

STAT 510 Homework 10

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April 11, 2017

1. $\mathbf{y} = \mathbf{X}\mathbf{b} + \boldsymbol{\epsilon}$, here we have

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \text{Var}(\boldsymbol{\epsilon}) = \boldsymbol{\Sigma} = \sigma^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

Then we have $\text{rank}(\mathbf{X}) = 2, n = 3, n - \text{rank}(\mathbf{X}) = 1$. Let $\mathbf{A}^T = [1 \quad -1 \quad 0]$, then we have

$$w = \mathbf{A}^T \mathbf{y} = y_1 - y_2, E(w) = 0, \text{Var}(w) = \mathbf{A}^T \boldsymbol{\Sigma} \mathbf{A} = \sigma^2$$

Then the log likelihood is

$$\ell(\sigma^2) = \log f(w|\sigma^2) = \log \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{w^2}{2\sigma^2}} = -\frac{w^2}{2\sigma^2} - \log \sigma - \frac{1}{2} \log(2\pi)$$

$$\frac{\partial}{\partial \sigma^2} \ell(\sigma^2) = 0 \Rightarrow \frac{w^2 - \sigma^2}{2\sigma^4} = 0 \Rightarrow \sigma^2 = w^2$$

We also have

$$\frac{\partial^2}{\partial (\sigma^2)^2} \ell(\sigma^2) \Big|_{\sigma^2=w^2} = -\frac{1}{w^4} < 0$$

Hence $\sigma^2 = w^2 = (y_1 - y_2)^2$ is MLE and is REML.

2. (a)

$$L(\lambda) = \prod_{i=1}^n f(y_i|\lambda) = \left(\prod_{i=1}^n \frac{1}{y_i!} \right) \exp(-n\lambda) \lambda^{\sum_{i=1}^n y_i}$$

(b)

$$\begin{aligned} \ell(\lambda) &= \log \prod_{i=1}^n f(y_i|\lambda) \\ &= \sum_{i=1}^n \log f(y_i|\lambda) \\ &= \log \lambda \sum_{i=1}^n y_i - n\lambda - \sum_{i=1}^n y_i! \end{aligned}$$

Thus score equation is

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \sum_{i=1}^n y_i - n = 0$$

(c)

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = 0 \Rightarrow \lambda = \frac{\sum_{i=1}^n y_i}{n}$$

(d)

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^n y_i}{\lambda^2}$$

Thus

$$\left. \frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} \right|_{\lambda = \frac{\sum_{i=1}^n y_i}{n}} = -\frac{1}{\sum_{i=1}^n y_i} < 0$$

(e) $y_i \stackrel{\text{iid}}{\sim} \text{Poisson}(\lambda) \Rightarrow \sum_{i=1}^n y_i \sim \text{Poisson}(n\lambda)$

$$I(\lambda) = -E \left[\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} \right] = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

(f)

$$I(\lambda)^{-1} = \frac{\lambda}{n}$$

(g)

$$I(\lambda)^{-1} = \frac{\lambda}{n}$$

$$\text{Var}(\hat{\lambda}) = \text{Var} \left(\sum_{i=1}^n y_i / n \right) = \frac{1}{n^2} \text{Var} \left(\sum_{i=1}^n y_i \right) = \frac{1}{n^2} n\lambda = \frac{\lambda}{n}$$

They are equal.

(h)

$$\text{Var}(\hat{\lambda}) = \frac{\hat{\lambda}}{n} = \frac{\sum_{i=1}^n y_i}{n^2}$$

3. (a)

$$\begin{aligned} \ell(\boldsymbol{\lambda}) &= \log \prod_{i=1}^2 \prod_{j=1}^7 f(y_{ij} | \boldsymbol{\lambda}) \\ &= \sum_{i=1}^2 \sum_{j=1}^7 \log f(y_{ij} | \boldsymbol{\lambda}) \\ &= \sum_{i=1}^2 \sum_{j=1}^7 (y_{ij} \log \lambda_i - \lambda_i - \log y_{ij}!) \end{aligned}$$

Then we have

$$\frac{\partial \ell(\boldsymbol{\lambda})}{\partial \lambda_i} = \frac{\sum_{j=1}^7 y_{ij}}{\lambda_i} - 7 = 0 \Rightarrow \lambda_i = \frac{\sum_{j=1}^7 y_{ij}}{7}$$

$$\text{Thue } \hat{\boldsymbol{\lambda}} = \left(\frac{\sum_{j=1}^7 y_{1j}}{7}, \frac{\sum_{j=1}^7 y_{2j}}{7} \right).$$

Also

$$\left. \frac{\partial^2 \ell(\boldsymbol{\lambda})}{\partial \lambda_i} \right|_{\hat{\boldsymbol{\lambda}}} = -\frac{1}{\hat{\lambda}_i} < 0, \left. \frac{\partial^2 \ell(\boldsymbol{\lambda})}{\partial \lambda_1 \partial \lambda_2} \right|_{\hat{\boldsymbol{\lambda}}} = 0$$

Thus $H = \begin{bmatrix} -\frac{1}{\hat{\lambda}_1} & 0 \\ 0 & -\frac{1}{\hat{\lambda}_2} \end{bmatrix}$ is non-positive definite. Hence $\hat{\lambda} = \left(\sum_{j=1}^7 y_{1j}/7, \sum_{j=1}^7 y_{2j}/7 \right)$ is MLE.

Then

$$\ell(\hat{\lambda}) = \sum_{i=1}^2 \sum_{j=1}^7 (y_{ij} \log \bar{y}_{i\cdot} - \bar{y}_{i\cdot} - \log y_{ij}!) = -37.10781, k = 2, n = 14$$

Hence

$$\text{AIC} = -2\ell(\hat{\lambda}) + 2k = 78.21563$$

(b)

$$\text{BIC} = -2\ell(\hat{\lambda}) + k \log(n) = 79.49374$$

(c) Like wise, we have

$$\ell(\hat{\lambda}) = \sum_{i=1}^2 \sum_{j=1}^7 (y_{ij} \log \bar{y}_{\cdot j} - \bar{y}_{\cdot j} - \log y_{ij}!) = -40.30237, k = 1, n = 14$$

Hence

$$\text{AIC} = -2\ell(\hat{\lambda}) + 2k = 82.60474$$

(d)

$$\text{BIC} = -2\ell(\hat{\lambda}) + k \log(n) = 83.2438$$

(e) Model (1).

(f) Model (1).

(g)

$$-2 \log \Lambda = 2(\ell(\hat{\lambda}) - \ell(\hat{\hat{\lambda}})) = 6.389118$$

(h)

$$p = 1 - P(\chi_1^2 \leq 6.389118) = 0.01148221$$

(i)

$$\hat{Var}(\hat{\lambda}_1 - \hat{\lambda}_2) = \hat{Var}\left(\frac{\hat{\lambda}_1 - \hat{\lambda}_2}{7}\right) = \frac{\hat{\lambda}_1 + \hat{\lambda}_2}{7}$$

Then

$$\text{Wald Statistic} = \frac{(\hat{\lambda}_1 - \hat{\lambda}_2)^2}{(\hat{\lambda}_1 + \hat{\lambda}_2)/7} = \frac{7(\bar{y}_{1\cdot} - \bar{y}_{2\cdot})^2}{\bar{y}_{1\cdot} + \bar{y}_{2\cdot}} = 6.351648$$

(j)

$$p = P(\chi_2^2 \geq 6.351648) = 0.04175968$$