STAT 542 Homework 3

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1. Support of f_Y is $\mathcal{Y} = \{\frac{k}{k+1} | k = 0, 1, 2, \cdots\}$.

$P(Y = y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{y/(1-y)}$, $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots$.

Thus, pmf of Y

$$f_Y(y) = \begin{cases} \frac{1}{3}(\frac{2}{3})^{y/(1-y)} & y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \cdots \\ 0 & \text{otherwise} \end{cases}$$

2. (a)

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x)\mathrm{d}x \\ =&\int_{-\infty}^{0} \frac{1}{2}\lambda \mathrm{e}^{\lambda x}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{2}\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x \\ =&\frac{1}{2}\mathrm{e}^{\lambda x}\Big|_{-\infty}^{0} - \frac{1}{2}\mathrm{e}^{-\lambda x}\Big|_{0}^{\infty} \\ =&\frac{1}{2} - (-\frac{1}{2}) = 1 \end{aligned}$$

$f(x) \geq 0$ for $x \in \mathbb{R}$ and the integral is 1, thus $f(x)$ is a cdf.

(b) For $t \leq 0$,

$$\begin{aligned} P(X < t) &= \int_{-\infty}^{0} f(x)\mathrm{d}x \\ &= \int_{-\infty}^{t} \frac{1}{2}\lambda \mathrm{e}^{\lambda x}\mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^{\lambda x}\Big|_{-\infty}^{t} = \frac{1}{2}\mathrm{e}^{\lambda t} \end{aligned}$$

For $t > 0$,

$$\begin{aligned} P(X < t) &= \int_{-\infty}^{t} f(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} \frac{1}{2}\lambda \mathrm{e}^{\lambda x}\mathrm{d}x + \int_{0}^{t} \frac{1}{2}\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-\lambda x}\Big|_{0}^{t} \\ &= 1 - \left(\frac{1}{2}\mathrm{e}^{-\lambda t} - \frac{1}{2}\right) = 1 - \frac{1}{2}\mathrm{e}^{-\lambda t} \end{aligned}$$

Thus

$$P(X < t) = \begin{cases} \frac{1}{2}\mathrm{e}^{\lambda t} & t \le 0 \\ 1 - \frac{1}{2}\mathrm{e}^{-\lambda t} & t > 0 \end{cases}$$

(c) For $t \ge 0$

$$\begin{aligned} P(|X| < t) = P(-t < X < t) &= \int_{-t}^{0} \frac{1}{2}\lambda \mathrm{e}^{\lambda x}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{2}\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^{\lambda x}\Big|_{-t}^{0} - \frac{1}{2}\mathrm{e}^{\lambda x}\Big|_{0}^{t} \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-\lambda t} - \left(\frac{1}{2}\mathrm{e}^{-\lambda t} - \frac{1}{2}\right) \\ &= 1 - \mathrm{e}^{-\lambda t} \end{aligned}$$

For $t < 0$, $P(|X| < t) = 0$. Thus

$$P(|X| < t) = \begin{cases} 0 & t < 0 \\ 1 - \mathrm{e}^{-\lambda t} & t \ge 0 \end{cases}$$

3. (b) Let $A_1 = (-1, 0]$, $A_2 = (0, 1)$. $g_1(x) = 1 - x^2$, $g_1^{-1}(y) = -\sqrt{1-y}$. $g_2(x) = 1 - x^2$, $g_2^{-1}(y) = \sqrt{1-y}$. g_1 and g_2 are monotone on A_1 and A_2, thus

$$\begin{aligned} f_Y(y) &= f_X(g_1^{-1}(y))|{g_1^{-1}}'(y)| + f_X(g_2^{-1}(y))|{g_2^{-1}}'(y)| \\ &= \frac{3}{8}(1-\sqrt{1-y})^2\frac{1}{2\sqrt{1-y}} + \frac{3}{8}(1+\sqrt{1-y})^2\frac{1}{2\sqrt{1-y}} \\ &= \frac{3}{8}\frac{2-y}{\sqrt{1-y}} \end{aligned}$$

(c) Let $A_1 = (-1, 0]$, $A_2 = (0, 1)$. $g_1(x) = 1 - x^2$, $g_1^{-1}(y) = -\sqrt{1-y}$. $g_2(x) = 1 - x$, $g_2^{-1}(y) = 1 - y$. g_1 and g_2 are monotone on A_1 and A_2, thus

$$\begin{aligned} f_Y(y) &= f_X(g_1^{-1}(y))|{g_1^{-1}}'(y)| + f_X(g_2^{-1}(y))|{g_2^{-1}}'(y)| \\ &= \frac{3}{8}(1-\sqrt{1-y})^2\frac{1}{2\sqrt{1-y}} + \frac{3}{8}(1+1-y)^2|-1| \\ &= \frac{3}{16}(1-\sqrt{1-y})^2\frac{1}{\sqrt{1-y}} + \frac{3}{8}(2-y)^2 \end{aligned}$$

4. The cdf of X is

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4}(x-1)^2 & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

Let $Y = u(X) = F_X(X)$. Then for $y \le 0$,

$$P(Y \le y) = P(Y = 0) = P(X \le 1) = F_X(1) = 0$$

For $0 < y \le 1$,

$$P(Y \le y) = P(X \le 2\sqrt{y} + 1) = F_X(2\sqrt{y} + 1) = y$$

For $y > 1$,

$$P(Y \le y) = 1$$

Thus the cdf of Y is

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y \le 1 \\ 1 & y > 1 \end{cases}$$

Hence, $Y \sim \text{unif}(0,1)$.

5. (a) Let $y = \frac{\beta}{\sqrt{2}}y$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \int_0^{\infty} \frac{4}{\beta^3\sqrt{\pi}} x^2 \mathrm{e}^{-x^2/\beta^2}\mathrm{d}x \\
&= \int_0^{\infty} \frac{4}{\beta^3\sqrt{\pi}} \frac{\beta^2}{2} y^2 \mathrm{e}^{-y^2/2} \frac{\beta}{\sqrt{2}}\mathrm{d}y \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} y^2 \mathrm{e}^{-y^2/2}\mathrm{d}y = 1
\end{aligned}$$

The integral is 1 and $f(x) \ge 0$ for $x \in \mathbb{R}$, thus $f(x)$ is a pdf.

(b) Let $y = \frac{\beta}{\sqrt{2}}y$. Then

$$\begin{aligned}
E(X) = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x &= \int_0^{\infty} \frac{4}{\beta^3\sqrt{\pi}} x^3 \mathrm{e}^{-x^2/\beta^2}\mathrm{d}x \\
&= \int_0^{\infty} \frac{4}{\beta^3\sqrt{\pi}} \frac{\beta^3}{2\sqrt{2}} y^2 \mathrm{e}^{-y^2/2} \frac{\beta}{\sqrt{2}}\mathrm{d}y \\
&= \frac{\beta}{\sqrt{\pi}} \int_0^{\infty} y^3 \mathrm{e}^{-y^2/2}\mathrm{d}y \\
&= \frac{\beta}{\sqrt{\pi}} \int_0^{\infty} y^2 \mathrm{d}\left(-\mathrm{e}^{-y^2/2}\right) \\
&= \frac{\beta}{\sqrt{\pi}} \left[-y^2\mathrm{e}^{-y^2/2}\Big|_0^{\infty} + \int_0^{\infty} \mathrm{e}^{-y^2/2}\mathrm{d}y^2 \right] \\
&= \frac{\beta}{\sqrt{\pi}} \int_0^{\infty} \mathrm{e}^{-t/2}\mathrm{d}t = -2\mathrm{e}^{-t/2}\Big|_0^{\infty} = \frac{\beta}{\sqrt{\pi}} \cdot 2 = \frac{2\beta}{\sqrt{\pi}}
\end{aligned}$$

In the second last line, $\lim_{y\to\infty} y^2\mathrm{e}^{-y^2/2} = 0$

6.

$$\begin{aligned}
E(Y) = EX^2 &= \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2}x^2(1+x)\mathrm{d}x \\
&= \frac{1}{6}x^3 + \frac{1}{8}x^4 \Big|_{-1}^{1} \\
&= \frac{1}{3}
\end{aligned}$$

$$
\begin{aligned}
E(Y^2) = E(X^4) &= \int_{-\infty}^{\infty} x^4 f(x)\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2}x^4(1+x)\mathrm{d}x \\
&= \frac{1}{10}x^5 + \frac{1}{12}x^6\Big|_{-1}^{1} \\
&= \frac{1}{5} \\
Var(Y) = E(Y^2) - (EY)^2 &= \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}
\end{aligned}
$$

7. (a)

$$
P(X = x) = \begin{cases} (\frac{1}{2})^{x-1}\frac{1}{2} = (\frac{1}{2})^x & x = 1, 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}
$$

(b) $X \sim$ Geometric($\frac{1}{2}$), thus $EX = \frac{1}{1/2} = 2$.

(c) $X_m = X - 1$, $X_f = 1$. Thus $E(X_m) = E(X-1) = EX - 1 = 1$, $E(X_f) = 1$.

8. (a) $F_Y(y) = P(Y \leq y) = P(F^{-1}(U) \leq y) = P(U \leq F(y)) = F(y)$. (bacause $0 \leq F(y) \leq 1$) Thus Y and X have the same distribution.

(b) For $u > 0$, cdf of Z

$$
\begin{aligned}
F_Z(u) &= \int_0^u f_Z(z)\mathrm{d}z = \int_0^u 2ze^{-z^2}\mathrm{d}z \\
&= \int_0^u \mathrm{e}^{-z^2}\mathrm{d}z^2 = \int_0^{u^2} \mathrm{e}^{-t}\mathrm{d}t \\
&= -\mathrm{e}^{-t}\Big|_0^{u^2} \\
&= 1 - \mathrm{e}^{-u^2}
\end{aligned}
$$

On $(0, \infty)$, we have $F_Z : (0, \infty) \mapsto (0, 1)$ and it is monotone.So the inverse $F_Z^{-1} : (0, 1) \mapsto (0, \infty)$. $F_Z^{-1}(u) = \sqrt{-\log(1-u)}$. Let $Z = \sqrt{-\log(1-U)}$, where $U \sim \text{unif}(0, 1)$, then Z distributes according to f_Z. With the randonly generated numbers distributed unifornly on $(0, 1)$, we can plugin these numbers into $\sqrt{-\log(1-(\cdot))}$ to get the numbers distributed according to f_Z.