STAT 510 Homework 5

Yifan Zhu

February 14, 2017

1. (a) $\mu_1 = \texttt{Intercept} \Rightarrow \text{BLUE}(\mu_1) = \text{BLUE}(\texttt{Intercept}) = 351$

(b) $\mu_2 = \texttt{Intercept} + \texttt{dose2} \Rightarrow \text{BLUE}(\mu_2) = \text{BLUE}(\texttt{Intercept}) + \text{BLUE}(\texttt{dose2}) = 351 - 10 = 341$

(c) $\widehat{Var}(\hat{\mu}_2) = \frac{\hat{\sigma}^2}{2}, \widehat{Var}(\texttt{Intercept}) = \widehat{Var}(\hat{\mu}_1) = \frac{\hat{\sigma}^2}{2} = \widehat{Var}(\hat{\mu}_2)$, then $se(\hat{\mu}_2) = se(\texttt{Intercept}) = 6.576$.

(d) $t = \frac{\hat{\mu}_1 - \hat{\mu}_2}{se(\hat{\mu}_1 - \hat{\mu}_2)} = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{\hat{\sigma}^2}} = \frac{-\texttt{dose2}}{se(\texttt{dose2})} = \frac{10}{9.301} = 1.075$.
Distribution is $t_5(\mu_1 - \mu_2)$.
p-value is 0.331406. Since p-value is large, there is no significant evidence that $\mu_1 = \mu_2$.

(e) $F = \left(\frac{\hat{\mu}_3 - \hat{\mu_4}}{se(\hat{\mu}_3 - \hat{\mu}_4)}\right)^2 = \left(\frac{-6+17}{9.301}\right)^2 = 1.399$.

(f) $F = \frac{(1038.5 - 432.5)/3}{432.5/5} = 2.3353$. The degrees of freedom is $(3, 5)$.
The p-value is 0.1907591, thus there is no siginificant evidence that full model can fit better than simple regression model. Thus a simple regression model is adequate here.

(g)

$$
\begin{aligned}
\mu_1 &= \beta_0 + 0\beta_1 \\
\mu_2 &= \beta_0 + 2\beta_1 \\
\mu_3 &= \beta_0 + 4\beta_1 \\
\mu_4 &= \beta_0 + 8\beta_1 \\
\mu_5 &= \beta_0 + 16\beta_1
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
\mu_1 - \mu_2 &= 2\beta_1 \\
\mu_3 - \mu_2 &= 2\beta_1 \\
\mu_4 - \mu_3 &= 4\beta_1 \\
\mu_5 - \mu_4 &= 8\beta_1
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
\mu_1 - \mu_2 &= \mu_3 - \mu_2 \\
2(\mu_3 - \mu_2) &= \mu_4 - \mu_3 \\
2(\mu_4 - \mu_3) &= \mu_5 - \mu_4
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
\mu_1 - 2\mu_2 - \mu_3 &= 0 \\
2\mu_2 - 3\mu_2 + \mu_4 &= 0 \\
2\mu_3 - 3\mu_4 + \mu_5 &= 0
\end{aligned}
$$

$\iff$

$$\begin{bmatrix}1 & -2 & -1 & 0 & 0\\ 0 & 2 & -3 & 1 & 0\\ 0 & 0 & 2 & -3 & 1\end{bmatrix}\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\end{bmatrix} = \mathbf{0}$$

Thus

$$\boldsymbol{C} = \begin{bmatrix}1 & -2 & -1 & 0 & 0\\ 0 & 2 & -3 & 1 & 0\\ 0 & 0 & 2 & -3 & 1\end{bmatrix}, \boldsymbol{d} = \mathbf{0}$$

(h)

	Df	Sum Sq	Mean Sq	F Value	Pr(>F)
d	1	5899.6	5899.6	67.4	0.0004245
dose	3	607	202.3	2.3	0.1907591
Residuals	5	432.5	87.5		

2. $\boldsymbol{X} = \boldsymbol{X}\boldsymbol{B}^{-1}\boldsymbol{B} \Rightarrow$ Every column of $\boldsymbol{X}$ is linear combination of columns of $\boldsymbol{X}\boldsymbol{B}^{-1} \Rightarrow \mathcal{C}(\boldsymbol{X}) \subset \mathcal{C}(\boldsymbol{X}\boldsymbol{B}^{-1})$.

$\boldsymbol{X}\boldsymbol{B}^{-1} = \boldsymbol{X}\boldsymbol{B}^{-1} \Rightarrow$ Every column of $\boldsymbol{X}\boldsymbol{B}^{-1}$ is linear combination of columns of $\boldsymbol{X} \Rightarrow \mathcal{C}(\boldsymbol{X}\boldsymbol{B}^{-1}) \subset \mathcal{C}(\boldsymbol{X})$.

Thus $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X}\boldsymbol{B}^{-1})$.

3.

$$\boldsymbol{P_X} = \boldsymbol{P_W} \iff \boldsymbol{P_X} - \boldsymbol{P_W} = \mathbf{0} \iff (\boldsymbol{P_X} - \boldsymbol{P_W})^T(\boldsymbol{P_X} - \boldsymbol{P_W})$$

$$\begin{aligned}(\boldsymbol{P_X} - \boldsymbol{P_W})^T(\boldsymbol{P_X} - \boldsymbol{P_W}) &= (\boldsymbol{P_X^T} - \boldsymbol{P_W^T})(\boldsymbol{P_X} - \boldsymbol{P_W})\\ &= \boldsymbol{P_X}\boldsymbol{P_X} - \boldsymbol{P_X}\boldsymbol{P_W} - \boldsymbol{P_W}\boldsymbol{P_X} + \boldsymbol{P_W}\boldsymbol{P_W}\\ &= \boldsymbol{P_X} - \boldsymbol{P_X}\boldsymbol{P_W} - \boldsymbol{P_W}\boldsymbol{P_X} + \boldsymbol{P_W}\end{aligned}$$

As $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(W)$, we have $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{B}$. Then $\boldsymbol{P_X}\boldsymbol{P_W} = \boldsymbol{P_X}\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^-\boldsymbol{W}^T = \boldsymbol{P_X}\boldsymbol{X}\boldsymbol{B}(\boldsymbol{W}^T\boldsymbol{W})^-\boldsymbol{W}^T = \boldsymbol{X}\boldsymbol{B}(\boldsymbol{W}^T\boldsymbol{W})^T\boldsymbol{W}^T = \boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^-\boldsymbol{W}^T = \boldsymbol{P_W}$. In the same way we can prove $\boldsymbol{P_W}\boldsymbol{P_X} = \boldsymbol{P_X}$. Then

$$(\boldsymbol{P_X} - \boldsymbol{P_W})^T(\boldsymbol{P_X} - \boldsymbol{P_W}) = \boldsymbol{P_X} - \boldsymbol{P_W} - \boldsymbol{P_X} + \boldsymbol{P_W} = \mathbf{0}$$

Thus $\boldsymbol{P_X} = \boldsymbol{P_W}$.

4. (a)

$$\boldsymbol{X} = \begin{bmatrix}1 & 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1\end{bmatrix}$$

(b) Let $\boldsymbol{A} = \begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\end{bmatrix}$, then $\boldsymbol{A}\boldsymbol{X} = \begin{bmatrix}0 & 0 & 0 & 1 & -1\end{bmatrix}$. $\boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta} = \tau_1 - \tau_2$, hence it is estimable.

(c)

$$\boldsymbol{X}^* = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Each column of $\boldsymbol{X}^*$ is linear combination of columns of $\boldsymbol{X}$. We can also get every column of $\boldsymbol{X}$ with linear combination of columns of $\boldsymbol{X}^*$. Thus $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X}^*)$. Also, columns of $\boldsymbol{X}^*$ are orthogonal.

(d)

$$\boldsymbol{\beta}^* = \begin{bmatrix} \mu + \lambda_1 + \frac{\tau_1+\tau_2}{2} \\ \mu + \lambda_1 + \frac{\tau_1+\tau_2}{2} \\ \frac{\tau_1-\tau_2}{2} \end{bmatrix}$$

(e) $\boldsymbol{X}^* = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix}$, then

$$(X^*)^T \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T \\ \boldsymbol{x}_3^T \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1^T \boldsymbol{x}_1 & 0 & 0 \\ 0 & \boldsymbol{x}_2^T \boldsymbol{x}_2 & 0 \\ 0 & 0 & \boldsymbol{x}_3^T \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$(\boldsymbol{X}^*)^T \boldsymbol{X}^* = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$$

$$\boldsymbol{P_X} = \boldsymbol{P_{X^*}} = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3\end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T \\ \boldsymbol{x}_3^T \end{bmatrix}$$

$$= \frac{1}{4}\boldsymbol{x}_1\boldsymbol{x}_1^T + \frac{1}{4}\boldsymbol{x}_2\boldsymbol{x}_2^T + \frac{1}{8}\boldsymbol{x}_3\boldsymbol{x}_3^T$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$+ \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 3/8 & 1/8 & 1/8 & 1/8 & 1/8 & -1/8 & -1/8 \\ 3/8 & 3/8 & 1/8 & 1/8 & 1/8 & 1/8 & -1/8 & -1/8 \\ 1/8 & 1/8 & 3/8 & 3/8 & -1/8 & -1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 3/8 & 3/8 & -1/8 & -1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & -1/8 & -1/8 & 3/8 & 3/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & -1/8 & -1/8 & 3/8 & 3/8 & 1/8 & 1/8 \\ -1/8 & -1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/8 & 3/8 \\ -1/8 & -1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/8 & 3/8 \end{bmatrix}$$

$$\boldsymbol{AP_X} = \begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & -1/4 \end{bmatrix}$$

Then

$$\text{OLSE}(\tau_1 - \tau_2) = \boldsymbol{AP_X y} = \frac{1}{4}y_{111} + \frac{1}{4}y_{112} - \frac{1}{4}y_{121} - \frac{1}{4}y_{122} + \frac{1}{4}y_{211} + \frac{1}{4}y_{212} - \frac{1}{4}y_{221} - \frac{1}{4}y_{222}$$