Solutions to
Introductory Real Analysis
by Kolmogorov & Fomin

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To all who wish to become a mathematician

Contents

Preface

I am trying to work on all problems given in Kolmogorov's *Introductory Real Analysis*, as all "future analysts" should do so. In order to motivate myself, I have even created a web page for this matter. I hope someone may find these solutions useful and am eager to hear from anyone who has read this. Well, needless to say, I have to work it through first. I hope I could.

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1

Set Theory

"The set concept plays a key role in modern mathematics."
– Kolmogorov

1 Sets and Functions

Problem 1. Prove that if $A \cup B = A$ and $A \cap B = A$, then $A = B$.

Proof.
Since we have

$$\begin{aligned} A \cap B = A &\quad \Rightarrow \quad B \subset A \\ A \cup B = A &\quad \Rightarrow \quad A \subset B \end{aligned}$$

Thus, it is obvious that $A = B$.

$\square$

Problem 2. Show that in general $(A - B) \cup B \neq A$.

Proof.
(1) If $B \subset A$, $(A - B) \cup B = A$.
(2) If $B \not\subset A$,

$$A - B = \{x |\ x \in A \ \&\ x \notin B\}$$

Thus, for any point in $(A - B) \cup B$, say x, falls in two cases: either $x \in B$ or $x \in A \ \&\ x \notin B$. For the former case, since $B \not\subset A$, there exist $x \in B \ \&\ x \notin A$. Hence we have established the fact that $\exists\ x \in (A - B) \cup B$,*s.t.* $x \notin A$. $\square$

Problem 3. Let $A = \{2, 4, \ldots, 2n, \ldots\}$ and $B = \{3, 6, \ldots, 3n, \ldots\}$. Find $A \cap B$ and $A - B$.

Proof.
$A \cap B = \{x|\ x = 6n,\ n \in \mathbb{N}\}$, and $A - B = \{x|\ x = 2n,\ x \neq 6n,\ n \in \mathbb{N}\}$.
□

Problem 4. Prove that
a) $(A - B) \cap C = (A \cap C) - (B \cap C)$;
b) $A \Delta B = (A \cup B) - (A \cap B)$.

Proof.
a)
"⇒":
If $x \in (A - B) \cap C$, it leads to $x \in (A - B)$ & $x \in C$. Thus, $x \in A$, $x \notin B$, and $x \in C$. Hence, $x \in A \cap B$ & $x \notin B \cap C$. That is $x \in (A \cap C) - (B \cap C)$. Converse statement is similar to show.
b)
"⇒":
Since $A \Delta B = (A - B) \cup (B - A)$, and for $x \in A \Delta B$, $x \in A \cup B$and $x \notin A \cap B$. Hence we have $x \in (A \cup B) - (A \cap B)$.
□

Problem 5. Prove that

$$\bigcup_\alpha A_\alpha - \bigcup_\alpha B_\alpha \subset \bigcup_\alpha (A_\alpha - B_\alpha).$$

Proof.
Suppose $x \in \bigcup\limits_\alpha A_\alpha - \bigcup\limits_\alpha B_\alpha$, $x \in \bigcup\limits_\alpha A_\alpha$ and $x \notin \bigcup\limits_\alpha B_\alpha$. Then for some α_0, $x \in A_{\alpha_0}$ and $\forall \alpha$, $x \notin B_\alpha$. Thus, $x \in (A_{\alpha_0} - B_{\alpha_0})$. $\therefore x \in \bigcup\limits_\alpha (A_\alpha - B_\alpha)$.
□

Problem 6. Let A_n be the set of all positive integers divisible by n. Find the sets
a) $\bigcup\limits_{n=2}^{\infty} A_n$; b) $\bigcap\limits_{n=2}^{\infty} A_n$.

Proof.
a) $\bigcup\limits_{n=2}^{\infty} A_n = \mathbb{N}$

b) $\bigcap\limits_{n=2}^{\infty} A_n = \emptyset$

$\square$

Problem 7. Find
a) $\bigcup\limits_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$; b) $\bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.

Proof.
a) (a, b); b) $[a, b]$.

$\square$

Problem 8. Let A_α be the set of points lying on the curve

$$y = \frac{1}{x^\alpha} \quad (0 < x < \infty).$$

What is

$$\bigcap_{\alpha \geq 1} A_\alpha ?$$

Proof. No idea. But I guess ∞.

$\square$

Problem 9. Let $y = f(x) =< x >$ for all real x, where $< x >$ is the fractional part of x. Prove that every closed interval of length 1 has the same image under f. What is this image? Is f one-to-one? What is the preimage of the interval $\frac{1}{4} \leq y \leq \frac{3}{4}$? Partition the real line into classes of points with the same image.

Proof.

$\square$

Problem 10. Given a set M, let $\mathscr{R}$ be the set of all ordered pairs on the form (a, a) with $a \in M$, and let aRb if and only if $(a, b) \in \mathscr{R}$. Interpret the relation R.

Proof.

$\square$

Problem 11. Give an example of a binary relation which is
a) Reflexive and symmetric, but not transitive;
b) Reflexive, but neither symmetric nor transitive;
c) Symmetric, but neither reflexive nor transitive;
d) Transitive, but neither reflexive nor symmetric.

Proof.

$\square$

2 Equivalence of Sets. The Power of a Set

Problem 1. Prove that a set with an uncountable subset is itself uncountable.

Proof.
Let $A \subset X$ be an uncountable subset. Therefore, $m(A) = c$. Since $m(A) \leq m(X)$. $m(X) \geq c$ must follow. Hereby we complete the proof. $\square$

Problem 2. Let M be any infinite set and A any countable set. Prove that $M \sim M \cup A$.

Proof.
First consider the case that M is countable. We can therefore list all the elements: $m_1, m_2, \ldots$ and clearly there is a bijection between $m_i \longleftrightarrow i$. And $M \sim \mathbb{N}$. For $M \cup A$, we can list the elements:
Obviously, all elements in $M \cup A$ can be made 1-1 correspondence with $\mathbb{N}$.

$$\begin{tikzcd}
m_1 \arrow[d] & m_2 \arrow[d] & m_3 \arrow[d] & \ldots \\
a_1 \arrow[ur] & a_2 \arrow[ur] & a_3 \arrow[ur] & \ldots
\end{tikzcd}$$

And $M \sim \mathbb{N} \sim M \cup A$. We then consider the case that M is uncountable. Hence $m(M) = c$ and $m(M \cup A) = c$. Both of then have the power of continuum. Therefore they are equivalent. $\square$

Problem 3. Prove that each of the following sets is countable:
a) The set of all numbers with two distinct decimal expansions (like $0.5000\dots$ and $0.4999\dots$);
b) The set of all rational points in the plane (i.e., points with rational coordinates);
c) The set of all rational intervals (i.e., intervals with rational end points);
d) The set of all polynomials with rational coefficients.

Proof.
a) Consider the decimal expansion of numbers with 4 and 9. Since this is an infinite-digit-number, one of these numbers must be repeated infinitely after some certain digit. WLOG assume the number ends in all nines.

$$d = 0.d_1 d_2 d_3 \dots\dots d_k d_{k+1} d_{k+2} \dots\dots$$

where $d_{k+1} = d_{k+2} = \cdots = 9$

$$d_i = \begin{cases} 4, \text{for some } i \text{ if } i \leq k \\ 9, \text{for some } i \text{ if } i \leq k, \text{ and all } i \text{ if } i > k \end{cases} \tag{1.1}$$

Consider the set $D_k = \{d|\ d_{k+1}, d_{k+2}, \dots \mathit{are\ all\ nines}\}$, and $Card(D_k)$ is therefore determined by the previous k digits, and there are 2^k possibilities. Then the set of interest

$$D(4,9) = \bigcup_{k=1}^{\infty} D_k$$

and it is the union of countably many sets, each of which has at most finite elements. Thus $D(4,9)$ is countable. We have hereby proves only one set containing 4 and 9 and ending in all nines. With base 10 number system, we could have the permutation $P_2^9 = 72$ possibilities including $D(4,9)$. Consider them all could still give us at most countable set.
b) We wish to show that $\mathbb{Q} \times \mathbb{Q}$ is countable. Since $\mathbb{Q}$ is countable, we could list all the elements in ascending order such that:

$$a_1, a_2, a_3, \dots$$

and similarly for the y-coordinates we have

$$b_1, b_2, b_3, \dots$$

$$\begin{tikzcd}
(a_1,b_3) \arrow[dr] & & & \\
(a_1,b_2) \arrow[u] & (a_2,b_2) \arrow[dr] & & \\
(a_1,b_1) \arrow[r] & (a_2,b_1) \arrow[ul] & (a_3,b_1) \arrow[r] & \cdots
\end{tikzcd}$$

We can thus list all coordinates (or follow the path shown in the graph)(Well, I have to admit that this is rather a poor drawing, but Tikz package are too painful to learn.)

$$(a_1,b_1),(a_2,b_1),(a_1,b_2),(a_1,b_3),(a_2,b_2),(a_3,b_1),\ldots$$

And of course $\mathbb{Q}\times\mathbb{Q}$ is therefore countable.
c) Let a and b be the two end points of the interval, and a, $b\in\mathbb{Q}$. Similar to b), we could list them and order them, which shows that all rational intervals are countable.
d) Denote polynomials as

$$p = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots$$

where p_i are all rationals. Let the set $P_i = \{p_i |\ p_i \in \mathbb{Q}\}$ and clearly it is countable. Let the set

$$P = \bigcup_{i=0}^{\infty} P_i$$

which is countably union of countable sets. Thus P is countable. □

Problem 4. A number α is called algebraic if it is a root of a polynomial equation with rational coefficients. Prove that the set of all algebraic numbers is countable.

Proof.
Since there are at most countable number of polynomial equations, each of which has at most n roots, if it is a polynomial of degree n. Therefore the roots are countable and therefore is all algebraic numbers. □

Problem 5. Prove the existence of uncountably many *transcendental* numbers, i.e., numbers which are not algebraic.

Proof. Since we have countably many algebraic numbers, and there are uncountably in $\mathbb{R}$. It follows that there must be countably transcendental numbers. $\square$

Problem 6. Prove that the set of all real functions (more generally, functions taking values in a set containing at least two elements) defined on a set M is of power greater than the power of M. In particular, prove that the power of the set of *real* functions (continuous and discontinuous) defined in the interval$[0, 1]$ is greater than c.

Proof.

$\square$

Problem 7. Give an indirect proof of the equivalence of the closed interval $[a, b]$, the open interval (a, b) and the half-open interval $[a, b)$ or $(a, b]$.
Hint. Use Theorem 7. (**Cantor-Bernstein Theorem**: *Given any two sets A and B, suppose A contains a subset A_1 equivalent to B, while B contains a subset B_1 equivalent to A. Then A and B are equivalent.*)

Proof.
I have no idea how to prove this one "indirectly", yet having a "direct" proof echoing that of "equivalence of $(0, 1)$ and $[0, 1]$". It goes as follows.
Consider a sequence in (a, b),say (x_n). We can form a bijection such that

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ a & b & x_1 & x_2 & x_3 & \cdots \end{array}$$

In other words, we let $x_{n-2} \longleftrightarrow x_n$, if $n > 2$. And we let the rest elements in (a, b) point to itself, i.e., $x \longleftrightarrow x$. Thus, the bijection between $[a, b]$ and (a, b) is established and we have the equivalence. Similar proof could be used on half-open intervals.
However, I cannot think of an "indirect" proof using Cantor-Bernstein Theorem. $\square$

Problem 8. Prove that the union of a finite or countable number of sets each of power c is itself of power c.

Proof.

□

Problem 9. Prove that each of the following sets has the power of the continuum:
a) The set of all infinite sequences of positive integers;
b) The set of all ordered n-tuples of real numbers;
c) The set of all infinite sequences of real numbers.

Proof.
a) Suppose that the set of all infinite sequences of positive integers has power of $\aleph_0$. Denote this set as $A = \{\alpha_i | \ i \in \mathbb{N}\}$. Since it is countable, we could list all the elements:

$$\begin{aligned} &\alpha_1 = (a_{11}, a_{12}, a_{13}, \dots) \\ &\alpha_2 = (a_{21}, a_{22}, a_{23}, \dots) \\ &\dots \end{aligned}$$

We could therefore have a sequence of integers

$$d = (d_1, d_2, d_3, \dots)$$

such that

$$d_i = \begin{cases} 0, a_{ii} \neq 0 \\ 2, a_{ii} = 0 \end{cases}$$

It is obvious that d is different from any of the element in A, and thus we have a contradiction.
b) For any given n numbers to form an ordered tuple, we could have at most $n!$ numbers of results ($P(n, k) = \frac{n!}{(n-k)!}$). But choosing those n elements from uncountable $\mathbb{R}$ has uncountable permutations. Therefore the set is countable.
c) Since we have the result from a), and all positive integers form a subset of $\mathbb{R}$. Therefore set of all infinite sequences of real numbers is also uncountable.

□

Problem 10. Develop a contradiction inherent in the notion of the "set of all sets which are not members of themselves."

Hint. Is this set a member of itself?

Comment. Thus we will be careful to avoid sets which are "too big," like the "set of all sets."

Proof.
This is also known as Russell's paradox. We prove as follows. Define two sets:

$$A = \{x|\ x \in x\}$$
$$B = \{x|\ x \notin x\}$$

We wish to ask whether $B \in B$. There are only two possibilities: either $B \in B$, or $B \notin B$. First suppose the former, then we have set B as an element of B, and thus $B \notin B$, which is a contradiction. Then consider the latter, and we have B not as an element of B. Hence B must in A, and it follows $B \in B$, which is also a contradiction. Therefore there does not exist a "set of all sets". □

3 Ordered Sets and Ordinal Numbers

Problem 1. Exhibit both a partial ordering and a simple ordering of the set of all complex numbers.

Proof.
Denote complex numbers as a pair $x = (a, b)$, where $a = Re(x)$ and $b = Im(x)$. Define a partial ordering as the following: $x_1 < x_2$, if $a_1 < a_2$. We could also define the simple ordering (total order) as: $x_1 < x_2$, if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. □

Problem 2. What is the minimal element of the set of all subsets of a given set X, partially ordered by set inclusion. What is the maximal element?

Proof.
Clearly, the minimal element is $\emptyset$ and the maximal is X itself. Since $\emptyset$ is subset of any set, and X is the only set that contains all elements in itself. □

Problem 3. A partially ordered set M is said to be a *directed set* if, given any two elements $a, b \in M$, there is an element $c \in M$ such that $a \leq c$, $b \leq c$. Are the partially ordered sets in Examples 1-4, Sec. 3.1 all directed sets?

Proof.
a) False. Since $a \leq b$ if and only if $a = b$. Let any two elements in M, and x, y are not necessarily to be equal. If we wish to find a c, such that $x \leq c$ and $y \leq c$, we must have $x = c = y$. Hence it is not a directed set.
b) True. Since M contains all CTS function, there must one function $h(x)$ such that given any two other functions f and g in M, $f \leq h$ and $g \leq h$ must hold. Hence it is a directed set.
c) True. Since $\mathcal{M} = \{M_i\}$ is the set of all subsets of M, the maximal element must be M itself. Thus, any two sets must be subsets of M. It is a directed set.
d) False. Suppose for a contradiction that there exist a c such that any a, b as integers, and c is divisible by both a and b. However, there must also exist integer $x =$ □

Problem 4. By the *greatest lower bound* of two elements a and b of a partially ordered set M, we mean an element $c \in M$ such that $c \leq a$, $c \leq b$ and there is no element $d \in M$ such that $c < d \leq a$, $d \leq b$. Similarly, by the *least upper bound* of a and b, we mean an element $c \in M$ such that $a \leq c$, $b \leq c$ and there is no element $d \in M$ such that $a \leq d < c$, $b < d$. by a *lattice* is meant a partially ordered set any two element of which have both a greatest lower bound and a least upper bound. Prove that the set of all subsets of a given set X, partially ordered by set inclusion, is a lattice. What is the set-theoretic meaning of the greatest lower bound and least upper bound of two elements of this set?

Proof.

□

Problem 5. Prove that an order-preserving mapping of one ordered set onto another is automatically an isomomrphism.

Proof.

□

4 System of Sets

2

Metric Spaces

"One of the most important operations in mathematical analysis is the taking of limits."

– Kolmogorov

5 Basic Concepts

6 Convergence. Open and Closed Sets

Problem 1.

Proof.

□

Problem 2. Prove that every contact point of a set M is either a limit point of M or an isolated point of M.

Proof.

□

Problem 3. Prove that if $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $\rho(x_n, y_n) \to \rho(x, y)$.

Proof.
Fix $\varepsilon > 0$, $\exists\ N_1 \in \mathbb{N}$, *s.t.* $n > N_1$, $\rho(x_n, x) < \frac{\varepsilon}{2}$. Also $\exists\ N_2 \in \mathbb{N}$, *s.t.* $n > N_2$, $\rho(y_n, y) < \frac{\varepsilon}{2}$. Pick $N = \max(N_1, N_2)$, such that $n > N$, (by 1.a, p.45) we have $|\rho(x_n, y_n) - \rho(x, y)| \leq \rho(x_n, x) + \rho(y_n, y) < \frac{\varepsilon}{2}$.

□

Problem 4. Let f be a mapping of one metric space X into another metric space Y. Prove that f is continuous at a point x_0 if and only if the sequence $\{y_n\} = \{f(x_n)\}$ converges to $y = f(x_0)$ whenever the sequence $\{x_n\}$ converges to x_0.

Proof.
"$\Rightarrow$":
Fix $\varepsilon > 0$, $\exists\ \delta > 0$, *s.t.* $\rho(x_n, x_0) < \delta$ implies $\rho'(f(x_n), f(x_0))$. Hence $(y_n) \to y$, as $n \to \infty$.
"$\Leftarrow$"
Fix $\varepsilon > 0$, $\exists\ \delta > 0$, $\rho'(y_n, y_0) < \varepsilon$ whenever $\rho(x_n, x_0) < \delta$. It implies $\rho'(f(x_n), f(x_0)) < \varepsilon$. Therefore, f is continuous at x_0. □

Problem 5. Prove that
a) The closure of any set M is a closed set;
b) $[M]$ is the smallest closed set containing M.

Proof.
a) Let $x \in [[M]]$. Fix $\varepsilon > 0$, there exist $x_1 \in [M]$, *s.t.* $x_1 \in O_\varepsilon(x)$. Consider $O_{\varepsilon_1}(x_1)$, where $\varepsilon_1 = \varepsilon - \rho(x, x_1)$. Since $O_{\varepsilon_1}(x_1) \subset O_\varepsilon(x)$, and there exists $x_2 \in O_{\varepsilon_1}(x_1)$ and $x_2 \in O_\varepsilon(x)$. Hence $x \in [M]$.

Since we have $x \in [[M]]$ and then it is also in $[M]$. We could have $[[M]] \subset [M]$. With $[M] \subset [[M]]$, we have $[[M]] = [M]$.
b) Suppose $\exists\ x \in [M]$, and $x \notin M$. Therefore $O_\varepsilon(x) \cap M = \emptyset$. Since also $x \in [M]$, x must a contact a point and thus any neighborhood of x contains at least one point of M, which is a contradiction. Therefore $[M]$ is the smallest set containing M. □

Problem 6. Is the union of infinitely many closed sets necessarily closed? How about the intersection of infinitely many open sets? Give examples.

Proof.

□

Problem 7. Prove directly the the point $\frac{1}{4}$ belongs to the Cantor set F, although it is not an end of any of the open intervals deleted in constructing F. (Hint: The point $\frac{1}{4}$ divides the interval $[0, 1]$ in the ratio $1 : 3$. It also divides the interval $[0, \frac{1}{3}]$ left after deleting $(\frac{1}{3}, \frac{2}{3})$ in the ratio $3 : 1$, and so on.)

Proof.

□

7 Complete Metric Spaces

8 Contraction Mappings

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Topological Spaces

"Metric spaces are topological spaces of a rather special (although very important) kind."

– Kolmogorov

9 Basic Concepts

10 Compactness

11 Compactness in Metric Spaces

12 Real Functions on Metric and Topological Spaces

4

Linear Spaces

"One of the most important concepts in mathematics is that of a linear space, which will play a key role in the rest of this book."

– Kolmogorov