STAT 557 Homework 3

Yifan Zhu

November 2, 2017

1. (a) Under the alternative hypothesis, $\hat{m}_{A,ij} = Y_{ij}$.
Under the null hypothesis,

$$\hat{m}_{0,i1} = \frac{9}{16}n_i$$
$$\hat{m}_{0,i2} = \frac{3}{16}n_i$$
$$\hat{m}_{0,i3} = \frac{3}{16}n_i$$
$$\hat{m}_{0,i4} = \frac{1}{16}n_i$$

where $n_i = \sum_{j=1}^{4} Y_{ij}$.
Thus we have

$$G^2 = 2\sum_{i=1}^{3}\sum_{j=1}^{4} Y_{ij}\log(\hat{m}_{A,ij}/\hat{m}_{0,ij}) = 3.143858$$

The degrees of freedom is **9** and p-value is **0.9583171**. With large p-value we conclude that the model fits the data well.

(b) Let the parameters for multinomial distribution be region i be $p_{ai}p_{bi}, (1-p_{ai})p_{bi}, p_{ai}(1-p_{bi}), (1-p_{ai})(1-p_{bi})$. Then

$$\begin{aligned}
\ell &= \sum_{i=1}^{3}\log n_i! \\
&+ \sum_{i=1}^{3}\left[Y_{i1}\log(p_{ai}p_{bi}) + Y_{i2}\log((1-p_{ai})p_{bi}) + Y_{i3}\log(p_{ai}(1-p_{bi})) + Y_{i4}\log((1-p_{ai})(1-p_{bi}))\right] \\
&- \sum_{i=1}^{3}\sum_{j=1}^{4}\log Y_{ij}!
\end{aligned}$$

Let $\frac{\partial \ell}{\partial p_{ai}} = \frac{\partial \ell}{\partial p_{bi}} = 0$, we have

$$\hat{p}_{ai} = \frac{Y_{i1}+Y_{i3}}{n_i}, \hat{p}_{bi} = \frac{Y_{i1}+Y_{i2}}{n_i}$$

And

$$\hat{m}_{0,i1} = \hat{p}_{ai}\hat{p}_{bi}n_i$$
$$\hat{m}_{0,i2} = (1-\hat{p}_{ai})\hat{p}_{bi}n_i$$
$$\hat{m}_{0,i3} = \hat{p}_{ai}(1-\hat{p}_{bi})n_i$$
$$\hat{m}_{0,i4} = (1-\hat{p}_{ai})(1-\hat{p}_{bi})n_i$$

Then

$$G^2 = 1.133048$$

The degrees of freedom is **3** and the p-value is **0.7691027**. With large p-value we conclude that the model fits the data well.

(c) Set $p_{ai} = p_a$ for $i = 1, 2, 3$, then we have

$$\begin{aligned}\ell = &\sum_{i=1}^{3} \log n_i! \\ &+ \sum_{i=1}^{3} [Y_{i1} \log(p_a p_b) + Y_{i2} \log((1-p_a)p_b) + Y_{i3} \log(p_a(1-p_b)) + Y_{i4} \log((1-p_a)(1-p_b))] \\ &- \sum_{i=1}^{3}\sum_{j=1}^{4} \log Y_{ij}!\end{aligned}$$

Let $\frac{\partial \ell}{\partial p_a} = \frac{\partial \ell}{\partial p_b} = 0$, we have

$$\hat{p}_a = \frac{\sum_{i=1}^{3}(Y_{i1} + Y_{i3})}{n}, \hat{p}_b = \frac{\sum_{i=1}^{3}(Y_{i1} + Y_{i2})}{n}$$

where $n = \sum_{i=1}^{3} n_i = \sum_{i=1}^{3}\sum_{j=1}^{4} Y_{ij}$. Then

$$G^2 = 1.628537$$

The degrees of freedom is **7** and the p-value is **0.9775233**. With large p-value we conclude that the model fits the data well.

(d) Let the model for (a), (b), (c) be Model A, Model B and Model C. And the model for alternative hypothesis be the saturated model Model S, then parameter space of Model A is in Model C in Model B in Model S. The deviance table is shown below

Comparison	**d.f.**	**deviance value**	**p-value**
Model A vs Model C	2	1.515321	0.4687618
Model C vs Model B	4	0.4954887	0.9739387
Model B vs Model S	3	1.133048	0.7691027
Model A vs Model S	9	3.143858	0.9583171

From the table we see Model A already fit the data pretty well, and finer models do not fit significantly better. Thus we can use Model A.

(e) Model A, B and C all provide an adequate description of the data. The simplest one is Model A. Large p-value shows a good fit of data and finer models do not fit significantly better because of the large p-value.

2.

$$f(y_i|\mu) = \frac{\mu^{y_i}}{y_i!} \mathrm{e}^{-\mu}$$

then

$$\ell = \log \prod_{i=1}^{n} f(y_i|\mu) = \log \mu \sum_{i=1}^{n} y_i - n\mu - \sum_{i=1}^{n} \log y_i! = n\bar{y} \log \mu - n\mu - \sum_{i=1}^{n} \log y_i!$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$. Then

$$\begin{aligned}\frac{\partial \ell}{\partial \mu} &= \frac{n\bar{y}}{\mu} - n = Q \\ \frac{\partial^2 \ell}{\partial \mu^2} &= -\frac{n\bar{y}}{\mu^2} = -H\end{aligned}$$

Hence

$$I = E[H] = E\left[-\frac{\partial^2 \ell}{\partial \mu^2}\right] = \frac{nEy_i}{\mu^2} = \frac{n}{\mu}$$

Using Fisher scoring algorithm,

$$\hat{\mu}^{(1)} = \hat{\mu}^{(0)} + [\hat{I}^{(0)}]^{-1}\hat{Q}^{(0)} = \hat{\mu}^{(0)} + \frac{\hat{\mu}^{(0)}}{n}\left(\frac{n\bar{y}}{\hat{\mu}^{(0)}} - n\right) = \bar{y}$$

Now

$$\hat{Q}^{(1)} = \frac{n\bar{y}}{\hat{\mu}^{(1)}} - n = 0$$

thus it converges after one step.

Using Newton-Raphson, we have

$$\hat{\mu}^{(s+1)} = \hat{\mu}^{(s)} + [\hat{H}^{(s)}]^{-1}\hat{Q}^{(s)} = \hat{\mu}^{(s)} + \frac{(\hat{\mu}^{(s)})^2}{n\bar{y}}\left(\frac{n\bar{y}}{\hat{\mu}^{(s)}} - n\right) = 2\hat{\mu}^{(s)} - \frac{(\hat{\mu}^{(s)})^2}{\bar{y}}$$

Then

$$\hat{\mu}^{(s+1)} - \hat{\mu}^{(s)} = \left[2 - \frac{\hat{\mu}^{(s+1)} + \hat{\mu}^{(s)}}{\bar{y}}\right](\hat{\mu}^{(s)} - \hat{\mu}^{(s-1)})$$

Because

$$\hat{\mu}^{(s+1)} = 2\hat{\mu}^{(s)} - \frac{(\hat{\mu}^{(s)})^2}{\bar{y}} = -\frac{1}{\bar{y}}(\hat{\mu}^{(s)} - \bar{y})^2 + \bar{y}$$

when 0 ¡ $\hat{\mu}^{(0)} < 2\bar{y}$, we have $0 < \hat{\mu}^{(s)} \leq \bar{y}$. Then

$$\left|2 - \frac{\hat{\mu}^{(s+1)} + \hat{\mu}^{(s)}}{\bar{y}}\right| \leq 1, \forall s \geq 1$$

Hence with initial value $\hat{\mu}^{(0)} \in (0, 2\bar{y})$, Newton-Raphson converges. When $\hat{\mu}^{(0)} > 2\bar{y}$, we have $\hat{\mu}^{(1)} < 0$. Then it does not converge.

3. (a)

$$f(y|k,\mu) = \exp(\log f(y|k,\mu))$$

and

$$\log f(y|k,\mu) = \log\Gamma(y+k) - \log\Gamma(k) - \log G(y+1) + k\log k - k\log(\mu+k) + y\left(\log(1 - \frac{k}{\mu+k})\right)$$

$$= y\log\left(1 - \frac{k}{\mu+k}\right) - \log(\mu+k) + [k\log k + \log\Gamma(y+k) - \log\Gamma(k) - \log\Gamma(y+1)]$$

Here $T(\mu) = \log\left(1 - \frac{k}{\mu+k}\right)$, $A(\mu) = \log(\mu+k)$, $h(y) = k\log k + \log\Gamma(y+k) - \log\Gamma(k) - \log\Gamma(y+1)$. $f(t|k,\mu) = \exp(yT(\mu) - A(\mu) + h(y))$.It is a member of the exponential family.

(b) It is not a member of exponential family. Because in this case $log\Gamma(y+k)$ cannot be expressed as $yg(\mu,k)$ for some function g. Hence it is not a member of exponential family.

4. (a) Estimates and standard errors of β_0 and β_1 is shown below.

Parameter	**Estimate**	**Standard Error**
β_0	-61.3183	12.0224
β_1	2.2110	0.4309

(b) The Wald Chi-squared statistic is given by

$$\frac{(\hat{\beta}_1^2)}{\widehat{Var}(\hat{\beta}_1)} = 26.3345$$

The degrees of freedom is 1 and p-value is < 0.0001. With small p-value we reject the null hypothesis and conclude that there is significant evidence that $\beta_1 \neq 0$ and the expected proportion of female born is non-constant with respect to the incubation temperature.

(c) The odds ratio is estimated by

$$e^{\hat{\beta}_1} = 9.125$$

And the confidence interval is to plug in confidence interval of $\hat{\beta}_1$ to $\exp(\cdot)$. The 95% confidence interval is

$$(3.922, 21.232)$$

(d) Let

$$\eta = \beta_0 + 28\beta_1$$

And we have

$$\log\left(\frac{\pi}{1-\pi}\right) = \eta \Rightarrow \pi = \frac{e^{\eta}}{1+e^{\eta}}$$

Then the estimate would be

$$\hat{\eta} = \hat{\beta}_0 + 28\hat{\beta}_1 = 0.5897$$
$$\hat{\pi} = \frac{e^{\hat{\eta}}}{1+e^{\hat{\eta}}} = 0.6433$$

Using Delta Method to obtain the standard error of $\hat{\pi}$.

$$\frac{\partial \pi}{\partial \beta_0} = \frac{\partial \pi}{\partial \eta}\frac{\partial \eta}{\partial \beta_0} = \frac{e^{\eta}}{(1+e^{\eta})^2}$$
$$\frac{\partial \pi}{\partial \beta_1} = \frac{\partial \pi}{\partial \eta}\frac{\partial \eta}{\partial \beta_1} = \frac{28e^{\eta}}{(1+e^{\eta})^2}$$

Let

$$D = \begin{bmatrix} \frac{\partial \pi}{\partial \beta_0} & \frac{\partial \pi}{\partial \beta_1} \end{bmatrix}$$

then $\hat{D} = D|_{\eta=\hat{\eta}}$ and

$$\widehat{Var}(\pi) = \hat{D}\widehat{Var}([\hat{\beta}_0\, \hat{\beta}_1])\hat{D}^T = 0.0027$$

and

$$s.e.(\hat{\pi}) = \sqrt{\widehat{Var}(\hat{\pi})} = 0.052$$

(e) When $\pi = 0.5$, we have

$$0 = \beta_0 + t\beta_1 \Rightarrow t = -\frac{\beta_0}{\beta_1}$$

Then the estimate

$$\hat{t} = -\frac{\hat{\beta}_0}{\hat{\beta}_1} = 27.7333$$

Using Delta Method to obtain the standard error of $\hat{t}$.

$$\frac{\partial t}{\partial \beta_0} = -\frac{1}{\beta_1}$$
$$\frac{\partial t}{\partial \beta_1} = \frac{\beta_0}{\beta_1^2}$$

Let

$$D = \begin{bmatrix} \frac{\partial t}{\partial \beta_0} & \frac{\partial t}{\partial \beta_1} \end{bmatrix}$$

then $\hat{D} = D|_{\beta_0=\hat{\beta}_0,\beta_1=\hat{\beta}_1}$ and

$$\widehat{Var}(\hat{t}) = \hat{D}\widehat{Var}([\hat{\beta}_0\ \hat{\beta}_1])\hat{D}^T = 0.0111582$$

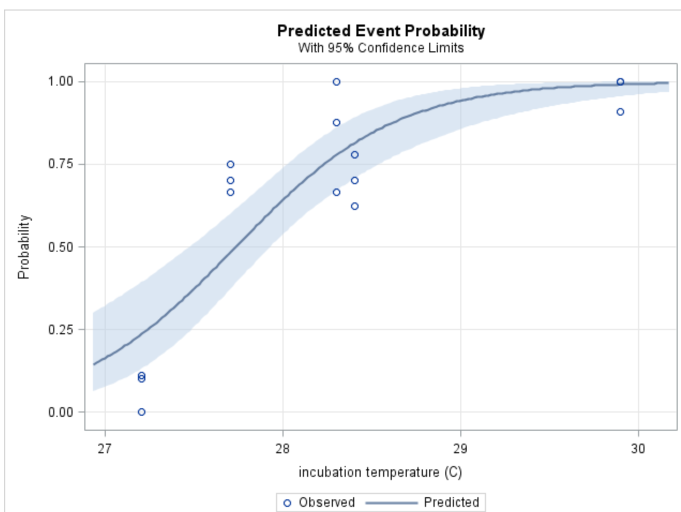
and

$$s.e.(\hat{t}) = \sqrt{\widehat{Var}(\hat{t})} = 0.1056324$$

Hence the 95% confidence interval is

$$(\hat{t} - 1.96s.e.(\hat{t}), \hat{t} + 1.96s.e.(\hat{t})) = (27.52625, 27.94033)$$

(f) The plot is shown below



And the Hosmer-Lemeshow test chi-squared statistic is **14.9522**. The degrees of freedom is 3 and the p-value is **0.0019**. The p-value is small and we tend to say the fit is not good. From the plot we can also know the fit is not quite good.

(g) For `teggnm.txt`, estimates and standard errors of β_0 and β_1 is shown below.

Parameter	**Estimate**	**Standard Error**
β_0	-41.7419	14.5197
β_1	1.4696	0.5126

(h) Repeat what we did in (e). When $\pi = 0.5$, we have

$$0 = \beta_0 + t\beta_1 \Rightarrow t = -\frac{\beta_0}{\beta_1}$$

Then the estimate

$$\hat{t} = -\frac{\hat{\beta}_0}{\hat{\beta}_1} = 28.40358$$

Using Delta Method to obtain the standard error of $\hat{t}$.

$$\frac{\partial t}{\partial \beta_0} = -\frac{1}{\beta_1}$$
$$\frac{\partial t}{\partial \beta_1} = \frac{\beta_0}{\beta_1^2}$$

Let

$$D = \begin{bmatrix} \frac{\partial t}{\partial \beta_0} & \frac{\partial t}{\partial \beta_1} \end{bmatrix}$$

then $\hat{D} = D|_{\beta_0=\hat{\beta}_0, \beta_1=\hat{\beta}_1}$ and

$$\widehat{Var}(\hat{t}) = \hat{D}\widehat{Var}([\hat{\beta}_0\, \hat{\beta}_1])\hat{D}^T = 0.09115882$$

and

$$s.e.(\hat{t}) = \sqrt{\widehat{Var}(\hat{t})} = 0.3019252$$

Hence the 95% confidence interval is

$$(\hat{t} - 1.96 s.e.(\hat{t}), \hat{t} + 1.96 s.e.(\hat{t})) = (27.81181, 28.99535)$$

(i) From the results above we have

$$\hat{t}_{il} - \hat{t}_{nm} = -0.6702911$$

and

$$\widehat{Var}(\hat{t}_{il} - \hat{t}_{nm}) = \widehat{Var}(\hat{t}_{il}) + \widehat{Var}(\hat{t}_{nm}) = 0.102317$$

So the asymptotic normal test statistic is

$$z = \frac{\hat{t}_{il} - \hat{t}_{nm}}{\sqrt{\widehat{Var}(\hat{t}_{il}) + \widehat{Var}(\hat{t}_{nm})}} = -2.095509$$

and the absolute value $|z|$ is greater than 1.96. Hence we fail to reject the null hypothesis and conclude that there is no significant difference between the incubation temperature at with 50% of the eggs produce females for Illinois and New Mexico.

(j) Using the logit and complimentary log-log link, we get the AIC and BIC for Illinois and New Mexico. The results are shown below.

Illinois:

	AIC	BIC
logit	53.836	59.662
cloglog	61.8852	63.3013

New Mexico:

	AIC	BIC
logit	26.866	29.798
cloglog	28.5217	28.9161

For Illinois, the AIC and BIC are both higher when using complementary log-log link, thus it is not a better model than logistic regression. For New Mexico, the AIC is a bit higher and BIC is a bit lower when using complementary log-log. So it is not a better model than logistic regression for New Mexico either.

(k) We use a link function like

$$\log\left(\frac{(1-\pi)^{-\alpha}-1}{\alpha}\right)$$

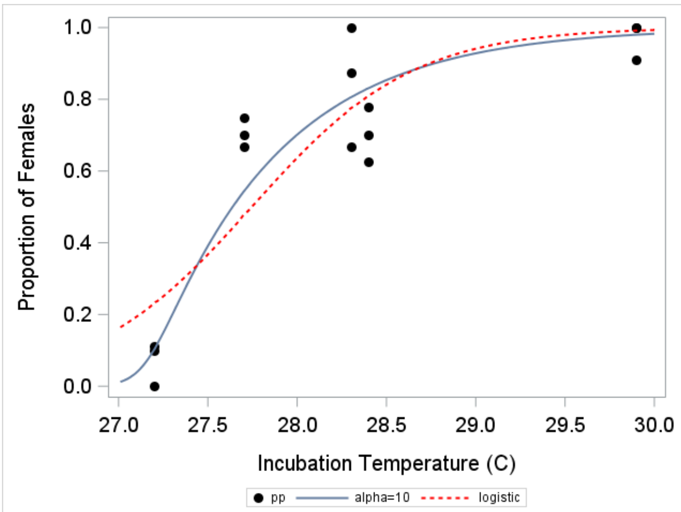
and set $\alpha = 10$. The AIC and BIC for Illinois and New Mexico using this model is

	AIC	BIC
Illinois	45.7833	47.1994
New Mexico	22.4850	22.8794

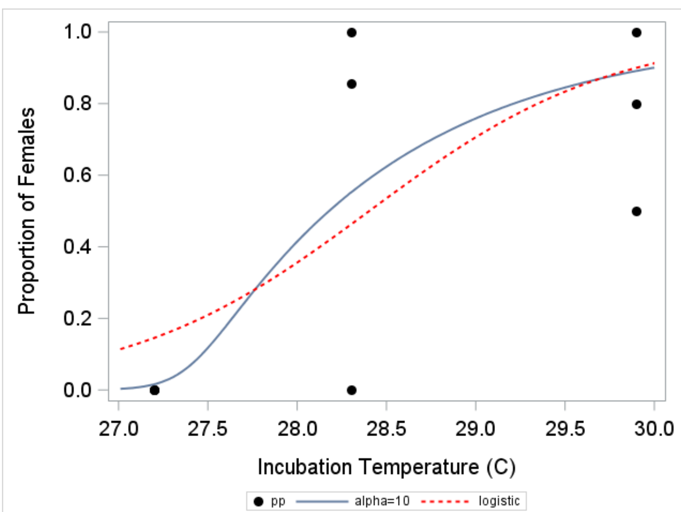
We can see AIC and BIC are both smaller using this model, so it should be better in describing the data.

Furthermore, the fitted curve seems to fit the data better with this model compared to the logistic regression.

Illinois:



New Mexico:



Similar to what we did in (e), When $\pi = 0.5$, we have

$$\eta_0 = \beta_0 + t\beta_1 \Rightarrow t = \frac{\eta_0 - \beta_0}{\beta_1}$$

where

$$\eta_0 = \log\left(\frac{2^{10} - 1}{10}\right)$$

Then the estimate

$$\hat{t} = -\frac{\hat{\beta}_0}{\hat{\beta}_1}$$

Using Delta Method to obtain the standard error of $\hat{t}$.

$$\frac{\partial t}{\partial \beta_0} = -\frac{1}{\beta_1}$$
$$\frac{\partial t}{\partial \beta_1} = \frac{\beta_0 - \eta_0}{\beta_1^2}$$

Let

$$D = \begin{bmatrix} \frac{\partial t}{\partial \beta_0} & \frac{\partial t}{\partial \beta_1} \end{bmatrix}$$

then $\hat{D} = D|_{\beta_0 = \hat{\beta}_0, \beta_1 = \hat{\beta}_1}$ and

$$\widehat{Var}(\hat{t}) = \hat{D}\widehat{Var}([\hat{\beta}_0\ \hat{\beta}_1])\hat{D}^T$$

and

$$s.e.(\hat{t}) = \sqrt{\widehat{Var}(\hat{t})}$$

Hence the 95% confidence interval is

$$(\hat{t} - 1.96 s.e.(\hat{t}), \hat{t} + 1.96 s.e.(\hat{t}))$$

The estimates and confidence interval for Illinois and New Mexico is shown in the table below.

	Estimate	95% CI
Illinois	27.63837	(27.50222, 27.77453)
New Mexico	28.17839	(27.67799, 28.67880)

5. (a) ω is the proportion of people that never drink water in population. μ is the mean milk consumption in people that drink milk in population.

(b)

$$E(X) = \sum_{x=0}^{\infty} xP(X = x) = 0 \cdot \omega + (1 - \omega)\sum_{x=0}^{\infty} x\frac{\mathrm{e}^{-\mu}\mu^x}{x!} = (1 - \omega)\mu$$
$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X = x) = 0^2 \cdot \omega + (1 - \omega)\sum_{x=0}^{\infty} x^2\frac{\mathrm{e}^{-\mu}\mu^x}{x!} = (1 - \omega)(\mu^2 + \mu)$$
$$Var(X) = E(X^2) - (E(X))^2 = (1 - \omega)(\omega\mu^2 + \mu)$$

(c) $\log \mu = \eta_1$, $\log(\omega) = \eta_2$. Thus $\mu = \exp(\eta_1), \omega = \exp(\eta_2)$. And we have

$$\begin{aligned}
\hat{\eta}_1 &= 0.3388 \\
\hat{\eta}_2 &= -1.5638 \\
s.e.(\hat{\eta}_1) &= 0.0296 \\
s.e.(\hat{\eta}_2) &= 0.0877
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\mu} &= \exp(\hat{\eta}_1) = 1.403263 \\
s.e.(\hat{\mu}) &= \left|\frac{\partial \mu}{\partial \eta_1}\right|_{\eta_1 = \hat{\eta}_1} s.e.(\hat{\eta}_1) = 0.04148793 \\
\hat{\omega} &= \exp(\eta_2) = 0.2093341 \\
s.e.(\hat{\omega}) &= \left|\frac{\partial \omega}{\partial \eta_2}\right|_{\eta_2 = \hat{\eta}_2} s.e.(\hat{\eta}_2) = 0.01835863
\end{aligned}$$

(d) The mean consumption is

$$m = (1 - \omega)\mu$$

Then we have

$$\hat{m} = (1 - \hat{\omega})\hat{\mu} = 1.109474$$

And

$$\begin{aligned}
\frac{\partial m}{\partial \eta_1} &= \frac{\partial m}{\partial \mu}\frac{\partial \mu}{\partial \eta_1} = (1 - \omega)\exp(\eta_1) = (1 - \omega)\mu \\
\frac{\partial m}{\partial \eta_2} &= \frac{\partial m}{\partial \omega}\frac{\partial \omega}{\partial \eta_2} = -\mu \exp(\eta_2) = -\mu\omega
\end{aligned}$$

Let $D = \begin{bmatrix} \frac{\partial m}{\partial \eta_1} & \frac{\partial m}{\partial \eta_2} \end{bmatrix}$, then by Delta Method

$$s.e.(\hat{m}) = \sqrt{D\widehat{Var}([\hat{\eta}_1\, \hat{\eta}_2])D^T} = 0.0274856$$

Hence 95% confidence interval is

$$(\hat{m} - 1.96 s.e.(\hat{m}), \hat{m} + 1.96 s.e.(\hat{m})) = (1.055602, 1.163345)$$

(e) The probability is

$$p = P(X = 0) = \omega + (1 - \omega)\mathrm{e}^{-\mu}$$

Then we have

$$\hat{p} = \hat{\omega} + (1 - \hat{\omega})\mathrm{e}^{-\hat{\mu}} = 0.4036842$$

And

$$\begin{aligned}
\frac{\partial p}{\partial \eta_1} &= \frac{\partial p}{\partial \mu}\frac{\partial \mu}{\partial \eta_1} = -(1 - \omega)\mathrm{e}^{-\mu}\exp(\eta_1) = -(1 - \omega)\mathrm{e}^{-\mu}\mu \\
\frac{\partial p}{\partial \eta_2} &= \frac{\partial p}{\partial \omega}\frac{\partial \omega}{\partial \eta_2} = -(1 - \exp(-\mu))\exp(\eta_2) = (1 - \exp(-\mu))\omega
\end{aligned}$$

Let $D = \begin{bmatrix} \frac{\partial m}{\partial \eta_1} & \frac{\partial m}{\partial \eta_2} \end{bmatrix}$, then by Delta Method

$$s.e.(\hat{p}) = \sqrt{D\widehat{Var}([\hat{\eta}_1\, \hat{\eta}_2])D^T} = 0.01125595$$

Hence 95% confidence interval is

$$(\hat{p} - 1.96 s.e.(\hat{p}), \hat{p} + 1.96 s.e.(\hat{p})) = (0.3816226, 0.4257459)$$

(f) The proportion of population that never drinks milk is ω, thus the 95% confidence interval is

$$(\hat{\omega} - 1.96 s.e.(\hat{\omega}), \hat{\omega} + 1.96 s.e.(\hat{\omega})) = (0.1733512, 0.2453170)$$

(g) When fitting the Poisson model, the last category has a expected value less than 2, thus we need to combine it with category corresponding to number of glasses being 5 and then compute the Pearson statistic. The chi-squared statistic is then **153.7398**, degrees of freedom is **4** and p-value is < **0.0001**.

Fitting the ZIP model, the last category then has a expected value of 4.8, which is close to 5. So we did not combine categories in this case. The chi-squared statistic is **38.04**, degrees of freedom is **4** and p-value is < **0.0001**.

6. (a) π is the probability that a person randomly selected from the mixture of populations gives a zero response. μ does not have a clear meaning in the context.

(b)

$$\begin{aligned}
E(X) &= \frac{1-\pi}{1-\mathrm{e}^{-\mu}} \sum_{x=1}^{\infty} x \frac{\mathrm{e}^{-\mu}\mu^x}{\Gamma(x+1)} = \frac{(1-\pi)\mu}{1-\mathrm{e}^{-\mu}} \\
E(X^2) &= \frac{1-\pi}{1-\mathrm{e}^{-\mu}} \sum_{x=1}^{\infty} x^2 \frac{\mathrm{e}^{-\mu}\mu^x}{\Gamma(x+1)} = \frac{(1-\pi)(\mu+\mu^2)}{1-\mathrm{e}^{-\mu}} \\
Var(X) &= E(X^2) - (E(X))^2 = \left(\frac{1-\pi}{1-\mathrm{e}^{-\mu}}\right)\left(\mu + \left(\frac{\pi - \mathrm{e}^{-\mu}}{1-\mathrm{e}^{-\mu}}\right)\mu^2\right)
\end{aligned}$$

(c) $\log\mu = \eta_1$, $\log(\pi/(1-\pi)) = \eta_2$. Thus $\mu = \exp(\eta_1), \pi = 1/(1+\exp(-\eta_2))$. And we have

$$\begin{aligned}
\hat{\eta}_1 &= 0.3388 \\
\hat{\eta}_2 &= -0.3901 \\
s.e.(\hat{\eta}_1) &= 0.02957 \\
s.e.(\hat{\eta}_2) &= 0.04676
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\mu} &= \exp(\hat{\eta}_1) = 1.4032 \\
s.e.(\hat{\mu}) &= \left|\frac{\partial\mu}{\partial\eta_1}\right|_{\eta_1=\hat{\eta}_1} s.e.(\hat{\eta}_1) = 0.04149 \\
\hat{\pi} &= 1/(1+\exp(-\eta_2)) = 0.4037 \\
s.e.(\hat{\pi}) &= \left|\frac{\partial\pi}{\partial\eta_2}\right|_{\eta_2=\hat{\eta}_2} s.e.(\hat{\eta}_2) = 0.01126
\end{aligned}$$

(d) The 95% confidence interval for $\hat{\pi}$ is

$$(0.3771, 0.4303)$$

(e) This is not a estimable quantity.

(f) Fitting the Poisson Hurdle model, the last category then has a expected value of 4.8, which is close to 5. So we did not combine categories in this case. The chi-squared statistic is **38.04**, degrees of freedom is **4** and p-value is < **0.0001**.

7. (a) ω is the proportion of people that never drink water in population. μ is the mean milk consumption in people that drink milk in population.

(b)

$$E(X) = (1-\omega)\mu$$
$$Var(X) = (1-\omega)[\mu(1+\mu\kappa) + \omega\mu^2]$$

(c) $\log\mu = \eta_1$, $\log(\omega/(1-\omega)) = \eta_2$. Thus $\mu = \exp(\eta_1), \omega = 1/(1+\exp(-\eta_2))$. κ is still κ. And we have

$$\begin{aligned}
\hat{\eta}_1 &= 0.1962 \\
\hat{\eta}_2 &= -2.3358 \\
s.e.(\hat{\eta}_1) &= 0.0561 \\
s.e.(\hat{\eta}_2) &= 0.5752
\end{aligned}$$

Hence

$$\begin{aligned}
&\hat{\mu} = \exp(\hat{\eta}_1) = 1.21677 \\
&s.e.(\hat{\mu}) = \left|\frac{\partial\mu}{\partial\eta_1}\right|_{\eta_1=\hat{\eta}_1} s.e.(\hat{\eta}_1) = 0.06821514 \\
&\hat{\omega} = 1/(1+\exp(-\eta_2)) = 0.0882011 \\
&s.e.(\hat{\omega}) = \left|\frac{\partial\omega}{\partial\eta_2}\right|_{\eta_2=\hat{\eta}_2} s.e.(\hat{\eta}_2) = 0.04625889
\end{aligned}$$

(d) The mean consumption is

$$m = (1-\omega)\mu$$

Then we have

$$\hat{m} = (1-\hat{\omega})\hat{\mu} = 1.10945$$

And

$$\begin{aligned}
\frac{\partial m}{\partial\eta_1} &= \frac{\partial m}{\partial\mu}\frac{\partial\mu}{\partial\eta_1} = (1-\omega)\exp(\eta_1) = (1-\omega)\mu \\
\frac{\partial m}{\partial\eta_2} &= \frac{\partial m}{\partial\omega}\frac{\partial\omega}{\partial\eta_2} = -\mu\frac{\exp(-\eta_2)}{(1+\exp(-\eta_2))^2} = -\mu\omega^2\exp(-\eta_2)
\end{aligned}$$

Let $D = \begin{bmatrix}\frac{\partial m}{\partial\eta_1} & \frac{\partial m}{\partial\eta_2}\end{bmatrix}$, then by Delta Method

$$s.e.(\hat{m}) = \sqrt{D\widehat{Var}([\hat{\eta}_1\,\hat{\eta}_2])D^T} = 0.02871818$$

Hence 95% confidence interval is

$$(\hat{m} - 1.96 s.e.(\hat{m}), \hat{m} + 1.96 s.e.(\hat{m})) = (1.053162, 1.165737)$$

(e) The probability is

$$p = P(X=0) = \omega + (1-\omega)\left(\frac{1}{1+\kappa\mu}\right)^{1/\kappa}$$

Then we have

$$\hat{p} = \hat{\omega} + (1-\hat{\omega})\left(\frac{1}{1+\hat{\kappa}\hat{\mu}}\right)^{1/\hat{\kappa}} = 0.4036878$$

And let $c = \left(\frac{1}{1+\kappa\mu}\right)^{1/\kappa}$

$$\begin{aligned}
\frac{\partial p}{\partial \eta_1} &= -(1-\omega)\frac{\mu}{1+\kappa\mu}c \\
\frac{\partial p}{\partial \eta_2} &= (1-c)\omega^2 \exp(-\eta_2) \\
\frac{\partial p}{\partial \kappa} &= (1-\omega)\left(-\frac{\mu}{\kappa(1+\kappa\mu)} + \frac{1}{\kappa^2}\log(1+\kappa\mu)\right)c
\end{aligned}$$

Let $D = \begin{bmatrix} \frac{\partial m}{\partial \eta_1} & \frac{\partial m}{\partial \eta_2} & \frac{\partial p}{\partial \kappa} \end{bmatrix}$, then by Delta Method

$$s.e.(\hat{p}) = \sqrt{D\widehat{Var}([\hat{\eta}_1\,\hat{\eta}_2\,\hat{\kappa}])D^T} = 0.01125215$$

Hence 95% confidence interval is

$$(\hat{p} - 1.96 s.e.(\hat{p}), \hat{p} + 1.96 s.e.(\hat{p})) = (0.38163360.4257420)$$

(f) The proportion of population that never drinks milk is ω, we first compute the 95% confidence interval of $\hat{\eta}_2$

$$(\hat{\eta}_2 - 1.96 s.e.(\hat{\eta}_2), \hat{\eta}_2 + 1.96 s.e.(\hat{\eta}_2)) = (-3.4632 - 1.2084)$$

Then plug it into $\omega = 1/(1+\exp(-\eta_2))$, we have the 95% confidence interval for $\hat{\omega}$

$$(0.03037762, 0.22998435)$$

(g) Fitting the ZINB model, all categories have expected values being greater than 5 and no combining is needed. The chi-squared statistic is **2.059263**, degrees of freedom is **3** and p-value is **0.5601979**.

8. (a) π is the probability that a person randomly selected from the mixture of populations gives a zero response. μ does not have a clear meaning in the context.

(b)

$$\begin{aligned}
E(X) &= \left(\frac{1-\pi}{1-\theta}\right)\mu \\
Var(X) &= \left(\frac{1-\pi}{1-\theta}\right)\mu(1+\mu+\mu\kappa) - \left(\frac{1-\pi}{1-\theta}\right)^2\mu^2
\end{aligned}$$

where

$$\theta = \left(\frac{1}{1+\kappa\mu}\right)^{1/\kappa}$$

(c) $\log\mu = \eta_1$, $\log(\pi/(1-\pi)) = \eta_2$. Thus $\mu = \exp(\eta_1), \pi = 1/(1+\exp(-\eta_2))$. κ is still κ. And we have

$$\begin{aligned}
\hat{\eta}_1 &= 0.1962 \\
\hat{\eta}_2 &= -0.3901 \\
s.e.(\hat{\eta}_1) &= 0.05606 \\
s.e.(\hat{\eta}_2) &= 0.04676
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\mu} &= \exp(\hat{\eta}_1) = 1.2168 \\
s.e.(\hat{\mu}) &= \left|\frac{\partial \mu}{\partial \eta_1}\right|_{\eta_1=\hat{\eta}_1} s.e.(\hat{\eta}_1) = 0.06821 \\
\hat{\pi} &= 1/(1+\exp(-\eta_2)) = 0.4037 \\
s.e.(\hat{\pi}) &= \left|\frac{\partial \pi}{\partial \eta_2}\right|_{\eta_2=\hat{\eta}_2} s.e.(\hat{\eta}_2) = 0.01126
\end{aligned}$$

(d) The 95% confidence interval for $\hat{\pi}$ is

$$(0.3771, 0.4303)$$

(e) This is not a estimable quantity.

(f) Fitting the Negtive Binomial Hurdle model, all categories have expected values being greater than 5 and no combining is needed. The chi-squared statistic is **2.06**, degrees of freedom is **3** and p-value is **0.5604**.