<u>Recall</u> $\mathcal{A}$ abelian category

$C^{\cdot}(\mathcal{A})$ (co)chain complexes over $\mathcal{A}$

$$D(\mathcal{A}) = C^{\cdot}(\mathcal{A}) \underbrace{[q\text{-iso}]^{-1}}$$

– $f : X^{\cdot} \to Y^{\cdot}$ is quasi-iso $\Leftrightarrow$ $H^i(f) : H^i(X^{\cdot}) \xrightarrow{\sim} H^i(Y^{\cdot})$

<u>Def</u> $\mathcal{C} \in \mathrm{Cat}$ $\quad \underline{W \subset \mathrm{Mor}(\mathcal{C})}$

If $\exists\ \mathcal{C}[W^{-1}] \in \mathrm{Cat}$

$+\ \mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$ functor

s.t. $\forall F : \mathcal{C} \to \mathcal{D}$

s.t. $\underline{F(W) \subset \mathrm{Iso}(\mathcal{D})}$

$\exists !\ F' : \mathcal{C}[W^{-1}] \to \mathcal{D}$ s.t. $\quad F = F' \circ \gamma$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "F'"] \\ & \mathcal{D} \end{tikzcd}$$

$\mathcal{C}[W^{-1}]$ = <u>localization</u> of $\mathcal{C}$ w.r.t. W

<u>Lemma</u> $\forall \mathcal{C}, \forall W,\quad \mathcal{C}[W^{-1}]$ always exists

Pf/ $\mathrm{Obj}(\mathcal{C}[W^{-1}]) = \mathrm{Obj}(\mathcal{C})$ $\quad (X \xrightarrow{f} Y) \mapsto (X \xleftarrow{=} X \xrightarrow{f} Y)$

$\leadsto$ $\mathrm{Obj}(\mathcal{C}[W^{-1}]) = \mathrm{Obj}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X,Y) = \left\{ \Gamma = \left(X = X_0 \leftarrow X_1 \to X_2 \leftarrow X_3 \to X_4 \cdots X_{2n-2} \leftarrow X_{2n-1} \to X_{2n} = Y \right) \right\} \Big/ \sim$$

where $(X_{2i+1} \to X_{2i+2}) \in \mathrm{Mor}(\mathcal{C})$

$(X_{2i} \leftarrow X_{2i+1}) \in W \cup \{id\}$

where

$$\begin{tikzcd}
 & X_1 \arrow[r] \arrow[dl] \arrow[d] & X_2 \arrow[d] & \cdots \arrow[l] \arrow[r] & X_{2n-2} \arrow[d] & X_{2n-1} \arrow[l] \arrow[d] \arrow[dr] & \\
X_0 & X_1' \arrow[l] \arrow[r] & X_2' & \cdots \arrow[l] \arrow[r] & X_{2n-2}' & X_{2n-1}' \arrow[l] \arrow[r] & X_{2n}
\end{tikzcd}$$

$\Rightarrow \Gamma \sim \Gamma'$ $\square$.

Pb 1) $\mathcal{C}[W^{-1}]$ need not be small

2) Hard to compute Hom.

Basic homotopical algebra

$\mathcal{C} \in \mathrm{Cat}$

Def 1) $X \xrightarrow{f} Y$, $U \xrightarrow{g} V$ $\in \mathrm{Mor}(\mathcal{C})$, we say that f is a retract of g

if $\exists$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & U \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r] \arrow[rr, bend right, equal] & V \arrow[r] & Y
\end{tikzcd}$$

2) $X, U \in \mathcal{C}$ X is a retract of U if id_X is a retract of id_U.

Def $F \subset \mathrm{Mor}(\mathcal{C})$

1) F is <u>stable under retracts</u> if $\forall g \in F$, $\forall f$ retract of g, $f \in F$.

2) F - - — - <u>pushouts</u> if $\forall$
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r] & V \end{tikzcd}$$
pushout square, $f \in F \Rightarrow g \in F$

2') (pullbacks) (pullback square) $g \in F \Rightarrow f \in F$.

3) F - — — - <u>transfinite compositions</u>

if $\forall$ functor $X : I \to \mathcal{C}$ s.t.

1) I <u>well-ordered</u> with initial element 0
(I = ordinal)

2) $\forall i \in I \setminus \{0\}$, $\operatorname{colim}_{j<i} X(j)$ exists

and $\operatorname{colim}_{j<i} X(j) \to X(i) \in F$.

Then $\operatorname{colim}_{i \in I} X(i)$ exists, and $\left(X(0) \to \operatorname{colim}_{i \in I} X(i) \right) \in F$.

4) F is <u>saturated</u> if it satisfies 1) 2) 3)

5) F satisfies <u>"2 out of 3" property</u> if

$\forall$
$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] & & \cdot \end{tikzcd}$$
$\in \mathcal{C}$, 2 of $\{f, g, h\}$ are in F

$\Rightarrow$ so is the third

Rk If F contains all id's & satisfies 2) 3)

then F is stable under small sums

$$\coprod_{i\in I} X_i \xrightarrow{\coprod_{i\in I} u_i} \coprod_{i\in I} Y_i$$

Def $A \xrightarrow{i} B \in \mathcal{C}$

$X \xrightarrow{p} Y \in \mathcal{C}$

If $\forall$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

$\exists h$ s.t. $hi = a$ & $ph = b$

We say - h is a filler

- i has the left lifting property w.r.t. p (LLP)

- p has the right LP w.r.t. i (RLP).

- $F \subset \mathrm{Mor}(\mathcal{C})$

$L(F) = \{ i \in \mathrm{Mor}(\mathcal{C}) \mid \forall f \in F,\ i \text{ has LLP w.r.t. } f \}$

$R(F) = \{ p \in \mathrm{Mor}(\mathcal{C}) \mid \forall f \in F,\ p \text{ has RLP w.r.t. } f \}$.

Lemma $F, G \in \mathrm{Mor}(\mathcal{C})$

- $F \subset R(G) \iff G \subset L(F)$

- $F \subset G \Rightarrow L(G) \subset L(F)$

$R(G) \subset R(F)$.

- $R(F) = R(L(R(F)))$

  $L(F) = L(R(L(F)))$

- $L(F)$ is saturated.

  $R(F)$ is cosaturated (i.e. saturated in $\mathcal{C}^{op}$).

<u>Lemma</u> (retract Lemma)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd} \in \mathcal{C} \qquad f = pi$$

1) $f \in R(i) \Rightarrow f$ is a retract of p

2) $f \in L(p) \Rightarrow$ - — — i

D/ 1)
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left] & T \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

2) work in $\mathcal{C}^{op}$. $\square$.

<u>Ex</u> $\mathcal{C} = Set$ $\quad \phi \xrightarrow{i} \{*\}$.

$R(i) = \{\text{surjections}\}$.

$L(R(i)) \overset{AC}{=} \{\text{injections}\}$ = smallest saturated class containing i.

<u>Def</u> A <u>weak factorization system</u> in $\mathcal{C}$ is $(A, B) \subset Mor(\mathcal{C})$

s.t. a) A & B are stable under retracts

b) $\mathcal{A} \subset \mathcal{L}(\mathcal{B})$

c) $\forall\ X \xrightarrow{f} Y \in \mathcal{C}$ $\exists$ factorization $f = pi$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "i"'] & & Y \\ & T \arrow[ur, dashed, "p"'] & \end{tikzcd}$$

$i \in \mathcal{A}$

$p \in \mathcal{B}$.

<u>Lemma</u> $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ adjoint functors

$(\mathcal{A}, \mathcal{B})$ WFS $\quad (\mathcal{A}', \mathcal{B}')$ WFS

Then $F(\mathcal{A}) \subset \mathcal{A}'$

$\iff G(\mathcal{B}') \subset \mathcal{B}$.

<u>Recall</u> κ cardinal, E = poset, $E \neq \emptyset$.

E is <u>κ-filtered</u> if

$\forall\ J \in Set$, $\# J < \kappa$, $\forall\ (x_j)_{j \in J} \in E$

$\exists x \in E$, $\forall j \in J$, $x \geq x_j$

<u>Prop</u> (small object argument)

- $\mathcal{C}$ = locally small cat. + small colimits
- $I \subset Mor(\mathcal{C})$ set of morphisms, κ cardinal, s.t.

  $\forall\ k \xrightarrow{i} l \in I$, $Hom_{\mathcal{C}}(k, -) : \mathcal{C} \to Set$

  commutes with colimits indexed by κ-filtered ordinals

Then - $(L(R(I)), R(I))$ is a WFS

- $L(R(I))$ = smallest saturated class containing I.

D/ Hovey, Model categories, 2.1.14. P32.

Idea: $X \xrightarrow{f} Y \in \mathcal{C}$ $\quad \lambda = \kappa$-filtered ordinal

define $Z^f : \lambda \to \mathcal{C}$ s.t. - $Z^f_0 = X$

- f induces $Z^f \to Y$

- $E_f = \operatorname{colim} Z^f \to Y$

$X \nearrow E_f$

$Z^f_\lambda \to Z^f_{\lambda+1} \in \amalg(i)$

+ transfinite induction. $\square$

<u>Cor</u> A = small cat, $\mathcal{C} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ presheaves

$I \subset \mathrm{Mor}(\mathcal{C})$ small set

Then $(L(R(I)), R(I))$ is a WFS on $\mathcal{C}$.

<u>Def</u> 1) A <u>(closed)</u> <u>model category</u>

= locally small category $\mathcal{C}$

+ 3 classes of morphisms W, Fib, Cof $\subset \mathrm{Mor}(\mathcal{C})$

s.t. i) $\mathcal{C}$ has finite limits/colimits

ii) W satisfies "2 out of 3" property

iii) $(\underline{\mathrm{Cof}}, \mathrm{Fib} \cap W)$ and $(\underline{\mathrm{Cof} \cap W}, \mathrm{Fib})$ are WFS.

2) W = {weak equivalences}

$\mathrm{Fib} = \{ \text{fibrations} \}$ $\quad$ $\mathrm{Fib} \cap W = \{ \text{trivial / acyclic fibrations} \}$

$\mathrm{Cof} = \{ \text{cofibrations} \}$. $\quad$ $\mathrm{Cof} \cap W = \{ \text{trivial / acyclic cofibrations} \}$

3) $\emptyset \in \mathcal{C}$ initial

$* \in \mathcal{C}$ final

$X \in \mathcal{C}$ <u>fibrant</u> $\iff$ $\begin{array}{c} X \\ \downarrow \\ * \end{array} \in \mathrm{Fib}$.

<u>cofibrant</u> $\iff$ $\begin{array}{c} \emptyset \\ \downarrow \\ X \end{array} \in \mathrm{Cof}$.

<u>Rk</u> 1) $\mathrm{Iso}(\mathcal{C}) \subset \mathrm{Fib} \cap \mathrm{Cof} \cap W$

2) $\forall\, X \xrightarrow{f} Y \in \mathcal{C}$ $\quad \exists$

$$\begin{tikzcd} & P \arrow[dr, "p(f)"', "\mathrm{Fib}\cap W"] & \\ X \arrow[ur, "c(f)", "\mathrm{Cof}"'] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$$\begin{tikzcd} & \tilde{I} \arrow[dr, "\mathrm{Fib}"] & \\ X \arrow[ur, "\mathrm{Cof}\cap W"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Sometimes we require: - $\mathcal{C}$ has small limits / colimits

- functorial factorizations

(all examples satisfy these)

3) <u>W</u> is essential

Cof & Fib are auxillary.

4) Model category has redundancies:

<u>Lemma</u> $\mathrm{Cof} = L(\mathrm{Fib} \cap W)$

$\mathrm{Fib} = R(\mathrm{Cof} \cap W)$.

D/ Factorization + retract Lemma. $\square$.

<u>Ex</u> 1) $\mathcal{C} \in \mathrm{Cat}$ + finite lim, colim

- $W = \mathrm{Iso}$, $\mathrm{Fib} = \mathrm{Cof} = \mathrm{Mor}(\mathcal{C})$.

- $W = \mathrm{Fib} = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Cof} = \mathrm{Iso}(\mathcal{C})$

- $W = \mathrm{Cof} = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Fib} = \mathrm{Iso}(\mathcal{C})$

2) $A \in \mathrm{Cat}$ $\mathcal{C} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ presheaves

$W = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Cof} = \mathrm{Mono}(\mathcal{C})$ $\mathrm{Fib} = R(\mathrm{Cof} \cap W)$.

D/ Let $I = \{ K \to L \in \mathrm{Mono}(\mathcal{C}) \mid L = \text{quotient of a representable} \}$

$AC \Rightarrow \mathrm{Mono}(\mathcal{C}) = L(R(I))$

$\xRightarrow{\text{small object argument}}$ $(\mathrm{Mono}(\mathcal{C}), R(I))$ WFS. $\square$.

<u>Lemma</u> $\mathcal{C}$ model cat.

1) $\mathcal{C}^{op}$ model cat: $W = W(\mathcal{C})$

$\mathrm{Cof} = \mathrm{Fib}(\mathcal{C})$

$\mathrm{Fib} = \mathrm{Cof}(\mathcal{C})$

2) $\mathcal{C}/X$ model cat: $\mathrm{Cof} = \mathrm{Cof}(\mathcal{C}) \cap \mathrm{Mor}(\mathcal{C}/X)$
etc.

3) $X \backslash \mathcal{C}$ model cat.

4) Cof is stable under p.o.

Fib is stable under p.b.

5). Cof, Fib & W are closed under compositions.

<u>Prop</u> (Ken Brown's lemma)

- $\mathcal{C}$ model cat
- $\mathcal{D} \in \mathrm{Cat}$, $V \subset \mathrm{Mor}(\mathcal{D})$ s.t. - $\mathrm{Iso}(\mathcal{D}) \subset V$
  w.e.
  - V satisfies "2/3"
- $F : \mathcal{C} \to \mathcal{D}$ sends trivial cofibrations between cofibrant objects to V.

Then F sends w.e. — — — — — — to V.

D/ Let $X \xrightarrow{f} Y \in W$ X, Y cofibrant

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & Y \arrow[d, "j"', "\in \mathrm{Cof}"] \\
X \arrow[r, "i"] & X \amalg Y
\end{tikzcd}$$

$\in \mathrm{Cof}$

$X \amalg Y \xrightarrow{(f, \mathrm{id}_Y)} Y$ factors as $X \amalg Y \xrightarrow{k} T \xrightarrow{p} Y$, $\mathrm{Cof} \ni k$, $p \in \mathrm{Fib} \cap W$.

T cofibrant

$\therefore \in \mathrm{Cof} \cap W$

$\in \mathrm{Cof} \cap W$

$$\Rightarrow \quad \begin{tikzcd} X \arrow[rr, "ki", "\in Cof \cap W"'] \arrow[dr, "f"'] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "kj", "\in Cof \cap W"'] \arrow[dr, equal] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

$$\Rightarrow \left.\begin{array}{l} F(X \xrightarrow{ki} T) \in V \\ F(Y \xrightarrow{kj} T) \in V \\ F(p) \in V \end{array}\right) \Rightarrow F(f) = F(pki) \in V. \quad \square.$$

Homotopy category

Def $\mathcal{C}$ model cat,

1) - $A \in \mathcal{C}$

$$\begin{tikzcd} A \amalg A \arrow[r, "(\partial_0, \partial_1)", "\in Cof"'] \arrow[rr, bend left, "id_A \amalg id_A"] & IA \arrow[r, "\sigma", "\in W"'] & A \end{tikzcd}$$

IA is a cylinder object of A

- $X \in \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "s", "\in W"'] \arrow[rr, bend left, "(id_X, id_X)"] & X^I \arrow[r, "(d^0, d^1)", "\in Fib"'] & X \times X \end{tikzcd}$$

X^I is a cocylinder / path object of X

2) $f_0, f_1 : A \longrightarrow X \in Mor(\mathcal{C})$

A <u>left homotopy</u> $f_0 \to f_1$ = $\left|\begin{array}{l} \text{cylinder object } IA \\ + \ h : IA \to X \end{array}\right.$

s.t. $i = 0, 1$, $h \partial_i = f_i$

<u>right homotopy</u> $f_0 \to f_1$ = $\left|\begin{array}{l} \text{cocylinder object } X^I \\ + \ k : A \to X^I \end{array}\right.$

s.t. $i = 0, 1$, $d^i k = f_i$.

<u>Lemma</u> <u>A cofibrant</u> <u>X fibrant</u> $A \underset{f_1}{\overset{f_0}{\rightrightarrows}} X$. TFAE

1) $\exists$ left homotopy $f_0 \to f_1$

2) $\exists$ right homotopy $f_0 \to f_1$

3) $\forall IA$ cyl. obj, $\exists\, h : IA \to X$ s.t. $h \partial_i = f_i$

4) $\forall X^I$ cocyl. obj, $\exists\, k : A \to X^I$ s.t. $d^i k = f_i$

D/ Suffices 1) $\Rightarrow$ 4)

$A \xrightarrow[\partial_1]{\in W} IA$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r, "\in Cof"'] \arrow[ur, phantom, "\ulcorner"] & A \amalg A \end{tikzcd}$$

$\Rightarrow$ <u>$\partial_1 \in Cof \cap W$</u>.

similarly $X^I \xrightarrow[\in Fib]{(d^0, d^1)} X \times X$.

$\Rightarrow$
$$\begin{tikzcd} A \arrow[r, "sf_1"] \arrow[d, "\partial_1"'] & X^I \arrow[d, "(d^0, d^1)"] \\ IA \arrow[r, "(h, f_1\sigma)"'] \arrow[ur, dashed, "\exists K"] & X\times X \end{tikzcd}$$

put $k = K\partial_0$

$\Rightarrow d^1 k = d^1 K \partial_0 = f_1 \sigma \partial_0 = f_1 \mathrm{Id}_A = f_1$

$d^0 k = d^0 K \partial_0 = h\partial_0 = f_0$. $\square$

<u>Lemma</u> A cofibrant, X fibrant. Then "$\exists$ left htp" (or right htp) is an equivalence relation on $\mathrm{Hom}(A,X)$

Pf/ - Reflexive: clear

- Symmetry: follows from the equivalence 1) $\Rightarrow$ 3).

- Transitivity: $u, v, w \in \mathrm{Hom}(A,X)$

$$\begin{tikzcd} A\amalg A \arrow[r, "(\partial_0,\partial_1)"] \arrow[dr, "(\partial_0', \partial_1')"'] & IA \arrow[r, "\sigma"] & A \\ & I'A \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

$h: IA \to X$

$h': I'A \to X$

s.t. $\begin{cases} h\partial_0 = u \\ h\partial_1 = v = h'\partial_0' \\ h\partial_1' = w \end{cases}$

$$\begin{tikzcd} A \arrow[r, "\partial_1"] \arrow[d, "\partial_0'"'] & IA \arrow[d, "e"] \arrow[ddr, bend left, "\sigma"] & \\ I'A \arrow[r, "e'"'] & I''A \arrow[ul, phantom, "\ulcorner"] & \\ & & A \end{tikzcd}$$

$\exists !\, \sigma'' : I''A \to A$

s.t. $\begin{cases} \sigma'' e = \sigma \\ \sigma'' e' = \sigma' \end{cases}$

$$\begin{tikzcd} I'A \arrow[r, "e'"] \arrow[drr, "\sigma'"'] & I''A \arrow[dr, dashed, "\exists ! \sigma''"] & \\ & & A \end{tikzcd} \qquad \text{s.t. } \sigma'' e' = \sigma'$$

$$\begin{tikzcd} A \arrow[r, "{(id, id)}"] \arrow[d, "\partial_0'"'] & A \amalg A \arrow[r, "{(\partial_0, \partial_1)}", "\in Cof"'] \arrow[d, "id_A \amalg \partial_0'"', "\in Cof"] & IA \arrow[d, "e"] \\ I'A \arrow[r] & A \amalg I'A \arrow[r, "{(e\partial_0, e')}", "\in Cof"'] \arrow[ul, phantom, "\ulcorner"] & I''A \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$\Rightarrow (e\partial_0, e'\partial_1') \in Cof$

$\Rightarrow A \amalg A \xrightarrow{(e\partial_0, e'\partial_1')} I''A \xrightarrow{\sigma''} A$ cylinder object.

Define $h'' : I''A \to X$ s.t. $h'' e = h$

$h'' e' = h'$

$\Rightarrow h'' \partial_0'' = u$

$h'' \partial_1'' = w$. $\square$

<u>Def</u> $\mathcal{C}_c \subset \mathcal{C}$ full subcat of cofib objects

$\mathcal{C}_f \subset \mathcal{C}$ — — — fib objects

$\mathcal{C}_{cf} = \mathcal{C}_c \cap \mathcal{C}_f$

<u>Th</u>

$$\begin{tikzcd} & \mathcal{C}_c[W^{-1}] \arrow[dr, "\sim"] & \\ \mathcal{C}_{cf}[W^{-1}] \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{C}[W^{-1}] \\ & \mathcal{C}_f[W^{-1}] \arrow[ur, "\sim"'] & \end{tikzcd}$$

$$\searrow^{\sim} \mathcal{C}_f[W^{-1}] \nearrow^{\sim}$$

D/ <u>Assume</u> $\exists$ <u>func. cofibrant replacement</u> Q

i.e. $\emptyset \underset{\in Cof}{\longrightarrow} QX \underset{\in Fib \cap W}{\xrightarrow{q_X}} X$

(The general case is true but harder)

Show that $\mathcal{C}_c[W^{-1}] \simeq \mathcal{C}[W^{-1}]$

$\mathcal{C}_c \xrightarrow{i} \mathcal{C}$ preserves $W \Rightarrow$ induces $\mathcal{C}_c[W^{-1}] \xrightarrow{i[W^{-1}]} \mathcal{C}[W^{-1}]$

$\Rightarrow Q : \mathcal{C} \to \mathcal{C}_c$ preserves $W \Rightarrow$ induces

$$Q[W^{-1}] : \mathcal{C}[W^{-1}] \to \mathcal{C}_c[W^{-1}].$$

$Q \circ i \xrightarrow{\in W} \mathrm{id}_{\mathcal{C}_c}$ $\quad\quad$ $i[W^{-1}] \circ Q[W^{-1}] \simeq \mathrm{id}_{\mathcal{C}_c[W^{-1}]}$

$\Rightarrow$

$i \circ Q \xrightarrow{\in W} \mathrm{id}_{\mathcal{C}}$ $\quad\quad$ $Q[W^{-1}] \circ i[W^{-1}] \simeq \mathrm{id}_{\mathcal{C}[W^{-1}]}$.

$\Rightarrow Q[W^{-1}]$ is an inverse of $i[W^{-1}]$. $\quad\square$.

<u>Def</u> $[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X) / \sim_{\text{left homotopy}}$

$\Rightarrow [-,-] : \mathcal{C}_c^{op} \times \mathcal{C}_f \to \mathrm{Set}$.

$\Rightarrow [-,-] : \underline{\mathcal{C}_c}^{op} \times \underline{\mathcal{C}_f} \to Set.$

<u>Prop</u> The functor $[-,-]$ preserves W

$\Rightarrow$ induces functor $[-,-] : \mathcal{C}_c^{op}[W^{-1}] \times \mathcal{C}_f[W^{-1}] \to Set$

P/ Let $A, B \in \mathcal{C}_c$, $A \xrightarrow{i} B \in W$, $X \in \mathcal{C}_f$

Need to show: $i^* : [B, X] \to [A, X]$ bijective.

Ken Brown's Lemma $\Rightarrow$ WMA $i \in Cof \cap W$

<u>Surjectivity</u>:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\exists g"] & * \end{tikzcd}$$

$\forall f : A \to X$

$\exists g : B \to X \quad f = gi$

$\Rightarrow i^*[g] = [f]$

<u>Injectivity</u> Let $f, g : B \to X$ s.t. $[fi] = [gi] \in [A, X]$

$\Rightarrow \exists \left| \begin{array}{l} \text{cocylinder } X^I \xrightarrow{(d^0, d^1)} X \times X \\ \text{map } A \xrightarrow{k} X^I \end{array} \right.$ s.t. $d^0 k = fi$, $d^1 k = gi$

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"'] & X^I \arrow[d, "(d^0, d^1)"] \\ B \arrow[r, "(f,g)"'] \arrow[ur, dashed, "\exists K"] & X \times X \end{tikzcd}$$

$\Rightarrow [f] = [g] \in [B, X]$

Similarly, $\forall \left| \begin{array}{l} A \in \mathcal{C}_c \\ X \xrightarrow{p} Y \in \mathcal{C}_f \cap W \end{array} \right.$ $p_* : [A, X] \xrightarrow{\sim} [A, Y]$

$\square$.

Th Let $\mathcal{C}_{cf}/\sim$ be the quotient category

Obj = cofibrant - fibrant objects in $\mathcal{C}$

$\mathrm{Hom}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)/\sim_{htp} = [A,B]$

Then $\exists$ iso of cat. $\underline{\mathcal{C}_{cf}/\sim} \simeq \underline{\mathcal{C}[W^{-1}]}$.

D/ Let $f,g : A \to X \in \mathcal{C}_{cf}$, $f \sim g$

$$\Rightarrow \exists \left| \begin{array}{l} \text{cylinder } A \amalg A \xrightarrow{(\partial_0,\partial_1)} IA \underset{\in W}{\xrightarrow{\sigma}} A \\ h : A \to X \end{array} \right. \quad \text{s.t.} \quad \begin{array}{l} h\partial_0 = f \\ h\partial_1 = g . \end{array}$$

$$\left| \begin{array}{l} \sigma\partial_0 = \mathrm{Id}_A = \sigma\partial_1 \\ \sigma \in \mathrm{Iso}(\mathcal{C}[W^{-1}]) \end{array} \right. \Rightarrow [\partial_0] = [\partial_1] \in \mathcal{C}[W^{-1}]$$

$$\Rightarrow [f] = [g] \in \mathcal{C}[W^{-1}]$$

i.e. $\forall$ htp in $\mathcal{C}_{cf}$ induces an iso in $\mathcal{C}[W^{-1}]$

But $\forall A,B \in \mathcal{C}_{cf}$ $\quad \forall u : A \to B \in W$

$\forall X \in \mathcal{C}_{cf}$ $\quad \Rightarrow u^* : [B,X] \simeq [A,X]$

$\overset{\text{Yoneda}}{\Rightarrow} [u] \in \mathrm{Iso}(\underline{\mathcal{C}_{cf}/\sim})$

i.e. $\forall \; u : A \to B \in \mathcal{C}_{cf}$

$u \in \underline{W} \iff u$ is a $\underline{\text{htp eq.}}$

$\Rightarrow \mathcal{C}_{cf}/\sim \; \simeq \mathcal{C}[W^{-1}]$. $\square$.

Cor $A \in \mathcal{C}_c$ $X \in \mathcal{C}_f$, $\exists$ iso of functors $[A, X] \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(A, X)$

$$\mathcal{C}_c^{op}[W^{-1}] \times \mathcal{C}_f^{op}[W^{-1}] \to Set$$

Def $\mathcal{C}_{cf}/\sim \; \simeq \mathcal{C}[W^{-1}]$ = the <u>homotopy category</u> of $\mathcal{C}$.

Cor $\mathcal{C}$ model cat, $\mathcal{C} \to \mathcal{C}[W^{-1}]$

- $\mathcal{C}$ locally small $\Rightarrow$ $\mathcal{C}[W^{-1}]$ locally small

- $f \in W \iff [f] \in \mathrm{Iso}(\mathcal{C}[W^{-1}])$

Derived functors

Def $\mathcal{C}$ = model cat

$\mathcal{D}$ = cat.

$F : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[dl, "LF"', shift right] \arrow[dl, bend left, "\Phi"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathcal{D} & \end{tikzcd}$$

1) A <u>left derived functor</u> of F is | functor $LF : \mathcal{C}[W^{-1}] \to \mathcal{D}$
| nat. trans $a_X : LF(\gamma(X)) \to F(X)$

s.t. $\forall \Phi : \mathcal{C}[W^{-1}] \to \mathcal{D}$

$\forall$ nat. trans $\alpha_X : \Phi(\gamma(X)) \to F(X)$

$\exists !$ nat. morphism $f_Y : \Phi(Y) \to LF(Y)$.

s.t. $\alpha_X = a_X f_{\gamma(X)}$.

(i.e. LF = <u>right Kan extension</u> of F along γ)

2) A <u>right derived functor</u> of F is $\left|\begin{array}{l} RF : \mathcal{C}[W^{-1}] \longrightarrow \mathcal{D} \\ b_x : F(x) \longrightarrow RF(\gamma(x)) \end{array}\right.$

s.t. RF^{op} = Left derived functor of $F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$.

Let $F : \mathcal{C} \to \mathcal{D}$ sends $\mathrm{Cof} \cap W$ in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{D})$

KBL $\Rightarrow$ F sends W in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{D})$

$$\Rightarrow \quad \begin{tikzcd} \mathcal{C}_c \arrow[r, "F|_{\mathcal{C}_c}"] \arrow[d] & \mathcal{D} \\ \mathcal{C}_c[W^{-1}] \arrow[ur, dashed, "\exists F_c"'] & \end{tikzcd} .$$

$$\mathcal{C}_c[W^{-1}] \underset{Q}{\overset{i}{\rightleftarrows}} \mathcal{C}[W^{-1}] \quad (\simeq)$$

<u>Prop</u> $LF = F_c \circ Q : \mathcal{C}[W^{-1}] \to \mathcal{D}$

is a left derived functor of F.

<u>Cor</u> $\forall\ G : \mathcal{D} \longrightarrow \mathcal{E}$, GLF is a left derived functor of GF.

<u>Def</u> $\mathcal{C}, \mathcal{C}'$ model cat, $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$

$$\mathcal{C}' \xrightarrow{\gamma'} \underbrace{\mathcal{C}'[W'^{-1}]}$$

- $F : \mathcal{C} \to \mathcal{C}'$ preserve $\mathrm{Cof} \cap W$

$\Rightarrow$ $\gamma' F$ sends $\mathrm{Cof} \cap W$ in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{C}'[W'^{-1}])$

$\Rightarrow$ $LF := L(\gamma' F) : \mathcal{C}[W^{-1}] \to \mathcal{C}'[W'^{-1}]$

<u>total LDF</u>

- $F : \mathcal{C} \to \mathcal{C}'$ preserve $\mathrm{Fib} \cap W$

$\Rightarrow$ $RF := R(\gamma' F) : \mathcal{C}[W^{-1}] \to \mathcal{C}'[W'^{-1}]$

<u>total RDF</u>

<u>Prop</u> $LF' \circ LF \simeq L(F' \circ F)$

$R(F' \circ F) \simeq RF' \circ RF$.

<u>Def</u> $\mathcal{C}, \mathcal{C}'$ model cat.

$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ is a <u>Quillen adjunction</u>

if - F preserve Cof

- G preserve Fib.

$\Rightarrow$ F = <u>left Quillen functor</u>

G = <u>right — —</u>

<u>Lemma</u> (F, G) Quillen adj

$\Leftrightarrow$ F preserves Cof & Cof $\cap W$

$\Leftrightarrow$ G preserves Fib & Fib $\cap W$.

<u>Th</u> A Quillen adjunction (F, G)

induces an adjunction $LF : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{C}'[W'^{-1}] : RG$.

<u>Def</u> A Quillen adjunction (F, G) is a <u>Quillen equivalence</u>

if $\forall X \in \mathcal{C}_c$, $Y \in \mathcal{C}'_f$ $\quad f : FX \to Y \in W' \Leftrightarrow \varphi(f) : X \to GY \in W$

<u>Th</u> (F, G) Quillen eq $\Leftrightarrow$ (LF, RG) is an adjoint equivalence

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<u>Examples</u>

<u>Top. spaces</u>

<u>Th</u> $\mathcal{C} = \mathrm{Top}$. $\exists$ model str.

W = weak htp. eq. (i.e. $f : X \to Y$ s.t. $\forall n$, $\pi_n(f)$ iso)

Cof $= L(R(S^{n-1} \hookrightarrow D^n))$ = retracts of relative cell complexes

$$\mathrm{Cof} = L\left(R\left(\begin{array}{c} S^{n-1} \\ \downarrow \\ D^n \end{array}\right)\right) = \text{retracts of relative cell complexes}$$

$$\mathrm{Fib} = R\left(\begin{array}{cc} D^n & x \\ \downarrow & \downarrow \\ D^n \times [0,1] & (x,0 \end{array}\right) = \text{Serre fibrations.}$$

$\rightsquigarrow \mathrm{Ho}(\mathrm{Top})$.

<u>Chain complexes</u> R = ring $\mathcal{C} = C_\bullet(R)$

(unbounded) chain complexes

W = quasi-iso.

<u>Th</u> $\exists$ model structures on $C_\bullet(R)$

1) $\mathrm{Fib} = \{\text{degree-wise epimorphisms}\}$

$\mathrm{Cof} = L(\mathrm{Fib} \cap W)$

<u>(projective model structure)</u>

- $\mathcal{C}_f$ = all.

$C(\text{projectives})$

$\cup$

- $\mathcal{C}_c$ = "dg-projective complexes"

$= \{D \mid \forall A \text{ acyclic}, \ \underline{\mathrm{Hom}}(D, A) \text{ acyclic}\}$.

$\cup$

$C^-(\text{projectives})$

- X projective $\Leftrightarrow$ $\begin{cases} X \in \mathcal{C}_c \\ X \text{ acyclic}. \end{cases}$

- $\mathrm{Cof} = \{$degreewise split inclusion with cofibrant cokernel$\}$

  $\mathrm{Cof} \cap W \subset$ injection with projective cokernel.

2) $\mathrm{Cof} = \{$degreewise epimorphisms$\}$

   $\mathrm{Fib} = \{$surjections with fibrant kernel$\}$.

(injective model structure)

- $\mathcal{C}_c = $ all.
- $C^+(\text{injective}) \subset \mathcal{C}_f \subset C(\text{injectives})$

- $\mathrm{Fib} \cap W = $ surjection with injective kernel.

- X inj $\Leftrightarrow$ $\begin{cases} X \in \mathcal{C}_f \\ X \text{ acyclic}. \end{cases}$

Properties - $C(R)^{proj} \xleftrightarrow{id} C(R)^{inj}$ Quillen eq.

- $f : R \to R'$ $\quad C(R)^{proj} \rightleftarrows C(R')^{proj}$

  $\exists$ Quillen adj.

  $X \longmapsto X \otimes_R R'$

$$Y \longleftarrow Y$$

Quillen eq $\Leftrightarrow$ f iso.

$$C(R)^{inj} \rightleftarrows C(R')^{inj}$$

Quillen adj $\Leftrightarrow$ R' flat over R.

- $M, N \in R\text{-Mod}$ $\quad [M[n], N] = \mathrm{Ext}^n(M,N)$.

<u>Th</u> $\mathcal{A}$ = Grothendieck ab. cat.

$\mathcal{C} = C.(\mathcal{A})$ $\quad W = $ q-iso. $\quad \exists$ model structures

1) Cof = {mono} $\quad$ (<u>injective model structure</u>)

2) $\mathcal{T} \in \mathrm{Top}$ $\quad \mathcal{A} = \mathrm{Sh}(\mathcal{T}, \Lambda)$

Grothendieck top $\qquad \Lambda \in$ Rings.

$\mathcal{G}$ = generating family of $\mathcal{T}$.

$$\mathcal{C}_f = \{ K \mid \forall U \in \mathcal{G},\ \forall n,\ H^n(\Gamma(U,K)) \simeq \mathcal{H}^n(U,K) \}$$

hypercohomology

$\mathrm{Fib} = \{ p : K \to L \mid$ - $\forall U \in \mathcal{G}$, $\Gamma(U, K) \xrightarrow{p_\#} \Gamma(U, L)$

degreewise surj with fibrant kernel $\}$

$\mathrm{Fib} \cap W = \{$ - —— —— + $\forall U \in \mathcal{G}$,

$\Gamma(U, \ker p)$ acyclic $\}$

s.t. $\forall U \in \mathcal{G}$, $\Lambda(U)$ is fibrant. (projective model str.)

Simplicial sets