STAT 543 Homework 5

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1. Let $Y_i = \log X_i$. $y = \log x \Rightarrow x = \mathrm{e}^y$, $y \in \mathbb{R}$. Thus

$$f(y) = f_X(x(y)) \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = \alpha(\mathrm{e}^y)^{\alpha-1}\mathrm{e}^{-(\mathrm{e}^y)^\alpha}\mathrm{e}^y = \alpha\mathrm{e}^{\alpha y}\mathrm{e}^{-\mathrm{e}^{\alpha y}}$$

Hence Y_i's are in a scale family and there exist $Z_i \sim f(z) = \mathrm{e}^z\mathrm{e}^{-\mathrm{e}^z}$, such that $Y_i = \frac{1}{\alpha}Z_i$. Then

$$\frac{\log X_1}{\log X_2} = \frac{Y_1}{Y_2} = \frac{\frac{1}{\alpha}Z_1}{\frac{1}{\alpha}Z_2} = \frac{Z_1}{Z_2}$$

Hence it is independent of α and is ancillary.

2. (a) The parameter space $\Theta = \{(\theta, a\theta^2) : \theta > 0\}$ is a curve in $\mathbb{R}^2$. So it does not contain an two-dimensional open set.

(b)

$$\begin{aligned} f(\boldsymbol{x}) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi a\theta^2}}\mathrm{e}^{-\frac{(x_i-\theta)^2}{2a\theta^2}} \\ &= \left(\frac{1}{2\pi a\theta^2}\right)^{n/2}\mathrm{e}^{-\frac{\sum_{i=1}^n(x_i^2-2\theta x_i+\theta^2)}{2a\theta^2}} \\ &= \left(\frac{1}{2\pi a\theta^2}\right)^{n/2}\mathrm{e}^{-\frac{1}{2a}}\mathrm{e}^{-\frac{\sum_{i=1}^n x_i^2}{2a\theta^2}}\mathrm{e}^{\frac{\sum_{i=1}^n x_i}{a\theta}} \end{aligned}$$

By Factorization Theorem, $(\sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i)$ is sufficient statistic. We also know $\sum_{i=1}^n X_i = n\bar{X}$ and $\sum_{i=1}^n X_i^2 = (n-1)S^2 + n\bar{X}^2$. Hence $(\bar{X}, S^2)$ is also sufficient.
$E(\bar{X}) = \theta$, $E(S^2) = a\theta^2$. Thus $E(\bar{X}^2) = Var(\bar{X}) + (E(\bar{X}))^2 = \frac{a\theta^2}{n} + \theta^2 = \frac{a+n}{n}\theta^2$. Let

$$u(\bar{X}, S^2) = \frac{n}{n+a}\bar{X}^2 - \frac{S^2}{a}$$

Then

$$E(u(\bar{X}, S^2)) = \frac{n}{n+a}E(\bar{X}^2) - \frac{1}{a}E(S^2) = \theta^2 - \theta^2 = 0$$

However $P(u(\bar{X}, S^2) = 0) = P(\frac{n}{n+a}\bar{X}^2 - \frac{S^2}{a} = 0) \neq 1$. Thus it is not complete.

3. (a)

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= \frac{n!}{x_1!x_2!x_3!x_4!}\left(\frac{1}{2} + \frac{\theta}{4}\right)^{x_1}\left(\frac{1}{4}(1-\theta)\right)^{x_2+x_3}\left(\frac{\theta}{4}\right)^{x_4} \\ &= \frac{n!}{x_1!x_2!x_3!(n-x_1-x_2-x_3)!}\mathrm{e}^{x_1\log(1/2+\theta/4)+(x_2+x_3)\log((1-\theta)/4)+(n-x_1-x_2-x_3)\log(\theta/4)} \\ &= \frac{n!}{x_1!x_2!x_3!(n-x_1-x_2-x_3)!}\left(\frac{\theta}{4}\right)^n\mathrm{e}^{x_1(\log(\theta+2)-\log\theta)+(x_2+x_3)(\log(1-\theta)-\log\theta)} \end{aligned}$$

x_1, x_2, x_3 do not have linear constraint. But $(\log(\theta+2)-\log\theta, \log(1-\theta)-\log\theta, \log(1-\theta)-\log\theta)$ is a curve in $\mathbb{R}^3$. Thus it is a curved exponential family.

(b) From the result of (a), we have

$$f(x_1, x_2, x_3, x_4) = \frac{n!}{x_1!x_2!x_3!(n-x_1-x_2-x_3)!}\left(\frac{\theta}{4}\right)^n\left(\frac{\theta+2}{\theta}\right)^{x_1}\left(\frac{1-\theta}{\theta}\right)^{x_2+x_3}$$

By Factorization Theorem, (X_1, X_2+X_3) is sufficient statistic.

(c)

$$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})} = \frac{y_1!y_2!y_3!(n-y_1-y_2-y_3)!}{x_1!x_2!x_3!(n-x_1-x_2-x_3)!}\left(\frac{\theta+2}{\theta}\right)^{x_1-y_1}\left(\frac{1-\theta}{\theta}\right)^{(x_2+x_3)-(y_2+y_3)}$$

$\frac{f(\boldsymbol{x})}{f(\boldsymbol{y})}$ is a constant as a function of $\theta \iff x_1 = y_1$ and $x_2+x_3 = y_2+y_3$. Hence (X_1, X_2+X_3) is minimal sufficient statistic.

4. (a)

$$f(x) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}$$

By Factorization Theorem, X is sufficient.
However $E(X) = -\frac{\theta}{2}+\frac{\theta}{2} = 0$ and $P(X=0) = 1-\theta$. Thus X is not complete.

(b) $|X|$ is sufficient by Factorization Theorem. We also have $E(u(|X|)) = P(|X|=1)u(1) + P(|X|=0)u(0) = \theta u(1) + (1-\theta)u(0) = \theta(u(1)-u(0)) + u(0) = 0 \Rightarrow u(1) = u(0) = 0 \Rightarrow P(u(|X|)=0) = 1$. Thus $|X|$ is complete.

(c)

$$f(x|\theta) = \left(\frac{\theta}{2(1-\theta)}\right)^{|x|}(1-\theta)$$

Thus it is exponential family.

5. (a)

$$f(\boldsymbol{x}) = \prod_{i=1}^n \mathrm{e}^{-(x_i-\mu)}\mathbf{1}\{x_i > \mu\} = \mathrm{e}^{n\mu}\mathrm{e}^{-\sum_{i=1}^n x_i}\mathbf{1}\{\mu < x_{(1)}\}$$

By Factorization Theorem $X_{(1)}$ is sufficient.
Let $X_{(1)} = Y$, then we have

$$f(y) = n\mathrm{e}^{-n(y-\mu)},\ y > \mu$$

For any function $u(Y)$, $E(u(Y)) = \int_\mu^\infty u(y)n\mathrm{e}^{-n(y-\mu)}\mathrm{d}y = n\mathrm{e}^{n\mu}\int_\mu^\infty u(y)\mathrm{e}^{-ny}\mathrm{d}y = 0$ for any μ. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\int_\mu^\infty u(y)\mathrm{e}^{-ny}\mathrm{d}y = -u(\mu)\mathrm{e}^{-n\mu} = 0 \Rightarrow u \equiv 0$$

Thus $P(u(Y)=0) = 1$, $Y = X_{(1)}$ is complete.

(b) X_i's are from a location family, thus $X_i = Z_i + \mu$, where $Z_i \sim$ Exponemtial(1). $S^2 = \frac{1}{n-1}\sum_{i=1}^n(X_i-\bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^n(Z_i+\mu-\bar{Z}-\mu)^2 = \frac{1}{n-1}\sum_{i=1}^n(Z_i-\bar{Z})^2$. Hence it is independent of μ and is ancillary. By Basu's Theorem, because $X_{(1)}$ is complete and sufficient, we have $X_{(1)}$ and S^2 being independent.

6. $\phi(T) = E(h(X_1,\ldots,X_n)|T)$ is a function of T and $E(\phi(T)) = E(E(h(X_1,\ldots,X_n)|T)) = E(h(X_1,\ldots,X_n)) = \tau(\theta)$. Then $\phi(T)$ is an UE of $\tau(\theta)$. Because T is complete and sufficient, by Lehmann-Scheffe Theorem, $\phi(T)$ is UMVUE of $\tau(\theta)$.

7. (a)

$$\begin{aligned}
E_\theta(\delta-\theta)^2 &= E_\theta(a\bar{X}+b-\theta)^2\\
&= E_\theta(a^2\bar{X}^2+(b-\theta)^2+2a(b-\theta)\bar{X})\\
&= a^2E_\theta(\bar{X})+2a(b-\theta)E(\bar{X})+(b-\theta)^2\\
&= a^2(Var_\theta(\bar{X})+(E(\bar{X}))^2)+2a(b-\theta)E(\bar{X})+(b-\theta)^2\\
&= a^2\left(\frac{\sigma^2}{n}+\theta^2\right)+(a\theta)^2+2a\theta(b-\theta)+(b-\theta)^2\\
&= a^2\frac{\sigma^2}{n}+(a\theta+b-\theta)^2\\
&= a^2\frac{\sigma^2}{n}+(b-(1-a)\theta)^2
\end{aligned}$$

(b)

$$\delta^\pi = E(\theta|X_1,\ldots,X_n) = \frac{n\tau^2}{n\tau^2+\sigma^2}\bar{X}+\frac{\sigma^2}{n\tau^2+\sigma^2}\mu = (1-\eta)\bar{X}+\eta\mu$$

Plug in $a=1-\eta$ and $b=\eta\mu$ to the result in (a). We have

$$R(\theta,\delta^\pi) = (1-\eta)^2\frac{\sigma^2}{n}+(\eta\mu-(1-1+\eta)\theta)^2 = (1-\eta)^2\frac{\sigma^2}{n}+(\theta-\mu)^2\eta^2$$

(c)

$$B(\pi,\delta^\pi) = \int_{-\infty}^{\infty}R(\theta,\delta^\pi)\pi(\theta)\mathrm{d}\theta = (1-\eta)^2\frac{\sigma^2}{n}+\eta^2\int_{-\infty}^{\infty}(\theta-\mu)^2\pi(\theta)\mathrm{d}\theta = (1-\eta)^2\frac{\sigma^2}{n}+\eta^2\tau^2 \to \eta^2\tau^2,\ n\to\infty$$