# STAT 510 Homework 6

Yifan Zhu

February 28, 2017

**1.** (a)

$$\boldsymbol{c}_i^T \boldsymbol{\beta} = \boldsymbol{0} \iff (\boldsymbol{P}_{i+1} - \boldsymbol{P}_i)\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{0}$$

As $\text{rank}(\boldsymbol{P}_{j+1} - \boldsymbol{P}_j) = 1$ in this case, we can just pick one row of the matrix $(\boldsymbol{P}_{j+1} - \boldsymbol{P}_j)\boldsymbol{X}$ as our vector $\boldsymbol{c}_i^T$. Then

$$\begin{aligned}
\boldsymbol{c}_1^T &= \begin{bmatrix} 2 & 1 & 0 & -1 & 0 \end{bmatrix} \\
\boldsymbol{c}_2^T &= \begin{bmatrix} 2 & -1 & -2 & -1 & 2 \end{bmatrix} \\
\boldsymbol{c}_3^T &= \begin{bmatrix} 1 & -2 & 0 & 2 & -1 \end{bmatrix} \\
\boldsymbol{c}_4^T &= \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \end{bmatrix}
\end{aligned}$$

(b) As for all $\boldsymbol{c}_i^T$, we have $\boldsymbol{c}_i^T \boldsymbol{1} = 0$, then $\boldsymbol{c}_i^T \boldsymbol{\beta}$ are contrast.

(c) In this case, $(\boldsymbol{X}^T\boldsymbol{X})^- = \frac{1}{3}\boldsymbol{I}_5$, then $\boldsymbol{C}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{C}^T = \frac{1}{3}\boldsymbol{C}\boldsymbol{C}^T$, where $\boldsymbol{C}^T = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 & \boldsymbol{c}_4 \end{bmatrix}$. We also have $\boldsymbol{c}_i^T \boldsymbol{c}_j = 0$ for all $i \neq j$, thus $\boldsymbol{c}_i^T \boldsymbol{\beta}$ are orthogonal.

**2.** $\boldsymbol{H}$ is symmetric, then it can be decomposed as $\boldsymbol{H} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T$, where $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \dots & \boldsymbol{p}_n \end{bmatrix}$ is an orthogonal matrix and $\boldsymbol{\Lambda} = \text{diag}(\{\lambda_i\}_{i=1}^n)$.

If $\boldsymbol{H}$ is non-negative definite, then for all $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T\boldsymbol{x} \geq 0$$

Let $\boldsymbol{x} = \boldsymbol{p}_i$, then $\boldsymbol{P}^T\boldsymbol{p}_i = \begin{bmatrix} \boldsymbol{p}_1^T \\ \boldsymbol{p}_2^T \\ \vdots \boldsymbol{p}_n^T \end{bmatrix} \boldsymbol{p}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. It is a vector with all 0 except for the $i$-th index. Hence

$$\boldsymbol{p}_i^T\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T\boldsymbol{p}_i = \lambda_i \geq 0$$

If all $\lambda_i \geq 0$, then for all $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{\Lambda}\boldsymbol{y} = \sum_{i=1}^n \lambda_i y_i^2 \geq 0$$

Here $\boldsymbol{y} = \boldsymbol{P}^T\boldsymbol{x}$.

**3.** $y_i = \mu + |x_i|\epsilon_i \Rightarrow \frac{y_i}{|x_i|} = \frac{1}{|x_i|}\frac{\mu}{+}\epsilon_i$. Let $\boldsymbol{z} = \begin{bmatrix} \frac{y_i}{|x_i|} \end{bmatrix}$, $\boldsymbol{X} = \begin{bmatrix} \frac{1}{|x_i|} \end{bmatrix}$, $\boldsymbol{\epsilon} = [\epsilon_i]$, then

$$\boldsymbol{z} = \boldsymbol{X}\mu + \boldsymbol{\epsilon}$$

And it is a Gauss-Markov Model. Hence

$$\text{BLUE}(\mu) = (\boldsymbol{X}^T\boldsymbol{X})^{-}\boldsymbol{X}^T\boldsymbol{z} = \left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-1}\sum_{i=1}^{n}\frac{1}{|x_i|}\frac{y_i}{|x_i|} = \frac{\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\sum_{i=1}^{n}\frac{1}{x_i^2}}$$

**4.** (a) $\boldsymbol{\alpha} = \boldsymbol{B}\boldsymbol{\beta} = \begin{bmatrix}\boldsymbol{A}\boldsymbol{\beta}\\ \boldsymbol{B}\boldsymbol{\beta}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\alpha}_1\\ \boldsymbol{\alpha}_2\end{bmatrix}$. Thus

$$H_0 : \boldsymbol{\alpha}_2 = \boldsymbol{0},\ H_A = \boldsymbol{\alpha}_2 \neq \boldsymbol{0}$$

(b) When null hypothesis is true, we have $\boldsymbol{\alpha} = \boldsymbol{0}$. Then $\boldsymbol{W}\boldsymbol{\alpha} = \boldsymbol{W}_1\boldsymbol{\alpha}_1 + \boldsymbol{W}_2\boldsymbol{\alpha}_2 = \boldsymbol{W}_1\boldsymbol{\alpha}_1$. Then the model matrix I will use is $\boldsymbol{W}_1$.

(c) $\boldsymbol{C} = \begin{bmatrix}1/2 & 1/2 & -1/2 & -1/2\end{bmatrix}$.

(d) Let

$$\boldsymbol{A} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

We have

$$\boldsymbol{B} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1/2 & 1/2 & -1/2 & -1/2\end{bmatrix}$$

be a non-singular matrix.

(e)

$$\boldsymbol{B}^{-1} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 1 & -1 & -2\end{bmatrix}$$

Then

$$\boldsymbol{W}_1 = \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}$$

$\boldsymbol{W}_1$ here is a reduced model.

(f)

$$\text{SSE}_{\text{reduced}} = \boldsymbol{y}^T(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{W}_1})\boldsymbol{y} = 12$$
$$\text{SSE}_{\text{full}} = \boldsymbol{y}^T(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}})\boldsymbol{y} = 24$$

(g) $\text{DFSSE}_{\text{reduced}} = 10 - 3 = 7$, $\text{DFSSE}_{\text{full}} = 10 - 4 = 6$

(h)

$$F = \frac{(\text{SSE}_{\text{reduced}} - \text{SSE}_{\text{full}})/(\text{DFSSE}_{\text{reduced}} - \text{DFSSE}_{\text{full}})}{\text{SSE}_{\text{full}}/\text{DFSSE}_{\text{full}}} = \frac{24 - 12}{12/6} = 6$$