The limit and boundary characteristic classes in Borel-Moore motivic homology

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Algebraic cycles and Chow groups

- X separated of finite type over a field
- group of n-dimensional (raw) **algebraic cycles**

$$Z_n(X) = \bigoplus_W \mathbb{Z}[W]$$

  for $W \subset X$ subvariety of dimension n

- **Chow group** of algebraic n-dimensional cycles

$$CH_n(X) = Z_n(X)/\{\text{rational equivalence}\}$$

- Chow group of n-codimensional cycles

$$CH^n(X) = CH_{\dim(X)-n}(X)$$

- proper covariance, flat/lci contravariance

Chern classes in Chow groups, after Grothendieck

Theorem (Grothendieck)

E/X rank n vector bundle, $\xi = e(\mathcal{O}(-1)) \in CH^1(\mathbb{P}(E))$
Then cup-products with powers of ξ induce an isomorphism

$$(1, \xi, \cdots, \xi^{n-1}) : CH^*(X) \oplus \cdots \oplus CH^*(X) \simeq CH^*(\mathbb{P}(E))$$

and there are unique classes $c_i(E) \in CH^i(X)$ such that

$$\xi^n - c_1(E)\xi^{n-1} + \cdots + (-1)^n c_n(E) = 0$$

X/k smooth, **(total) Chern class**

$$c(X) = c(TX) = 1 + \sum_{i=1}^{\dim(X)} c_i(TX) \in CH_*(X)$$

Question: what about singular varieties?
$k = \mathbb{C}$: Chern-Schwartz-MacPherson class

The Nash blow-up

- X/k possibly singular, $\dim(X) = d$, $X_0 \subset X$ smooth locus
- $i : X \to N$ closed embedding, N/k smooth
- $\phi : X_0 \to Gr(d.i^*TN)_{|X_0}$ induced by TX_0
- **Nash blow-up** $\widetilde{X} = \overline{\phi(X_0)} \subset Gr(d.i^*TN)_{|X_0}$
- $\nu : \widetilde{X} \to X$ proper birational
- $\widetilde{\Omega}$ =restriction of the tautological bundle on $Gr(d.i^*TN)$
  =locally free rank d sheaf on $\widetilde{X}$
- $\nu^*\Omega^1_X \to \widetilde{\Omega}$ canonical sheaf map

- **Mather Chern class**

$$c_M(X) = \nu_*(c(\widetilde{\Omega}^\vee) \cap [\widetilde{X}]) \in CH_*(X)$$

- linearization:

$$c_M : Z_*(X) \to CH_*(X)$$
$$\sum n_i V_i \mapsto \sum n_i c_M(V_i)$$

- c_M is nice, but need modification for better functoriality

Constructible functions

- $X/\mathbb{C}$, $Cons(X)$ =group of **constructible functions** on X $=\mathbb{Z}$-linear combinations of 1_Z for $Z \subset X$ subvariety
- $X \mapsto Cons(X)$ covariant for $f : X \to Y$ proper: $p \in Y$,

$$f_* 1_W(p) = \chi(f^{-1}(p) \cap W)$$

χ =topological Euler characteristic

- **Local Euler obstruction**: $p \in X$,

$$Eu_p(X) = \int_{\nu^{-1}(p)} c(\widetilde{\Omega}^{\vee}_{|\nu^{-1}(p)}) \cap s(\nu^{-1}(p), \widetilde{X}) \in \mathbb{Z}$$

  (Gonzalez-Sprinberg-Verdier formula)

- linearization:

$$\begin{aligned} Eu : Z_*(X) &\xrightarrow{\sim} Cons(X) \\ \sum n_i V_i &\mapsto (p \mapsto \sum n_i Eu_p(V_i)) \end{aligned}$$

The Chern-Schwartz-MacPherson class

Theorem (MacPherson, Deligne-Grothendieck conjecture)

The **Chern-Schwartz-MacPherson (CSM) class**

$$c_X^{SM} : Cons(X) \xrightarrow{Eu^{-1}} Z_*(X) \xrightarrow{c_M} CH_*(X)$$

is the unique natural transformation $Cons(-) \to CH_*$ commuting with proper push-forwards such that for X smooth

$$c_X^{SM}(1_X) = c(TX)$$

- Riemann-Roch type formula
- MacPherson's original construction and proof use transcendental methods
- Aluffi: algebraic reconstruction of the CSM class

Category of compactifications

- **Category of compactifications** $Cpt(X)$:
  Objects: $X \xrightarrow{j} \overline{X} \xrightarrow{\pi} k$, j open immersion with dense image, π proper
  Morphisms: proper morphisms $\overline{X} \to \overline{X}'$ making the obvious diagram commute
- $Cpt(X)$ is non-empty (Nagata), cofiltered

Limit Chow groups

- **Limit Chow groups**:

$$lCH_*(X) = \lim_{\overline{X} \in Cpt(X)} CH_*(\overline{X})$$

- $\alpha \in lCH_*(X) \Leftrightarrow (\alpha_{\overline{X}} \in CH_*(\overline{X}))_{\overline{X} \in Cpt(X)}$, compatible with proper push-forwards
- canonical map $lCH_*(X) \to CH_*(X)$, isomorphism if X proper
- covariance: $f : X \to Y \Rightarrow f_* : lCH_*(X) \to lCH_*(Y)$
  In particular: $\deg : lCH_*(X) \to \mathbb{Z}$

The pro-CSM class

Theorem (Aluffi)

*Assume resolution of singularities and factorization of birational maps over k. There is a unique way to define classes $Ic^{SM}(X) \in ICH_*(X)$ (**pro-CSM classes**) such that*

- *(Additivity)* $X = \coprod U_i$ *stratification,* $\iota_i : U_i \to X$

$$Ic^{SM}(X) = \sum \iota_{i*} Ic^{SM}(U_i)$$

- *(Normalization)* X *smooth proper,* $D \subset X$ *snc divisor,* $j : X - D \to X$ *open complement,*

$$j_* Ic^{SM}(X - D) = c(\Omega^1_X(\log D)^\vee)$$

- $ICH_*(X) \to CH_*(X)$ sends Ic^{SM} to c^{SM}
- In the sequel, we focus on the 0-dimensional part

- **motivic stable homotopy category**

$$\mathbf{SH}(X) = L_{\mathbb{P}^1} L_{\mathbb{A}^1} Sh((Sm/X)_{Nis}, sSets)$$

- six functors compatible with étale realization

$$\mathbf{SH}_c(X) \to D^b_{ctf}(X_{et}, \Lambda)$$

- Chow groups are representable as Borel-Moore theories: $f : X \to k$, $\mathbf{H}\mathbb{Z}$ =motivic Eilenberg-Mac Lane spectrum

$$\mathbf{H}\mathbb{Z}^{BM}(X/k) = [\mathbb{1}_X, f^!\mathbf{H}\mathbb{Z}] \simeq CH_0(X)$$

- More generally, **limit Borel-Moore theories**

$$l\mathbb{E}^{BM}(X/k) = \lim_{\overline{X} \in Cpt(X)} \mathbb{E}^{BM}(\overline{X}/k)$$

The characteristic class

- Classical: SGA5, Kashiwara-Schapira, Abbes-Saito
- Motivic: Olsson, J.-Yang, Cisinski, J.
- Lu-Zheng: generalized trace map in the (2-)category of correspondences
- $f : X \to k, K \in \mathbf{SH}_c(X), \mathbb{1}_k \to \mathbb{E} \in \mathbf{SH}(k)$
  $\delta : X \to X \times_k X, \mathbb{D}(K) = \underline{Hom}(K, f^!\mathbb{1}_k)$
- $\mathbb{E}$**-valued characteristic class** $C_X(K, \mathbb{E}) \in \mathbb{E}^{BM}(X/k)$

$$\mathbb{1}_X \to \underline{Hom}(K, K) \simeq \delta^!(\mathbb{D}(K) \boxtimes_k K) \to \delta^*(\mathbb{D}(K) \boxtimes_k K)$$
$$= \mathbb{D}(K) \otimes_k K \simeq K \otimes_k \mathbb{D}(K) \to f^!\mathbb{1}_k \to f^!\mathbb{E}$$

Properties of the characteristic class

- Compatibility with proper push-forwards (**Lefschetz-Verdier formula**) and étale pullbacks
  special case: motivic Gauss-Bonnet formula (Levine, Déglise-J.-Khan)
- Additivity along distinguished triangles (May, Groth-Ponto-Shulman, J.-Yang)

Limit characteristic classes

Definition

Limit characteristic class (with/without compact support):

$$lC_X(K,\mathbb{E}) = (C_{\overline{X}}(j_*K,\mathbb{E}))_{X\xrightarrow{j}\overline{X}\in Cpt(X)} \in l\mathbb{E}^{BM}(X/k)$$

$$lC^{\mathrm{c}}_X(K,\mathbb{E}) = (C_{\overline{X}}(j_!K,\mathbb{E}))_{X\xrightarrow{j}\overline{X}\in Cpt(X)} \in l\mathbb{E}^{BM}(X/k)$$

Properties of limit characteristic classes

- IC and IC^{c} are additive along distinguished triangles
- (Push-forward formula) $f : Y \to X$,

$$f_* IC_Y(K) = IC_X(f_* K)$$

$$f_* IC^{\mathrm{c}}_Y(K) = IC^{\mathrm{c}}_X(f_! K)$$

- $I\mathbb{E}^{BM}(X/k) \to \mathbb{E}^{BM}(X/k)$ sends both $IC_X(K, \mathbb{E})$ and $IC^{\mathrm{c}}_X(K, \mathbb{E})$ to $C_X(K, \mathbb{E})$

Comparison with the pro-CSM class

Theorem (J.-Sun-Yang)

$$Ic_0^{SM}(X) = IC_X^{\mathrm{c}}(\mathbb{1}_X, \mathbf{H}\mathbb{Z}) \in ICH_0(X/k)$$

- Consequently, $IC_X^{\mathrm{c}}(\mathbb{1}_X, \mathbf{H}^{MW}\mathbb{Z}) \in ICH_0^{MW}(X/k)$ is a quadratic refinement of the pro-CSM class (independently defined by Azouri)

More comparison

Lemma (JSY)

$$IC_X(K, \mathbf{H}\mathbb{Z}) = IC^{\mathrm{c}}_X(K, \mathbf{H}\mathbb{Z})$$

- reflects the fact $\chi(M) = \chi_c(M)$ for any complex manifold M, or more generally for any stratified space with only even-dimensional strata (cf. Laumon for étale sheaves)
- fails for $\mathbf{H}^{MW}\mathbb{Z}$ in place of $\mathbf{H}\mathbb{Z}$ (Levine)

Definition

Boundary Borel-Moore theories

$$b\mathbb{E}^{BM}(X/k) = \lim_{\overline{X} \in Cpt(X)} \mathbb{E}^{BM}(\overline{X} - X/k)$$

- X proper $\Rightarrow b\mathbb{E}^{BM}(X/k) = 0$
- $f : Y \to X$ proper $\Rightarrow f_* : b\mathbb{E}^{BM}(Y/k) \to b\mathbb{E}^{BM}(X/k)$
- canonical map $b\mathbb{E}^{BM}(X/k) \to l\mathbb{E}^{BM}(X/k)$

In the sequel we focus on the case $\mathbb{E} = \mathbf{H}\mathbb{Z}$, i.e. the **boundary Chow group** of 0-cycles bCH_0 (Kato-Saito)

The Kato-Saito-Swan class

- $\mathrm{char}(k) = p$, U/k smooth connected, $\mathcal{F}/U$ smooth sheaf
- $f : V \to U$ finite Galois trivializing $\mathcal{F}$, $G = Gal(V/U)$, $\mathcal{F} \leftrightarrow M \in \Lambda[G] - \mathrm{Mod}$

Definition (Kato-Saito-Swan class)

$$\mathrm{Sw}_U^{\mathrm{KS}}(\mathcal{F}) = \frac{1}{|G|} f_* \sum_{\sigma \in G_{(p)} \setminus \{1\}} \left(\dim M^\sigma - \frac{\dim M^{\sigma^p}/M^\sigma}{p-1} \right) \cdot (V \times_U V \setminus \Delta_V, \Delta_V)_{\mathrm{b}}^{\log} \in bCH_0(U)$$

Theorem (Kato-Saito)

$$\chi_c(\mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(\Lambda) - \deg(\mathrm{Sw}_U^{KS}(\mathcal{F}))$$

- generalizes the Grothendieck-Ogg-Shafarevich formula to higher dimensions

The localized characteristic class

- Abbes-Saito for étale sheaves, J.-Yang in the motivic/quadratic setting
- $f : X \to k$ smooth, $i : Z \to X$ nowhere dense closed, $\delta : X \to X \times_k X$ with open complement γ

$$\delta^\Delta = \delta^!(- \otimes \gamma_* \mathbb{1}) : \mathbf{SH}(X \times X) \to \mathbf{SH}(X)$$

- $K \in \mathbf{SH}_c(X)$ such that $K_{|X-Z}$ is dualizable, then the class

$$\begin{aligned} i_*\mathbb{1}_Z \to i_* i^* \delta^\Delta \delta_* \mathbb{1}_X &\to i_* i^* \delta^\Delta(\mathbb{D}(K) \boxtimes_k K) \\ &\simeq \delta^\Delta(\mathbb{D}(K) \boxtimes_k K) \to \delta^\Delta \delta_* f^! \mathbf{H}\mathbb{Z} \end{aligned}$$

lifts to a unique class $C_X^Z(K) \in CH_0(Z)$, the **localized characteristic class**

Theorem (Abbes-Saito, J.-Yang)

$$C_X(K) = \mathrm{rk}(K) \cdot C_X(\mathbb{1}_X) + i_* C_X^Z(K)$$

Saito's uniqueness conjecture

Conjecture (T. Saito)

The three approaches

- *via* $\mathrm{Sw}^{KS}(\mathcal{F})$
- *via* $C_X^Z(K)$
- *via characteristic cycles*

yield the same Swan class for étale sheaves

- Umezaki-Yang-Zhao, Yang-Zhao: compare $\mathrm{Sw}^{\mathrm{KS}}(\mathcal{F})$ and $\mathrm{Sw}^{CC}(\mathcal{F})$

Functoriality of the localized characteristic class

X, Y smooth, $U = X - Z$, Cartesian diagram

$$\begin{array}{ccccc} V & \xrightarrow{l} & Y & \xleftarrow{k} & W \\ g\downarrow & & \downarrow f & & \downarrow h \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z. \end{array}$$

- (Pullback) g étale, $K \in \mathbf{SH}_c(X)$, $K_{|U}$ dualizable, $\mathrm{rk}(K) = 0 \Rightarrow$

$$h^! C_X^Z(K) = C_Y^W(f^* K) \in CH_0(W)$$

- (Push-forward) f proper, g smooth, $K \in \mathbf{SH}_c(Y)$, $K_{|V}$ dualizable, $\mathrm{rk}(K) = 0 \Rightarrow$

$$h_* C_Y^W(K) = C_X^Z(g_* K) \in CH_0(Z)$$

- Additivity along distinguished triangles

Boudary characteristic class

- Assume the category of smooth compactifications $Cpt^{Sm}(X)$ is cofinal in $Cpt(X)$
  (Known if $\dim(X) \leqslant 3$ or under resolution of singularities)
- $K \in \mathbf{SH}_c^{\mathrm{rig}}(X)$, dualizable

Definition (Boundary characteristic class (with compact support))

$$bC_X^{\mathrm{c}}(K) = \left(C_{\overline{X}}^{\overline{X}-X}(j_!K) - \mathrm{rk}(K) \cdot C_{\overline{X}}^{\overline{X}-X}(j_!\mathbb{1}_X)\right)_{X \xrightarrow{j} \overline{X} \in Cpt^{Sm}(X)}$$
$$\in bCH_0(X)$$

- well-defined by the pullback formula
- $bCH_0(X) \to ICH_0(X)$ sends $bC_X^{\mathrm{c}}(K)$ to $IC_X^{\mathrm{c}}(K) - \mathrm{rk}(K) \cdot IC_X^{\mathrm{c}}(\mathbb{1}_X)$

Comparison with the Kato-Saito-Swan class

Conjecture (JSY)

Under resolution of singularities, the following diagram commutes

$$\begin{tikzcd} K_0(\mathbf{SH}_c^{\mathrm{rig}}(X)) \arrow[r] \arrow[d, "bC_X^{\mathrm{c}}"'] & K_0(\mathbf{D}_{ctf}^{b,\mathrm{rig}}(X,\Lambda)) \arrow[d, "bC_X^{\mathrm{c}}"] \arrow[dl, "-\mathrm{Sw}_X^{KS}"'] \\ bCH_0(X) \arrow[r] & b\mathbf{H}_{\mathrm{et}}\Lambda^{BM}(X/k). \end{tikzcd}$$

Theorem (JSY)

The lower triangle in the above diagram commutes

- proof uses push-forward formula, Brauer theory and an argument of T. Saito

Thank you!