STAT 510 Homework 4

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1. (a)

$$\begin{bmatrix} \hat{\boldsymbol{y}} \\ \boldsymbol{y} - \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X y} \\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{y}$$

We know $\boldsymbol{y} \sim N(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$, thus $\begin{bmatrix} \hat{\boldsymbol{y}} \\ \boldsymbol{y} - \hat{\boldsymbol{y}} \end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\begin{aligned}
\mu &= \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{X\beta} = \begin{bmatrix} \boldsymbol{P_X X\beta} \\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{X\beta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X\beta} \\ \boldsymbol{0} \end{bmatrix} \\
\boldsymbol{\Sigma} &= \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \sigma^2 \boldsymbol{I} \begin{bmatrix} \boldsymbol{P_X^T} & (\boldsymbol{I} - \boldsymbol{P_X})^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{P_X P_X^T} & \boldsymbol{P_X}(\boldsymbol{I} - \boldsymbol{P_X})^T \\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{P_X^T} & (\boldsymbol{I} - \boldsymbol{P_X})(\boldsymbol{I} - \boldsymbol{P_X})^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{P_X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{bmatrix} \hat{\boldsymbol{y}} \\ \boldsymbol{y} - \hat{\boldsymbol{y}} \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{X\beta} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{P_X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \right)$$

(b) We have

$$\hat{\boldsymbol{y}}^T \hat{\boldsymbol{y}} = \boldsymbol{y}^T \boldsymbol{P_X^T P_X y} = \boldsymbol{y}^T \boldsymbol{P_X y}$$

So let $\boldsymbol{A} = \frac{1}{\sigma^2}\boldsymbol{P_X}$, we have $rank(\boldsymbol{A}) = rank(\boldsymbol{P_X}) = rank(\boldsymbol{X}) = r$. Also, $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}$ is positive definite, and

$$\boldsymbol{A\Sigma A\Sigma} = \frac{1}{\sigma^2}\boldsymbol{P_X}\sigma^2\boldsymbol{I}\frac{1}{\sigma^2}\boldsymbol{P_X}\sigma^2\boldsymbol{I} = \frac{1}{\sigma^2}\boldsymbol{P_X}\sigma^2\boldsymbol{I} = \boldsymbol{A\Sigma}$$

From $\boldsymbol{y} \sim N(\boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ we have

$$\boldsymbol{y}^T\boldsymbol{A y} = \frac{1}{\sigma^2}\boldsymbol{y}^T\boldsymbol{P_X y} \sim \chi_r^2\left(\boldsymbol{\beta}^T\boldsymbol{X}^T\frac{1}{\sigma^2}\boldsymbol{P_X X\beta}/2\right) = \chi_r^2\left(\frac{1}{\sigma^2}\boldsymbol{\beta}^2\boldsymbol{X}^T\boldsymbol{X\beta}/2\right)$$

Thus

$$\hat{\boldsymbol{y}}^T\hat{\boldsymbol{y}} \sim \sigma^2\chi_r^2\left(\frac{1}{\sigma^2}\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X\beta}/2\right)$$

2. (a)

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) By Gauss-Markov Theorem, BLUE here is OLSE, thus

$$\text{BLUE}(\beta_4) = \text{OLSE}(\beta_4) = \text{OLSE}(\beta_1+\beta_2+\beta_3+\beta_4) - \text{OLSE}(\beta_1+\beta_2+\beta_3) = 26.3 - 22.8 = 3.5$$

(c)

$$\hat{\text{Var}}(\beta_4) = \hat{\sigma}^2 \boldsymbol{C}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{C}^T = \boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{X}^T\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{P_X}\boldsymbol{A}^T$$

The matrix $\boldsymbol{A}$ we use here is $\begin{bmatrix} 0 & \cdots & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$, which is literally take the last row of a matrix and minus the last 5th row. And in this model, the estimated response variable is just sample means of each treatment. Then we have

$$\boldsymbol{A}\boldsymbol{P_X} = \begin{bmatrix} 0 & \cdots & 0 - 1/2 & -1/2 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Then we have $\boldsymbol{A}\boldsymbol{P_X}\boldsymbol{A}^T = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.
And

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^4 s_i^2(n_i - 1)}{\sum_{i=1}^4 n_i - 4} = 3.428571$$

Then

$$se(\hat{\beta}_4) = \sqrt{\frac{3}{4}\hat{\sigma}^2} = 1.603576$$

3. (a)

$$\begin{aligned} \boldsymbol{X}^T\boldsymbol{X} &= \begin{bmatrix} \boldsymbol{1}^T \\ \boldsymbol{x}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{x} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{1}^T\boldsymbol{1} & \boldsymbol{1}^T\boldsymbol{x} \\ \boldsymbol{x}^T\boldsymbol{1} & \boldsymbol{x}^T\boldsymbol{x} \end{bmatrix} \\ &= \begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} \end{aligned}$$

Take the inverse, we have

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \begin{bmatrix} \frac{\sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} & \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\ \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} & \frac{n}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \end{bmatrix}$$

And

$$\boldsymbol{X}^T\boldsymbol{y} = \begin{bmatrix} \boldsymbol{1}^T \\ \boldsymbol{x}^T \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} \boldsymbol{1}^T\boldsymbol{y} \\ \boldsymbol{x}^T\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix}$$

Then

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} \\
&= \begin{bmatrix} \frac{\sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} & \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\ \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} & \frac{n}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix} \\
&= \begin{bmatrix} \frac{(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i) - (\sum_{i=1}^n x_i)(\sum_{i=1}^n x_i y_i)}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\ \frac{n\sum_{i=1}^n x_i y_i - (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\bar{x^2}\bar{y} - \bar{x}\bar{xy}}{\bar{x^2} - (\bar{x})^2} \\ \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} \end{bmatrix}
\end{aligned}
$$

(b) $\begin{bmatrix} \mathbf{1} & \boldsymbol{x} - \bar{x}\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \boldsymbol{x} \end{bmatrix} \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix}$. Thus we choose $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix}$

(c) We have

$$
\boldsymbol{W}^T\boldsymbol{W} = \begin{bmatrix} \mathbf{1}^T \\ (\boldsymbol{x} - \bar{x}\mathbf{1})^T \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{x} - \bar{x}\mathbf{1} \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \sum_{i=1}^n (x_i - \bar{x}) \end{bmatrix}
$$

Also,

$$
\boldsymbol{W}^T\boldsymbol{y} = \begin{bmatrix} \mathbf{1}^T\boldsymbol{y} \\ (\boldsymbol{x} - \bar{x}\mathbf{1})^T\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n (x_i - \bar{x}) y_i \end{bmatrix}
$$

Hence

$$
\begin{aligned}
\hat{\boldsymbol{\alpha}} &= (\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T\boldsymbol{y} \\
&= \begin{bmatrix} 1/n & 0 \\ 0 & 1/\sum_{i=1}^n (x_i - \bar{x}) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n (x_i - \bar{x}) y_i \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{i=1}^n y_i}{n} \\ \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix} \\
&= \begin{bmatrix} \bar{y} \\ \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} \end{bmatrix}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sum_{i=1}^n y_i}{n} \\ \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{i=1}^n y_i/n - \bar{x}\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{y} \\ \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\bar{x^2}\bar{y} - (\bar{x})^2\bar{y} - \bar{x}\bar{xy} + (\bar{x})^2\bar{y}}{\bar{x^2} - (\bar{x})^2} \\ \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\bar{x^2}\bar{y} - \bar{x}\bar{xy}}{\bar{x^2} - (\bar{x})^2} \\ \frac{\bar{xy} - \bar{x}\bar{y}}{\bar{x^2} - (\bar{x})^2} \end{bmatrix}
\end{aligned}
$$

It matches (a).

4. (a) The null hypothesis tested is the difference in score students gave for different perceived instructor when ruling out any possible impact the actually instructor gave is zero.

(b) The matrix $(\boldsymbol{X}_R^T\boldsymbol{X}_R)^-$ we got is

$$\begin{bmatrix} 0.1 & -0.1 & -0.1 & 0.1 \\ -0.1 & 0.2 & 0.1 & -0.2 \\ -0.1 & 0.1 & 0.2 & -0.2 \\ 0.1 & -0.2 & -0.2 & 0.377 \end{bmatrix}$$

With we can have

$$\hat{\sigma}^2 = (10se(\text{Intercept}))^2 = 0.507$$

To calculate the variance of main effect of perceived instructor, we use $\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 & 1/2 \end{bmatrix}$ Hence

$$\hat{\text{Var}}(\text{main effect of perceived instructor}) = \hat{\sigma}^2\boldsymbol{C}(\boldsymbol{X}_R^T\boldsymbol{X}_R)^-\boldsymbol{C}^T = 0.047775$$

Thus the standard error is $\sqrt{0.0047775} = 0.2185749$.

Also, we have the estimate of main effect of perceived instructor to be $\begin{bmatrix} 0 & 0 & 1 & 1/2 \end{bmatrix}\hat{\boldsymbol{\beta}} = 0.8700 - 0.1831/2 = 0.77845$. And we know $t_{43-4,0.975} = t_{39,0.975} = 2.022691$. Thus the 95% confidence interval

$$[0.77845 - 0.2185749 \times 2.022691, 0.77845 + 0.2185749 \times 2.022691] = [0.3363, 1.2206]$$