint corresponding to the clique formed by the anchors.

REMARK 4.1. *The function $f_2(\bar{Y}) = f_2(PP^T)$, and it is clear that $f_2(PP^T) = f_1(P)$ in (2.2). Note that the functions f_2, g_u, g_l act only on $\bar{Y}$ and the locations of the anchors and sensors is completely hiding in the* hard, *nonconvex quadratic constraint $\bar{Y} = PP^T$. The problem (4.6) is a linear least squares problem with nonlinear constraints. The objective function is generally underdetermined. This can result in ill-conditioning problems, e.g. [10]. Therefore, reducing the number of variables helps with stability.*

4.1.2 SDP Relaxation of the Hard Quadratic Constraint

We now consider the hard quadratic constraint in (4.6)

$$\bar{Y} = \begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{21}^T \\ \bar{Y}_{21} & AA^T \end{pmatrix} = PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}, \quad (4.7)$$

where P is defined in (2.1). We study the standard current semidefinite relaxation in (4.5) with (4.4), or equivalently with $\bar{Y} \succeq PP^T$. We show that this is equivalent to the simpler $\bar{Y} \succeq 0$. We include details on problems and weaknesses with the relaxation. We first present a couple of lemmas, starting with the following well known result.

LEMMA 4.1. *Suppose that the partitioned symmetric matrix $\begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & AA^T \end{pmatrix} \succeq 0$. Then $Y_{21}^T = XA^T$, with $X = Y_{21}^T A(A^TA)^{-1}$.*

In the recent literature, e.g. [7, 6, 15], it is common practice to relax the hard constraint (4.7) to a tractable semidefinite constraint, $\bar{Y} \succeq PP^T$, or equivalently, $\bar{Y}_{11} \succeq XX^T$ with $\bar{Y}_{21} = AX^T$. The following lemma presents several characterizations for the resulting feasible set.

LEMMA 4.2. *Let $P, \bar{Y}$ be partitioned as in (2.1),(4.7),*

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad \bar{Y} = \begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{21}^T \\ \bar{Y}_{21} & \bar{Y}_{22} \end{pmatrix}.$$

Define the semidefinite relaxation of the hard quadratic constraint (4.7) as:

$$G(P,\bar{Y}) := PP^T - \bar{Y} \preceq 0, \quad \bar{Y}_{22} = AA^T, \quad P_2 = A. \quad (4.8)$$

By abuse of notation, we allow G to act on spaces of different dimensions. Then we get the following equivalent representations of the corresponding feasible set $\mathcal{F}_G$.

$$\mathcal{F}_G = \left\{(P,\bar{Y}) : G(P,\bar{Y}) \preceq 0, \bar{Y}_{22} = AA^T, P_2 = A\right\} \quad (4.8a)$$

$$\mathcal{F}_G = \left\{ (P,\bar{Y}) : \begin{array}{c} G(X,Y) \preceq 0, \bar{Y}_{11} = Y, \bar{Y}_{21} = AX^T, \\ \bar{Y}_{22} = AA^T, P = \begin{pmatrix} X \\ A \end{pmatrix} \end{array} \right\} \tag{4.8b}$$

$$\mathcal{F}_G = \left\{ (P,\bar{Y}) : \begin{array}{c} Z = \begin{pmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{pmatrix} \succeq 0, \\ \bar{Y} = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} Z \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}^T, \\ Z_{22} = I, X = Z_{21}^T, P = \begin{pmatrix} X \\ A \end{pmatrix} \end{array} \right\} \tag{4.8c}$$

$$\mathcal{F}_G = \left\{ (P,\bar{Y}) : \begin{array}{c} \bar{Y} \succeq 0, \bar{Y}_{22} = AA^T, \\ X = \bar{Y}_{21}^T A(A^TA)^{-1}, P = \begin{pmatrix} X \\ A \end{pmatrix} \end{array} \right\} \tag{4.8d}$$

Moreover, the function G is convex in the Löwner (semidefinite) partial order; and the feasible set $\mathcal{F}_G$ is a closed convex set.

Proof. Recall that the cone of positive semidefinite matrices is self-polar. Let $Q \succeq 0$ and $\phi_Q(P) = \operatorname{trace} QPP^T$. Convexity of G follows from positive semidefiniteness of the Hessian $\nabla^2\phi_Q(P) = I \otimes Q$, where $\otimes$ denotes the Kronecker product.

In addition,

$$0 \succeq G(P,\bar{Y}) = PP^T - \bar{Y} = \begin{pmatrix} XX^T - \bar{Y}_{11} & XA^T - \bar{Y}_{21}^T \\ AX^T - \bar{Y}_{21} & 0 \end{pmatrix}$$

holds if and only if

$$0 \succeq G(X,\bar{Y}_{11}) = XX^T - \bar{Y}_{11}, \text{ and } AX^T - \bar{Y}_{21} = 0.$$

This shows the equivalence with (4.8b). A Schur complement argument, with $\bar{Y}_{11} = Y$, shows the equivalence with $\begin{pmatrix} Y & X \\ X^T & I_r \end{pmatrix} \succeq 0$, i.e. with the set in (4.8c). The equivalence with (4.8d) follows from Lemma 4.1.

∎

Lemma 4.2 shows that we can treat the set of anchors as a set of sensors for which all the distances are known, i.e. the set of corresponding nodes is a clique. The fact that we have a clique and the diagonal $m \times m$ block AA^T in $\bar{Y}$ is rank deficient, $r < m$, means that the Slater constraint qualification, $\bar{Y} \succ 0$, cannot hold. Therefore, we can project onto the minimal cone containing the feasible set and thus reduce the size of the problem, see Lemma 4.2, (4.8c), i.e. the variable $\bar{Y} \in \mathcal{S}^{n+m}$ is reduced in size to $Z \in \mathcal{S}^{n+r}$. The reduction can be done by using any point in the relative interior of the minimal cone, e.g. any feasible point of maximum rank. The representation in (4.8c) illustrates this.

4.1.3 Current SDP Relaxation using Projection onto Minimal Cone

The above reduction to Y in Lemma 4.2, (4.8b), allows us to use the smaller dimensional semidefinite constrained variable

$$Z_s = \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix} \succeq 0 \in \mathcal{S}^{n+m}, \quad \bar{Y}_{11} = Y,\ \bar{Y}_{21} = AX^T. \tag{4.13}$$

This is what is introduced in e.g. [6].

REMARK 4.2. *Note that the mapping $Z_s = Z_s(X,Y) : \mathcal{M}^{n\times r} \times \mathcal{S}^n \to \mathcal{S}^{n+r}$ is* not *onto. This means that the Jacobian of the optimality conditions cannot be full rank, i.e. this formulation introduces instability into the model. A minor modification corrects this, i.e. the I constraint is added explicitly:*

$$Z = \begin{pmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{pmatrix} \succeq 0, \quad Z_{11} = I,\ \bar{Y}_{11} = Z_{22},\ \bar{Y}_{21} = AZ_{21}^T.$$

4.1.4 SDP Formulation Using EDM

The equivalent representations of the feasible set given in Lemma 4.2, in particular by (4.8c), show that ***SNL*** is an ***EDM*** problem $D = \mathcal{K}(\bar{Y})$, with the additional upper and lower bound constraints as well as the block constraint $D_{22} = \mathcal{K}(AA^T)$, or equivalently, $\bar{Y}_{22} = AA^T$.

We can now obtain an equivalent relaxation for ***SNL*** by using the ***EDM*** completion problem (4.6) and replacing the hard quadratic constraint with the simpler semidefinite constraint $\bar{Y} \succeq 0$. We then observe that the Slater constraint qualification (strict feasibility) fails. Therefore, we can project onto the minimal cone, i.e. onto the minimal face of the ***SDP*** cone that contains the feasible set; see [8, 2]. Let

$$U_A = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}. \tag{4.14}$$

We get an ***SDP*** relaxation of (4.6):

$$\begin{array}{ll} \min & f_3(Z) := \frac{1}{2}\|W \circ (\mathcal{K}(U_A Z U_A^T) - E)\|_F^2 \\ \text{subject to} & H_u \circ (\mathcal{K}(U_A Z U_A^T) - U^b) \le 0 \\ & H_l \circ (\mathcal{K}(U_A Z U_A^T) - L^b) \ge 0 \\ & Z_{22} = I_r \\ & Z \succeq 0. \end{array} \tag{4.15}$$

REMARK 4.3. *Note that we do not substitute the constraint on Z_{22} into Z, but leave it explicit. Though this does not change the feasible set, it does change the stability and the dual.*

4.2 Clique Reductions using Minimal Cone Projection

Now suppose that we have another clique of $p > r$ sensors where the exact distances are known and are used as constraints. Then every feasible matrix $\bar{Y} = PP^T$ that has a diagonal rank deficient $p \times p$ block, which implies that the Slater constraint qualification fails again.

We now see that we can again take advantage of the loss of the Slater constraint qualification.

LEMMA 4.3. *Suppose that the hypotheses and definitions from Lemma 4.2 hold; and suppose that there exists a set of sensors, without loss of generality $S_c := \{p^{t+1}, \ldots, p^n\}$, so that the distances $\|p^i - p^j\|$ are known for all $t+1 \le i,j \le n$; i.e. the graph of the partial **EDM** has two cliques, one clique corresponding to the set of known anchors, and the other to the set of sensors S_c. Let $P, \bar{Y}$ be partitioned as*

$$P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} X \\ A \end{pmatrix}, \quad \bar{Y} = \begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{21}^T & \bar{Y}_{31}^T \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{32}^T \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{pmatrix} = PP^T,$$

where $P_i = A_i, i = 2,3$, and $A_3 = A$ corresponds to the known anchors while $P_2 = A_2$ corresponds to the clique of *sensors and* $X = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ *corresponds to all the sensors. Let the* ***EDM*** , $E = \mathcal{K}(\bar{Y})$, *be correspondingly blocked*

$$E = \begin{pmatrix} E_1 & \cdot & \cdot \\ \cdot & E_2 & \cdot \\ \cdot & \cdot & E_3 \end{pmatrix},$$

so that $E_3 = \mathcal{K}(AA^T)$ *are the anchor-anchor squared distances, and* $E_2 = \mathcal{K}(P_2P_2^T)$ *are the squared distances between the sensors in the set* S_c. *Let*

$$B = \mathcal{K}^{\dagger}(E_2).$$

Then the following hold.

1. $Be = 0$ *and*
$$\bar{Y}_{22} = B + \bar{y}_2 e^T + e\bar{y}_2^T \succeq 0,$$
$$\text{for some } \bar{y}_2 \in \mathcal{R}(B) + \alpha e, \alpha \geq 0, \text{ with } \operatorname{rank}(\bar{Y}_{22}) \leq r. \tag{4.16}$$

2. *The feasible set* $\mathcal{F}_G$ *in Lemma 4.2 can be formulated as*
$$\mathcal{F}_G = \left\{ (P, \bar{Y}) : Z = \begin{pmatrix} Z_{11} & Z_{21}^T & Z_{31}^T \\ Z_{21} & Z_{22} & Z_{32}^T \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \succeq 0, \right.$$
$$\bar{Y} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix} Z \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix}^T,$$
$$\left. Z_{33} = I_r, X = \begin{pmatrix} Z_{31}^T \\ U_2 Z_{32}^T \end{pmatrix}, P = \begin{pmatrix} X \\ A \end{pmatrix} \right\}, \tag{4.17}$$
where $\hat{B} := B + 2ee^T = \begin{pmatrix} U_2 & \bar{U}_2 \end{pmatrix} \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_2 & \bar{U}_2 \end{pmatrix}^T$ *is the orthogonal diagonalization of* $\hat{B}$, *with* $D_2 \in \mathcal{S}_+^{r_2}$, $r_2 \leq r + 1$.

Proof. We proceed just as we did in Lemma 4.2, i.e. we reduce the problem by projecting onto a smaller face in order to obtain the Slater constraint qualification.

The equation for $\bar{Y}_{22}$ for some $\bar{y}_2$, given in (4.16), follows from the nullspace characterization in Lemma 3.1. Moreover, $\bar{Y}_{22} = P_2P_2^T$ implies that $\operatorname{rank}(Y_{22}) \leq r$, the embedding dimension. And, $\bar{Y}_{22} \succeq 0, Be = 0$ implies the inclusion $\bar{y}_2 \in \mathcal{R}(B) + \alpha e, \alpha \geq 0$. Moreover, we can shift $\bar{P}_2^T = P_2^T - \frac{1}{n-t}(P_2^T e)e^T$. Then for $B = \bar{P}_2\bar{P}_2^T$, we get $Be = 0$, i.e. this satisfies $B = \mathcal{K}^{\dagger}(E_2)$ and $\operatorname{rank}(B) \leq r$. Therefore, for any $Y = B + ye^T + ey^T \succeq 0$, we must have $y = \alpha e, \alpha \geq 0$. Therefore, $\hat{B}$ has the maximum rank, at most $r+1$, among all feasible matrices of the form $0 \preceq Y \in B + \mathcal{N}(\mathcal{K})$. $\hat{B}$ determines the smallest face containing all such feasible Y.

Define the linear transformation $H : \mathbb{R}^{n-t} \to \mathcal{S}^{n-t}$ by $H(y) = \bar{Y}_{22} + ye^T + ey^T$. Let $\mathcal{L} := \bar{Y}_{22} + \mathcal{R}(\mathcal{D}_e)$ and $\mathcal{F}_e$ denote the smallest face of $\mathcal{S}_+^{n-t}$ that contains $\mathcal{L}$. Since $\hat{B}$ is a feasible point of maximum rank, we get

$$\hat{B} = B + \mathcal{D}_e(\hat{y}_2) \in (\mathcal{L} \cap \operatorname{relint} \mathcal{F}_e).$$

Thus, we have

$$\begin{aligned} \mathcal{F}_e &= \{U_2 Z U_2^T : Z \in \mathcal{S}_+^{r_2}\} \\ &= \{Y \in \mathcal{S}_+^{n-t} : \operatorname{trace} Y(\bar{U}_2\bar{U}_2^T) = 0\}. \end{aligned}$$

Finally, we expand

$$\begin{aligned} &\begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{21}^T & \bar{Y}_{31}^T \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{32}^T \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{pmatrix} \\ &= \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{21}^T & Z_{31}^T \\ Z_{21} & Z_{22} & Z_{32}^T \\ Z_{31} & Z_{32} & I_r \end{pmatrix} \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix}^T \\ &= \begin{pmatrix} Z_{11} & Z_{21}^T U_2^T & Z_{31}^T A^T \\ U_2 Z_{21} & U_2 Z_{22} U_2^T & U_2 Z_{32}^T A^T \\ A Z_{31} & A Z_{32} U_2^T & AA^T \end{pmatrix}. \end{aligned}$$

■

We can apply Lemma 4.3 to further reduce the ***SDP*** relaxation. Suppose there are a group of sensors for which pairwise distances are all known. Without loss of generality, we assume the of sensors to be $\{p^{t+1}, \ldots, p^n\}$. Let E_2, $B = \mathcal{K}^{\dagger}(E_2)$, and U_2, be found using Lemma 4.3 and denote

$$U_{2A} := \begin{pmatrix} I_n & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix}, \tag{4.18}$$

In (4.15), we can replace U_A with U_{2A} and reach a reduced ***SDP*** formulation. Furthermore, we may generalize to the k clique cases for any positive integer k. We similarly define each $U_i, 2 \leq i \leq k$, and define

$$U_{kA} = \begin{pmatrix} I_n & 0 & \cdots & 0 & 0 \\ 0 & U_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U_k & 0 \\ 0 & 0 & \cdots & 0 & A \end{pmatrix}, \tag{4.19}$$

Then we can formulate a reduced ***SDP*** for k cliques:

$$\begin{array}{ll} \min & f_4(Z) := \frac{1}{2}\|W \circ (\mathcal{K}(U_{kA} Z U_{kA}^T) - E)\|_F^2 \\ \text{s.t.} & H_u \circ (\mathcal{K}(U_{kA} Z U_{kA}^T) - U^b) \leq 0 \\ & H_l \circ (\mathcal{K}(U_{kA} Z U_{kA}^T) - L^b) \geq 0 \\ & Z_{kk} = I_r \\ & Z \succeq 0 \end{array} \tag{4.20}$$

where Z_{kk} is the last r by r diagonal block of Z.

For a clique with r_e sensors, a U_i is constructed with r_e rows and at most $r+1$ columns. This implies the dimension of Z has been reduced by $r_e - r - 1$. So if $r = 2$, cliques larger than a triangle help reduce the dimension of Z.

5. CONCLUDING REMARKS

In this paper, we have analyzed the well known ***SNL*** problem from a new perspective. By considering the set of anchors as a clique in the underlying graph, the ***SNL*** problem can be studied using traditional ***EDM*** theory. Our main contributions follow from this ***EDM*** approach:

1. The Slater constraint qualification can fail for cliques and/or dense subgraphs in the underlying graph. If this happens, then we can project the feasible set of the ***SDP*** relaxation to the *minimal cone*. This projection improves the stability and can also reduce the size of the ***SDP*** significantly.

2. We used the ℓ_2 norm formulation instead of the ℓ_1 norm. This is a better fit for the data that we used. However, the quadratic objective makes the problem more difficult to solve.

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