STAT 543 Homework 1

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1. (a) $L(\theta) = \prod_{i=1}^n f(x_i|\theta) = \theta^n \prod_{i=1}^n x_i^{\theta-1}$, then $\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^n \log x_i$.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log L(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \log x_i$$

Then let $\frac{\mathrm{d}}{\mathrm{d}\theta}L(\theta)\big|_{\hat{\theta}} = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^n \log x_i}$.
And we also have

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}L(\theta) = -\frac{n}{\theta^2} < 0$$

Hence $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \log X_i}$ is the MLE of θ.
Let $Y_i = -\log X_i$, then the pdf of Y_i's is

$$f(y|\theta) = \theta(\mathrm{e}^{-y})^{\theta-1}|-\mathrm{e}^{-y}| = \theta\mathrm{e}^{-\theta y},\ y > 0$$

Thus $Y_i \overset{iid}{\sim} Exponential(1/\theta) \sim Gamma(1, 1/\theta)$. Then $-\sum_{i=1}^n \log X_i = \sum_{i=1}^n Y_i \sim Gamma(n, 1/\theta)$.
Denote $Z = \sum_{i=1}^n Y_i$, then $\hat{\theta} = \frac{n}{Z}$.
For the variance,

$$\begin{aligned}
E(\frac{1}{Z^2}) &= \int_0^\infty \frac{1}{z^2}\frac{\theta^n}{\Gamma(n)} z^{n-1}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta^n}{\Gamma(n)}\int_0^\infty z^{n-3}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-2)}{\theta^{n-2}}\int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} z^{(n-2)-1}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta^2}{(n-1)(n-2)}
\end{aligned}$$

$$\begin{aligned}
E(\frac{1}{Z}) &= \int_0^\infty \frac{1}{z}\frac{\theta^n}{\Gamma(n)} z^{n-1}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta^n}{\Gamma(n)}\int_0^\infty z^{n-2}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-1)}{\theta^{n-1}}\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} z^{(n-1)-1}\mathrm{e}^{-\theta z}\mathrm{d}z \\
&= \frac{\theta}{n-1}
\end{aligned}$$

Then

$$\begin{aligned}
Var(\hat{\theta}) &= Var(\frac{n}{Z}) = n^2 Var(\frac{1}{Z}) \\
&= n^2 \left[E(\frac{1}{Z^2}) - (E(\frac{1}{Z}))^2 \right] \\
&= n^2 \left(\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} \right) \\
&= \frac{n^2}{(n-1)^2(n-2)} \theta^2 \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

(b) We can see $X \sim Beta(\theta, 1)$, then $E(X) = \frac{\theta}{\theta+1}$. Let $E(X) = \mu_1'$, then

$$\frac{\theta}{1+\theta}\bigg|_{\hat{\theta}} = \bar{X}_n = \frac{\sum_{i=1}^n X_i}{n} \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^n X_i}{n - \sum_{i=1}^n X_i}$$

2.

$$L(\lambda, \mu) = \prod_{i=1}^n f(z_i, w_i | \lambda, \mu) = \prod_{i=1}^n I(w_i) \mathrm{e}^{-z_i(\lambda^{-1} + \mu^{-1})}$$

where $I(w) = \begin{cases} \lambda^{-1} & , w = 1 \\ \mu^{-1} & , w = 0 \end{cases}$.

Hence

$$\begin{aligned}
\log L(\lambda, \mu) &= \sum_{i=1}^n \log I(w_i) - \sum_{i=1}^n z_i(\lambda^{-1} + \mu^{-1}) \\
&= (\sum_{i=1}^n w_i) \log(\lambda^{-1}) + (n - \sum_{i=1}^n w_i) \log(\mu^{-1}) - \sum_{i=1}^n z_i(\lambda^{-1} + \mu^{-1}) \\
&= -[(\sum_{i=1}^n w_i) \log \lambda + (\sum_{i=1}^n z_i)\lambda^{-1}] - [(n - \sum_{i=1}^n w_i) \log \mu + (\sum_{i=1}^n z_i)\mu^{-1}]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L(\lambda, \mu)}{\partial \lambda} &= -(\sum_{i=1}^n w_i)\frac{1}{\lambda} + (\sum_{i=1}^n z_i)\frac{1}{\lambda^2} \\
\frac{\partial L(\lambda, \mu)}{\partial \mu} &= -(n - \sum_{i=1}^n w_i)\frac{1}{\mu} + (\sum_{i=1}^n z_i)\frac{1}{\mu^2}
\end{aligned}$$

Let $\frac{\partial L}{\partial \lambda}\big|_{\hat{\lambda}} = 0$ and $\frac{\partial L}{\partial \mu}\big|_{\hat{\mu}} = 0 \Rightarrow$

$$\begin{aligned}
\hat{\lambda} &= \frac{\sum_{i=1}^n z_i}{\sum_{i=1}^n w_i} \\
\hat{\mu} &= \frac{\sum_{i=1}^n z_i}{n - \sum_{i=1}^n w_i}
\end{aligned}$$

We also have

$$\left.\frac{\partial^2 L(\lambda,\mu)}{\partial\lambda^2}\right|_{\hat\lambda} = (\sum_{i=1}^n w_i)\frac{1}{\lambda^2} - 2(\sum_{i=1}^n z_i)\frac{1}{\lambda^3} = -\frac{(\sum_{i=1}^n w_i)^3}{(\sum_{i=1}^n z_i)^2}$$

$$\left.\frac{\partial^2 L(\lambda,\mu)}{\partial\mu^2}\right|_{\hat\mu} = (n-\sum_{i=1}^n w_i)\frac{1}{\mu^2} - 2(\sum_{i=1}^n z_i)\frac{1}{\mu^3} = -\frac{(n-\sum_{i=1}^n w_i)^3}{(\sum_{i=1}^n z_i)^2}$$

$$\frac{\partial^2 L(\lambda,\mu)}{\partial\lambda\partial\mu} = 0$$

Thus Hessian matrix is

$$H = \begin{bmatrix} -\frac{(\sum_{i=1}^n w_i)^3}{(\sum_{i=1}^n z_i)^2} & 0 \\ 0 & -\frac{(n-\sum_{i=1}^n w_i)^3}{(\sum_{i=1}^n z_i)^2} \end{bmatrix}$$

$h_{11} < 0$ and $|H| > 0$, hence $\hat\lambda = (\sum_{i=1}^n Z_i)/(\sum_{i=1}^n W_i)$ and $\hat\mu = (\sum_{i=1}^n Z_i)/(n - \sum_{i=1}^n W_i)$ is MLE of λ and μ.

3. (b) We know that $Y_i \sim N(\beta x_i, \sigma^2)$ and Y_i's are independent. Thus

$$L(\beta) = \prod_{i=1}^n f(y_i|\beta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(y_i-\beta x_i)^2}{2\sigma^2}}$$

Hence

$$\log L(\beta) = -n\sum_{i=1}^n \log(\sqrt{2\pi}\sigma) - \sum_{i=1}^n \frac{(y_i-\beta x_i)^2}{2\sigma^2}$$

$$\begin{aligned}\frac{\mathrm{d}L(\beta)}{\mathrm{d}\beta} &= -\frac{1}{2\sigma^2}\sum_{i=1}^n 2(y_i-\beta x_i)(-x_i) \\ &= \sum_{i=1}^n \frac{1}{\sigma^2}(y_i-\beta x_i)x_i \\ &= \frac{1}{\sigma^2}(\sum_{i=1}^n y_i x_i - \beta\sum_{i=1}^n x_i^2)\end{aligned}$$

Let $\frac{\mathrm{d}L(\beta)}{\mathrm{d}\beta}|_{\hat\beta} = 0 \Rightarrow$

$$\hat\beta = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

Also,

$$\frac{\mathrm{d}^2 L(\beta)}{\mathrm{d}\beta^2}|_{\hat\beta} = -\frac{\sum_{i=1}^n x_i^2}{\sigma^2} < 0$$

Hence, $\hat\beta = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}$ is the MLE of β.

$$E(\hat\beta) = E\left(\frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}\right) = \frac{1}{\sum_{i=1}^n x_i^2}\sum_{i=1}^n x_i E(Y_i) = \frac{1}{\sum_{i=1}^n x_i^2}\sum_{i=1}^n x_i x_i \beta = \beta$$

Thus $\hat\beta$ is unbiased estimator of β.

(c)

$$Var(\hat{\beta}) = Var(\frac{1}{\sum_{i=1}^n x_i^2}\sum_{i=1}^n x_i Y_i) = \frac{1}{(\sum_{i=1}^n x_i^2)^2}\sum_{i=1}^n x_i^2 Var(Y_i) = \sigma^2\frac{\sum_{i=1}^n x_i^2}{(\sum_{i=1}^n x_i^2)^2} = \frac{\sigma^2}{\sum_{i=1}^n x_i^2}$$

Hence $\hat{\beta} \sim N(\beta, \frac{\sigma^2}{\sum_{i=1}^n x_i^2})$.

4. (a) The pdf of Y

$$f(y) = \frac{n!}{(n-1)!}\frac{1}{\lambda}\mathrm{e}^{-y/\lambda}\left[1-(1-\mathrm{e}^{-y/\lambda})\right]^{n-1} = \frac{1}{\lambda/n}\mathrm{e}^{-\frac{y}{\lambda/n}}$$

Thus $Y \sim Exponential(\lambda/n) \Rightarrow E(nY) = nE(Y) = n\frac{\lambda}{n} = \lambda$. Hence $\hat{\lambda}_1 = n\min\{X_1, X_2, \cdots, X_n\}$ is an unbiased estimator of λ.

(b) Let $\hat{\lambda}_2 = \bar{X}_n = \sum_{i=1}^n X_i/n$. $E(\hat{\lambda}_2) = \frac{1}{n}\sum_{i=1}^n \lambda = \lambda$. Thus $\hat{\lambda}_2$ is an unbiased estimator of λ. And $Var(\hat{\lambda}_2) = n^2 Var(Y) = n^2(\lambda/n)^2 = \lambda^2$. $Var(\hat{\lambda}_2) = \frac{1}{n^2}\sum_{i=1}^n \lambda^2 = \lambda^2/n$. Hence $\hat{\lambda}_2$ has smaller variance and is a better unbiased estimator.

(c)

$$\hat{\lambda}_1 = 50.1 \times 12 = 601.2$$
$$\hat{\lambda}_2 = \bar{x} = 128.8$$

5. (a)

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f(y_i|\theta) \\ &= \prod_{i=1}^n (\mathrm{e}^{-y_i\theta} - \mathrm{e}^{-\theta(1+y_i)}) \\ &= \prod_{i=1}^n \mathrm{e}^{-y_i\theta}(1-\mathrm{e}{-}^{\theta}) \\ &= \mathrm{e}^{-n\theta\bar{y}_n}(1-\mathrm{e}^{-\theta})^n \\ &= \left[\mathrm{e}^{-\theta\bar{y}_n}(1-\mathrm{e}^{-\theta})\right]^n \end{aligned}$$

(b) When $\bar{Y}_n = 0$, $L(\theta) = (1-\mathrm{e}^{-\theta})^n$ is monotone increasing in $(0,\infty)$. Thus $L(\theta)$ do not have a minimum and MLE for θ does not exist.

(c)

$$\begin{aligned} &\log L(\theta) = n\left[-\theta\bar{y}_n + \log(1-\mathrm{e}^{-\theta})\right] \\ &\frac{\mathrm{d}}{\mathrm{d}\theta}\log L(\theta) = n\left[-\bar{y}_n + \frac{\mathrm{e}^{-\theta}}{1-\mathrm{e}^{-\theta}}\right] = n\left[\frac{-\bar{y}_n + (\bar{y}_n+1)\mathrm{e}^{-\theta}}{1-\mathrm{e}^{-\theta}}\right] \end{aligned}$$

Let $\frac{\mathrm{d}L(\theta)}{\mathrm{d}\theta}|_{\hat{\theta}} = 0 \Rightarrow \hat{\theta} = \log(\frac{1}{\bar{y}_n}+1)$ Also,

$$\frac{\mathrm{d}^2 L(\theta)}{\mathrm{d}\theta^2}|_{\hat{\theta}} = -\frac{\mathrm{e}^{-\hat{\theta}}}{(1-\mathrm{e}^{-\hat{\theta}})^2} < 0$$

Thus $\hat{\theta} = \log(\frac{1}{\bar{Y}_n}+1)$ is MLE of θ.