Solid abelian groups II

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Recall $S \in ED$ $S = \lim S_i$

$\rightsquigarrow \mathbb{Z}[S]^{\blacksquare} = \lim \mathbb{Z}[S_i] \rightsquigarrow \text{Solid} \subset \text{Cond}(Ab)$

$- \mathbb{Z}[S]^{\blacksquare} = \underline{\text{Hom}}(\underbrace{\underline{\text{Hom}}(\mathbb{Z}[S], \mathbb{Z})}_{= C(S,\mathbb{Z})}, \mathbb{Z})$ $S \in CH$

$- S \in \text{Prof} \Rightarrow \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]^{\blacksquare}$ inj

fails for $S \in CH$

Cor 1) Comp. proj. obj. in Solid $= \prod_I \mathbb{Z}$ for some I

2) Comp proj obj in $D(\text{Solid})$ = bounded complex termwise $\prod \mathbb{Z}$

3) $R\text{Hom}(-, \mathbb{Z}) : \text{Solid}_c^{op} \xrightarrow{\sim} Ab$

$$D_c(\text{Solid})^{op} \xrightarrow{\sim} D^b(Ab)$$

Q $\text{Cond}(Ab)_c = ?$

4) $C \in D(\text{Solid}) \Rightarrow \forall S \in ED,\ R\underline{\text{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} R\underline{\text{Hom}}(\mathbb{Z}[S], C)$

$\Rightarrow$ Refinement works:

$\exists - \otimes^{\blacksquare} -$ on Solid s.t. $(\forall A, B \in \text{Solid}\ A \otimes^{\blacksquare} B = (A \otimes B)^{\blacksquare})$

$(-)^{\blacksquare} : (\text{Cond}(Ab), \otimes) \longrightarrow (\text{Solid}, \otimes^{\blacksquare})$ sym. monoidal

$\forall A, B \in \text{Cond}(Ab)$ $(A \otimes B)^{\blacksquare} = (A^{\blacksquare}) \otimes^{\blacksquare} (B^{\blacksquare})$

$-\otimes^{\blacksquare}-$ commutes with colimits in each variable

4. Computation of $\otimes^{\blacksquare}$

1) $\prod_I \mathbb{Z} \otimes^{\blacksquare} \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}$

P/ $\prod_I \mathbb{Z} = \mathbb{Z}[S]^{\blacksquare}$ $\qquad \prod_J \mathbb{Z} = \mathbb{Z}[T]^{\blacksquare}$

(S = 1-pt comp. of I)

$$\mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \mathbb{Z}[T]^{\blacksquare} = (\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{\blacksquare}$$

$$= \mathbb{Z}[S\times T]^{\blacksquare} = \prod_{I\times J} \mathbb{Z}. \qquad \square.$$

2) $\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$, $\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_\ell = 0$

P/ $\mathbb{Z}_p = \mathbb{Z}[[x]]/x-p$

$$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_\ell = \mathbb{Z}[[x,y]]/(x-p, y-\ell) = \begin{cases} \mathbb{Z}_p & p=\ell \\ 0 & p\neq \ell. \end{cases} \qquad \square.$$

2') $\mathbb{Z}[[x]] \otimes^{\blacksquare} \mathbb{Z}[[y]] = \mathbb{Z}[[x,y]]$

3) $\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}[[x]] = \mathbb{Z}_p[[x]]$

4) $\mathbb{R}^{\blacksquare} = 0$.

P/ Yoneda $\Rightarrow$ suffices: $\underset{\substack{\| \\ R\mathrm{Hom}(\mathbb{R}, C)}}{R\mathrm{Hom}}(\mathbb{R}^{\blacksquare}, C) = 0 \quad \forall\, C \in D(\mathrm{Solid})$

WMA C bounded

$$R\mathrm{Hom}(\mathbb{R}, \bigoplus \prod \mathbb{Z}) = \bigoplus R\mathrm{Hom}(\mathbb{R}, \prod \mathbb{Z}) = 0.$$

$\uparrow$ pseudo-coh

5) S = finite CW complex

$\mathbb{Z}[S]^{\blacksquare, L}$ = complex computing singular homology $H_i(S, \mathbb{Z})$

$\rightsquigarrow$ $\mathbb{Z}[S]^{\blacksquare, L}$ = "good" homology theory of S $\rightarrow$ good formal property

$\shortparallel$

$R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}), \mathbb{Z})$

$\rightsquigarrow$ Same for any CW-complex

5. Alternative approach to Solid

$\mathcal{A}$ = ab. cat. generated by comp. proj. + ε. $(\forall X = \mathrm{coker}(A \to B))$

$\mathcal{A}^{cp} \subset \mathcal{A}$ subcat of comp. proj

$\Rightarrow$ Hom in $\mathcal{A}$ is completely determined formally by maps between comp. proj.

$\Rightarrow$ If $\mathcal{A}^{cp}$ is small then

$$\begin{array}{c} \mathcal{A} = \mathrm{Fun}^{add}((\mathcal{A}^{cp})^{op}, \mathrm{Ab}) \\ \rotatebox{90}{$\in$} \\ M \longmapsto (C \longmapsto \mathrm{Hom}(C, M)) \end{array}$$

Apply to $\mathcal{A}$ = Solid : comp. proj generator $\prod_I \mathbb{Z}$

each map $\prod_I \mathbb{Z} \to \prod_J \mathbb{Z}$ is dual

of $\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$

$\Rightarrow$ Solid $= \mathrm{Fun}^{add}$ (Free abelian group , Ab)

$\cap$?

(Cond(Ab))

$\otimes^{\blacksquare}$ easy to describe :

Comm fun $\otimes$ on Free Ab

6. Application

Whitehead problem : $A \in Ab$ $\quad \mathrm{Ext}^1(A, \mathbb{Z}) = 0 \overset{?}{\Rightarrow} A = \bigoplus_I \mathbb{Z}$ free ?

A (Shelah) Depends on axioms beyond ZFC

Th (In ZFC) $A \in Ab$, $\mathrm{Ext}^1_{Cond(Ab)}(A, \mathbb{Z}) = 0$

$\Rightarrow A = \bigoplus_I \mathbb{Z}$.

D/ $0 \to \bigoplus \mathbb{Z} \to \bigoplus \mathbb{Z} \to A \to 0$

$R\underline{\mathrm{Hom}}(-, \mathbb{Z}) \overset{0\to}{\Rightarrow} R\underline{\mathrm{Hom}}(A, \mathbb{Z}) \to R\underline{\mathrm{Hom}}(\bigoplus \mathbb{Z}, \mathbb{Z}) \to R\underline{\mathrm{Hom}}(\bigoplus \mathbb{Z}, \mathbb{Z})$

$0 \to \underline{\mathrm{Hom}}(A, \mathbb{Z}) \to \prod \mathbb{Z} \to \prod \mathbb{Z} \to 0$

$\Rightarrow A$ free . $\square$.