STAT 542 Homework 11

Yifan Zhu

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1. (a) Since $X_i \sim Bernolli(p_i)$, then $E(X_i - p_i) = 0$ and $Var(X_i - p_i) = Var(X_i) = p_i(1-p_i) \leq \frac{1}{4}$.

$$E(Y_n) = E(\frac{1}{n}\sum_{i=1}^{n}(X_i - p_i)) = 0$$

$$Var(Y_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i - p_i) = \frac{\sum_{i=1}^{n} p_i(1-p_i)}{n^2} \leq \frac{1}{4n}$$

Thus

$$E(Y_n^2) = Var(Y_n) + (E(Y_n))^2 = Var(Y_n) \leq \frac{1}{4n}$$

By Markov's Inequality,

$$\begin{aligned} P(|Y_n| > \epsilon) &= P(Y_n^2 > \epsilon^2) \\ &\leq \frac{E(Y_n^2)}{\epsilon^2} \\ &\leq \frac{1}{4n\epsilon^2} \end{aligned}$$

Hence $P(|Y_n| > \epsilon) \to 0$ as $n \to \infty$, $Y_n \xrightarrow{p} 0$.

(b) For any $\epsilon > 0$, $Z_n \xrightarrow{p} a$, $W_n \xrightarrow{p} b \Rightarrow P(|Z_n - a| > \epsilon/2) \to 0$ and $P(|W_n - b| > \epsilon/2) \to 0$ as $n \to \infty$. Also, $|Z_n + W_n - (a+b)| = |(Z_n - a) + (W_n - b)| \leq |Z_n - a| + |W_n - b|$. Thus $|Z_n + W_n - (a+b)| > \epsilon \Rightarrow |Z_n - a| + |W_n - b| > \epsilon \Rightarrow |Z_n - a| > \epsilon/2$ or $|W_n - b| > \epsilon/2$. Hence we have

$$P(|Z_n+W_n-(a+b)| > \epsilon) \leq P(\{|Z_n-a| > \epsilon/2\} \cup \{|W_n-b| > \epsilon/2\}) \leq P(|Z_n-a| > \epsilon/2) + P(|W_n-b| > \epsilon/2)$$

Hence $P(|Z_n + W_n - (a+b)| > \epsilon) \to 0$ as $n \to \infty$, $Z_n + W_n \xrightarrow{p} a + b$.

(c) For $p_i = \frac{i}{n}$,

$$\begin{aligned} Y_n &= \frac{1}{n}\sum_{i=1}^{n}(X_i - p_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} p_i \\ &= \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\frac{n(n-1)}{2n} \end{aligned}$$

Thus $\frac{1}{n}\sum_{i=1}^{n} X_i = Y_n + \frac{n-1}{2n}$. As $Y_n \xrightarrow{p} 0$, $\frac{n-1}{2n} \xrightarrow{p} \frac{1}{2}$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \frac{1}{2}$$

2. (a)

$$F(x) = \int_1^x f(t)\mathrm{d}t = \int_1^x t^{-2}\mathrm{d}t = 1 - \frac{1}{x},\ x > 1$$

(b)

$$\begin{aligned} F_{X_{(n)}}(y) &= P(X_{(n)} \le y) \\ &= \prod_{i=1}^{n} P(X_i \le y) \\ &= (1 - \frac{1}{y})^n,\ y > 1 \end{aligned}$$

If there exists a random variable Y such that $X_{(n)} \to Y$, then for any continuous point y of F_Y, we have

$$F_Y(y) = \lim_{n\to\infty} F_{X_{(n)}}(y) = \lim_{n\to\infty}(1 - \frac{1}{y})^n = 0.$$

It conflicts with the fact that any random variable Y will always have a continuous point y in its cdf for which $F_Y(y) > 0$. Thus $X_{(n)}$ does not have a limiting distribution.

(c)

$$\begin{aligned} F_{X_{(n)}/n}(y) &= P(\frac{X_{(n)}}{n} \le y) \\ &= P(X_{(n)} \le ny) \\ &= (1 - \frac{1}{ny})^n \to \mathrm{e}^{-\frac{1}{y}}, \quad n \to \infty \end{aligned}$$

Hence the liminting distribution is $f(y) = \mathrm{e}^{-\frac{1}{y}}$, $y > 0$, $\frac{X_{(n)}}{n} \xrightarrow{d} InverseExponential(1)$

3. Let $Y_n = \sum_{i=1}^{n}(Z_i + \frac{1}{n})/\sqrt{n}$,

$$\begin{aligned} M_{Y_n}(t) &= E(\mathrm{e}^{t\sum_{i=1}^{n}(Z_i+\frac{1}{n})/\sqrt{n}}) \\ &= E(\mathrm{e}^{\frac{t}{\sqrt{n}}\sum_{i=1}^{n} Z_i + \frac{t}{\sqrt{n}}}) \\ &= \prod_{i=1}^{n} E(\mathrm{e}^{\frac{t}{\sqrt{n}}Z_i})\mathrm{e}^{\frac{t}{\sqrt{n}}} \\ &= (\mathrm{e}^{\frac{1}{2}(\frac{t}{\sqrt{n}})^2})^n \mathrm{e}^{\frac{t}{\sqrt{n}}} \\ &= \mathrm{e}^{\frac{1}{2}t^2}\mathrm{e}^{\frac{t}{\sqrt{n}}} \end{aligned}$$

$M_{Y_n}(t) \to \mathrm{e}^{\frac{1}{2}t^2}$, thus $Y_n = \sum_{i=1}^{n}(Z_i + \frac{1}{n})/\sqrt{n} \xrightarrow{d} N(0,1)$.

4. From $Y_n \xrightarrow{p} c$ we know $Y_n \xrightarrow{d} c$.

On the other hand, if $Y_n \xrightarrow{d} c$, then the limiting distribution cdf will be $F(y) = \begin{cases} 1, & y \geq c \\ 0. & y < c \end{cases}$. Hence for any $\epsilon > 0$, $c+\epsilon$ and $c-\epsilon$ are continuous points, and then

$$\begin{aligned} P(|Y_n - c| > \epsilon) &= P(\{Y_n < c-\epsilon\} \cup \{Y_n > c+\epsilon\}) \\ &\leq P(Y_n \leq c-\epsilon) + P(Y_n > c+\epsilon) \\ &= F_{Y_n}(c-\epsilon) + 1 - F_{Y_n}(c+\epsilon) \to F(c-\epsilon) + 1 - F(c+\epsilon) = 0,\ n \to \infty \end{aligned}$$

Hence $\lim_{n\to\infty} P(|Y_n - c| > \epsilon) = 0$, thus $Y_n \xrightarrow{p} c$.

5. $X_i \sim Beta(1,\beta) \Rightarrow f_{X_1}(x) = \beta(1-x)^{\beta-1}, F_{X_1}(x) = 1-(1-x)^\beta$
When $\nu = \frac{1}{\beta}$,

$$\begin{aligned} P(n^\nu(1-X_{(n)}) > t) &= P(X_{(n)} < 1 - \frac{t}{n^\nu}) \\ &= (F_{X_1}(1-\frac{t}{n^\nu}))^n \\ &= (1-(1-1+\frac{t}{n^\nu})^\beta)^n \\ &= (1-\frac{t^\beta}{n^{\nu\beta}})^n \\ &= (1-\frac{t^\beta}{n})^n \to \mathrm{e}^{-t^\beta} \end{aligned}$$

Hence

$$P(n^\nu(1-X_{(n)}) \leq t) \to 1 - \mathrm{e}^{-t^\beta}$$

6. We know that $Y_n \stackrel{d}{=} \sum_{i=1}^n X_i$ where $X_i \sim \chi_1^2$. $E(X_i) = 1$, $Var(X_i) = 2$.

Then $\frac{Y_n - n}{\sqrt{2n}} = \frac{\sqrt{n}\frac{Y_n}{n} - 1}{\sqrt{2}} \stackrel{d}{=} \frac{\sqrt{n}(\frac{\sum_{i=1}^n X_i}{n} - 1)}{\sqrt{2}} = \frac{\sqrt{n}(\bar{X}_n - 1)}{\sqrt{2}} \xrightarrow{d} N(0,1)$. (By CLT) Hence $\frac{Y_n - n}{\sqrt{2n}} \xrightarrow{d} N(0,1)$

7. (a) $g(x) = \log x + x^2$ is a continuous function for $x > 1$. By WLLN, $\bar{X}_n \xrightarrow{p} \mu$, then by continuous mapping,

$$g(\bar{X}_n) = \log \bar{X}_n + \bar{X}_n^2 \xrightarrow{p} \log\mu + \mu^2$$

(b) By CLT, $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0,\sigma^2)$. $\bar{X}_n + \mu \xrightarrow{d} \mu + \mu = 2\mu$. By Slutsky's theorem, $\sqrt{n}(\bar{X}_n^2 - \mu^2) = \sqrt{n}(\bar{X}_n - \mu)(\bar{X}_n + \mu) \xrightarrow{d} N(0, 4\mu^2\sigma^2)$.

(c) $\sqrt{n}(\bar{X}_n^2 - \mu^2) \to N(0, 4\mu^2\sigma^2)$, $\log \bar{X}_n \xrightarrow{d} \log\mu$. By Slutsky's theorem, $\frac{\sqrt{n}(\bar{X}_n^2 - \mu^2)}{\log \bar{X}_n} \xrightarrow{d} N(0, \frac{4\mu^2\sigma^2}{(\log\mu)^2})$.