STAT 580 Homework 5

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1. (a)

$$\begin{aligned}
\ell(\theta) &= \log \prod_{i=1}^{n} p(x_i - \theta) \\
&= \sum_{i=1}^{n} \log p(x_i - \theta) \\
&= \sum_{i=1}^{n} \left(-\log \pi - \log(1 + (x_i - \theta)^2)\right) \\
&= -n \log \pi - \sum_{i=1}^{n} \log \left(1 + (\theta - x_i)^2\right)
\end{aligned}$$

$$\begin{aligned}
l'(\theta) &= -\sum_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2} \cdot 2(\theta - x_i) \\
&= -2 \sum_{i=1}^{n} \frac{\theta - x_i}{1 + (\theta - x_i)^2}
\end{aligned}$$

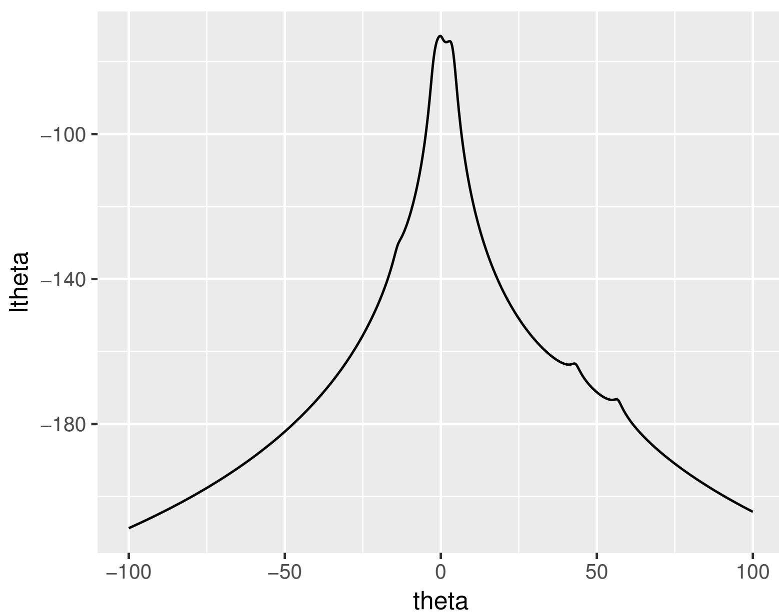
$$\begin{aligned}
l''(\theta) &= -2 \sum_{i=1}^{n} \frac{\left(1 + (\theta - x_i)^2\right) - 2(\theta - x_i)^2}{\left(1 + (\theta - x_i)^2\right)^2} \\
&= -2 \sum_{i=1}^{n} \frac{1 - (\theta - x_i)^2}{\left(1 + (\theta - x_i)^2\right)^2}
\end{aligned}$$

(b)

$$\begin{aligned}
I(\theta) &= -E_\theta(l''(\theta)) \\
&= n \int_{-\infty}^{\infty} \frac{(p'(x))^2}{p(x)} \mathrm{d}x \\
&= n \int_{-\infty}^{\infty} \frac{4x^2}{\pi(1 + x^2)^3} \mathrm{d}x \\
&= 2n \int_{0}^{\infty} \frac{4x^2}{\pi(1 + x^2)^3} \mathrm{d}x \\
&= 2n \frac{x(x^2 - 1) + (x^2 + 1) \arctan x}{2\pi(x^2 + 1)^2} \Bigg|_0^\infty \\
&= 2n \frac{\pi/2}{2\pi} \\
&= \frac{n}{2}
\end{aligned}$$

(c)

```r
library(ggplot2)



n <- length(x)

l <- function(theta){
  x <- c(-13.87, -2.53, -2.44, -2.40, -1.75, -1.34, -1.05, -0.23,
       -0.07, 0.27, 1.77, 2.76, 3.29, 3.47, 3.71, 3.80, 4.24, 4.53,
       43.21, 56.75)

  return(- n * log(pi) - sum(log(1 + (theta - x)^2)))
}

theta <- seq(-100, 100, 0.2)

ltheta <- sapply(theta, l)


qplot(x = theta, y = ltheta, geom = "line")
```



(d)

```r
dltheta <- function(theta){
  x <- c(-13.87, -2.53, -2.44, -2.40, -1.75, -1.34, -1.05, -0.23,
       -0.07, 0.27, 1.77, 2.76, 3.29, 3.47, 3.71, 3.80, 4.24, 4.53,
       43.21, 56.75)
  return(-2 * sum((theta - x)/(1 + (theta - x)^2)))
}

ddltheta <- function(theta){
  x <- c(-13.87, -2.53, -2.44, -2.40, -1.75, -1.34, -1.05, -0.23,
       -0.07, 0.27, 1.77, 2.76, 3.29, 3.47, 3.71, 3.80, 4.24, 4.53,
```

```r
      43.21, 56.75)
  return(-2 * sum((1 - (theta - x)^2)/(1 + (theta - x)^2)^2))
}


Newton <- function(thetat){
  thetat_new <- thetat - dltheta(thetat)/ddltheta(thetat)
  condition <- (abs((thetat_new - thetat)/(thetat + 0.00005)) >
      0.0001)
  while (condition){
    thetat <- thetat_new
    thetat_new <- thetat - dltheta(thetat)/ddltheta(thetat)
    condition <- (abs((thetat_new - thetat)/(thetat + 0.00005)) >
        0.0001)
    if (is.na(condition))
      return("Not Converge")
  }
  return(thetat)
}

FisherS <- function(thetat){
  x <- c(-13.87, -2.53, -2.44, -2.40, -1.75, -1.34, -1.05, -0.23,
      -0.07, 0.27, 1.77, 2.76, 3.29, 3.47, 3.71, 3.80, 4.24, 4.53,
      43.21, 56.75)
  I <- length(x)/2
  thetat_new <- thetat + dltheta(thetat)/I
  while (abs((thetat_new - thetat)/(thetat + 0.00005)) > 0.0001){
    thetat <- thetat_new
    thetat_new <- thetat + dltheta(thetat)/I
  }
  return(thetat)
}

start <- c(-11, -1, 0, 1.4, 4.1, 4.8, 7, 8, 38)

# (d) Newton Method
for (thetat in start){
  thetahat <- Newton(thetat)
  print(thetahat)
}

# (e) First use Fisher Scoring and then refinr using Newton
for (thetat in start){
  thetahat <- FisherS(thetat)
  thetahat <- Newton(thetahat)
  print(thetahat)
}
```

Results:

-11	-1	0	1.4	4.1
Not Converge	-0.1922865	-0.1922865	1.713569	2.817473

4.8	7	8	38
Not Converge	41.04099	Not Converge	42.79593

(e) Results:

-11	-1	0	1.4	4.1
-0.1922866	-0.1922866	-0.1922866	-0.1922866	2.817474

4.8	7	8	38
2.817474	2.817474	2.817474	2.817474

In this way the algorithm is more stable. We do nat have non-convergence case and the results is not sensitive about the starting points.

2. (a)

```
y <- c(47, 76, 97, 107, 123, 139, 152, 159, 191, 201, 200, 207)
x <- rep(c(0.02, 0.06, 0.11, 0.22, 0.56, 1.10), each = 2)

o <- lm(y~x)
summary(o)
```

Then we have

$$\hat{\theta}_1 = \frac{1}{\hat{\beta}_0} = 1/103.49 = 0.00966,\ \hat{\theta}_2 = \frac{\hat{\beta}_1}{\hat{\beta}_0} = 110.42/103.49 = 1.066963$$

(b)

$$\begin{aligned}
g(\boldsymbol{\theta}) &= -\sum_{i=1}^{n}\left(y_i - \frac{\theta_1 x_i}{x_i+\theta_2}\right)^2 \\
\frac{\partial^2 g}{\partial \theta_1} &= 2\sum_{i=1}^{n}\left(y_i - \frac{\theta_1 x_i}{x_i+\theta_2}\right)\frac{x_i}{x_i+\theta_2} \\
\frac{\partial g}{\partial \theta_2} &= -2\sum_{i=1}^{n}\left(y_i - \frac{\theta_1 x_i}{x_i+\theta_2}\right)\frac{\theta_1 x_i}{(x_i+\theta_2)^2} \\
\frac{\partial^2 g}{\partial \theta_1^2} &= -2\sum_{i=1}^{n}\frac{x_i^2}{(x_i+\theta_2)^2} \\
\frac{\partial^2 g}{\partial \theta_2^2} &= 2\sum_{i=1}^{n}\left(\frac{2\theta_1 x_i y_i}{(x_i+\theta_2)^3} - \frac{3\theta_1^2 x_i^2}{(x_i+\theta_2)^4}\right) \\
\frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} &= 2\sum_{i=1}^{n}\left(\frac{2\theta_1 x_q^2}{(x_i+\theta_2)^3} - \frac{x_i y_i}{(x_i+\theta_2)^2}\right)
\end{aligned}$$

Then

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \left(\frac{\partial^2 g}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}\right)^{-1} \left.\frac{\partial g}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_t}$$

Codes:

```
library(MASS)

y <- c(47, 76, 97, 107, 123, 139, 152, 159, 191, 201, 200, 207)
x <- rep(c(0.02, 0.06, 0.11, 0.22, 0.56, 1.10), each = 2)

dg <- function(theta, y, x){
  dg1 <- 2* sum((y - (theta[1]*x/(x + theta[2]))) * (x / (x + theta
      [2])))
```

```r
  dg2 <- -2 * sum((y - (theta[1] * x/ (x + theta[2]))) * (theta[1] *
       x / (x + theta[2])^2))
  return(c(dg1, dg2))
}

ddg <- function(theta, y, x){
  ddg11 <- -2 * sum(x^2/(x + theta[2])^2)
  ddg22 <- 2 * sum(2*theta[1]*x*y/(x + theta[2])^3 - 3 * (theta[1] *
       x)^2/(x + theta[2])^4)
  ddg12 <- 2 * sum(2*theta[1]*x^2/(x + theta[2])^3 - x*y/(x + theta
      [2])^2)
  return(matrix(c(ddg11, ddg12, ddg12, ddg22), nrow = 2))
}

Newton <- function(thetat, y, x){
  thetat_new <- thetat - ginv(ddg(thetat,y, x))%*%dg(thetat,y,x)
  condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2) +
       0.00005)) > 0.000001)
  while (condition){
    thetat <- thetat_new
    thetat_new <- thetat - ginv(ddg(thetat,y, x))%*%dg(thetat,y,x)
    condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2)
         + 0.00005)) > 0.000001)
    if (is.na(condition))
      return("Not Converge")
  }
  return(thetat)
}
```

(c)

```r
library(MASS)

y <- c(47, 76, 97, 107, 123, 139, 152, 159, 191, 201, 200, 207)
x <- rep(c(0.02, 0.06, 0.11, 0.22, 0.56, 1.10), each = 2)

g <- function(theta, y, x){
  return(-sum((y - theta[1]*x/(x + theta[2]))^2))
}

dg <- function(theta, y, x){
  dg1 <- 2* sum((y - (theta[1]*x/(x + theta[2]))) * (x / (x + theta
      [2])))
  dg2 <- -2 * sum((y - (theta[1] * x/ (x + theta[2]))) * (theta[1] *
       x / (x + theta[2])^2))
  return(c(dg1, dg2))
}

SA <- function(thetat, y, x, alpha){
  thetat_new <- thetat + alpha * dg(thetat, y, x)
  condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2) +
       0.00005)) > 0.000001)
  while (condition){
    thetat <- thetat_new
    thetat_new <- thetat + alpha * dg(thetat, y, x)
    while (g(thetat_new, y, x) < g(thetat, y, x)){
      alpha <- alpha/2
      thetat_new <- thetat + alpha * dg(thetat, y, x)
```

```
    }
    condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2)
         + 0.00005)) > 0.000001)
    if (is.na(condition))
      return("Not␣Converge")
  }
  return(thetat)
}
```

(d)

$$
\begin{aligned}
f_i(\boldsymbol{\theta}) &= \frac{\theta_1 x_i}{x_i + \theta_2} \\
\frac{\partial f_i}{\partial \theta_1} &= \frac{x_i}{x_i + \theta_2} \\
\frac{\partial f_i}{\partial \theta_2} &= -\frac{\theta_1 x_i}{(x_i + \theta_2)^2}
\end{aligned}
$$

```
library(MASS)

y <- c(47, 76, 97, 107, 123, 139, 152, 159, 191, 201, 200, 207)
x <- rep(c(0.02, 0.06, 0.11, 0.22, 0.56, 1.10), each = 2)

f <- function(theta,x){
  return(theta[1]*x/(x + theta[2]))
}

A <- function(theta, x){
  df1 <- x/(x + theta[2])
  df2 <- - theta[1]*x/(x + theta[2])^2
  return(matrix(c(df1, df2), ncol = 2))
}

Z <- function(theta, y, x){
  return(y - f(theta,x))
}

GN <- function(thetat, y, x){
  At <- A(thetat, x)
  Zt <- Z(thetat, y, x)
  thetat_new <- thetat + ginv(t(At)%*%At)%*%t(At)%*%Zt
  condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2) +
       0.00005)) > 0.000001)
  while (condition){
    thetat <- thetat_new
    At <- A(thetat, x)
    Zt <- Z(thetat, y, x)
    thetat_new <- thetat + ginv(t(At)%*%At)%*%t(At)%*%Zt
    condition <- (sqrt(sum((thetat_new - thetat)^2)/(sum((thetat)^2)
         + 0.00005)) > 0.000001)
    if (is.na(condition))
      return("Not␣Converge")
  }
  return(thetat)
}
```