STAT 542 Homework 4

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1. (a)

$$\begin{aligned}
M_X(t) = E(\mathrm{e}^{Xt}) &= \int_0^c \mathrm{e}^{xt}\frac{1}{c}\mathrm{d}x \\
&= \frac{1}{c}\frac{1}{t}\mathrm{e}^{tx}\Big|_0^c = \frac{1}{ct}(\mathrm{e}^{ct}-1) \qquad t\in\mathbb{R}
\end{aligned}$$

(b)

$$\begin{aligned}
M_X(t) = E(\mathrm{e}^{Xt}) &= \int_0^c \mathrm{e}^{xt}\frac{2x}{c^2}\mathrm{d}x \\
&= \frac{2}{c^2}\int_0^c x\mathrm{d}\left(\frac{1}{t}\mathrm{e}^{xt}\right) \\
&= \frac{2}{c^2}\left[x\frac{1}{t}\mathrm{e}^{xt}\Big|_0^c - \int_0^c \frac{1}{t}\mathrm{e}^{xt}\mathrm{d}x\right] \\
&= \frac{2}{c^2}\left(\frac{c}{t}\mathrm{e}^{ct} - \frac{1}{t^2}\mathrm{e}^{ct} + \frac{1}{t^2}\right) \\
&= \frac{2}{ct}\mathrm{e}^{ct} - \frac{2}{c^2t^2}\mathrm{e}^{ct} + \frac{2}{c^2t^2} \qquad t\in\mathbb{R}
\end{aligned}$$

(c)

$$\begin{aligned}
M_X(t) = E(\mathrm{e}^{Xt}) &= \int_{-\infty}^{\infty} \mathrm{e}^{xt}\frac{1}{2\beta}\mathrm{e}^{-|x-\alpha|/\beta}\mathrm{d}x \\
&= \frac{1}{2\beta}\int_{-\infty}^{\alpha}\mathrm{e}^{xt}\mathrm{e}^{(x-\alpha)/\beta}\mathrm{d}x + \frac{1}{2\beta}\int_{\alpha}^{\infty}\mathrm{e}^{xt}\mathrm{e}^{-(x-\alpha)/\beta}\mathrm{d}x \\
&= \frac{\mathrm{e}^{-\alpha/\beta}}{2\beta}\int_{-\infty}^{\alpha}\mathrm{e}^{(t+\frac{1}{\beta})x}\mathrm{d}x + \frac{\mathrm{e}^{\alpha/\beta}}{2\beta}\int_{\alpha}^{\infty}\mathrm{e}^{(t-\frac{1}{\beta})x}\mathrm{d}x \\
&= \frac{\mathrm{e}^{-\alpha/\beta}}{2\beta}\frac{1}{t+\frac{1}{\beta}}\mathrm{e}^{(t+\frac{1}{\beta})\alpha} - \frac{\mathrm{e}^{\alpha/\beta}}{2\beta}\frac{1}{t-\frac{1}{\beta}}\mathrm{e}^{(t-\frac{1}{\beta})\alpha} \\
&= \frac{\mathrm{e}^{\alpha t}}{2\beta t+2} - \frac{\mathrm{e}^{\alpha t}}{2\beta t-2} = \frac{\mathrm{e}^{\alpha t}}{1-\beta^2t^2} \qquad t\in(-\frac{1}{\beta},\frac{1}{\beta})
\end{aligned}$$

2. For any mgf, $M_X(t) = E(\mathrm{e}^{Xt}) \Rightarrow M_X(0) = E(\mathrm{e}0) = E(1) = 1$. However, $\frac{t}{1-t}|_{t=0} = 0$. Thus it cannot be an mgf.

3. (a)

$$
\begin{aligned}
M_X(t) = E(\mathrm{e}^{Xt}) &= \int_{-\infty}^{infty} \mathrm{e}^{tx}\mathrm{d}F_X(x) \\
&\geq \int_a^{\infty} \mathrm{e}^{tx}\mathrm{d}F_X(x) \\
&\geq \mathrm{e}^{at}\int_a^{\infty} \mathrm{d}F_X(x) \qquad \text{(when } t>0,\ \mathrm{e}^{xt} \text{ increases as } x \text{ increases)} \\
&= \mathrm{e}^{at}P(X \geq a)
\end{aligned}
$$

Thus $P(X \geq a) \leq \mathrm{e}^{-at}M_X(t)$.

(b)

$$
\begin{aligned}
M_X(t) = E(\mathrm{e}^{Xt}) &= \int_{-\infty}^{infty} \mathrm{e}^{tx}\mathrm{d}F_X(x) \\
&\geq \int_{-\infty}^{a} \mathrm{e}^{tx}\mathrm{d}F_X(x) \\
&\geq \mathrm{e}^{at}\int_{-\infty}^{a} \mathrm{d}F_X(x) \qquad \text{(when } t<0,\ \mathrm{e}^{xt} \text{ decreases as } x \text{ increases)} \\
&= \mathrm{e}^{at}P(X \leq a)
\end{aligned}
$$

Thus $P(X \leq a) \leq \mathrm{e}^{-at}M_X(t)$.

(c) Add these conditions on $h(t,x)$: (i) $h(t,x) \geq 0$; (ii) $\{x|h(t,x) \geq 1\} \subset \{x|x \geq 0\}$ for all $t \geq 0$. Then by Marcov's Inequality,

$$
P(X \geq 0) \geq P(h(t,X) \geq 1) \geq \frac{E(h(t,X))}{1} = E(h(t,X))
$$

4. (a)

$$
\begin{aligned}
E(Y) = E(\mathrm{e}^{Xr}) &= \int_{-\infty}^{\infty} \mathrm{e}^{xr}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x \\
&= \frac{\mathrm{e}^{r^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}(x-r)^2}\mathrm{d}x \\
&= \frac{\mathrm{e}^{r^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}u^2}\mathrm{d}u \qquad (u = x-r) \\
&= \mathrm{e}^{r^2/2}
\end{aligned}
$$

(b) For any $t > 0$,

$$
\begin{aligned}
M_Y(t) = E(\mathrm{e}^{Yt}) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{e}^x t}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{t\mathrm{e}^x - x^2/2}\mathrm{d}x
\end{aligned}
$$

Because $\frac{\mathrm{e}^x}{x^2} \to \infty$ as $x \to \infty$, for any $t > 0$, there would be an constant $x_t > 0$ such that $\frac{\mathrm{e}^x}{x^2} > \frac{1}{t}$. Thus for any $x > x_t$, we have $t\mathrm{e}^x - x^2/2 > t\frac{1}{t}x^2 - x^2/2 = x^2/2$. So

$$
\geq \frac{1}{\sqrt{2\pi}}\int_{x_t}^{\infty} \mathrm{e}^{x^2/2}\mathrm{d}x = \infty
$$

Hence, we can not find a interval containing 0 such that $E(\mathrm{e}^{Yt})$ exists. Thus $M_Y(t)$ dose not exist.

5. Let $g(x) = \mathrm{e}^{tx}$. We have $g''(x) = t^2\mathrm{e}^{tx} \geq 0$, thus $g(x)$ is a convex function. So by Jensen's Inequality,

$$E(g(X)) \geq g(E(X)) \Rightarrow E(\mathrm{e}^{Xt}) \geq \mathrm{e}^{\mu t} \Rightarrow M_X(t) \geq \mathrm{e}^{\mu t}$$

6.

$$\begin{aligned}
M_X(t) &= \sum_{x=1}^{\infty} \mathrm{e}^{tx} p(1-p)^{1-x} \\
&= p\mathrm{e}^t \sum_{x=1}^{\infty} \mathrm{e}^{t(x-1)}(1-p)^{x-1} \\
&= p\mathrm{e}^t \sum_{u=0}^{\infty} [\mathrm{e}^t(1-p)]^u = \frac{p\mathrm{e}^t}{1-\mathrm{e}^t(1-p)} \qquad |\mathrm{e}^t(1-p)| < 1 \Rightarrow t < -\log(1-p) \\
M_X'(t) &= p\frac{\mathrm{e}^t(1-\mathrm{e}^t(1-p)) - \mathrm{e}^t(-(1-p)\mathrm{e}^t)}{(1-\mathrm{e}^t(1-p))^2} \\
&= \frac{p\mathrm{e}^t}{(1-\mathrm{e}^t(1-p))^2} \\
M_X''(t) &= \left(\frac{p\mathrm{e}^t}{(1-\mathrm{e}^t(1-p))^2}\right)' \\
&= p\frac{\mathrm{e}^t(1-\mathrm{e}^2(1-p))^2 - \mathrm{e}^t \cdot 2(1-\mathrm{e}^t(1-p))(-(1-p)\mathrm{e}^t)}{(1-\mathrm{e}^t(1-p))^4} \\
&= \frac{p\mathrm{e}^t(1+\mathrm{e}^t(1-p))}{(1-\mathrm{e}^t(1-p))^3}
\end{aligned}$$

Thus

$$E(X) = M_X'(0) = \frac{1}{p},\ E(X^2) = M_X''(0) = \frac{2-p}{p^2},\ Var(X) = E(X^2) - (E(X))^2 = \frac{1-p}{p^2}$$

7. (a) We need

$$P = \frac{\binom{95}{K}\binom{5}{0}}{\binom{100}{K}} \leq 1$$

When $K = 36$, $P = 0.1013$; when $K = 37$, $P = 0.0934$. Thus $K = 37$ is the smallest K.

(b) We need

$$P = \binom{K}{5}(0.95)^5(0.05)^0 \leq 1$$

When $K = 44$, $P = 0.1047$; when $K = 45$, $P = 0.0994$. Thus $K = 45$ is the smallest K.

(c)

$$P(\{\text{the manufactuer accepts the lot with 5 defective parts}\}) = \frac{\binom{95}{10}\binom{5}{0}}{\binom{100}{10}} = 0.584$$

$$P(\{\text{the manufactuer accepts the lot with 10 defective parts}\}) = \frac{\binom{90}{10}\binom{5}{0}}{\binom{100}{10}} = 0.330$$

$$P(\{\text{the manufactuer accepts the lot with 15 defective parts}\}) = \frac{\binom{85}{10}\binom{5}{0}}{\binom{100}{10}} = 0.181$$

8. (a) For $s < c$, $P(S = s) = P(X = s) = f(s)$. For $s = c$, $P(S = s) = f(c) + P(X > c) = f(c) + 1 - P(X \leq c) = f(c) + 1 - F(c)$. Thus

$$f_S(x) = \begin{cases} f(x) & x \in \{0, 1, 2, \cdots, c-1\} \\ f(c) + 1 - F(c) & x \in \{c\} \\ 0 & \text{otherwise} \end{cases}$$

Then the expectation

$$E(S = \sum_{x=0}^{c-1} xf(x) + c(f(c) + 1 - F(c)) = \sum_{x=0}^{c} xf(x) + c(1 - F(c))$$

(b) $Y = d_2 S - d_1 c$. Then

$$E(Y) = d_2 E(S) - d_1 c = d_2 \sum_{x=0}^{c} xf(x) - d_2 cF(c) + (d_2 - d_1)c$$

(c) We need the $E(Y)$ be bigger for c than for $c+1$, so

$$\begin{aligned} &d_2 \sum_{x=0}^{c} xf(x) - d_2 cF(c) + (d_2 - d_1)c \geq d_2 \sum_{x=0}^{c+1} xf(x) - d_2(c+1)F(c+1) + (d_2 - d_1)(c+1) \\ \Rightarrow & d_2(c+1)F(c+1) - d_2 cF(c) \geq d_2(c+1)f(c+1) + (d_2 - d_1) \\ \Rightarrow & d_2(c+1)F(c+1) - d_2 cF(c) - d_2(c+1)F(c+1) + d_2(c+1)F(c) \geq d_2 - d_1 \\ \Rightarrow & d_2 F(c) \geq d_2 - d_1 \\ \Rightarrow & F(c) \geq \frac{d_2 - d_1}{d_2} \end{aligned}$$

Hence, c should be the smallest integer satistiying the inequality above.