STAT 510 Homework 7

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1. (a)

$$\begin{aligned}
E(MS_{ou(trt,xu)}) &= E\left(\frac{1}{tn(m-1)}\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk}-\bar{y}_{ij\cdot})^2\right)\\
&= \frac{1}{tn(m-1)}E\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(\mu+\tau_i+u_{ij}+e_{ijk}-\mu-\tau_i-u_{ij}-\bar{e}_{ij\cdot})^2\right)\\
&= \frac{1}{tn(m-1)}E\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(e_{ijk}-\bar{e}_{ij\cdot})^2\right)
\end{aligned}$$

As $e_ijk \sim N(0,\sigma_e^2)$, then

$$\begin{aligned}
E\left(MS_{ou(trt,xu)}\right) &= \frac{1}{tn(m-1)}\sum_{i=1}^{r}\sum_{j=1}^{n}(m-1)\sigma_e^2\\
&= \frac{1}{tn(m-1)}tn(m-1)\sigma_e^2\\
&= \sigma_e^2
\end{aligned}$$

(b) Let $\boldsymbol{A} = (\boldsymbol{I}-\boldsymbol{P}_3)$, where $\boldsymbol{P}_3 = \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{1}\boldsymbol{1}^T_{m\times m}$ is a projection matrix. Then $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y} = \sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk}-\bar{y}_{ij\cdot})^2$. We also have

$$\Sigma = \sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes\boldsymbol{1}\boldsymbol{1}^T_{m\times m}+\sigma_e^2\boldsymbol{I}_{tnm\times tnm} = \sigma_u^2 m\boldsymbol{P}_3+\sigma_e^2\boldsymbol{I}$$

Then

$$\begin{aligned}
&\boldsymbol{A\Sigma} = (\boldsymbol{I}-\boldsymbol{P}_3)(\sigma_u^2 m\boldsymbol{P}_3+\sigma_e^2\boldsymbol{I}) = \sigma_e^2(\boldsymbol{I}-\boldsymbol{P}_3) = \sigma_e^2(\boldsymbol{I}-\frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes\boldsymbol{I}_{m\times m})\\
\Rightarrow&\mathrm{tr}(\boldsymbol{A\Sigma}) = tnm\sigma_e^2(1-\frac{1}{m}) = \sigma_e^2 tn(m-1)
\end{aligned}$$

We know that $E(\boldsymbol{y})\subset\mathcal{C}(\boldsymbol{P}_2)\subset\mathcal{C}(\boldsymbol{P}_3)$. Thus $(\boldsymbol{I}-\boldsymbol{P}_3)E(\boldsymbol{y}) = \boldsymbol{0}\Rightarrow E(\boldsymbol{y})^T\boldsymbol{A}E(\boldsymbol{y}) = 0$. Hence,

$$E(\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}) = tn(m-1)\sigma_e^2 \Rightarrow E(MS_{ou(trt,xu)}) = \frac{1}{tn(m-1)}tn(m-1)\sigma_e^2 = \sigma_e^2$$

2. (a) $d_j = y_{1j}-y_{2j} = (\mu_1+u_j+e_{1j})-(\mu_2+u_j+e_{2j}) = (\mu_1-\mu_2)+(e_{1j}-e_{2j})$. We have $e_{ij}\overset{iid}{\sim}N(0,\sigma_e^2)$, then $d_j\overset{iid}{\sim}N(\mu_1-\mu_2,2\sigma_e^2)$.

(b)

$$T = \frac{\bar{d}}{\sqrt{\frac{1}{380}\sum_{j=1}^{20}(d_j - \bar{d})^2}}, \text{ where } \bar{d} = \frac{1}{20}\sum_{j=1}^{20} d_j$$

(c)

$$T \sim t_{19}(\mu_1 - \mu_2)$$