Grothendieck - Witt ring. / Witt ring

Plan:

- For field F, $\operatorname{char} F \neq 2$, define $GW(F)$, $W(F)$.
- For $F \hookrightarrow K$ finite field extension, $r^*: W(F) \to W(K)$.
  for $K \xrightarrow{s} F$ nonzero F-linear functional, $s_*: W(K) \to W(F)$.
  (in particular, K/F separable, choose $s = \mathrm{tr}$.)

Q: When r^* injective? surjective?

- Find a non-rational, unirational complex threefold.
  conic bundle / $\mathbb{P}^2$

Let F be a field, $\operatorname{char} F \neq 2$.

Let $M(F) := \{$ nonsingular quadratic form (V, B). $/F\}$ / isometry.

2 operations on $M(F)$:

$$V_1 \perp V_2 := (V_1 \oplus V_2, B_1 + B_2).$$
$$V_1 \otimes V_2 := (V_1 \otimes V_2, B_1 \cdot B_2).$$

$(M(F), \perp)$ is commutative, cancellation, monoid, but not a group!

Let $GW(F) := M(F) \times M(F) / \sim$, $(V, U) \sim (V', U')$
$\Leftrightarrow V \perp U' = U \perp V' \in M(F)$.

$\Rightarrow (GW(F), \perp)$ is an abelian group.
$(GW(F), \perp, \otimes)$ is a commutative ring, called the Grothendieck-Witt ring.

Define $W(F) := GW(F) / \mathbb{Z} \cdot H$, $H = \langle 1, -1 \rangle_F$.

Example:
Let A be a finite dim. F-alg, $\mathrm{tr}_{A/F}: A \to F$ trace map.
Let $\mathrm{tr}_A : A \times A \to F$, $(x, y) \mapsto \mathrm{tr}_{A/F}(x \cdot y)$.
then (A, tr_A) is a quadratic form.
In particular, Let K/F finite separable extension. then (K, tr_K) is non-degenerate.

Rmk: $M(F) \xrightarrow{i} GW(F)$ has universal property:

$$\begin{tikzcd} M(F) \arrow[r, "i"] \arrow[dr, "f"'] & GW(F) \arrow[d, dashed, "\exists!"] \\ & G \end{tikzcd}$$

G: abelian gp., f: morphism of monoid.

Let k/F be a finite field extension.

Define $r: \mathcal{M}(F) \to \mathcal{M}(k) \subset GW(k)$.
$(V, q) \mapsto (V_k, q_k)$, $V_k = V \otimes_F k$.
$q_k(v \otimes k) = k^2 \cdot q(v)$.

$\Rightarrow \hat{r}^*: GW(F) \to GW(k)$.
$\Rightarrow r^*: W(F) \to W(k)$, b/c $r^*\langle 1, -1\rangle_F = \langle 1, -1\rangle_k$.

Rmk: $\hat{r}^*$, r^* may not be injective in general!
For example, consider $\mathbb{C}/\mathbb{R}$, then.
$\hat{r}^*: GW(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow GW(\mathbb{C}) = \mathbb{Z}$.
$r^*: \underbrace{W(\mathbb{R}) = \mathbb{Z} \longrightarrow W(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}}$.

However, we have
Thm: A) $r: F^*/F^{*2} \to k^*/k^{*2}$ injective.
B) $r^*: W(F) \to W(k)$ injective.
C). $r^*(\text{anistropic form}) = \text{anisotropic form}$
D). $[k:F]$ is odd.

(V, q). quadratic form. $\to \nexists\, v \neq 0 \in V$, $q(v) = 0$.

Then $D \Rightarrow C \Rightarrow B \Rightarrow A$, but $A \not\Rightarrow B$, $B \not\Rightarrow C$, $A+B+C \not\Rightarrow D$.

Pf: C) $\Rightarrow$ B) : $W(F) \Leftrightarrow \{\text{antisotropic form}/F\}$ as sets.
B) $\Rightarrow$ A) : Suppose r is not injective,
then $\exists\ a \in F^* \cap k^{*2} - F^{*2}$
$\Rightarrow \langle 1, -a\rangle_F$ is antisotropic in $W(F)$.
but $\langle 1, -a\rangle_k = \langle 1, -1\rangle_k = 0 \in W(k)$.

Scharlau's transfer map.
Let $s: k \to F$ be a nonzero F-linear function.
For any $(U, B) \in \mathcal{M}(k)$, define
$sB: U \times U \xrightarrow{B} k \xrightarrow{s} F$.
then $(U, sB) \in \mathcal{M}(F)$.
$\Rightarrow s_*: GW(k) \to GW(F)$ is a group homomorphism
b/c $s_*(U_1 \perp U_2) = s_*U_1 \perp s_*U_2$.

Prop: $s_*((\hat{r}^*V) \otimes_k U) \simeq V \otimes_F s_*U$.

Cor: $s_*H_k = s_*((\hat{r}^*H_F) \otimes_k \langle 1\rangle_k) \simeq \underset{\overset{\shortparallel !}{\langle 1, -1\rangle_F}}{H_F} \otimes_F s_*\langle 1\rangle_k \simeq \dim_F k \cdot H_F$.
$\Rightarrow s_*: W(k) \to W(F)$ is a group homomorphism.

Let k/F be a finite Galois extension with $\mathrm{Gal}(k/F) =: G$.
G acts on $GW(k)$: $GW(k) \times G \to GW(k)$.

$$((V,q),\sigma) \mapsto (V^\sigma, q^\sigma).$$

$$k \times V^\sigma \to V^\sigma,\ (k,v) \mapsto k * v = \sigma(k)\cdot v$$

$$q^\sigma = \sigma^{-1} \circ q.$$

Thm: For any k-quadratic form (V,q).

$$k \otimes_F tr_*(q) \simeq \perp_{\sigma \in G} q^\sigma.$$

Define: Define

$$f: k \otimes_F V \to \perp_{\sigma \in G} V^\sigma$$

$$k \otimes v \mapsto \sum_{\sigma \in G} k * v.$$

Check f is an K-isometry.

Thm: Assume $[k:F] = 2m+1$, then

$$r^*: W(F) \xrightarrow{\sim} W(k)^G.$$

Proof: k/F Galois $\Rightarrow$ separable $\Rightarrow$ $k = F(x)$ for some $x \in k$.
Define $s: k \to F$, $s(1) = 1$, $s(x) = s(x^2) = \cdots = s(x^{2m}) = 0$.
$tr = tr_k$ nonsingular $\Rightarrow \exists\, a \in k^*$, $s(y) = tr(a\cdot y)$, $\forall y \in k$.
For any $(V,q)/k$, $s_*(q) = tr_*(a\cdot q)$.
If $q \in W(k)^G$, then

$$r^* s_*(q) = k \otimes_F tr_*(a\cdot q) = \perp_{\sigma \in G} (a\cdot q)^\sigma = (\perp_{\sigma \in G} \langle a\rangle^\sigma)\cdot q$$

Let $q = \langle 1\rangle_k$, then,

$$q = r^* s_* \langle 1\rangle_k \underset{\uparrow}{=} r^* \langle 1\rangle_F = \langle 1\rangle_k.$$

$[k:F] =$ odd, $\Rightarrow s_*\langle 1\rangle_k = \langle 1\rangle_F \perp n H_F$.

$\Rightarrow r^* s_* = id_{W(k)^G}$. $\Rightarrow r^*$ surjective.
$[k:F]$ odd $\Rightarrow r^*$ injective.
$\Rightarrow r^*$ isom.

Let K/k be a field extension, we say K is unirational $/k$ if

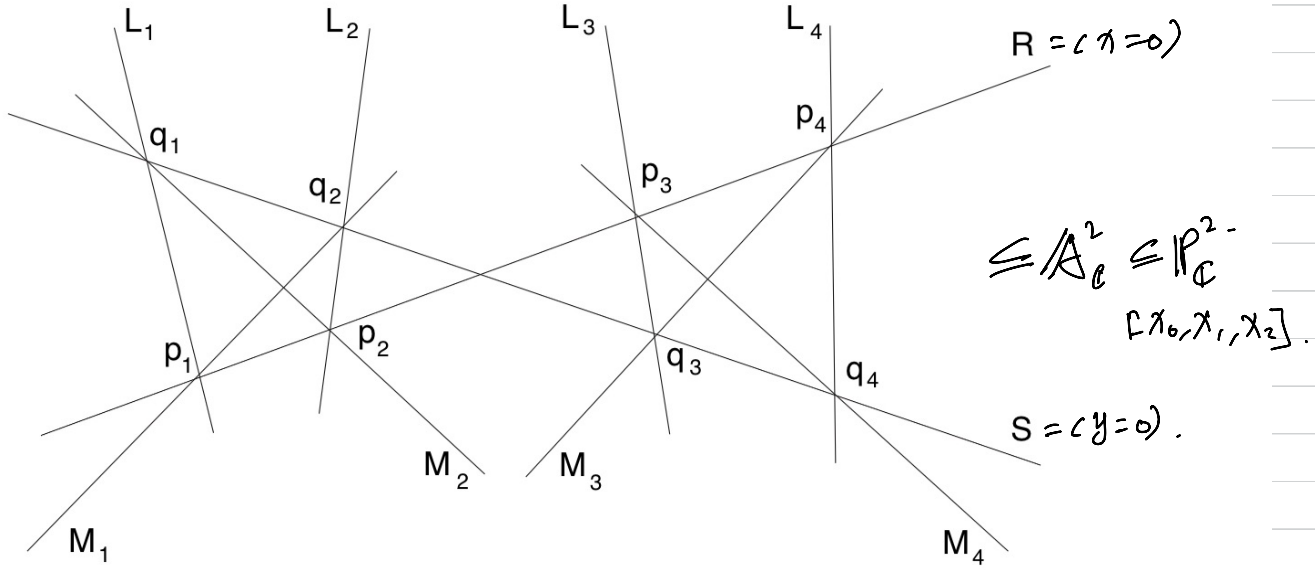
$$k \subset K \subset k(t_1, \cdots, t_n).$$

Lemma: WLOG, we can alway chose $n = \mathrm{tr.deg}\, K/k$.

Q: Let K be unirational $/k$, is K rational $/k$?

$n=1$. ✓ Lüroth's theorem
$n=2$ ✓ Castelnuovo's theorem. (char $=0$ or separable unirational).
$n=3$ ✗. $\exists$ non-rational unirational complex threefold, Ojanguren.

conic bundle $/ \mathbb{P}^2$.



$x = X_1/X_0,\ y = X_2/X_0.$

$g_1 = l_1 \cdot l_2 \cdot m_1 \cdot m_2,\quad g_2 = l_3 \cdot l_4 \cdot m_3 \cdot m_4.$

$f = x \cdot y.$

$K = \mathbb{C}(x, y).$, L = function field of $C = (X_0^2 - f X_1^2 - g_1 g_2 X_2^2 = 0)$.

L is unirational / $\mathbb{C}$:

$$\mathbb{C} \subseteq L \subseteq L(\sqrt{f}) = K(\sqrt{f})(C).$$

C has a rational point P / $K(\sqrt{f})$ $\Longrightarrow$ C is rational / $K(\sqrt{f})$.

$C - P \xrightarrow{\sim} \mathbb{P}^1$

$$\Longrightarrow K(\sqrt{f})(C) = K(\sqrt{f})(t) = \mathbb{C}(x, y, \sqrt{xy}, t) = \mathbb{C}(x, \sqrt{xy}, t) \simeq \mathbb{C}(t_1, t_2, t_3).$$

L is not rational / $\mathbb{C}$:

First, introduce unramified Witt group $W_{nr}(K)$., K/k field ext.

Let $v \in \mathcal{V}_K = \{$discrete valuation of rank 1 of K, trivial on $k\}$.

$\mathcal{O}_v = \{a \in K \mid v(a) \geq 0\}$. DVR.

$\mathfrak{B}_v = \{a \in K \mid v(a) > 0\}$. maximal ideal.

$k(v) = \mathcal{O}_v / \mathfrak{B}_v$. residue field.

$\rightsquigarrow \partial_v : W(K) \to W(k(v)).$

Define $W_{nr}(K) = \bigcap_{v \in \mathcal{V}_K} \ker \partial_v$.

Fact:
- $\forall n$, $W_{nr}(k(t_1, \cdots, t_n)) = W(k)$.
- $\forall m > 0$, $I^m_{nr}(K) = I(K)^m \cap W_{nr}(K)$, $I(K) = \ker(W(K) \xrightarrow{rk} \mathbb{Z}/2)$

To prove L not rational, we want to show $I^2_{nr}(L) \neq 0$.

Let $\alpha_1 = \langle 1, -f \rangle \langle 1, -g_1 \rangle$, $\alpha_2 = \langle 1, -f \rangle \langle 1, -g_2 \rangle$.

then
- $\mathrm{Ram}\, \alpha_1 = \{ v \in \mathcal{V}_k \mid \partial_v(\alpha_1) \neq 0 \} \neq \phi$.
  for $v \in \mathrm{Ram}\, \alpha_1$, $C_v = L_1, L_2, M_1, M_2, p_1, p_2, q_1, q_2$.
- $\mathrm{Ram}\, \alpha_2 \neq \phi$.
  for $v \in \mathrm{Ram}\, \alpha_2$, $C_v = L_3, L_4, M_3, M_4, p_3, p_4, q_3, q_4$.

$\Longrightarrow$. $\mathrm{Ram}\, \alpha_1 \cap \mathrm{Ram}\, \alpha_2 = \phi \Longrightarrow \alpha_1 - \alpha_2 \in I^2_{nr}(L)$.

Then, only need to show $\alpha_1 - \alpha_2 \neq 0 \in W_{nr}(L)$.

$\Longrightarrow$ L is not rational $/ k$.