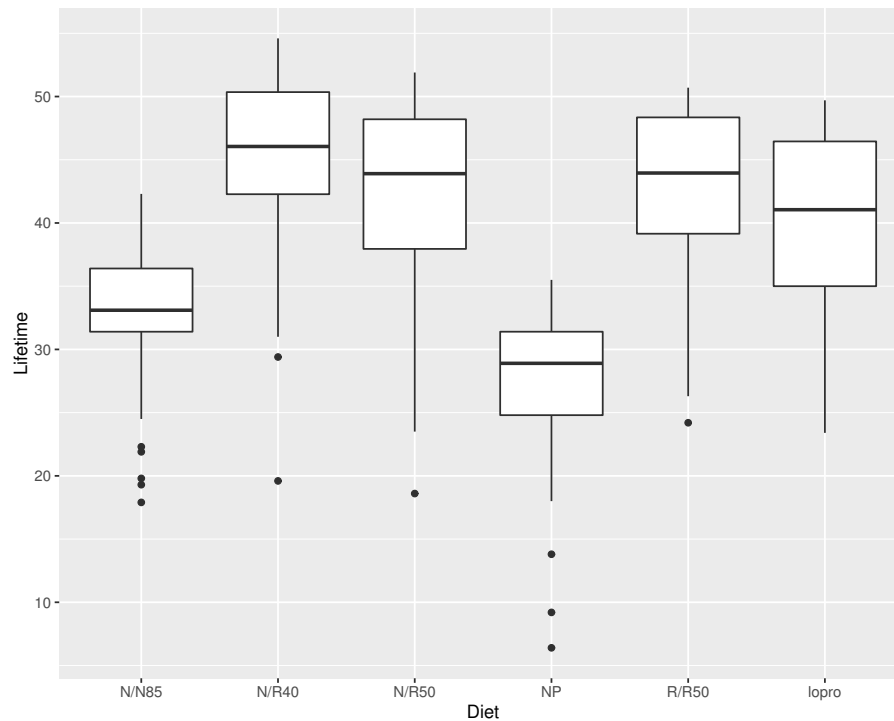


# STAT 510 Homework 3

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1. (a)



(b)

$$SSE_{full} = 15297$$

(c)

$$\hat{\sigma}^2 = \frac{SSE_{full}}{n - r} = \frac{15297}{343} = 44.60$$

(d)

$$SSE_{reduced} = 15511$$

(e)

$$F = \frac{(SSE_{reduced} - SSE_{full}) / (DFE_{reduced} - DFE_{full})}{SSE_{full} / DFE_{full}} = \frac{(15511 - 15297) / 1}{15297 / 343} = 4.8$$

(f) This test is to test the among the mice having these six diet plans, wheatehr the mice having diet plan N/R50 and N/R50 loopro have the same population life time.

- (g) If the parameter vector  $\beta = [\beta_1 \ \beta_2 \ \beta_2 \ \beta_4 \ \beta_5 \ \beta_6]$  are in the order for N/N85, N/R40, N/R50, NP, R/R50 and N/R50 loopro, then

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, d = 0$$

Then

$$F = \frac{(C\hat{\beta} - d)^T (C(X^T X)^{-1} C^T)^{-1} (C\hat{\beta} - d) / \text{rank}(C)}{\hat{\sigma}^2} = 4.8$$

where  $X$  is the model matrix for full model.

The F statistic is the same as is computed in (e).

2. Let  $X = \begin{bmatrix} 1 & 1 \end{bmatrix}$ , then  $A = X^T X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  is symmetric. Let  $G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , then

$$AGA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$$

Thus  $G$  is a generalized inverse of the symmetric matrix  $A$ , but it is not symmetric.

3. (a)

$$\text{rank}(X^T X) \leq \text{rank}(X)$$

$$\text{rank}(X) = \text{rank}(P_X) = \text{rank}(X(X^T X)^{-1} X^T X) \leq \text{rank}(X^T X)$$

$$\Rightarrow \text{rank}(X) = \text{rank}(X^T X)$$

- (b)

$$\text{rank}(X) = \text{rank}(P_X X) \leq \text{rank}(P_X)$$

$$\text{rank}(P_X) = \text{rank}(X(X^T X)^{-1} X) \leq \text{rank}(X)$$

$$\Rightarrow \text{rank}(X) = \text{rank}(P_X)$$

- (c) First we can prove  $\text{rank}(AP_X) = \text{rank}(AX)$ .

$$\text{rank}(AX) = \text{rank}(AP_X X) \leq \text{rank}(AP_X)$$

$$\text{rank}(AP_X) = \text{rank}(AX(X^T X)^{-1} X^T) \leq \text{rank}(AX)$$

$$\Rightarrow \text{rank}(AX) = \text{rank}(AP_X)$$

Then we have

$$C(X^T X)^{-1} C^T = AX(X^T X)^{-1} X^T A^T = AP_X A^T = AP_X P_X A^T$$

Thus  $\text{rank}(C(X^T X)^{-1} C^T) = \text{rank}(AP_X (AP_X)^T) = \text{rank}(AP_X) = \text{rank}(AX) = \text{rank}(C) = q$ .

- (d) From (c) we know  $\text{rank}(AX) = \text{rank}(AP_X) = \text{rank}(A)$ .

4. (a) Let the parameter vector in such an order

$$\begin{bmatrix} (S1, F1, 25\%) \\ (S2, F1, 25\%) \\ (S1, F2, 25\%) \\ (S2, F2, 25\%) \\ (S1, F1, 50\%) \\ (S2, F1, 50\%) \\ (S1, F2, 50\%) \\ (S2, F2, 50\%) \\ (S1, F1, 75\%) \\ (S2, F1, 75\%) \\ (S1, F2, 75\%) \\ (S2, F2, 75\%) \end{bmatrix}$$

Then

$$\mathbf{C}_{(a)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0], \hat{\mu}_{F2,S1,50\%} = \mathbf{C}_{(a)}\hat{\beta} = 233.5$$

(b)

$$\mathbf{C}_{(b)} = \begin{bmatrix} 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 \end{bmatrix}, \begin{bmatrix} \hat{\mu}_{F1,...} \\ \hat{\mu}_{F2,...} \end{bmatrix} = \mathbf{C}_{(b)}\hat{\beta} = \begin{bmatrix} 214.75 \\ 181.08 \end{bmatrix}$$

(c)

$$\mathbf{C}_{(c)} = [0 \ 1/2 \ 0 \ 1/2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \hat{\mu}_{.,S2,25\%} = \mathbf{C}_{(c)}\hat{\beta} = 156.25$$

(d)

$$se_{(c)} = \sqrt{\hat{\sigma}^2 \mathbf{C}_{(c)}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{C}_{(c)}^T} = 8.20$$

(e)

$$\begin{aligned} H_0 : \bar{\mu}_{F1,...} &= \bar{\mu}_{F2,...} \quad (\mathbf{C}\beta - d = 0) \\ H_a : \bar{\mu}_{F1,...} &\neq \bar{\mu}_{F2,...} \quad (\mathbf{C}\beta - d \neq 0) \end{aligned}$$

$$\begin{aligned} \mathbf{C} &= [1/6 \ 1/6 \ -1/6 \ -1/6 \ 1/6 \ 1/6 \ -1/6 \ -1/6 \ 1/6 \ 1/6 \ -1/6 \ -1/6], d = 0 \\ F &= \frac{(\mathbf{C}\hat{\beta} - d)^T (\mathbf{C}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{C}^T)^{-1} (\mathbf{C}\hat{\beta} - d) / \text{rank}(\mathbf{C})}{\hat{\sigma}^2} = 25.3 \end{aligned}$$

$$\text{p-value} = 0.0003$$

Conclusion : Since p-value is small, we decide to reject the null hypothesis and conclude that there is filler type main effect for the fabric loss.

(f)

$$\begin{aligned} H_0 : & \text{there is no three way interaction.} \quad (\mathbf{C}\beta - d = 0) \\ H_a : & \text{there is three way interaction.} \quad (\mathbf{C}\beta - d \neq 0) \end{aligned}$$

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \end{bmatrix}, d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ F &= \frac{(\mathbf{C}\hat{\beta} - d)^T (\mathbf{C}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{C}^T)^{-1} (\mathbf{C}\hat{\beta} - d) / \text{rank}(\mathbf{C})}{\hat{\sigma}^2} = 0.89 \end{aligned}$$

$$\text{p-value} = 0.44$$

Conclusion : Since p-value is big, we fail to reject the null hypothesis and conclude that there is no three way interaction among the three factors for fabric loss.

(g)

$$\begin{aligned} H_0 : & \text{there is no two way interaction between filler type and filler proportion.} \quad (\mathbf{C}\beta - d = 0) \\ H_a : & \text{there is two way interaction between filler type and filler proportion.} \quad (\mathbf{C}\beta - d \neq 0) \end{aligned}$$

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}, d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ F &= \frac{(\mathbf{C}\hat{\beta} - d)^T (\mathbf{C}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{C}^T)^{-1} (\mathbf{C}\hat{\beta} - d) / \text{rank}(\mathbf{C})}{\hat{\sigma}^2} = 6.57 \end{aligned}$$

$$\text{p-value} = 0.01$$

Conclusion : Since p-value is big, we decide to reject the null hypothesis and conclude that there is two way interaction between filler type and filler proportion for fabric loss.