STAT 542 Homework 8

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November 3, 2016

1. (a)

$$\begin{aligned} f(x,y) &= f(y|x)f(x) \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(y-x)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}} \\ &= \frac{1}{2\pi}\mathrm{e}^{-\frac{(y-x)^2+x^2}{2}} \end{aligned}$$

(b)

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{-\frac{(y-x)^2+x^2}{2}}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{-\frac{2x^2-2xy+y^2}{2}}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{-\frac{(x-\frac{1}{2}y)^2+\frac{1}{2}y^2}{2}}\mathrm{d}x \\ &= \frac{\sqrt{1/2}}{\sqrt{2\pi}}\mathrm{e}^{\frac{1}{4}y^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}}\mathrm{e}^{-\frac{(x-\frac{1}{2}y)^2}{2(\sqrt{\frac{1}{2}})^2}}\mathrm{d}x \\ &= \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\frac{1}{4}y^2} \end{aligned}$$

Thus $Y \sim N(0,2)$

(c)

$$\begin{aligned} f(w) &= \frac{1}{\sqrt{2\pi}\sqrt{2}}\mathrm{e}^{-\frac{w^2}{2(\sqrt{2})^2}} \\ f(v|w) &= \frac{f(v,w)}{f(w)} \\ &= \frac{1}{2\pi}\mathrm{e}^{\frac{(w-v)^2+v^2}{2}} \Bigg/ \frac{1}{2\sqrt{\pi}}\mathrm{e}^{\frac{w^2}{4}} \\ &= \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{2(w-v)^2+2v^2-w^2}{4}} \\ &= \frac{1}{\sqrt{\pi}}\mathrm{e}^{(v-\frac{1}{2}w)^2} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}}\mathrm{e}^{\frac{(v-\frac{1}{2}w)^2}{2\cdot\frac{1}{2}}} \end{aligned}$$

Thus $V|W=w \sim N(\frac{1}{2}w, \frac{1}{2})$

2. $D \sim unif(8, 8.5)$, $T \sim unif(2/3, 5/6)$. $f(d,t) = f_D(d)f_T(t) = \frac{1}{8.5-8}\frac{1}{5/6-2/3} = 12$. Thus

$$P(D + T \le 9) = \int_{2/3}^{5/6}\int_{8}^{9-t} 12 \mathrm{d}d\mathrm{d}t = 1/2$$

3. (a) $P(X = 1) = 1/4$, $P(X = 2) = 1/2$, $P(X = 3) = 1/4$, $P(Y = 2) = 1/3$, $P(Y = 3) = 1/3$, $P(Y = 4) = 1/3$. Thus $P(X = 1)P(Y = 4) = 1/4 \times 1/3 = 1/12 \ne P(X = 1, Y = 4) = 0$. Hence, X, Y are dependent.

(b)

		U		
		1	2	3
	2	1/12	1/6	1/12
V	3	1/12	1/6	1/12
	4	1/12	1/6	1/6

4. $X \sim Geometric(p)$, $Y \sim Geometric(p)$.

When $v \ge 0$, $X - Y \ge 0 \Rightarrow U = \min(X, Y) = Y$. Thus

$$\begin{aligned}
P(U = u, V = v) &= P(Y = u, X - Y = v) \\
&= P(X = u + v, Y = u) = p(1-p)^{u+v-1}p(1-p)^{u-1} \\
&= p^2(1-p)^{2u+v-2} \qquad u = 1, 2, \cdots, \ v = 0, 1, 2, \cdots
\end{aligned}$$

When $v < 0$, $X - Y < 0 \Rightarrow U = \min(X, Y) = X$. Thus

$$\begin{aligned}
P(U = u, V = v) &= P(X = u, X - Y = v) \\
&= P(X = u, Y = u - v) \\
&= p(1-p)^{u-1}p(1-p)^{u-v-1} \\
&= p^2(1-p)^{2u-v-2} \qquad u = 1, 2, \cdots, \ v = -1, -2, \cdots
\end{aligned}$$

Thus

$$f(u, v) = p^2(1-p)^{2u+|v|-2}, \ u = 1, 2, \cdots, \ v = 0, \pm 1, \pm 2, \cdots$$

5.

$$\begin{aligned}
P(X + Y = k) &= \sum_{i=1}^{k-1} P(X = i, Y = k - i) \\
&= \sum_{i=1}^{k-1} P(X = i)P(Y = k - i) \\
&= \sum_{i=1}^{k-1} p(1-p)^{i-1}p(1-p)^{k-i-1} \\
&= \sum_{i=1}^{k-1} p^2(1-p)^{k-2} \\
&= (k-1)p^2(1-p)^{k-2}
\end{aligned}$$

$$
\begin{aligned}
P(X=x|X+Y=k) &= \frac{P(X=x, X+Y=k)}{P(X+Y=k)} \\
&= \frac{P(X=x, Y=k-x)}{P(X+Y=k)} \\
&= \frac{p(1-p)^{x-1}p(1-p)^{k-x-1}}{(k-1)p^2(1-p)^{k-2}} \\
&= \frac{p^2(1-p)^{k-2}}{(k-1)p^2(1-p)^{k-2}} \\
&= \frac{1}{k-1} \qquad x = 1, 2, \cdots, k-1
\end{aligned}
$$

6. From

$$
\begin{cases} u = x+y \\ v = x-y \end{cases}
$$

we have

$$
\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}
$$

Thus

$$
\det J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}
$$

We also have

$$
f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(y-\gamma)^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{(x-\mu)^2+(y-\gamma)^2}{2\sigma^2}}
$$

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(\frac{1}{2}(u+v), \frac{1}{2}(u-v))|\det J| \\
&= \frac{1}{2}\frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{(\frac{1}{2}(u+v)-\mu)^2+(\frac{1}{2}(u-v)-\gamma)^2}{2\sigma^2}} \\
&= \frac{1}{4\pi\sigma^2}\mathrm{e}^{-\frac{\frac{1}{2}u^2+\frac{1}{2}v^2-\mu(u+v)-\gamma(u-v)+\mu^2+\gamma^2}{2\sigma^2}} \\
&= \frac{1}{4\pi\sigma^2}\mathrm{e}^{-\frac{(u-(\mu+\gamma))^2+(v-(\mu-\gamma))^2}{4\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{(u-(\mu+\gamma))^2}{2(\sqrt{2}\sigma)^2}} \cdot \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{(v-(\mu-\gamma))^2}{2(\sqrt{2}\sigma)^2}} \\
&= g(u)f(v) = f_U(u)\cdot f_V(v)
\end{aligned}
$$

U, V are independent, and $U \sim N(\mu+\gamma, 2\sigma^2)$, $V \sim N(\mu-\gamma, 2\sigma^2)$

7.

$$
\begin{aligned}
Cov(XY, Y) &= E(XY^2) - E(XY)E(Y) \\
&= E(X)E(Y^2) - E(X)(E(Y))^2 \\
&= \mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y^2 \\
&= \mu_X\sigma_Y^2
\end{aligned}
$$

$$\begin{aligned}
Var(XY) &= E(X^2Y^2) - (E(XY))^2 \\
&= E(X^2)E(Y^2) - (E(X)E(Y))^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \\
&= \sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2 + \sigma_X^2\sigma_Y^2
\end{aligned}$$

$$\begin{aligned}
Corr(X,Y) &= \frac{Cov(XY,Y)}{\sqrt{Var(XY)}\sqrt{Var(Y)}} \\
&= \frac{\mu_X\sigma_Y^2}{\sqrt{\sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2 + \sigma_X^2\sigma_Y^2}\sigma_Y} \\
&= \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2 + \sigma_X^2\sigma_Y^2}}
\end{aligned}$$

8. (a) No. With different X_1 and X_3, according to the support of (X_1, X_2, X_3), X_2 has different support.

(b)

$$f_{X_2}(x_2) = \int_{x_2}^{1}\int_{0}^{x_2} f(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_3 = 6x_2(1-x_2) = \frac{1}{B(2,2)}x_2^{2-1}(1-x_2)^{2-1}$$

Thus $X_2 \sim Beta(2,2)$

(c) Under the condition $X_2 = x_2$, the support of (X_1, X_2) is a rectangle $(0, x_2) \times (x_2, 1)$.

$$f(x_1, x_2|x_3) = \frac{f(x_1,x_2,x_3)}{f_{X_2}(x_2)} = \frac{6}{6x_2(1-x_2)}$$

(d) Given $X_2 = x_2$, the support of (X_1, X_2) is a rectangle $(0, x_2) \times (x_2, 1)$, and the joint pdf can be written as $g(x_1)h(x_3)$ when we view x_2 as a constant.

$$f(x_1, x_2|x_3) = \frac{1}{x_2}\frac{1}{1-x_2} = f(x_1|x_2)f(x_3|x_2)$$

Hence X_1 and X_3 are independent conditioning on $X_2 = x_2$. And $X_1|X_2 = x_2 \sim unif(0, x_2)$, $X_3|X_2 = x_2 \sim unif(x_2, 1)$.

(e) X_1 and X_3 are independent given $X_2 = x_2$, thus

$$\begin{aligned}
Cov(X_1|X_2 = x_2, X_3|X_2 = x_2) &= E(X_1X_3|X_2 = x_2) - E(X_1|X_2 = x_2)E(X_3|X_2 = x_2) \\
&= E(X_1|X_2 = x_2)E(X_3|X_2 = x_2) - E(X_1|X_2 = x_2)E(X_3|X_2 = x_2) \\
&= 0
\end{aligned}$$

9. X and Y are independent, thus $f(x,y) = f_X(x)f_Y(y) = 1$.

$$F_Z(z) = P(Z \le z) = P(\sqrt{X+Y} \le z) = P(X + Y \le z^2) = \int_0^{z^2}\int_0^{z^2-x}\mathrm{d}y\mathrm{d}x = \frac{1}{2}z^4$$

Hence $f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}F_Z(z) = 2z^3$, $z > 0$.