STAT 542 Homework 6

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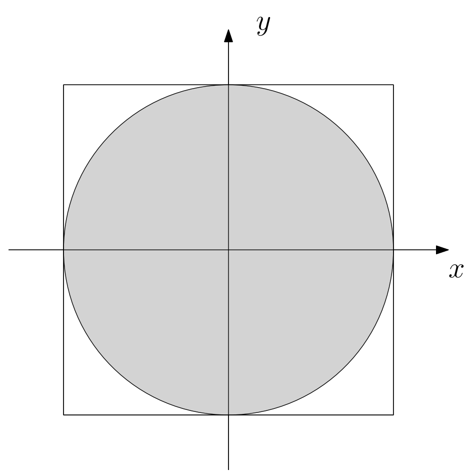
1. (a) When θ is the location parameter, denote $F(x)$ is the cdf with $\theta = 0$, i.e. $F(x) = F(x|0)$. Then $F(x|\theta) = F(x - \theta)$. cdf is non-decreasing, thus for $\theta_1 > \theta_2$ and any $x \in \mathbb{R}$, we have $x - \theta_1 < x - \theta_2$ and $F(x - \theta_1) \leq F(x - \theta_2)$. Hence, $F(x|\theta_1) \leq F(x|\theta_2)$, a location family is stochastically ordered in terms of location family.

(b) When θ is the scale parameter, denote $F(x)$ is the cdf with $\theta = 0$, i.e. $F(x) = F(x|0)$. Then $F(x|\theta) = F(x/\theta)$. cdf is non-decreasing, thus for $\theta_1 > \theta_2 > 0$ and any $x \in [0, \infty)$, we have $x/\theta_1 < x - \theta_2$ and $F(x/\theta_1) \leq F(x/\theta_2)$. For $x < 0$, we have $F(x) = 0$, thus $F(x/\theta_1) = F(x/\theta_2) = 0$. Hence, $F(x|\theta_1) \leq F(x|\theta_2)$, $x \in \mathbb{R}$, a scale family is stochastically ordered in terms of location family.

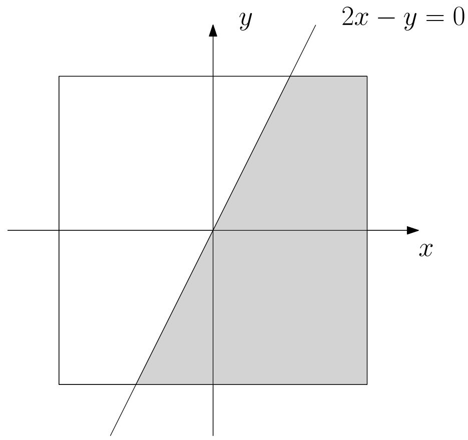
2. (a) Let $D = \{(x, y)|x^2 + y^2 < 1\} \cap (-1, 1) \times (-1, 1) = \{(x, y)|x^2 + y^2 < 1\}$. Then

$$P(X^2 + Y^2 < 1) = \iint_D \frac{1}{4} \mathrm{d}x\mathrm{d}y = \frac{\pi}{4}$$



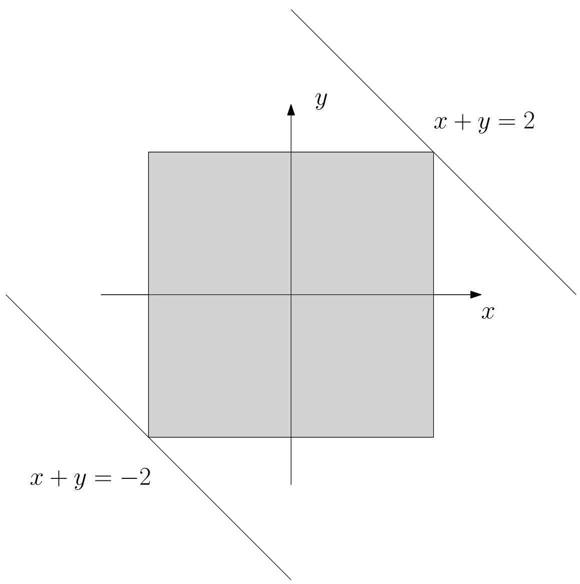
(b) Let $D = \{(x, y)|2x - y > 0\} \cap (-1, 1) \times (-1, 1)$. Then by symmetric

$$P(2X - Y > 0) = \iint_D \frac{1}{4} \mathrm{d}x\mathrm{d}y = \frac{1}{2}$$



Let $D = \{(x,y)|x+y<2\} \cap \{(x,y)|x+y>-2\} \cap (-1,1)\times(-1,1) = (-1,1)\times(-1,1)$. Then

$$P(|X+Y|<2) = \iint_D \frac{1}{4}\mathrm{d}x\mathrm{d}y = 1$$



3. (a)

$$\begin{aligned}
&\int_0^1 \int_0^2 C(x+2y)\mathrm{d}x\mathrm{d}y \\
=&C\left(\frac{1}{2}x^2 + 2yx\Big|_0^2\right)\mathrm{d}y \\
=&C\,(2+4y)\,\mathrm{d}y \\
=&C\left(2y+2y^2\big|_0^1\right) \\
=&4C = 1 \qquad \Rightarrow C = \frac{1}{4}
\end{aligned}$$

(b)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{1}{4}(x+2y)\mathrm{d}y \\ &= \frac{1}{4}\left(xy + y^2\Big|_0^1\right) \\ &= \frac{1}{4}(x+1) \end{aligned}$$

(c) For $(x,y) \in \{(x,y)|x \le 0\} \cup \{(x,y)|y \le 0\}$, $F(x,y) = 0$.
For $(x,y) \in \{(x,y)|x \ge 2\} \cup \{(x,y)|y \ge 1\}$, $F(x,y) = 1$.
For $(x,y) \in \{(x,y)|0 < x < 2\} \cap \{(x,y)|0 < y < 1\}$, $F(x,y) = \int_0^y \int_0^x \frac{1}{4}(s+2t)\mathrm{d}s\mathrm{d}t = \frac{1}{8}x^2y + \frac{1}{4}xy^2$.
For $(x,y) \in \{(x,y)|0 < x < 2\} \cap \{(x,y)|y \ge 1\}$, $F(x,y) = \int_0^1 \int_0^x \frac{1}{4}(s+2t)\mathrm{d}s\mathrm{d}t = \frac{1}{8}x^2 + \frac{1}{4}x$.
For $(x,y) \in \{(x,y)|x \ge 2\} \cap \{(x,y)|0 < y < 1\}$, $F(x,y) = \int_0^y \int_0^2 \frac{1}{4}(s+2t)\mathrm{d}s\mathrm{d}t = \frac{1}{2}y^2 + \frac{1}{2}y$.

(d) $g(x) = \frac{9}{(x+1)^2}$ is monotone on $(0,2)$. Support of Z is then $(1,9)$. $g^{-1}(z) = \frac{3}{\sqrt{z}} - 1$, $\left|\frac{\mathrm{d}}{\mathrm{d}z}g^{-1}(z)\right| = \frac{3}{2}z^{-3/2}$. Thus

$$f_Z(z) = f_X(g^{-1}(z))\left|\frac{\mathrm{d}}{\mathrm{d}z}g^{-1}(z)\right| = \frac{9}{8}z^{-2},\ 1 < z < 9$$

4. (a)

$$\begin{aligned} P(X > \sqrt{Y}) &= \int_0^1 \int_0^{x^2} (x+y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 (x^3 + \frac{1}{2}x^4)\mathrm{d}x \\ &= \frac{1}{4}x^4 + \frac{1}{10}x^5\Big|_0^1 \\ &= \frac{7}{20} \end{aligned}$$

(b)

$$\begin{aligned} P(X^2 < Y < X) &= \int_0^1 \int_{x^2}^{x} 2x\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 2x(x - x^2)\mathrm{d}x \\ &= \frac{2}{3}x^3 + \frac{1}{2}x^4\Big|_0^1 \\ &= \frac{1}{6} \end{aligned}$$

5. (a) $f(x,y) = P(X = x, Y = y)$. Then $f(1,3) = 1/12$, $f(1,5) = 1/12$, $f(1,8) = 1/12$, $f(3,3) = 1/12$, $f(3,5) = 1/12$, $f(3,8) = 1/12$, $f(5,5) = 2/12 = 1/6$, $f(5,8) = 1/12$, $f(8,8) = 3/12 = 1/4$. For other (x,y), $f(x,y) = 0$.

		x			
		1	3	5	8
y	3	1/12	1/12	0	0
	5	1/12	1/12	1/6	0
	8	1/12	1/12	1/12	1/4

(b) $f_X(1) = f(1,3) + f(1,5) + f(1,8) = 3/12 = 1/4$, $f_X(3) = f(3,3) + f(3,5) + f(3,8) = 3/12 = 1/4$, $f_X(5) = f(5,5) + f(5,8) = 1/6 + 1/12 = 1/4$, $f_X(8) = 1/4$. For other x, $f_X(x) = 0$.

x	1	3	5	8
	1/4	1/4	1/4	1/4

$f_Y(3) = f(1,3) + f(3,3) = 2/12 = 1/6$, $f_X(5) = f(1,5) + f(3,5) + f(5,5) = 2/12 + 1/6 = 1/3$, $f_X(8) = f(1,8) + f(3,8) + f(5,8) + f(8,8) = 3/12 + 1/4 = 1/2$. For other y, $f_Y(y) = 0$.

y	3	5	8
	1/6	1/3	1/2

(c) $E(X) = 1 \cdot f_X(1) + 3 \cdot f_X(3) + 5 \cdot f_X(5) + 8 \cdot f_X(8) = \frac{17}{4}$.
$E(Y-X) = 2 \cdot f(1,3) + 4 \cdot f(1,5) + 7 \cdot f(1,8) + 2 \cdot f(3,5) + 5 \cdot f(3,8) + 3 \cdot f(5,8) = \frac{23}{12}$

(d) $E(Y) = 3f_Y(3) + 5f_Y(5) + 8f_Y(8) = 1/2 + 5/3 + 4 = \frac{37}{6}$.
$E(XY) = 3f(1,3)+5f(1,5)+8f(1,8)+9f(3,3)+15f(3,5)+24f(3,8)+25f(5,5)+40f(5,8)+64f(8,8) = \frac{173}{6}$.
Then $Cov(X,Y) = E(XY) - E(X)E(Y) = 173/6 - (37/6) \cdot (17/4) = \frac{21}{8} = 2.625$

6.

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\begin{aligned}
&Cov(X_1 - 2X_2 + 8, 3X_1 + X_2)\\
=&3Cov(X_1, X_1) + Cov(X_1, X_2) - 6Cov(X_1, X_2) - 2Cov(X_2, X_2)\\
=&3\sigma_1^2 - 5\sigma_{12} - 2\sigma_2^2
\end{aligned}
$$

7. (a) If $P(X > Y) = 0$, then $E(X-Y) = \iint_{\mathbb{R}^2}(x-y)f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\{(x,y)|x\leq y\}}(x-y)f(x,y)\mathrm{d}x\mathrm{d}y \leq 0$. This contradicts the assumption that $EX > EY$. Thus when $EX > EY$, we have $P(X > Y) > 0$.

(b) $F(x,y) = \max\{F_X(x), F_Y(y)\}$ is not a legistimate cdf.
From the definition, we have $F(x,y) \geq F_X(x)$, $F(x,y) \geq F_Y(y)$. For a fixed y such that $F_Y(y) > 0$, let $x \to -\infty$, $\lim_{x\to-\infty} F(x,y) \geq \lim_{x\to-\infty} F_Y(y) = F_Y(y) > 0$. Thus $F(x,y)$ is not a legistimate cdf.

(c) $F(x,y) = \min\{F_X(x), F_Y(y)\}$ is a legistimate cdf.
$g(x,y) = \min\{x,y\}$ is a continuous function for (x,y) in $\mathbb{R}^2$. Thus we can take the limit inside.

i. $\lim_{x\to-\infty} F(x,y) = \lim_{x\to-\infty} \min\{F_X(x), F_Y(y)\} = \min\{\lim_{x\to-\infty} F_X(x), \lim_{x\to-\infty} F_Y(y)\} = \min\{0, F_Y(y)\} = 0$.
$\lim_{y\to-\infty} F(x,y) = \lim_{y\to-\infty} \min\{F_X(x), F_Y(y)\} = \min\{\lim_{y\to-\infty} F_X(x), \lim_{y\to-\infty} F_Y(y)\} = \min\{F_X(x), 0\} = 0$.

ii. $\lim_{x,y\to\infty} F(x,y) = \lim_{x,y\to\infty} \min\{F_X(x), F_Y(y)\} = \min\{\lim_{x,y\to\infty} F_X(x), \lim_{x,y\to\infty} F_Y(y)\} = \min\{1,1\} = 1$.

iii. $\lim_{h\to0^+} F(x+h,y) = \lim_{h\to0^+} \min\{F_X(x+h), F_Y(y)\} = \min\{\lim_{h\to0^+} F_X(x+h), \lim_{h\to0^+} F_Y(y)\} = \min\{F_X(x), F_Y(y)\} = F(x,y)$. In the same way, we can prove $\lim_{h\to0^+} F(x,y+h) = F(x,y)$.

iv. When $F_Y(y) \leq F_X(x) \leq F_X(x+\Delta_1) \leq F_Y(y+\Delta_2)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_X(x+\Delta_1) - F_Y(y) - F_X(x) + F_Y(y) = F_X(x+\Delta_1) - F_X(x) \geq 0$.

When $F_X(x) \leq F_Y(y) \leq F_Y(y+\Delta_2) \leq F_X(x+\Delta_1)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_Y(y+\Delta_2) - F_Y(y) - F_X(x) + F_X(x) = F_Y(y+\Delta_2) - F_Y(y) \geq 0$.

When $F_X(x) \le F_Y(y) \le F_X(x+\Delta_1) \le F_Y(y+\Delta_2)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_X(x+\Delta_1) - F_Y(y) - F_X(x) + F_X(x) = F_X(x+\Delta_1) - F_Y(y) \ge 0$.

When $F_Y(y) \le F_X(x) \le F_Y(y+\Delta_2) \le F_X(x+\Delta_1)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_Y(y+\Delta_2) - F_Y(y) - F_X(x) + F_Y(y) = F_Y(y+\Delta_2) - F_X(x) \ge 0$.

When $F_Y(y) \le F_Y(y+\Delta_2) \le F_X(x) \le F_X(x+\Delta_1)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_Y(y+\Delta_2) - F_Y(y) - F_Y(y+\Delta_2) + F_Y(y) = 0$.

When $F_X(x) \le F_X(x+\Delta_1) \le F_Y(y) \le F_Y(y+\Delta_2)$,
$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_X(x+\Delta_1) - F_X(x) - F_X(x+\Delta_1) + F_X(x) = 0$.
Thus in any situation, we all have

$$F(x+\Delta_1, y+\Delta_2) - F(x+\Delta_1, y) - F(x, y+\Delta_2) + F(x,y) = F_X(x+\Delta_1) - F_X(x) - F_X(x+\Delta_1) + F_X(x) \ge 0$$

Thus $F(x,y) = \min\{F_X(x), F_Y(y)\}$ is a legistimate cdf.