STAT 510 Homework 8

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1. $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ are orthonormal eigenvectors of $\boldsymbol{\Sigma}$, then let $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \ldots & \boldsymbol{p}_n \end{bmatrix}$, we have $\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{I}$. Also, $\boldsymbol{\Sigma}\boldsymbol{p}_i = \lambda_i \boldsymbol{p}_i \Rightarrow \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \ldots & \boldsymbol{p}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \ldots & \boldsymbol{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Rightarrow$ $\boldsymbol{\Sigma}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda} \Rightarrow \boldsymbol{P}^T\boldsymbol{\Sigma}\boldsymbol{P} = \boldsymbol{\Lambda}$. Thus we have $\begin{bmatrix} \boldsymbol{p}_1^T & \boldsymbol{p}_2^T & \ldots & \boldsymbol{p}_n^T \end{bmatrix}^T \boldsymbol{y} = \boldsymbol{P}^T\boldsymbol{y}$ and $E\left(\boldsymbol{P}^T\boldsymbol{y}\right) = \boldsymbol{P}^T E\left(\boldsymbol{y}\right) = \boldsymbol{0}$, $Var\left(\boldsymbol{P}^T\boldsymbol{y}\right) = \boldsymbol{P}^T Var(\boldsymbol{y})\boldsymbol{P} = \boldsymbol{P}^T\boldsymbol{\Sigma}\boldsymbol{P} = \boldsymbol{\Lambda}$. Hence

$$\boldsymbol{P}^T\boldsymbol{y} \sim N\left(\boldsymbol{0}, \boldsymbol{\Sigma} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\right)$$

2. (a) $y_{ijk} = \mu + \alpha_i + u_{ij} + e_{ijk}$. Here u_{ij} and e_{ijk} are independent random variables and $u_{ij} \sim N(0, \sigma_u^2)$ and $e_{ijk} \sim N(0, \sigma_e^2)$.

$$\begin{aligned}
\boldsymbol{y} &= \begin{bmatrix} y_{111} & y_{112} & y_{121} & \cdots & y_{142} & y_{211} & \cdots & y_{342} \end{bmatrix}^T \\
\boldsymbol{\beta} &= \begin{bmatrix} \mu & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T \\
\boldsymbol{u} &= \begin{bmatrix} u_{11} & \cdots & u_{14} & u_{21} & \cdots & u_{34} \end{bmatrix}^T \\
\boldsymbol{\epsilon} &= \begin{bmatrix} e_{111} & e_{112} & e_{121} & \cdots & e_{142} & e_{211} & \cdots & e_{342} \end{bmatrix}^T \\
\boldsymbol{X} &= \begin{bmatrix} \boldsymbol{1}_{24\times 1}, \boldsymbol{I}_{3\times 3} \otimes \boldsymbol{1}_{8\times 1} \end{bmatrix} \\
\boldsymbol{Z} &= \begin{bmatrix} \boldsymbol{I}_{12\times 12} \otimes \boldsymbol{1}_{2\times 1} \end{bmatrix} \\
\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\epsilon} \end{bmatrix} &\sim N\left(\boldsymbol{0}, \begin{bmatrix} \sigma_u^2 \boldsymbol{I}_{12\times 12} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_e^2 \boldsymbol{I}_{24\times 24} \end{bmatrix}\right)
\end{aligned}$$

(b)

Source	DF	Sums of Squares	Mean Squares	Expected Mean Squares
temperature	2	$\sum_{i=1}^3 \sum_{j=1}^4 \sum_{k=1}^2 (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$	$SS/2$	$\sigma_e^2 + 2\sigma_u^2 + 4\sum_{i=1}^3 (\alpha_i - \bar{\alpha}_\cdot)^2$
cooler(temperature)	9	$\sum_{i=1}^3 \sum_{j=1}^4 \sum_{k=1}^2 (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})^2$	$SS/9$	$\sigma_e^2 + 2\sigma_u^2$
meat(cooler, temperature)	12	$\sum_{i=1}^3 \sum_{j=1}^4 \sum_{k=1}^2 (\bar{y}_{ijk} - \bar{y}_{ij\cdot})^2$	$SS/12$	σ_e^2
c.total	23	$\sum_{i=1}^3 \sum_{j=1}^4 \sum_{k=1}^2 (y_{ijk} - \bar{y}_{\cdot\cdot\cdot})^2$		

(c)

$$T = \frac{2(\bar{y}_{1\cdot\cdot} - \bar{y}_{2\cdot\cdot})}{\sqrt{MS_{\text{cooler(temperature)}}}}$$

(d) df = 9

(e) Non-central parameter is

$$\frac{2(\alpha_1 - \alpha_2)}{\sqrt{\sigma_e^2 + 2\sigma_u^2}}$$

3. (a) Meat.

(b)

Source	DF
temperature	2
preservative	1
temperature × preservative	2
cooler(temperature)	9
error	9
c.total	23

(c) cooler(temperature)

(d) error

4. (a)

$$\begin{aligned} Cov(y_{ij1}, y_{ij2}) &= Cov(g_i + t_{ij} + e_{ij1}, g_i + t_{ij} + e_{ij2}) \\ &= Var(g_i) + Var(t_{ij}) \\ &= \sigma_g^2 + \sigma_t^2 \end{aligned}$$

$$\begin{aligned} Var(y_{ijk}) &= Var(g_i + t_{ij} + e_{ijk}) \\ &= Var(g_i) + Var(t_{ij}) + Var(e_{ijk}) \\ &= \sigma + g^2 + \sigma_t^2 + \sigma_e^2 \end{aligned}$$

$$\begin{aligned} Corr(y_{ij1}, y_{ij2}) &= \frac{Cov(y_{ij1}, y_{ij2})}{\sqrt{Vary_{ij1}}\sqrt{Var(y_{ij2})}} \\ &= \frac{\sigma_g^2 + \sigma_t^2}{\sigma_g^2 + \sigma_t^2 + \sigma_e^2} \end{aligned}$$

(b) $H_0 : \gamma_1 + \bar{\phi}_{\cdot 1} = \gamma_2 + \bar{\phi}_{\cdot 2}$

(c)

$$\begin{aligned} &\boldsymbol{\beta} = \begin{bmatrix} \mu & \omega_1 & \omega_2 & \omega_3 & \gamma_1 & \gamma_2 & \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} & \phi_{31} & \phi_{32} \end{bmatrix}^T \\ &\boldsymbol{u} = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & t_{11} & t_{12} & t_{13} & t_{21} & t_{22} & t_{23} & t_{31} & t_{32} & t_{33} & t_{41} & t_{42} & t_{43} \end{bmatrix}^T \\ &\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{24\times 1}, \mathbf{1}_{4\times 1} \otimes \boldsymbol{I}_{3\times 3} \otimes \mathbf{1}_{2\times 1}, \mathbf{1}_{12\times 1} \otimes \boldsymbol{I}_{2\times 2}, \mathbf{1}_{4\times 1} \otimes \boldsymbol{I}_{6\times 6} \end{bmatrix} \\ &\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{I}_{4\times 4} \otimes \mathbf{1}_{6\times 1}, \boldsymbol{I}_{12\times 12} \otimes \mathbf{1}_{2\times 1} \end{bmatrix} \end{aligned}$$