STAT 510 Homework 11

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1. (a) For this model, we have the model matrix

$$\boldsymbol{X} = \left[\mathbf{1}_{n_{11}+n_{12}+n_{21}+n_{22}}, \begin{bmatrix}\mathbf{1}_{n_{11}+n_{12}} \\ \mathbf{0}_{n_{21}+n_{22}}\end{bmatrix}, \begin{bmatrix}\mathbf{0}_{n_{11}+n_{12}} \\ \mathbf{1}_{n_{21}+n_{22}}\end{bmatrix}, \begin{bmatrix}\mathbf{1}_{n_{11}} \\ \mathbf{0}_{n_{12}} \\ \mathbf{1}_{n_{21}} \\ \mathbf{0}_{n_{22}}\end{bmatrix}, \begin{bmatrix}\mathbf{0}_{n_{11}} \\ \mathbf{1}_{n_{12}} \\ \mathbf{0}_{n_{21}} \\ \mathbf{1}_{n_{22}}\end{bmatrix}\right]$$

To make it full column rank with the same column space, then

$$\boldsymbol{X} = \left[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\right], \text{ where } \boldsymbol{x}_1 = \begin{bmatrix}\mathbf{1}_{n_{11}+n_{12}} \\ \mathbf{0}_{n_{21}+n_{22}}\end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix}\mathbf{0}_{n_{11}+n_{12}} \\ \mathbf{1}_{n_{21}+n_{22}}\end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix}\mathbf{1}_{n_{11}} \\ \mathbf{0}_{n_{12}} \\ \mathbf{1}_{n_{21}} \\ \mathbf{0}_{n_{22}}\end{bmatrix}$$

(b) We already have $\boldsymbol{x}_1 \perp \boldsymbol{x}_2$, thus let $\boldsymbol{w}_1 = \boldsymbol{x}_1$, $\boldsymbol{w}_2 = \boldsymbol{x}_2$. For $\boldsymbol{w}_2$, first we know

$$\boldsymbol{P}_{[\boldsymbol{w}_1, \boldsymbol{w}_2]} = \boldsymbol{P}_{[\boldsymbol{x}_1, \boldsymbol{x}_2]} = \begin{bmatrix}\frac{1}{n_{11}+n_{12}}\mathbf{1}\mathbf{1}^T_{n_{11}+n_{12}} & \mathbf{0} \\ \mathbf{0} & \frac{1}{n_{21}+n_{22}}\mathbf{1}\mathbf{1}^T_{n_{21}+n_{22}}\end{bmatrix}$$

Thus we have

$$\boldsymbol{w}_3 = (\boldsymbol{I} - \boldsymbol{P}_{[\boldsymbol{x}_1, \boldsymbol{x}_2]})\boldsymbol{x}_3 = \begin{bmatrix}\mathbf{1}_{n_{11}} \\ \mathbf{0}_{n_{12}} \\ \mathbf{1}_{n_{21}} \\ \mathbf{0}_{n_{22}}\end{bmatrix} - \begin{bmatrix}\frac{n_{11}}{n_{11}+n_{12}}\mathbf{1}_{n_{11}+n_{12}} \\ \frac{n_{21}}{n_{21}+n_{22}}\mathbf{1}_{n_{21}+n_{22}}\end{bmatrix} = \begin{bmatrix}\frac{n_{12}}{n_{11}+n_{12}}\mathbf{1}_{n_{11}} \\ -\frac{n_{11}}{n_{11}+n_{12}}\mathbf{1}_{n_{12}} \\ \frac{n_{22}}{n_{21}+n_{22}}\mathbf{1}_{n_{21}} \\ -\frac{n_{21}}{n_{21}+n_{22}}\mathbf{1}_{n_{22}}\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\mathbf{1}_2 \\ -\frac{1}{5}\mathbf{1}_8 \\ \frac{2}{5}\mathbf{1}_6 \\ -\frac{3}{5}\mathbf{1}_4\end{bmatrix}$$

$$\boldsymbol{W}^T\boldsymbol{W} = \begin{bmatrix}\boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3^T\end{bmatrix}\begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_1^T\boldsymbol{w}_1 & 0 & 0 \\ 0 & \boldsymbol{w}_2^T\boldsymbol{w}_2 & 0 \\ 0 & 0 & \boldsymbol{w}_3^T\boldsymbol{w}_3\end{bmatrix}$$

$$\boldsymbol{w}_1^T\boldsymbol{w}_1 = n_{11} + n_{12} = 10$$

$$\boldsymbol{w}_2^T\boldsymbol{w}_2 = n_{21} + n_{22} = 10$$

$$\boldsymbol{w}_3^T\boldsymbol{w}_3 = \left(\frac{n_{12}}{n_{11}+n_{12}}\right)^2 n_{11} + \left(\frac{n_{11}}{n_{11}+n_{12}}\right)^2 n_{12} + \left(\frac{n_{22}}{n_{21}+n_{22}}\right)^2 n_{21} + \left(\frac{n_{21}}{n_{21}+n_{22}}\right)^2 n_{22}$$

$$= \left(\frac{4}{5}\right)^2 \times 2 + \left(\frac{1}{5}\right)^2 \times 8 + \left(\frac{2}{5}\right)^2 \times 6 + \left(\frac{3}{5}\right)^2 \times 4 = 4$$

$$\boldsymbol{W}^T\boldsymbol{y} = \begin{bmatrix}\boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3^T\end{bmatrix}\boldsymbol{y} = \begin{bmatrix}\boldsymbol{w}_1^T\boldsymbol{y} \\ \boldsymbol{w}_2^T\boldsymbol{y} \\ \boldsymbol{w}_3^T\boldsymbol{y}\end{bmatrix}$$

$$\boldsymbol{w}_1^T\boldsymbol{y} = n_{11}\bar{y}_{11\cdot} + n_{12}\bar{y}_{12\cdot} = 2 \cdot 3 + 8 \cdot 5 = 46$$

$$\boldsymbol{w}_2^T\boldsymbol{y} = n_{21}\bar{y}_{21\cdot} + n_{22}\bar{y}_{22\cdot} = 6 \cdot 7 + 4 \cdot 3 = 54$$

$$\boldsymbol{w}_2^T\boldsymbol{y} = \frac{n_{12}n_{11}}{n_{11}+n_{12}}\bar{y}_{11\cdot} - \frac{n_{11}n_{12}}{n_{11}+n_{12}}\bar{y}_{12\cdot} + \frac{n_{22}n_{21}}{n_{21}+n_{22}}\bar{y}_{21\cdot} - \frac{n_{21}n_{22}}{n_{21}+n_{22}}\bar{y}_{22\cdot} = 6.4$$

Then

$$(\boldsymbol{W}^T\boldsymbol{W})^-\boldsymbol{W}^T\boldsymbol{y} = \begin{bmatrix} 1/\boldsymbol{w}_1^T\boldsymbol{w} & 0\boldsymbol{0} & \\ 0 & 1/\boldsymbol{w}_2^T\boldsymbol{w}_2 & 0 \\ 0 & 0 & 1/\boldsymbol{w}_3^T\boldsymbol{w}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^T\boldsymbol{y} \\ \boldsymbol{w}_2^T\boldsymbol{y} \\ \boldsymbol{w}_3^T\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \frac{\boldsymbol{w}_1^T\boldsymbol{y}}{\boldsymbol{w}_1^T\boldsymbol{w}_1} \\ \frac{\boldsymbol{w}_2^T\boldsymbol{y}}{\boldsymbol{w}_2^T\boldsymbol{w}_2} \\ \frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3} \end{bmatrix} = \begin{bmatrix} 4.6 \\ 5.4 \\ 1.6 \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^-\boldsymbol{W}^T\boldsymbol{y} &= \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{w}_1^T\boldsymbol{y}}{\boldsymbol{w}_1^T\boldsymbol{w}_1} \\ \frac{\boldsymbol{w}_2^T\boldsymbol{y}}{\boldsymbol{w}_2^T\boldsymbol{w}_2} \\ \frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3} \end{bmatrix} \\
&= \frac{\boldsymbol{w}_1^T\boldsymbol{y}}{\boldsymbol{w}_1^T\boldsymbol{w}_1}\boldsymbol{w}_1 + \frac{\boldsymbol{w}_2^T\boldsymbol{y}}{\boldsymbol{w}_2^T\boldsymbol{w}_2}\boldsymbol{w}_2 + \frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3}\boldsymbol{w}_3 \\
&= \begin{bmatrix} \left(\frac{\boldsymbol{w}_1^T\boldsymbol{y}}{\boldsymbol{w}_1^T\boldsymbol{w}_1} + \frac{n_{12}}{n_{11}+n_{12}}\frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3}\right)\boldsymbol{1}_{n_{11}} \\ \left(\frac{\boldsymbol{w}_1^T\boldsymbol{y}}{\boldsymbol{w}_1^T\boldsymbol{w}_1} - \frac{n_{11}}{n_{11}-n_{12}}\frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3}\right)\boldsymbol{1}_{n_{12}} \\ \left(\frac{\boldsymbol{w}_2^T\boldsymbol{y}}{\boldsymbol{w}_2^T\boldsymbol{w}_2} + \frac{n_{22}}{n_{21}+n_{22}}\frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3}\right)\boldsymbol{1}_{n_{21}} \\ \left(\frac{\boldsymbol{w}_2^T\boldsymbol{y}}{\boldsymbol{w}_2^T\boldsymbol{w}_2} - \frac{n_{21}}{n_{21}-n_{22}}\frac{\boldsymbol{w}_3^T\boldsymbol{y}}{\boldsymbol{w}_3^T\boldsymbol{w}_3}\right)\boldsymbol{1}_{n_{22}} \end{bmatrix} \\
&= \begin{bmatrix} (4.6+0.8\times 1.6)\boldsymbol{1}_2 \\ (4.6-0.2\times 1.6)\boldsymbol{1}_8 \\ (5.4+0.4\times 1.6)\boldsymbol{1}_6 \\ (5.4-0.6\times 1.6)\boldsymbol{1}_4 \end{bmatrix} \\
&= \begin{bmatrix} 5.88\boldsymbol{1}_2 \\ 4.28\boldsymbol{1}_8 \\ 6.04\boldsymbol{1}_6 \\ 4.44\boldsymbol{1}_4 \end{bmatrix} = \boldsymbol{P_W}\boldsymbol{y} = \boldsymbol{P_X}\boldsymbol{y}
\end{aligned}$$

(c)

$$\boldsymbol{P}_{[\boldsymbol{x}_1,\boldsymbol{x}_3]}\boldsymbol{y} = \begin{bmatrix} \frac{n_{11}+\bar{y}_{11\cdot}+n_{12}\bar{y}_{12\cdot}}{n_{11}+n_{12}}\boldsymbol{1}_{n_{11}+n_{12}} \\ \frac{n_{21}+\bar{y}_{21\cdot}+n_{22}\bar{y}_{22\cdot}}{n_{21}+n_{22}}\boldsymbol{1}_{n_{21}+n_{22}} \end{bmatrix} = \begin{bmatrix} 4.6\boldsymbol{1}_{10} \\ 5.4\boldsymbol{1}_{10} \end{bmatrix}$$

Then

$$\begin{aligned}
SS(B|\boldsymbol{1},A) &= \boldsymbol{y}^T(\boldsymbol{P_X} - \boldsymbol{P}_{[\boldsymbol{x}_1,\boldsymbol{x}_2]})\boldsymbol{y} \\
&= \boldsymbol{y}^T\boldsymbol{P_X}\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{P}_{[\boldsymbol{x}_1,\boldsymbol{x}_2]}\boldsymbol{y} \\
&= (\boldsymbol{P_X}\boldsymbol{y})^T(\boldsymbol{P_X}\boldsymbol{y}) - (\boldsymbol{P}_{[\boldsymbol{x}_1,\boldsymbol{x}_2]}\boldsymbol{y})^T(\boldsymbol{P}_{[\boldsymbol{x}_1\boldsymbol{x}_2]}\boldsymbol{y}) \\
&= (5.88^2\times 2 + 4.28^2\times 8 + 6.04^2\times 6 + 4.44^2\times 4) - (4.6^2\times 10 + 5.4^2\times 10) \\
&= 10.24
\end{aligned}$$

2. (a) $\boldsymbol{\mu} = [\mu_{ik}]$, $i = 1,2,3$, $k = 1,\ldots,4$. $\boldsymbol{w} = [w_{ij}]$ $i = 1,2,3$, $j = 1,\ldots,5$. $bme = [e_{ijk}]$, $i = 1,2,3$, $j = 1,\ldots,5$, $k = 1,\ldots,4$. Then

$$\boldsymbol{y} = (\boldsymbol{I}_{3\times 3}\otimes\boldsymbol{1}_5\otimes\boldsymbol{I}_{4\times 4})\boldsymbol{\mu} + (\boldsymbol{I}_{15\times 15}\otimes\boldsymbol{1}_4)\boldsymbol{w} + \boldsymbol{I}_{60\times 60}\boldsymbol{e}$$

Then

$$Var(\boldsymbol{y}) = (\boldsymbol{I}_{15\times 15}\otimes\boldsymbol{1}_4)Var(\boldsymbol{w})(\boldsymbol{I}_{15\times 15}\otimes\boldsymbol{1}_4)^T + Var(\boldsymbol{e}) = \sigma_w^2(\boldsymbol{I}_{15\times 15}\otimes\boldsymbol{1}_4\boldsymbol{1}_4^T) + \sigma_e^2\boldsymbol{I}_{60\times 60}$$

(b) Test statistic $F = 7.1152$.
Degree of freedom are $(6, 36)$.
p-value < 0.0001.
Since p$<$ 0.0001 is a very small number less than 0.05. We reject the null hypothesis anc conclude that there is significant evidence that there is drug-by-time interactions.

(c) $H_0 : \mu_{24} - \mu_{14} = \mu_{34} - \mu_{14} = 0$
We have

$$\boldsymbol{C}\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\mu}_{24} - \hat{\mu}_{14} \\ \hat{\mu}_{34} - \hat{\mu}_{14} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_2 + \hat{\gamma}_{24} \\ \hat{\alpha}_3 + \hat{\gamma}_{14} \end{bmatrix} = \begin{bmatrix} 5.6 \\ 1.6 \end{bmatrix}$$

Also

$$\begin{aligned} Var\left(\boldsymbol{C}\hat{\boldsymbol{\beta}}\right) &= Var\left(\begin{bmatrix} \bar{y}_{2\cdot4} - \bar{y}_{1\cdot4} \\ \bar{y}_{3\cdot4} - \bar{y}_{1\cdot4} \end{bmatrix}\right) \\ &= \begin{bmatrix} Var(\bar{y}_{2\cdot4}) + Var(\bar{y}_{1\cdot4}) & Var(\bar{y}_{1\cdot4}) \\ Var(\bar{y}_{1\cdot4}) & Var(\bar{y}_{3\cdot4}) + Var(\bar{y}_{1\cdot4}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5}\sigma_w^2 + \frac{2}{5}\sigma_e^2 & \frac{1}{5}\sigma_w^2 + \frac{1}{5}\sigma_e^2 \\ \frac{1}{5}\sigma_w^2 + \frac{1}{5}\sigma_e^2 & \frac{2}{5}\sigma_w^2 + \frac{2}{5}\sigma_e^2 \end{bmatrix} \\ &= \frac{1}{5}(\sigma_w^2 + \sigma_e^2) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

Hence

$$(\widehat{Var}(\boldsymbol{C}\hat{\boldsymbol{\beta}}))^{-1} = \frac{5}{\hat{\sigma}_w^2 + \hat{\sigma}_e^2} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Then the test statistic

$$F = \frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}})^T(\widehat{Var}(\boldsymbol{C}\hat{\boldsymbol{\beta}}))^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}})}{q} = \frac{5}{2(\hat{\sigma}_e^2 + \hat{\sigma}_w^2)} \begin{bmatrix} 5.6 & 1.6 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 5.6 \\ 1.6 \end{bmatrix} = 1.109$$

For degrees of freedom, because

$$\hat{\sigma}_w^2 + \hat{\sigma}_e^2 = \frac{1}{4}(4\hat{\sigma}_w^2 + \hat{\sigma}_e^2) + \frac{3}{4}\hat{\sigma}_e^2 = \frac{1}{4}MS_{woman(drug)} + \frac{3}{4}MS_{error}$$

Thus

$$df = \frac{(\frac{1}{4}MS_{woman(drug)} + \frac{3}{4}MS_{error})^2}{\frac{(\frac{1}{4})^2 MS^2_{woman(drug)}}{12} + \frac{(\frac{3}{4})^2 MS^2_{error}}{36}} = 17.06$$

Then p-value

$$p = P(F_{2,17.06} > 1.109) = 0.35$$

The p-value > 0.05. We fail to reject the null hypothesis and conclude that there is no significant evidence that the mean heart rate 15 minutes after treatment are not the same for all three drugs.

(d) We know

$$\begin{aligned} &\hat{\mu}_{14} - \hat{\mu}_{24} = -5.6 \\ &SE(\hat{\mu}_{14} - \hat{\mu}_{24}) = \sqrt{\frac{2}{5}(\hat{\sigma}_e^2 + \hat{\sigma}_w^2)} = 3.87 \end{aligned}$$

Then

$$CI = (\hat{\mu}_{14} - \hat{\mu}_{24} - t_{17.06,0.975} \cdot SE, \hat{\mu}_{14} + \hat{\mu}_{24} - t_{17.06,0.975} \cdot SE) = (-13.766, 2.566)$$

3. (a) $\sigma = 6.12$, $\rho = 0.7769$.

(b) $AIC = 317.92$, $BIC = 344.12$.

(c) $\sigma = 6.00$, $\rho = 0.8278$.

(d) $AIC = 313.94$, $BIC = 340.14$

(e) $\sigma = 6.10$, $\delta_2 = 1.085, \delta_3 = 0.995$, $\delta_4 = 0.928$, $\rho_{12} = 0.850.\rho_{13} = 0.889$, $\rho_{14} = 0.625$, $\rho_{23} = 0.870$, $\rho_{24} = 0.631$, $\rho_{34} = 0.794$.

(f) $AIC = 322.85$, $BIC = 364.01$

(g) AR1 is prefered. The model has smaller AIC and BIC.

(h) $\hat{\mu}_{14} - \hat{\mu}_{24} = -5.6$, $SE(\hat{\mu}_{14} - \hat{\mu}_{24}) = \sqrt{\frac{2}{5}6^2} = 3.79$.

$$CI = (-5.6 - z_{0.975} \times 3.79, -5.6 + z_{0.975} \times 3.79) = (-13.03, 1.83)$$

4. (a)

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{31} \end{bmatrix}$$

$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 1/3 & -1/6 & 0 & 0 \\ -1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$, $\boldsymbol{\beta} = (\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y} = \begin{bmatrix} 50.33 \\ 53.67 \end{bmatrix} \Rightarrow \hat{\mu}_1 + \hat{\mu}_2 = 50.33 + 53.67 = 104.$

(b) From $Var(y_{i1}) = Var(y_{i2}) = 4$ and $Corr(y_{i1}, y_{i2}) = 0.5 \Rightarrow \sigma_u^2 = 2, \sigma_e^2 = 2$.

Thus $\boldsymbol{G} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\boldsymbol{Z}^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{y} - \boldsymbol{X\beta} = \begin{bmatrix} 0.67 \\ 0.33 \\ -2.33 \\ 1.67 \end{bmatrix}$. Hence

$$BLUP(\boldsymbol{u}) = \boldsymbol{G}\boldsymbol{Z}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{X\beta}) = \begin{bmatrix} 0.333 \\ -1.167 \\ 0.833 \end{bmatrix}$$

Hence $y_{22} = 53.67 - 1.167 = 52.5$, $y_{23} = 53.67 + 0.833 = 54.5$.