STAT 510 Homework 7

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1. (a)

$$\begin{aligned} E(MS_{ou(trt,xu)}) &= E\left(\frac{1}{tn(m-1)}\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk}-\bar{y}_{ij\cdot})^2\right) \\ &= \frac{1}{tn(m-1)}E\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(\mu+\tau_i+u_{ij}+e_{ijk}-\mu-\tau_i-u_{ij}-\bar{e}_{ij\cdot})^2\right) \\ &= \frac{1}{tn(m-1)}E\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(e_{ijk}-\bar{e}_{ij\cdot})^2\right) \end{aligned}$$

As $e_ijk \sim N(0,\sigma_e^2)$, then

$$\begin{aligned} E\left(MS_{ou(trt,xu)}\right) &= \frac{1}{tn(m-1)}\sum_{i=1}^{r}\sum_{j=1}^{n}(m-1)\sigma_e^2 \\ &= \frac{1}{tn(m-1)}tn(m-1)\sigma_e^2 \\ &= \sigma_e^2 \end{aligned}$$

(b) Let $\boldsymbol{A} = (\boldsymbol{I}-\boldsymbol{P}_3)$, where $\boldsymbol{P}_3 = \frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{1}\boldsymbol{1}^T_{m\times m}$ is a projection matrix. Then $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y} = \sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk}-\bar{y}_{ij\cdot})^2$. We also have

$$\Sigma = \sigma_u^2\boldsymbol{I}_{tn\times tn}\otimes \boldsymbol{1}\boldsymbol{1}^T_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm} = \sigma_u^2 m\boldsymbol{P}_3 + \sigma_e^2\boldsymbol{I}$$

Then

$$\begin{aligned} &\boldsymbol{A\Sigma} = (\boldsymbol{I}-\boldsymbol{P}_3)(\sigma_u^2 m\boldsymbol{P}_3+\sigma_e^2\boldsymbol{I}) = \sigma_e^2(\boldsymbol{I}-\boldsymbol{P}_3) = \sigma_e^2(\boldsymbol{I}-\frac{1}{m}\boldsymbol{I}_{tn\times tn}\otimes\boldsymbol{I}_{m\times m}) \\ \Rightarrow &\operatorname{tr}(\boldsymbol{A\Sigma}) = tnm\sigma_e^2(1-\frac{1}{m}) = \sigma_e^2 tn(m-1) \end{aligned}$$

We know that $E(\boldsymbol{y}) \subset \mathcal{C}(\boldsymbol{P}_2) \subset \mathcal{C}(\boldsymbol{P}_3)$. Thus $(\boldsymbol{I}-\boldsymbol{P}_3)E(\boldsymbol{y}) = \boldsymbol{0} \Rightarrow E(\boldsymbol{y})^T\boldsymbol{A}E(\boldsymbol{y}) = 0$. Hence,

$$E(\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}) = tn(m-1)\sigma_e^2 \Rightarrow E(MS_{ou(trt,xu)}) = \frac{1}{tn(m-1)}tn(m-1)\sigma_e^2 = \sigma_e^2$$

2. (a) $d_j = y_{1j}-y_{2j} = (\mu_1+u_j+e_{1j})-(\mu_2+u_j+e_{2j}) = (\mu_1-\mu_2)+(e_{1j}-e_{2j})$. We have $e_{ij} \overset{iid}{\sim} N(0,\sigma_e^2)$, then $d_j \overset{iid}{\sim} N(\mu_1-\mu_2, 2\sigma_e^2)$.

(b)

$$T = \frac{\bar{d}}{\sqrt{\frac{1}{380}\sum_{j=1}^{20}(d_j - \bar{d})^2}}, \text{ where } \bar{d} = \frac{1}{20}\sum_{j=1}^{20} d_j$$

(c)

$$T \sim t_{19}(\mu_1 - \mu_2)$$

(d)

$$CI = \left[\bar{a} - \bar{b} - t_{38,0.975}\sqrt{\frac{\sum_{j=1}^{20}(a_j - \bar{a})^2 + \sum_{j=1}^{2}(b_j - \bar{b})^2}{380}}, \bar{a} - \bar{b} + t_{38,0.975}\sqrt{\frac{\sum_{j=1}^{20}(a_j - \bar{a})^2 + \sum_{j=1}^{2}(b_j - \bar{b})^2}{380}}\right]$$

(e)

$$\hat{\sigma}_e^2 = \frac{\sum_{j=1}^{20}(d_j - \bar{d})^2}{38}$$

$$\hat{\sigma}_e^2 + \hat{\sigma}_u^2 = \frac{\sum_{j=1}^{20}(a_j - \bar{a})^2 + \sum_{j=1}^{20}(b_j - \bar{b})^2}{38} \Rightarrow \hat{\sigma}_u^2 = \frac{\sum_{j=1}^{20}(a_j - \bar{a})^2 + \sum_{j=1}^{20}(b_j - \bar{b})^2 - \sum_{j=1}^{20}(d_j - \bar{d})^2}{38}$$

(f)

$$\boldsymbol{X} = \begin{bmatrix} 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 2\sigma_e^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_e^2 & & & \\ & & & \sigma_e^2 + \sigma_u^2 & & \\ & & & & \ddots & \\ & & & & & \sigma_e^2 + \sigma_u^2 \end{bmatrix}$$

Then

$$\begin{aligned} \hat{\mu_1} - \hat{\mu_2} &= \begin{bmatrix} 1 & -1 \end{bmatrix} (\boldsymbol{X}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{X})^{-} \boldsymbol{X}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{y} \\ &= \frac{1}{\frac{1}{\sigma_e^2 + \sigma_u^2} + \frac{1}{\sigma_e^2}} \left(\frac{1}{\sigma_e^2}\bar{d} + \frac{1}{\sigma_u^2 + \sigma_e^2}(\bar{a} - \bar{b}) \right) \\ &= \frac{1}{\sigma_u^2 + 2\sigma_e^2} \left((\sigma_u^2 + \sigma_e^2)\bar{d} + \sigma_e^2(\bar{a} - \bar{b}) \right) \end{aligned}$$

3.

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{20\times 1} \otimes \boldsymbol{I}_{2\times 2} \\ \begin{bmatrix} \mathbf{1}_{20\times 1} & \mathbf{0}_{20\times 1} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{20\times 1} & \mathbf{1}_{20\times 1} \end{bmatrix} \end{bmatrix}$$

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{I}_{20\times 20} \otimes \mathbf{1}_{2\times 1} & \mathbf{0}_{40\times 40} \\ \mathbf{0}_{40\times 20} & \boldsymbol{I}_{40\times 40} \end{bmatrix}$$

4.

$$\begin{aligned}
\mathrm{BLUE}_1(\mu) &= \frac{y_1+y_2+y_3+y_4}{4} \\
\mathrm{Var}(\mathrm{BLUE}_1(\mu)) &= \frac{1}{16}\mathrm{Var}(y_1+y_2+y_3+y_4) \\
=&\frac{1}{16}\left(\sum_{i=1}^{4}\mathrm{Var}(y_i)+\sum_{i\neq j}\mathrm{Cov}(y_i,y_j)\right) \\
=&\frac{1}{16}(5\times 4+1\times 4\times 3)=2 \\
\mathrm{BLUE}_2(\mu) &= y_5 \\
\mathrm{Var}(\mathrm{BLUE}_2(\mu)) &= \mathrm{Var}(y_5)=4
\end{aligned}$$

We know $\mathrm{BLUE}_1(\mu)$ and $\mathrm{BLUE}_2(\mu)$ are independent, then

$$\mathrm{BLUE}(\mu)=\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}}\mathrm{BLUE}_1(\mu)+\frac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}}\mathrm{BLUE}_2(\mu)=\frac{2}{3}\frac{y_1+y_2+y_3+y_4}{4}+\frac{1}{3}y_5=\frac{1}{6}(y_1+y_2+y_3+y_4)+\frac{1}{3}y_5$$