STAT 542 Homework 7

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1. (a) $X \sim \text{exponential}(1)$, then

$$P(Y=k) = P(k-1 \le X < k) = \int_{k-1}^{k} \mathrm{e}^{-x}\mathrm{d}x = -\mathrm{e}^{-k} + \mathrm{e}^{-(k-1)} = (\mathrm{e}^{-1})^{k-1}(1-\mathrm{e}^{-1})$$

Hence $Y \sim Geometric(1-\mathrm{e}^{-1})$.

(b) $P(X-4=x|Y\ge 5) = P(X-4=x|\lfloor X \rfloor \ge 4) = P(X=x+4|X\ge 4) = \frac{\mathrm{e}^{-(x+4)}}{\mathrm{e}^{-4}} = \mathrm{e}^{-x}$. Thus $X-4|Y\ge 5 \sim \text{exponential}(1)$.

2.

$$\begin{aligned} f(y,\lambda) &= f(\lambda)\cdot f(y|\lambda) \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\mathrm{e}^{-\lambda/\beta}\cdot\frac{\lambda^y}{y!}\mathrm{e}^{-\lambda} \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\lambda^{y+\alpha-1}\mathrm{e}^{-\lambda(1+\frac{1}{\beta})} \qquad \lambda>0, y=0,1,2,\cdots \end{aligned}$$

$$\begin{aligned} f(y) &= \int_0^\infty \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\lambda^{y+\alpha-1}\mathrm{e}^{-\lambda(1+\frac{1}{\beta})}\mathrm{d}\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \lambda^{y+\alpha-1}\mathrm{e}^{-\lambda(1+\frac{1}{\beta})}\mathrm{d}\lambda \\ &= \frac{\Gamma(y+\alpha)}{y!\Gamma(\alpha)\beta^\alpha}\left(\frac{\beta}{1+\beta}\right)^{y+\alpha} \\ &= \frac{(y+\alpha-1)!}{y!(\alpha-1)!\beta^\alpha}\left(\frac{\beta}{1+\beta}\right)^{y+\alpha} \\ &= \binom{y+\alpha-1}{y}\left(\frac{\beta}{1+\beta}\right)^{y}\left(\frac{1}{1+\beta}\right)^{\alpha} \end{aligned}$$

Hence $Y \sim Negative-Binomial(\alpha, 1/(1+\beta))$.

$E(Y) = \frac{(1-p)r}{p} = \frac{\beta}{1+\beta}(\beta+1)\alpha = \alpha\beta$.

$Var(Y) = \frac{(1-p)r}{p^2} = \frac{\beta}{1+\beta}(\beta+1)^2\alpha = \alpha\beta(1+\beta)$.

3. (a) $Cov(X,C) = E(CX) - E(X)E(C) = CE(X) - CE(X) = 0$

(b) $f(y|x) = \frac{f(x,y)}{f_X(x)}$, thus for $f_X(x)>0$,

$$E[g(X)h(Y)|X=x] = \sum_y g(x)h(y)f(y|x) = g(x)\sum_y h(y)f(y|x) = g(x)E[g(Y)|X=x]$$

4. (a)

$$\begin{aligned}
Var[X] &= E[Var[X|P]] + Var[E[X|P]] \\
&= E[nP(1-P)] + Var[nP] \\
&= nE[P] - nE[p^2] + n^2 Var[P] \\
&= nE[P] - n(Var[P] + (E[P])^2) + n^2 Var[P] \\
&= nE[P] - n(E[P])^2 + (n^2 - n)Var[P] \\
&= n\frac{\alpha}{\alpha+\beta} - n\frac{\alpha^2}{(\alpha+\beta)^2} + (n^2-n)\frac{\alpha\beta}{(\alpha+\beta)(1+\alpha+\beta)} \\
&= \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}$$

(b)

$$E[X] = E[E[X|P]] = E[nP] = \frac{n\alpha}{\alpha+\beta}$$

Thus $E[W] = n\tilde{p} = \frac{n\alpha}{\alpha+\beta} \Rightarrow \tilde{p} = \frac{\alpha}{\alpha+\beta}$.

$$Var[W] = n\tilde{p}(1-\tilde{p}) = \frac{n\alpha\beta}{(\alpha+\beta)^2}$$

Hence we have

$$\frac{Var[X]}{Var[W]} = \frac{\alpha+\beta+n}{\alpha+\beta+1} > 1$$

5. (a)

$$\begin{aligned}
f_X(x) &= \int_0^{1-x} f(x,y)\mathrm{d}y \\
&= \int_0^{1-x} 3(x+y)\mathrm{d}y \\
&= 3(xy + \frac{1}{2}y^2)\Big|_0^{1-x} \\
&= \frac{3}{2}(1-x^2)
\end{aligned}$$

(b)

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2(x+y)}{1-x^2},\ 0 < y < 1-x$$

(c)

$$E[Y|X=x] = \int_0^{1-x} yf(y|X)\mathrm{d}y = \int_0^{1-x} \frac{2(x+y)y}{1-x^2}\mathrm{d}y = \frac{2}{1-x^2}(\frac{1}{2}xy^2 + \frac{1}{3}y^3)\Big|_0^{1-x} = \frac{(1-x)(x+2)}{3(x+1)}$$

(d) By symmetry of the support and pdf, $E(X|Y=y) = \frac{(1-y)(y+2)}{3(y+1)}$.

(e)

$$\begin{aligned}
E[XY - Y|X] &= 2E[XY|X] - E[Y|X] \\
&= 2XE[Y|X] - E[Y|X] \\
&= (2X-1)\frac{(1-X)(X+2)}{3(X+1)}
\end{aligned}$$

$$
\begin{aligned}
E[E[XY-Y|X]] &= E[\frac{(2X-1)(1-X)(X+2)}{3(X+1)}] \\
&= \int_0^1 \frac{(2x-1)(1-x)(x+2)}{3(x+1)}\frac{3}{2}(1-x^2)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (2x-1)(1-x)^2(x+2)\mathrm{d}x \\
&= -\frac{7}{40}
\end{aligned}
$$

6. (a)

$$
\begin{aligned}
M_{X,Y}(t_1,t_2) &= E(\mathrm{e}^{t_1X+t_2Y}) \\
&= \int_0^\infty \left(\int_0^y \mathrm{e}^{t_1x+t_2y}\mathrm{e}^{-y}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^\infty \mathrm{e}^{t_2y-y}\frac{1}{t_1}(\mathrm{e}^{t_1y}-1)\mathrm{d}y \\
&= \frac{1}{t_1}\int_0^\infty (\mathrm{e}^{t_1+t_2-1}y - \mathrm{e}^{t_2-1}y)\mathrm{d}y \\
&= \frac{1}{(t_2-1)(t_1+t_2-1)} \qquad t_1+t_2<1,\ t_2<1
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\frac{\partial}{\partial t_1}M_{X,Y}(t_1,t_2) = -\frac{1}{(t_2-1)(t_1+t_2-1)^2} \Rightarrow E[X]=1 \\
&\frac{\partial}{\partial t_2}M_{X,Y}(t_1,t_2) = -\frac{1}{(t_2-1)(t_1+t_2-1)^2} - \frac{1}{(t_1+t_2-1)(t_2-1)^2} \Rightarrow E[Y]=2 \\
&\frac{\partial^2}{\partial t_1^2}M_{X,Y}(t_1,t_2) = \frac{2}{(t_2-1)(t_1+t_2-1)^3} \Rightarrow E[X^2]=2 \\
&\frac{\partial^2}{\partial t_2^2}M_{X,Y}(t_1,t_2) = \frac{2}{(t_2-1)(t_1+t_2-1)^3} + \frac{2}{(t_1+t_2-1)^2(t_2-1)^2} + \frac{2}{(t_2-1)^3(t_1+t_2-1)} \Rightarrow E[Y^2]=6 \\
&\frac{\partial^2}{\partial t_2\partial t_1}M_{X,Y}(t_1,t_2) = \frac{1}{(t_2-1)^2(t_1+t_2-1)^2} + \frac{2}{(t_2-1)(t_1+t_2-1)^3} \Rightarrow E[XY]=3
\end{aligned}
$$

Thus,

$$\sigma_X^2 = 2-1^2 = 1,\ \sigma_Y^2 = 6-2^2 = 2,\ \sigma_{XY} = 3-1\cdot 2 = 1$$

(c) From the joint mgf, the mgf of X and Y are

$$
\begin{aligned}
M_X(t) &= M_{X,Y}(t,0) = -\frac{1}{t-1} \\
M_Y(t) &= M_{X,Y}(0,t) = \frac{1}{(t-1)^2}
\end{aligned}
$$

The pdf of X and Y are

$$
\begin{aligned}
f_X(x) &= \int_x^\infty \mathrm{e}^{-y}\mathrm{d}y = \mathrm{e}^{-x} \\
f_Y(y) &= \int_0^y \mathrm{e}^{-y}\mathrm{d}x = y\mathrm{e}^{-y}
\end{aligned}
$$

Thus the mgf are

$$
\begin{aligned}
M_X(t) &= \int_0^\infty \mathrm{e}^{tx}\mathrm{e}^{-x}\mathrm{d}x \\
&= \frac{1}{1-t}\mathrm{e}^{(1-t)x}\Big|_0^\infty = -\frac{1}{t-1} \\
M_Y(y) &= \int_0^\infty y\mathrm{e}^{ty}\mathrm{e}^{-y}\mathrm{d}y \\
&= \frac{y}{t-1}\mathrm{e}^{(t-1)y}\Big|_0^\infty - \frac{1}{t-1}\int_0^\infty \mathrm{e}^{(t-1)y}\mathrm{d}y \\
&= -\frac{1}{t-1}\frac{1}{t-1}\mathrm{e}^{(t-1)y}\Big|_0^\infty = \frac{1}{(t-1)^2}
\end{aligned}
$$

Same as the mgf's based on the joint mgf.