STAT 543 Homework 1

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January 18, 2017

1. (a)

$$\mu_1' = E(X) \Rightarrow \mu_1' = \frac{3(1-p)}{p}$$

Hence

$$p\mu_1' = 3(1-p) \Rightarrow p = \frac{3}{\mu_1' + 3} = \frac{3}{\frac{\sum_{i=1}^n X_i}{n} + 3}$$

(b)

$$\mu_1' = E(X) = \mu$$
$$\mu_2' = E(X^2) = (E(X))^2 + Var(X) = \mu^2 + 2\sigma^2$$

Hence

$$\mu = \mu_1' = \frac{\sum_{i=1}^n X_i}{n}$$
$$\sigma = \sqrt{\frac{\mu_2' - \mu_1'^2}{2}} = \sqrt{\frac{\sum_{i=1}^n X_i^2}{2n} - \frac{(\sum_{i=1}^n X_i)^2}{2n^2}}$$

2. For a given x, $\hat{\theta}$ maximize $f(x|\theta)$, thus

When $x = 0$, $\hat{\theta} = 1$.

When $x = 1$, $\hat{\theta} = 1$.

When $x = 2$, $\hat{\theta} = 2, 3$.

When $x = 3$, $\hat{\theta} = 3$.

When $x = 4$, $\hat{\theta} = 3$.

3. For two events A_1 and A_2, $A_1 \cap A_2$ is true if and only if A_1 is true and A_2 is true. When at least one of A_1 and A_2 is false then $A_1 \cap A_2$ is false. Then When both of them are true, $I(A_1) = I(A_2) = 1$ and $I(A_1 \cap A_2) = 1 \Rightarrow I(A_1 \cap A_2) = I(A_1)I(A_2)$. When at least one of them is false, then $I(A_1)I(A_2) = 0$ and $I(A_1 \cap A_2) = 0$. Hence we have

$$I(A_1 \cap A_2) = I(A_1)I(A_2)$$

Thus,

$$
\begin{aligned}
I(B) &= I(\bigcap_{i=1}^{n-1} A_i \cap A_n) \\
&= I(\bigcap_{i=1}^{n-1} A_i) I(A_n) \\
&= I(\bigcap_{i=1}^{n-2} A_i) I(A_{n-1}) I(A_n) \\
&= \cdots \\
&= \prod_{i=1}^{n} I(A_i)
\end{aligned}
$$

4. (a) $S_\theta = (0, +\infty)$ and $x_1, x_2, \cdots, x_n > 0$, hence $I(x_1, x_2, \cdots, x_n \in S_\theta) = 1$. Then

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathrm{e}^{-x_i/\theta} = \frac{1}{\theta^n} \mathrm{e}^{-\sum_{i=1}^{n} x_i/\theta}$$

Then

$$\log L(\theta) = -n \log \theta - \frac{\sum_{i=1}^{n} x_i}{\theta}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log L(\theta) \bigg|_{\theta=\hat{\theta}} = 0 \Rightarrow -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \bigg|_{\theta=\hat{\theta}} = 0 \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}_n$$

(b) $S_\theta = (0, \theta]$. As $x_1, x_2, \cdots, x_n > 0$, then $I(x_1, x_2, \cdots, x_n \in S_\theta) = I(x_1, x_2, \cdots, x_n \leq \theta) = I(\max_{1 \leq i \leq n} x_i \leq \theta)$. Hence

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} f(x_i|\theta) I(x_1, x_2, \cdots, x_n \in S_\theta) \\
&= \prod_{i=1}^{n} \frac{2x_i}{\theta^2} I(\max_{1 \leq i \leq n} x_i \leq \theta) \\
&= \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n} x_i I(\max_{1 \leq i \leq n} x_i \leq \theta) \\
&= \begin{cases} \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n} x_i & , \theta \geq \max_{1 \leq i \leq n} x_i \\ 0 & , \theta < \max_{1 \leq i \leq n} x_i \end{cases}
\end{aligned}
$$

We can see $L(\max_{1 \leq i \leq n} x_i) > 0$ and $L(\theta)$ goes down as θ increasing when θ is greater than $\max_{1 \leq i \leq n} x_i$, thus it is the point when $L(\theta)$ is the largest. Hence

$$\hat{\theta} = \max_{1 \leq i \leq n} X_i$$

5. (b) $S_\theta = [\theta, \infty)$. Then $I(x_1, x_2, \cdots, x_n \in S_\theta) = I(x_1, x_2, \cdots, x_n \geq \theta) = I(\min_{1 \leq i \leq n} x_i \geq \theta)$.

Hence

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i|\theta) I(x_1, x_2, \cdots, x_n \in S_\theta) \\ &= \prod_{i=1}^{n} \theta x_i^{-2} I(\min_{1\le i\le n} x_i \ge \theta) \\ &= \theta^n (\prod_{i=1}^{n} x_i)^{-2} I(\min_{1\le i\le n} x_i \ge \theta) \\ &= \begin{cases} \theta^n (\prod_{i=1}^{n} x_i)^{-2} & ,\theta \le \min_{1\le i\le n} x_i \\ 0 & ,\theta > \min_{1\le i\le n} x_i \end{cases} \end{aligned}$$

We can see $L(\min_{1\le i\le n} x_i) > 0$ and $L(\theta)$ goes up as θ increasing when θ is less than $\min_{1\le i\le n} x_i$, thus it is the point when $L(\theta)$ is the largest. Hence

$$\hat{\theta} = \min_{1\le i\le n} X_i$$

(c)

$$E(X) = \int_\theta^\infty x\theta x^{-2} \mathrm{d}x = \int_\theta^\infty \theta x^{-1} \mathrm{d}x = \theta \log x \Big|_\theta^\infty = \infty$$

Hence the MME does not exist.