(P1) k is a field.

Main Thm $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$, f poly. function

$f(0)=0$, 0 is isolated zero. then

→ EKL class

$\deg_0^{\mathbb{A}^1}(f) = W_0(f)$ in $GW(k)$.

Morel's Thm $[\mathbb{1}_k, \mathbb{1}_k]_{SH(k)} \simeq GW(k)$.

Goal of Today:

1, define $W_0(f)$ → purely algebraic.

easy computation of $W(f)$.

if f has a simple zero at 0.

follow E-L's construction.

2, define $\deg_0^{\mathbb{A}^1} f$. → idea of Hoyois

$\begin{cases} \text{global formula: in terms of local degrees} \\ \text{local computation.} \end{cases}$

$\deg f \overset{?}{=} \mathrm{Jac}(f_0)$

P2

In Topology. $f: \mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$.

$$\deg f_y = \sum_{x \in f^{-1}(y)} \operatorname{sign}\left(\operatorname{Jac}(f_x)\right).$$

---

k: field. $\quad \mathbb{P}^n_k = \operatorname{Proj} k[X_0, X_1, \cdots, X_n]$.

$P_A = k[x_1, \cdots, x_n]$. $\quad m_0 = (x_1, \cdots, x_n)$

$\rightsquigarrow P_y$, $P_m \to$ ideal.

$f: \mathbb{A}^n_k \to \mathbb{A}^n_k$, $(x_1, \cdots, x_n) \longmapsto (f_1, f_2, \cdots, f_n)$.

$\forall i$, $f_i \in P_A$.

Assume f has an isolated zero at x

x is a closed pt of $\mathbb{A}^n_k$.

($x = 0$, most cases)

$Q_x(f)$, $x = 0$, $Q_0(f) = Q$.

$$:= P_{m_x} / (f_1, f_2, \cdots, f_n)$$

$\mathbb{A}^n_k \xhookrightarrow{\text{open}} \mathbb{P}^n_k \longleftarrow \mathbb{P}^{n+1}_k$.

P3

$Q_0(f) \neq Q$ is Artinian Local ring.

Socle of $Q_0(f)$ is 1-dim'l

Lemma. (Schja-Storch 1975)

$$\mathrm{Socle}(Q_0(f)) = (E)$$

$E \to$ distinguished socle element.

$E := \det(a_{ij})$, $a_{ij} \in P_x$.

$\neq 0$

$$f_i(x) = f_i(0) + \sum_{j=1}^{n} a_{ij} x_j$$

$$E \in \mathrm{ann}(m)$$

Def (EKL) $\phi : Q \to k$ is a k-linear fcn.

$Q = Q_0(f)$

define. $\beta_\phi : Q \times Q \to k$

$$\beta_\phi(a_1, a_2) = \phi(a_1 a_2)$$

If $\phi(E) \neq 0$, then β_ϕ is non-degenerate.

Lemma: ϕ_1, ϕ_2 k-linear s.t. $\phi_1(E) = \phi_2(E)$ in $k/(k^*)^2$.

then $\beta_{\phi_1} \simeq \beta_{\phi_2}$

(P4)

Def (EKL class)

$w = \langle \beta_\phi \rangle \in GW(k)$

β_ϕ is the sym. bil. forms asso. to ϕ s.t. $\phi(E) = 1$.

Rmk: β_ϕ is indep. of choice of ϕ.

Lemma: If f has a simple zero at 0. then

$$w_0(f) = \langle \underbrace{\det \frac{\partial f_i}{\partial x_j}(0)}_{= J} \rangle$$

Morel's $\mathbb{A}^1$-Brouwer degree.

Thm (Morel) $[(\mathbb{P}^1_k)^{\wedge n}, (\mathbb{P}^1_k)^{\wedge n}]_{SH(k)} \xrightarrow{\simeq} GW(k)$.

note that $(\mathbb{P}^1_k)^{\wedge n} \underset{\mathbb{A}^1\text{-w.e.}}{\simeq} \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \simeq Th(\mathcal{O}^n_{\mathrm{Spec}\, k})$

Th: Thom space.

P5

Prop (global formula in terms of local degrees)

$f: \mathbb{P}^n_k \to \mathbb{P}^n_k$ finite. with $f^{-1}(\mathbb{A}^n_k) = \mathbb{A}^n_k$.

$\overset{\text{induce}}{\rightsquigarrow} \bar{f} : \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \longrightarrow \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \rightsquigarrow$ pass to $SH(k)$.

$\rightsquigarrow \deg^{\mathbb{A}^1}(\bar{f}) \in GW(k)$ ✓.

Then we have, $\forall y \in \mathbb{A}^n_k$, y is a k-pt.

$$\deg^{\mathbb{A}^1}(\bar{f}) = \sum_{x \in f^{-1}(y)} \deg^{\mathbb{A}^1}_x(f)$$

Q: What is $\deg^{\mathbb{A}^1}_x(f)$?

$f: \mathbb{A}^n_k \to \mathbb{A}^n_k$

x closed pt of $\mathbb{A}^n_k$, $y = f(x)$ is a k-pt.

Goal: find $f_x : \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k / \mathbb{P}^{n-1}_k$.

P6 Choose U open, $x \in U$.

$$f : U - \{x\} \to \mathbb{A}^n_k - \{f(x)\}, \quad f(x) = y$$

$$\begin{tikzcd}
U - \{x\} \arrow[r] \arrow[d] & \mathbb{P}^n_k - \{x\} \arrow[d] \\
U \arrow[r] & \mathbb{P}^n_k
\end{tikzcd}$$

homotopy $\rightsquigarrow$ equivalence

$$\begin{tikzcd}
U/U-\{x\} \arrow[r, "h.e."] \arrow[d] & \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \arrow[r, equal, "\overset{\text{Purity}}{\sim}"] & \mathrm{Th}(\mathcal{O}^n_{k(x)}) \\
 & & \mathbb{P}^n_{k(x)}/\mathbb{P}^{n-1}_{k(x)} \arrow[u, "\cong"']
\end{tikzcd}$$

$k(x)$: res. field of x.

$$f_x : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \xrightarrow{\text{collapse map}} \mathbb{P}^n_k/\mathbb{P}^n_k - \{x\} \overset{\sim}{=} U/U-\{x\} \xrightarrow{f|_U} \mathbb{P}^n_k/\mathbb{P}^n_k - \{y\} \overset{\sim}{=} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$$

$$\mathbb{P}^n_k/\mathbb{P}^n_k - \{y\} \xrightarrow{\mathbb{A}^1\text{-h.e.}} \mathrm{Th}(\mathcal{O}^n_{k(y)})$$

y is a k-pt. $k(y) = k$.

In case x is a k-pt.

$$\deg^{\mathbb{A}^1}_x(f) = ?$$

P7

denote $\mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \simeq \mathrm{Th}\, \mathcal{O}^n_{k(x)} \cong \mathbb{P}^n_{k(x)} / \mathbb{P}^{n-1}_{k(x)}$

$$\begin{tikzcd} \mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \arrow[r, bend right, "\gamma"'] & \mathbb{P}^n_{k(x)} / \mathbb{P}^{n-1}_{k(x)} \end{tikzcd}$$

Lemma: (Hoyois, K-W) $\quad x \in \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$.

x is a k-pt. Then C_x is $\mathbb{A}^1$-homotopy-equiv. to identity.

$$C_x : \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \xrightarrow[\text{map}]{\text{collapse}} \mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \xrightarrow[\gamma]{\simeq} \mathbb{P}^n_k / \mathbb{P}^{n-1}_k.$$

Proof: "reduce to the case $x = 0 \in \mathbb{A}^n_k$".

denote x by $[1, a_1, \dots, a_n]$.

Construct $g : \mathbb{P}^n_k \to \mathbb{P}^n_k$.

$$g([x_0, x_1, \dots, x_n]) = [x_0, x_1 + a_1 x_0, \dots, x_n + a_n x_0].$$

$\rightsquigarrow$ comm. diag.

$$\begin{tikzcd} \mathbb{P}^n_k / \mathbb{P}^n_k - \{0\} \arrow[r, "\gamma", "\cong"'] \arrow[d, "g"] & \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \arrow[d, "1"] \\ \mathbb{P}^n_k / \mathbb{P}^n_k - \{x\} \arrow[r, "\gamma", "\cong"'] & \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \end{tikzcd}$$

P8

$$\begin{array}{ccc}
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k & \xrightarrow[\text{map}]{\text{collapse}} & \mathbb{P}^n_k/\mathbb{P}^n_k-\{0\} \\
\downarrow \bar{g} & & \downarrow g \\
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k & \xrightarrow{\text{c.m.}} & \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\}
\end{array}$$

$\bar{g}$ & $\mathrm{Id}_{\mathbb{P}^n_k}$ has a naive homotopy.

$$[x_0,\dots,x_n]\times t \to [x_0, x_1+a_1 t x_0, \dots, x_n+a_n t x_0],$$

$t=0,1$

Prop: x is a k-pt. We have

$$\deg^{\mathbb{A}^1}_x f := \deg f_x = \deg f'_x.$$

where $f'_x: \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq U/U-\{x\} \xrightarrow{f|_U} \mathbb{A}^n_k/\mathbb{A}^n_k-\{y\} \simeq \mathrm{Th}\,\mathcal{O}^n_k$

$\mathrm{Th}\,\mathcal{O}^n_k \cong \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$

Pf: $f'_x c_x = c_{f(x)} f_x$

$k$pt

(P9)

Pf of global degree formula.

$$\begin{tikzcd}
 & \bigvee_{x\in f^{-1}(y)} \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \arrow[d, phantom, "\cong"] \arrow[dr] & \\
\mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \arrow[ur, "f_x"] \arrow[drr, bend right] & \mathbb{P}^n_k/\mathbb{P}^n_k-f^{-1}(y) \arrow[l] \arrow[r] \arrow[u, "\mathrm{Th}"'] & \mathbb{P}^n_k/\mathbb{P}^n_k-\{y\} \arrow[d, phantom, "\cong"] \\
 & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[u] \arrow[ul, "\cong", "x \text{ is a } k\text{-pt}"'] \arrow[r, "\bar{f}"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k
\end{tikzcd}$$

Apply $[\mathbb{P}^n_k/\mathbb{P}^n_{k-1}, -]_{SH(k)}$ to above diagram.

$$\bar{f}^* : \mathrm{End}(\mathbb{1}_k) \longrightarrow \mathrm{End}(\mathbb{1}_k).$$

$$\mathrm{End}(\mathbb{1}_k) = [\mathbb{1}_k, \mathbb{1}_k]_{SH(k)}$$

$$1 \longmapsto \deg \bar{f}.$$

$$\longmapsto \sum_{x\in f^{-1}(y)} \deg^{\mathbb{A}^1}_x(f).$$

Hoyois: A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula.

$p : V \xrightarrow[\text{v.b.}]{} \underline{X}$, $X \in \underline{\mathrm{Sch}_k}$.

$\phi : V \to V$ linear automorphism.

$s : X \to V$ zero section.

Thom transformation

$\Sigma^V := p_\# s_* : SH(X) \rightleftarrows SH(X) : s^! p^* =: \Sigma^{-V}$

$f : X \to B$ smooth, B : Base scheme.

$\beta_\phi = \underline{\omega} \in \mathrm{End}(\mathbb{1}_X) := [\mathbb{1}_X, \mathbb{1}_X]_{SH(X)}$

Define $\int_X \omega \, dx = \mathrm{tr}(f_\# \omega)$

$: \mathrm{End}(\mathbb{1}_X) \to \mathrm{End}(\mathbb{1}_B)$

<u>Thm</u> $k \subset L$ finite sep. field ext. $\mathrm{End}(\mathbb{1}_L) = GW(L)$

V/L $\dim V < +\infty$. $\phi : V \to V \in \mathrm{Aut}(V)$

$\omega := \beta\phi$. Then

$$\int_L \frac{\langle\phi\rangle}{\langle\beta\phi\rangle} dx = \mathrm{Tr}_{L|k} \langle \det(\phi) \rangle$$

$X = \mathrm{Spec}\, k(x)$

<u>Lemma:</u> $p : X \to B$ finite étale. $B = \mathrm{Spec}\, k$.

$\omega \in \mathrm{End}(\mathbb{1}_X)$. Then $\int_X \omega\, dx \in \mathrm{End}(\mathbb{1}_B)$

~~is identified with~~

$$\mathbb{1}_B \xrightarrow{\eta} p_* \mathbb{1}_X \simeq p_\# \mathbb{1}_X \xrightarrow{p_\#(\omega)} p_\# \mathbb{1}_X \xrightarrow{\varepsilon} \mathbb{1}_B .$$

$\eta : \mathbb{1}_k \to p_* p^* \mathbb{1}_k \simeq p_* \mathbb{1}_{k(x)}$

$\varepsilon : p_! p^! \to 1$

$E \xrightarrow{Vb} \mathrm{Spec}\, k.$

$$\underline{Th_p \mathbb{1}_{p^*E}} : p_! \Sigma^{p^*E} p^! \simeq \Sigma^E p_! p^! \xrightarrow{\varepsilon} \Sigma^E$$

Prop: $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$. x closed pt.

$f(x) = y$ y k-pt. isolated, x in $f^{-1}(y)$.

If f is étale at x, then

$$\deg^{\mathbb{A}^1}_x f = \mathrm{Tr}_{k(x)|k} \langle J(x) \rangle$$

idea: describe f_x as in the lemma above.

where ϕ is Jacobian. (given by.)

Proof: $df(x): T_x \mathbb{A}^n_k \longrightarrow f^* T_{f(x)} \mathbb{A}^n_k$.

$\rightsquigarrow$ a map on Thom space

$$\begin{array}{ccccc}
U/U-\{x\} & & \xrightarrow{\quad f|_U \quad} & & \mathbb{P}^n_k/\mathbb{P}^n_k-\{y\} \\
\cong\downarrow & & & & \uparrow\simeq \\
\mathrm{Th}T_xU \cong \mathrm{Th}\,T_x \underset{U}{\mathbb{A}^n_k} & & \xrightarrow{\mathrm{Th}_f(df(x))} & & \mathrm{Th}\,T_{f(x)}\mathbb{A}^n_k \quad {=}y \text{ is a } k\text{-pt.} \\
\updownarrow & & & & \updownarrow \wr\!\wr \\
\mathrm{Th}\,\mathcal{O}^n_{k(x)} & \xrightarrow{\mathrm{Th}_{k(x)}\left(\frac{\partial f_i}{\partial x_j}\right)} & \mathrm{Th}\,\mathcal{O}^n_{k(x)} & \xrightarrow{\mathrm{Th}_p \mathbb{1}_{p^*\mathcal{O}^n_k}} & \mathrm{Th}\,\mathcal{O}^n_k
\end{array}$$

$$
\begin{tikzcd}
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[r] \arrow[rrrrr, bend left, "f_x"] & \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \arrow[r, "\cong"] & U/U-\{x\} \arrow[r, "f|_U"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{P}^n_k/\mathbb{P}^n_k-\{y\} \arrow[r, "\cong"] \arrow[ddr, "\cong"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \\
 & & \mathrm{Th}\,T_xU \arrow[d, phantom, "\cong" rotate=90] & & \\
 & & \mathrm{Th}\,\mathcal{O}^n_{k(x)} \arrow[r, "\mathrm{Th}_{k(x)}\left(\frac{\partial f_i}{\partial x_j}\right)"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Th}\,\mathcal{O}^n_{k(x)} \arrow[r, "\mathrm{Th}\,p_\# p^* \mathcal{O}^n_k"] \arrow[r, "\varepsilon"'] & \mathrm{Th}\,\mathcal{O}^n_k \arrow[d, "\cong"] \\
 & & \mathbb{P}^n_{k(x)}/\mathbb{P}^{n-1}_{k(x)} & & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k
\end{tikzcd}
$$

$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge \gamma$

$W = \langle k, p \rangle$

Lemma
(Hoyois, $k=W$)

satisfy Lemma of Hoyois.

$$
\begin{cases}
\deg^{\mathbb{A}^1}(f) = \sum_{x \in f^{-1}(y)} \deg^{\mathbb{A}^1}_x(f) & \text{— degree side} \\
\deg^{\mathbb{A}^1}_x(f) = \mathrm{Tr}_{k(x)/k}\left(\langle J(x) \rangle\right) = w_x(f)
\end{cases}
$$

— EKL side.