STAT 543 Homework 7

Yifan Zhu

April 20, 2017

1. (a) $\theta_1 = \sigma_X^2, \theta_2 = \sigma_Y^2$, then $\Theta_0 = \{(\theta_1, \theta_2) : \theta_2/\theta_1 = \lambda_0\}$, $\Theta = (0, \infty) \times (0, \infty)$.

$$\begin{aligned} L(\boldsymbol{\theta}) &= f(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{\theta}) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_1}} \mathrm{e}^{-\frac{x_i^2}{2\theta_1}} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\theta_2}} \mathrm{e}^{-\frac{y_i^2}{2\theta_2}} \\ &= (2\pi)^{-\frac{(n+m)}{2}} \theta_1^{-n/2} \theta_2^{-m/2} \mathrm{e}^{-\sum_{i=1}^{n} \frac{x_i^2}{2\theta_1} - \sum_{i=1}^{m} \frac{y_i^2}{2\theta_2}} \end{aligned}$$

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= \log L(\boldsymbol{\theta}) \\ &= -\frac{n+m}{2} \log(2\pi) - \frac{n}{2} \log \theta_1 - \frac{m}{2} \log \theta_2 - \sum_{i=1}^{n} \frac{x_i^2}{2\theta_1} - \sum_{i=1}^{m} \frac{y_i^2}{2\theta_2} \end{aligned}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \theta_1} &= -\frac{n}{2} \frac{1}{\theta_1} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta_1^2} \\ \frac{\partial \ell}{\partial \theta_2} &= -\frac{m}{2} \frac{1}{\theta_2} + \frac{\sum_{i=1}^{m} y_i^2}{2\theta_2^2} \end{aligned}$$

Let $\frac{\partial \ell}{\partial \boldsymbol{\theta}} = \mathbf{0}$, we have

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n} x_i^2}{n},\ \hat{\theta}_2 = \frac{\sum_{i=1}^{m} y_i^2}{m}$$

Also we have

$$\begin{aligned} \left.\frac{\partial^2 \ell}{\partial \theta_1^2}\right|_{\hat{\boldsymbol{\theta}}} &= \frac{n}{2\hat{\theta}_1^2} - \frac{\sum_{i=1}^{n} x_i^2}{\hat{\theta}_1^3} = -\frac{n}{\hat{\theta}_1^2} \\ \left.\frac{\partial^2 \ell}{\partial \theta_2^2}\right|_{\hat{\boldsymbol{\theta}}} &= \frac{m}{2\hat{\theta}_2^2} - \frac{\sum_{i=1}^{m} y_i^2}{\hat{\theta}_2^3} = -\frac{m}{\hat{\theta}_2^2} \\ \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} &= 0 \end{aligned}$$

Thus $H = \frac{\partial^2 \ell}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}$ is negetive definite, then

$$\max_{\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta}) = L(\hat{\boldsymbol{\theta}}) = (2\pi)^{-\frac{n+m}{2}} \hat{\theta}_1^{-n/2} \hat{\theta}_2^{-m/2} \mathrm{e}^{-\frac{n+m}{2}}$$

In Θ_0,

$$\ell(\boldsymbol{\theta}) = -\frac{n+m}{2}\log(2\pi) - \frac{m}{2}\log\lambda_0 - \frac{n}{2}\log\theta_1 - \frac{m}{2}\log\theta_1 - \sum_{i=1}^{n}\frac{x_i^2}{2}\frac{1}{\theta_1} - \sum_{i=1}^{n}\frac{y_i^2}{2\lambda_0}\frac{1}{\theta_1}$$

$$\frac{\partial\ell}{\partial\theta_1} = -\frac{n+m}{2}\frac{1}{\theta_1} + \left(\sum_{i=1}^{n}\frac{x_i^2}{2} + \sum_{i=1}^{m}\frac{y_i^2}{2\lambda_0}\right)\frac{1}{\theta_1^2}$$

Let $\frac{\partial\ell}{\partial\theta_1} = 0$, we have

$$\tilde{\theta}_1 = \frac{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2/\lambda_0}{n+m}$$

Also

$$\frac{\partial^2\ell}{\partial\theta_1^2} = \frac{n+m}{2}\frac{1}{\theta_1^2} - \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{m} y_i^2/\lambda_0\right)\frac{1}{\theta_1^3} = -\frac{n+m}{2}\frac{1}{\tilde{\theta}_1^2} < 0$$

Thus

$$\max_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta}) = L((\tilde{\theta}_1, \lambda_0\tilde{\theta}_1)) = (2\pi)^{-\frac{n+m}{2}}\tilde{\theta}_1^{-\frac{n+m}{2}}\lambda_0^{-\frac{m}{2}}\mathrm{e}^{-\frac{n+m}{2}}$$

Hence we have

$$\lambda = \frac{\max_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta})} = \frac{\lambda_0^{-m/2}\tilde{\theta}_1^{-(n+m)/2}}{\hat{\theta}_1^{-n/2}\hat{\theta}_2^{-m/2}} = \frac{\hat{\theta}_1^{n/2}\hat{\theta}_2^{m/2}}{\lambda_0^{m/2}\tilde{\theta}_1^{(m+n)/2}}$$

Then

$$\begin{aligned}
&\lambda < k \\
\iff & \frac{\hat{\theta}_1^{n}\hat{\theta}_2^{m}}{\lambda_0^{m}\tilde{\theta}_1^{(m+n)}} < k^2 \\
\iff & \frac{\left(\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{m} Y_i^2/\lambda_0\right)^{n+m}}{\left(\sum_{i=1}^{n} X_i^2\right)^n\left(\sum_{i=1}^{m} Y_i^2/\lambda_0\right)^m} > k_1 \\
\iff & \left(1 + \frac{\sum_{i=1}^{m} Y_i^2/\lambda_0}{\sum_{i=1}^{n} X_i^2}\right)^n\left(1 + \frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{m} Y_i^m/\lambda_0}\right)^m > k_2
\end{aligned}$$

Let $T = \frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{m} Y_i^2/\lambda_0}$.
Then

$$\lambda < k \iff (1+T^{-1})^n(1+T)^m > k_2$$

Under H_0, $T = \frac{n}{m}\cdot\frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2/\sigma_X^2}{\frac{1}{m}\sum_{i=1}^{m} Y_i^2/(\lambda_0\sigma_X^2)} = \frac{n}{m}F \sim \frac{n}{m}F_{n,m}$. Then

$$P_{H_0}(\lambda < k) = P_{H_0}((1+\frac{m}{n}F^{-1})^n(1+\frac{n}{m}F)^m > k_2) = \alpha \Rightarrow k_2 = c_{n,m,\alpha}$$

Thus the LRT is

$$\Phi(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} 1 & , (1+T^{-1})^n(1+T)^m > c_{m,n,\alpha} \\ 0 & , (1+T^{-1})^n(1+T)^m < c_{m,n,\alpha} \end{cases}$$

(b) Rejection region in terms of F is derived in part (a). We have

$$\left\{(x,y) : \left(1+\frac{m}{n}F^{-1}\right)^n\left(1+\frac{n}{m}F\right)^m > c_{n,m,\alpha}\right\}$$

(c) The accept region is

$$\left\{(x,y) : \left(1+T^{-1}\right)^n (1+T)^m \le c_{n,m,\alpha}\right\}$$

And for function $g(t) = (1+t^{-1})^n(1+t)^m$, we have $g'(t) = \frac{(1+t^{-1})^n(1+t)^{m-1}(mt-n)}{t}$. Hence g first decreases then increases as t increases. The turning point is n/m. Then

$$\begin{aligned}
&\left(1+T^{-1}\right)^n (1+T)^m \le c_{n,m,\alpha} \\
\iff & l_{n,m,\alpha\alpha} \le T \le u_{n,m,\alpha} \\
\iff & l_{n,m,\alpha} \le \frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^m Y_i^2/\lambda_0} \le u_{n,m,\alpha} \\
\iff & l_{n,m,\alpha}\frac{\sum_{i=1}^m Y_i^2}{\sum_{i=1}^n X_i^2} \le \lambda_0 \le u_{n,m,\alpha}\frac{\sum_{i=1}^m Y_i^2}{\sum_{i=1}^n X_i^2}
\end{aligned}$$

Hence the $1-\alpha$ CI is

$$\left[\frac{l_{n,m,\alpha}\sum_{i=1}^m Y_i^2}{\sum_{i=1}^n X_i^2}, \frac{u_{n,m,\alpha}\sum_{i=1}^m Y_i^2}{\sum_{i=1}^n X_i^2}\right]$$

2. $\frac{\partial}{\partial t}F_\theta(t) = f(t|\theta) \ge 0$, then

$$g(Q(t,\theta))\left|\frac{\partial}{\partial t}Q(t,\theta)\right| = 1 \cdot |f(t|\theta)| = f(t|\theta)$$

3. We have $F_T(T|\theta) = U \sim Unif(0,1)$,then

$$\begin{aligned}
E_{\theta_0}\Phi(T) &= P_{\theta_0}(T \notin A_{\theta_0}) \\
&= 1 - P_{\theta_0}(\alpha_1 \le F_T(T|\theta_0) \le 1-\alpha_2) \\
&= 1 - P(\alpha_1 \le U \le 1-\alpha_2) \\
&= 1 - (1-\alpha_2-\alpha_1) \\
&= 1 - (1-\alpha) = \alpha
\end{aligned}$$

Hence it is a simple test with level α.

By inverting the test, we have the $1-\alpha$ confidence region

$$\{\theta : \alpha_1 \le F_T(t|\theta) \le 1-\alpha_2\}$$

4. (a) $X \sim Beta(\theta,1)$, then $f_X(x) = \theta x^{\theta-1}$ and $F_X(x) = x^\theta$.
$P_\theta(\theta \in [Y/2, Y]) = P_\theta(-\frac{1}{2\log X} \le \theta \le -\frac{1}{\log X}) = P_\theta(\mathrm{e}^{-1/\theta} \le X \le \mathrm{e}^{-1/(2\theta)}) = F_X(\mathrm{e}^{-\frac{1}{2\theta}}) - F_X(\mathrm{e}^{-\frac{1}{\theta}}) = (\mathrm{e}^{-\frac{1}{2\theta}})^\theta - (\mathrm{e}^{-\frac{1}{\theta}})^\theta = \mathrm{e}^{-1/2} - \mathrm{e}^{-1} = 0.236$.

(b) Use $Q(X,\theta) = X^\theta$ as a pivot quantity. Then the $1-\alpha$ confidece interval is

$$\left\{\theta : \frac{\alpha}{2} \le X^\theta \le 1-\frac{\alpha}{2}\right\} = \left\{\theta : \log\frac{\alpha}{2} \le \theta\log X \le \log(1-\frac{\alpha}{2})\right\} = \left\{\theta : \frac{\log(1-\alpha/2)}{\log X} \le \theta \le \frac{\log(\alpha/2)}{\log X}\right\}$$

(c) They are both of the form $[\frac{a}{\log X}, \frac{b}{\log X}]$.

5. (a)

$$A_{\theta_0} = \left\{\boldsymbol{x} : \theta_0 - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x} \le \theta_0 + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}$$

$$CI = \left\{\theta : \bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \le \theta \le \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}$$

(b)

$$A_{\theta_0} = \left\{ \boldsymbol{x} : \bar{x} \geq \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

$$CI = \left\{ \theta : \theta \leq \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

(c)

$$A_{\theta_0} = \left\{ \boldsymbol{x} : \bar{x} \leq \theta_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} \right\}$$

$$CI = \left\{ \theta : \theta \geq \bar{x} - z_{1-\alpha} \frac{\sigma}{\sqrt{n}} \right\}$$