STAT 542 Homework 5

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1. (a) $X \sim Binomial(n = 2000, p = 0.01)$.

(b) $P(X = 100) = \binom{2000}{100}(0.01)^{100}(0.99)^{1900}$.

2. (a) For $X \sim Possion(\lambda)$, we have

$$P(X > 0) = 1 - P(X = 0) = 1 - \frac{\mathrm{e}^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

Then

$$f_{X_T} = P(X_T = x) = \frac{P(X = x)}{P(X > 0)} = \frac{\mathrm{e}^{-\lambda}\lambda^x/x!}{1 - \mathrm{e}^{-\lambda}} = \frac{\lambda^x}{x!(\mathrm{e}^{\lambda} - 1)}$$

$$E(X_T) = \sum_{x=1}^{\infty} x\frac{P(X = x)}{P(X > 0)} = \frac{E(X)}{P(X > 0)} = \frac{\lambda}{1 - \mathrm{e}^{-\lambda}}$$

We also have

$$E(X_T^2) = \sum_{x=1}^{\infty} x^2\frac{P(X = x)}{P(X > 0)} = \frac{E(X^2)}{P(X > 0)} = \frac{\lambda^2 + \lambda}{1 - \mathrm{e}^{-\lambda}}$$

Then

$$Var(X_T) = E(X_T^2) - (E(X_T))^2 = \frac{\lambda^2 + \lambda}{1 - \mathrm{e}^{-\lambda}} + \left(\frac{\lambda}{1 - \mathrm{e}^{-\lambda}}\right)^2 = \frac{-\mathrm{e}^{-\lambda}\lambda^2 + (1 - \mathrm{e}^{-\lambda})\lambda}{(1 - \mathrm{e}^{-\lambda})^2}$$

3. $X \sim Gamma(\alpha, \beta)$, then $f_X(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$.

$$EX^{\nu} = \int_0^{\infty} x^{\nu}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}\mathrm{e}^{-x/\beta}\mathrm{d}x = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty} x^{\alpha+\nu-1}\mathrm{e}^{-x/\beta}\mathrm{d}x = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\Gamma(\alpha+\nu)\beta^{\alpha+\nu} = \frac{\Gamma(\alpha+\nu)\beta^{\nu}}{\Gamma(\alpha)}$$

4.

$$
\begin{aligned}
&\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1}\mathrm{e}^{-z}\mathrm{d}z \\
=&\int_x^\infty \frac{1}{(\alpha-1)!} z^{\alpha-1}\mathrm{d}(-\mathrm{e}^{-z}) \\
=&-\frac{z^{\alpha-1}\mathrm{e}^{-z}}{(\alpha-1)!}\bigg|_x^\infty + (\alpha-1)\int_x^\infty \mathrm{e}^{-z}z^{\alpha-2}\mathrm{d}z \\
=&\frac{z^{\alpha-1}\mathrm{e}^{-x}}{(\alpha-1)!} + \frac{1}{(\alpha-2)!}\int_x^\infty \mathrm{e}^{-z}z^{\alpha-2}\mathrm{d}z \\
=&\frac{z^{\alpha-1}\mathrm{e}^{-x}}{(\alpha-1)!} + \frac{z^{\alpha-2}\mathrm{e}^{-x}}{(\alpha-2)!} + \frac{1}{(\alpha-3)!}\int_x^\infty \mathrm{e}^{-z}z^{\alpha-3}\mathrm{d}z \\
=&\cdots \\
=&\frac{z^{\alpha-1}\mathrm{e}^{-x}}{(\alpha-1)!} + \frac{z^{\alpha-2}\mathrm{e}^{-x}}{(\alpha-2)!} + \cdots + \frac{z^{1}\mathrm{e}^{-x}}{1!} + \frac{1}{0!}\int_x^\infty \mathrm{e}^{-z}z^{0}\mathrm{d}z \\
=&\sum_{k=0}^{\alpha-1}\frac{z^{\alpha-1}}{k!}\mathrm{e}^{-x}
\end{aligned}
$$

5. (a) $g(x) = x^{1/\gamma}$ is monotone on $(0, \infty)$. And $f_X(x) = \frac{1}{\beta}\mathrm{e}^{-x/\beta}$, $x > 0$. $g^{-1}(y) = y^\gamma$, $\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right| = \gamma y^{\gamma-1}$.

For $y > 0$,

$$
f_Y(y) = f_X(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right| = \frac{1}{\beta}\mathrm{e}^{-y^\gamma/\beta}\gamma y^{\gamma-1}
$$

For $y \le 0$

$$
f_Y(y) = 0
$$

Verify the pdf:

$$
\begin{aligned}
&\int_0^\infty \frac{1}{\beta}\mathrm{e}^{-y^\gamma/\beta}\gamma y^{\gamma-1}\mathrm{d}y \\
=&\int_0^\infty \mathrm{e}^{-y^\gamma/\beta}\mathrm{d}(y^\gamma/\beta)
\end{aligned}
$$

Let $t = y^\gamma/\beta$

$$
=\mathrm{e}^{-t}\mathrm{d}t = -\mathrm{e}^{-t}\Big|_0^\infty = 1
$$

$$
\begin{aligned}
E(Y) = E(X^{1/\gamma}) &= \int_0^\infty x^{1/\gamma+1-1}\frac{1}{\beta}\mathrm{e}^{-x/\beta}\mathrm{d}x \\
&= \frac{1}{\beta}\Gamma(1/\gamma+1)\beta^{1/\gamma+1} \\
&= \Gamma(1/\gamma+1)\beta^{1/\gamma}
\end{aligned}
$$

$$
\begin{aligned}
E(Y^2) = E(X^{2/\gamma}) &= \int_0^\infty x^{2/\gamma+1-1}\frac{1}{\beta}\mathrm{e}^{-x/\beta}\mathrm{d}x \\
&= \frac{1}{\beta}\Gamma(2/\gamma+1)\beta^{2/\gamma+1} \\
&= \Gamma(2/\gamma+1)\beta^{2/\gamma} \\
Var(Y) &= E(Y^2) - (E(Y))^2 \\
&= \Gamma(2/\gamma+1)\beta^{2/\gamma} - \left(\Gamma(1/\gamma+1)\right)^2\beta^{2/\gamma}
\end{aligned}
$$

(c) $g(x) = \frac{1}{x}$ is monotone on $(0,\infty)$. And $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\mathrm{e}^{-x/\beta}$, $x > 0$. $g^{-1}(y) = \frac{1}{y}$, $\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right| = \frac{1}{y^2}$.

For $y > 0$,

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right| \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{y^{\alpha-1}}\mathrm{e}^{-\frac{1}{\beta y}}\cdot\frac{1}{y^2} \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{y^{\alpha+1}}\mathrm{e}^{-1/\beta y}
\end{aligned}
$$

For $y \leq 0$

$$
f_Y(y) = 0
$$

Verify the pdf:

$$
\begin{aligned}
&\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{y^{\alpha+1}}\mathrm{e}^{-1/\beta y}\mathrm{d}y \\
= &-\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{y^{\alpha-1}}\mathrm{e}^{-\frac{1}{\beta}\frac{1}{y}}\mathrm{d}\left(\frac{1}{y}\right)
\end{aligned}
$$

Let $t = 1/y$

$$
= \int_0^\infty \frac{1}{\Gamma(a)\beta^\alpha}t^{\alpha-1}\mathrm{e}^{-t/\beta}\mathrm{d}t = 1
$$

$$
\begin{aligned}
E(Y) = E(X^{-1}) &= \int_0^\infty x^{-1} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \mathrm{e}^{-x/\beta} \mathrm{d}x \\
&= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{(\alpha-1)\beta}
\end{aligned}
$$

$$
\begin{aligned}
E(Y^2) = E(X^{-2}) &= \int_0^\infty x^{-2} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \mathrm{e}^{-x/\beta} \mathrm{d}x \\
&= \frac{\Gamma(\alpha-2)\beta^{\alpha-2}}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{(\alpha-1)(\alpha-2)\beta^2}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= E(Y^2) - (E(Y))^2 \\
&= \frac{1}{(\alpha-1)(\alpha-2)\beta^2} - \frac{1}{(\alpha-1)^2\beta^2} \\
&= \frac{1}{(\alpha-1)^2(\alpha-2)\beta^2}
\end{aligned}
$$

6. (a)

$$
P(X \geq \mu) = \int_\mu^\infty \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)} \mathrm{d}x
$$

Let $t = \frac{x-\mu}{\sigma}$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{\pi(t^2+1)} \mathrm{d}t \\
&= \frac{1}{\pi} \arctan(t)\Big|_0^\infty \\
&= \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}
\end{aligned}
$$

Then

$$
P(X \leq \mu) = 1 - P(X \geq \mu) = \frac{1}{2}
$$

(b)

$$
P(X \geq \mu + \sigma) = \int_{\mu+\sigma}^\infty \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)} \mathrm{d}x
$$

Let $t = \frac{x-\mu}{\sigma}$

$$
\begin{aligned}
&= \int_1^\infty \frac{1}{\pi(1+t^2)} \mathrm{d}t \\
&= \frac{1}{\pi} \arctan(t)\Big|_1^\infty \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= \frac{1}{4}
\end{aligned}
$$

$$P(X \le \mu - \sigma) = \int_{-\infty}^{\mu-\sigma} \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)} \mathrm{d}x$$

Let $t = \frac{x-\mu}{\sigma}$

$$\begin{aligned}
&= \int_{-\infty}^{-1} \frac{1}{\pi(1+t^2)} \mathrm{d}t \\
&= \frac{1}{\pi} \arctan(t)\Big|_{-\infty}^{-1} \\
&= \frac{1}{\pi}(-\frac{\pi}{4} - (-\frac{\pi}{2})) \\
&= \frac{1}{4}
\end{aligned}$$

7. μ and σ are not unique.

$$\begin{aligned}
&P(|X| < 2) \\
&\quad = P(-2 < X < 2) \\
&\quad = P(\frac{-2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{2-\mu}{\sigma}) \\
&\quad = \Phi(\frac{2-\mu}{\sigma}) - \Phi(\frac{-2-\mu}{\sigma}) \\
&\quad = \Phi(\frac{2-\mu}{\sigma}) + \Phi(\frac{2+\mu}{\sigma}) - 1 = \frac{1}{2}
\end{aligned}$$

$\Rightarrow$

$$\Phi(\frac{2-\mu}{\sigma}) + \Phi(\frac{2+\mu}{\sigma}) = \frac{3}{2}$$

For $\mu = 0$, we need

$$2\Phi(\frac{2}{\sigma}) = \frac{3}{2} \Rightarrow \Phi(\frac{2}{\sigma}) = \frac{3}{4}$$

Beacause $\Phi(x)$ is continous function whose range is $(0, 1)$, there must be a point x_1 such that $\Phi(x_1) = \frac{3}{4}$. Then $(\mu, \sigma) = (0, \frac{2}{x_1})$ satisfies the condition.

For $\mu = 1$, we need

$$\Phi(\frac{1}{\sigma}) + \Phi(\frac{3}{\sigma}) = \frac{3}{2}$$

Beacause $\Phi(x) + \Phi(3x)$ is continous function whose range is $(0, 2)$ ($\lim_{x\to-\infty} \Phi(x) + \Phi(3x) = 0$, $\lim_{x\to\infty} \Phi(x) + \Phi(3x) = 1 + 1 = 2$), there must be a point x_2 such that $\Phi(x_2) + \Phi(3x_2) = \frac{3}{2}$. Then $(\mu, \sigma) = (1, \frac{2}{x_2})$ satisfies the condition.

Hence we have at leat two sets of (μ, σ) that satisfies the condition.

8. (a) $S(t) = P(T > t) = 1 - P(T \le t) = 1 - F(t)$.

(b)

$$\begin{aligned}
h(t) &= \lim_{\delta\to 0}\frac{P(t<T<t+\delta|T>t)}{\delta}\\
&= \lim_{\delta\to 0}\frac{P(t<T<t+\delta)}{\delta P(T>t)}\\
&= \lim_{\delta\to 0}\frac{F(t+\delta)-F(t)}{\delta(1-F(t))}\\
&= \frac{f(t)}{1-F(t)}
\end{aligned}$$

$S(t)=1-F(t)\Rightarrow S'(t)=-F'(t)=-f(t)$, thus

$$h(t)=\frac{-S'(t)}{S(t)}\Rightarrow S'(t)+h(t)S(t)=0$$

From the differential equation $S'(t)+h(t)S(t)=0$, $S(0)=1$, we can solve

$$S(t)=\exp\left(-\int_0^t h(x)\mathrm{d}x\right)$$

(c) $T\sim\mathrm{Exp}(\beta)$, then

$$f(t)=\frac{1}{\beta}\mathrm{e}^{-t/\beta},\ F(t)=\int_0^t=\frac{1}{\beta}\mathrm{e}^{-x/\beta}\mathrm{d}x=-\mathrm{e}^{-x/\beta}|_0^t=1-\mathrm{e}^{-t/\beta}$$

Then

$$h(t)=\frac{f(t)}{1-F(t)}=\frac{\frac{1}{\beta}\mathrm{e}^{-t/\beta}}{\mathrm{e}^{-t/\beta}}=\frac{1}{\beta}$$

$h(t)$ is a constant.

(d) $T\sim\mathrm{Weibull}(\gamma,\beta)$, then

$$\begin{aligned}
f(t) &= \frac{\gamma}{\beta}t^{\gamma-1}\mathrm{e}^{-t^\gamma/\beta}\\
F(t) &= \int_0^t\frac{\gamma}{\beta}x^{\gamma-1}\mathrm{e}^{-x^\gamma/\beta}\mathrm{d}x\\
&= \int_0^t \mathrm{e}^{-x^\gamma/\beta}\mathrm{d}(x^\gamma/\beta)
\end{aligned}$$

Let $u=x^\gamma/\beta$

$$\begin{aligned}
&= \int_0^{t^\gamma/\beta}\mathrm{e}^{-u}\mathrm{u}\\
&= -\mathrm{e}^{-u}\Big|_0^{t^\gamma/\beta}\\
&= 1-\mathrm{e}^{-t^\gamma/\beta}
\end{aligned}$$

Then

$$h(t)=\frac{f(t)}{1-F(t)}=\frac{\frac{\gamma}{\beta}t^{\gamma-1}\mathrm{e}^{-t^\gamma/\beta}}{\mathrm{e}^{-t^\gamma/\beta}}=\frac{\gamma}{\beta}t^{\gamma-1}$$

When $\gamma>1$, $h(t)$ increases as t increases.