STAT 543 Homework 6

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1. $f(\boldsymbol{x}|\sigma_1) = \prod_{i=1}^n f(x_i|\sigma_1) = \frac{1}{(2\pi\sigma_1^2)^{n/2}} \mathrm{e}^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^k x_i^2}$, we also have $f(\boldsymbol{x}|\sigma_0) = \prod_{i=1}^n f(x_i|\sigma_0) = \frac{1}{(2\pi\sigma_0^2)^{n/2}} \mathrm{e}^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^k x_i^2}$, thus

$$
\begin{aligned}
& f(\boldsymbol{x}|\sigma_1) > k f(\boldsymbol{x}|\sigma_0) \\
\iff & \frac{1}{(2\pi\sigma_1^2)^{n/2}} \mathrm{e}^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^k x_i^2} > k \frac{1}{(2\pi\sigma_0^2)^{n/2}} \mathrm{e}^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^k x_i^2} \\
\iff & \mathrm{e}^{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_{i=1}^n x_i^2} > k \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{n/2} \\
\iff & \sum_{i=1}^n x_i > c, \text{ because } \frac{1}{2\sigma_{0^2}} - \frac{1}{2\sigma_1^2} > 0
\end{aligned}
$$

Also $\sum_{i=1}^n X_i$ follows a continuous distribution, then $\gamma = 0$ and the MP test is

$$
\Phi(\boldsymbol{X}) = \begin{cases} 1 & , \sum_{i=1}^n X_i > c \\ 0 & , \sum_{i=1}^n X_i < c \end{cases}
$$

$\alpha = E_{\sigma_0}(\Phi(\boldsymbol{X})) = P_{\sigma_0}\left(\sum_{i=1}^n X_i^2 > c\right) = P_{\sigma_0}\left(\sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 > \frac{c}{\sigma_0^2}\right) = P_{\sigma_0}\left(\chi_n^2 > \frac{c}{\sigma_0^2}\right) \Rightarrow P_{\sigma_0}\left(\chi_n^2 \le \frac{c}{\sigma_0^2}\right) = 1 - \alpha$. Hence

$$
\frac{c}{\sigma_0^2} = \chi^2_{n,1-\alpha} \Rightarrow c = \sigma_0^2 \chi^2_{n,1-\alpha}
$$

2. For every x, we have

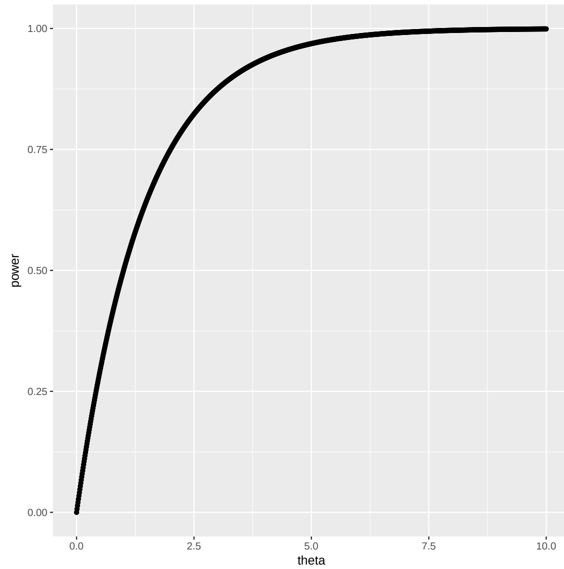
x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)/f(x\|H_0)$	6	5	4	3	2	1	0.84

By Neuman-Pearson Theorem, we reject H_0 when $f(x|H_1)/f(x|H_0)$ is large, which corresbond to when x is small. Beacause the Type I Error $\alpha = 0.04$, thus the test function is

$$
\Phi(X) = \begin{cases} 1 & , X = 1, 2, 3, 4 \\ 0 & , otherwise \end{cases}
$$

And Type II Error $\beta = P_{H_1}(\Phi(X) = 0) = P_{H_1}(X = 5, 6, 7) = 0.82$

3. (a) $\Pi_\Phi(\theta) = E_\theta(\Phi(X)) = P_\theta(\text{Reject } H_0) = P_\theta(X > 1/2) = \int_{1/2}^{1} \frac{\Gamma(\theta+1)}{\Gamma(1)\Gamma(\theta)} x^{\theta-1}(1-x)^{1-1}\mathrm{d}x = 1 - \left(\frac{1}{2}\right)^\theta$. The size is $\max_{\theta\in\Theta_0} \Pi_\Phi(\theta) = \max_{\theta\le 1} \left\{1 - \left(\frac{1}{2}\right)^\theta\right\} = 1/2$.



(b) $f(x|\theta) = \theta x^{\theta-1}, f(x|1) = 1, f(x|2) = 2x$, hence $f(x|2) > kf(x|1) \iff x > k/2$. X follows a continuous distribution, then $P_{\theta_0}(X > k/2) = 1 - k/2 = \alpha \Rightarrow k/2 = 1-\alpha$. Thus the MP test is

$$\Phi(X) = \begin{cases} 1 & , X > 1-\alpha \\ 0 & , otherwise \end{cases}$$

(c) Fix $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$, we have $f(x|\theta_i) = \theta_i x^{1-\theta_i}$. And $f(x|\theta_1) > kf(x|\theta_0) \iff \theta_1 x^{\theta_1 - 1} > k\theta_0 x^{\theta_0-1} \iff \frac{\theta_1}{\theta_0} x^{\theta_1-\theta_0} > k \iff x > c$. (Because $\theta_1 > \theta_0$). Let $\max_{\theta\in\Theta_0} P_\theta(X > c) = \max_{\theta\in\Theta_0} \left\{\int_c^1 \theta x^{\theta-1}\mathrm{d}x\right\} = \max_{\theta\in\Theta_0}\left\{1 - c^\theta\right\} = 1 - c = \alpha$. Thus $c = 1-\alpha$ and the test function

$$\Phi(X) = \begin{cases} 1 & , X > 1-\alpha \\ 0 & , otherwise \end{cases}$$

4. (a) For any $\theta_2 > \theta_1$,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \mathrm{e}^{\theta_1-\theta_2}\left(\frac{1+\mathrm{e}^{x-\theta_1}}{1+\mathrm{e}^{x-\theta_2}}\right)$$

And we have

$$\frac{1+\mathrm{e}^{x-\theta_1}}{1+\mathrm{e}^{x-\theta_2}} = \mathrm{e}^{\theta_2-\theta_1} + \frac{1-\mathrm{e}^{\theta_2-\theta_1}}{1+\mathrm{e}^{-\theta_2}\mathrm{e}^{x}}$$

$1 - \mathrm{e}^{\theta_2-\theta_1} < 0$, thus $\left(\frac{1+\mathrm{e}^{x-\theta_1}}{1+\mathrm{e}^{x-\theta_2}}\right)^2 \mathrm{e}^{\theta_1-\theta_2}$ will increase as x increases. Thus it has MLR.

(b) We know $\frac{f(x|1)}{f(x|0)}$ is increasing in x. Thus $\frac{f(x|1)}{f(x|0)} > k \iff x > c$. Then

$$P_{\theta_0}(X > c) = \int_c^\infty \frac{\mathrm{e}^x}{(1+\mathrm{e}^x)^2}\mathrm{d}x = -\frac{1}{1+\mathrm{e}^x}\Bigg|_c^\infty = \frac{1}{1+\mathrm{e}^c} = \alpha \Rightarrow c = \log(1-\alpha) - \log\alpha$$

Thus the test function is

$$\Phi(X) = \begin{cases} 1 & , X > \log(1-\alpha) - \log\alpha \\ 0 & , otherwise \end{cases}$$

For $\alpha = 0.2 \Rightarrow c = 1.386$ and Type II Error $\beta = 1 - P_{\theta_1}(X > 1.386) = 1 - \int_{1.386}^{\infty} \frac{e^{x-1}}{(1+e^{x-1})^2} dx = 1 - \int_{0.386} \frac{e^t}{(1+e^t)^2} dt = 1 - \frac{1}{1+e^{0.386}} = 0.595$.

(c) Beacause of MLR in x, the UMP test is the same as the MP test above.

$$\Phi(X) = \begin{cases} 1 & , X > \log(1-\alpha) - \log\alpha \\ 0 & , otherwise \end{cases}$$

(d) $f(\boldsymbol{x}|\lambda) = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda_i^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$. Then for $\lambda_2 > \lambda_1$

$$\frac{f(\boldsymbol{x}|\lambda_2)}{f(\boldsymbol{x}|\lambda_1)} = \frac{e^{-n\lambda_2}\lambda_2^{\sum_{i=1}^n x_i}}{e^{-n\lambda_1}\lambda_1^{\sum_{i=1}^n x_i}} = e^{n(\lambda_1-\lambda_2)}\left(\frac{\lambda_2}{\lambda_1}\right)^{\sum_{i=1}^n x_i}$$

Hence $\frac{f(\boldsymbol{x}|\lambda_2)}{f(\boldsymbol{x}|\lambda_1)}$ increase as $\sum_{i=1}^n x_i$ increases. Thus $\{f(\boldsymbol{x}|\lambda) : \lambda > 0\}$ have MLR in $\sum_{i=1}^n X_i$. Thus $\alpha = P_{\lambda_0}(\sum_{i=1}^n X_i > k) + \gamma P_{\lambda_0}(\sum_{i=1}^n X_i = k) \Rightarrow k = k_{n,\alpha,\lambda_0}$, $\gamma = \gamma_{n,\alpha,\lambda_0}$. Then the test function is

$$\Phi(X) = \begin{cases} 1 & , \sum_{i=1}^n X_i > k_{n,\alpha,\lambda_0} \\ \gamma_{n,\alpha,\lambda_0} & , \sum_{i=1}^n X_i = k_{n,\alpha,\lambda_0} \\ 0 & , otherwise \end{cases}$$

(e)

$$P\left(\sum_{i=1}^n X_i > k \middle| \lambda = 1\right) = P\left(\sqrt{n}\frac{\sum_{i=1}^n (X_i - 1)}{n} > \frac{k-n}{\sqrt{n}} \middle| \lambda = 1\right) \approx 0.05$$
$$P\left(\sum_{i=1}^n X_i > k \middle| \lambda = 2\right) = P\left(\sqrt{n}\frac{\sum_{i=1}^n (X_i - 2)}{n} > \frac{k-2n}{\sqrt{2n}} \middle| \lambda = 2\right) \approx 0.9$$

Thus we have

$$\frac{k-n}{\sqrt{n}} = 1.645$$
$$\frac{k-2n}{\sqrt{2n}} = -1.28$$

And we can get $n = 12$ and $k = 17.7$.

5. (a) $P(Y_n \geq 1|\theta = 0) = 0$, then $\alpha = P(Y_1 \geq k|\theta = 0) = \int_k^1 n(1-y_1)^{n-1}dy_1 = (1-k)^n$. Thus we use $k = 1 - \alpha^{1/n}$.

(b) When $\theta \leq k - 1$, $\Pi(\theta) = 0$.
When $k - 1 \leq \theta \leq 0$, $\Pi(\theta) = \int_k^{\theta+1} n(1-(y_1-\theta))^{n-1}dy_1 = (1-k+\theta)^n$.
When $0 < \theta \leq k$, $\Pi(\theta) = \int_k^{\theta+1} n(1-(y_1-\theta))^{n-1}dy_1 + \int_\theta^k \int_1^{\theta+1} n(n-1)(y_n-y_1)^{n-2}dy_n dy_1 = \alpha + 1 - (1-\theta)^n$.
When $k < \theta$, $\Pi(\theta) = 1$.

(c)

$$\frac{f(\boldsymbol{x}|\theta)}{f(\boldsymbol{x}|0)} = \frac{\mathbf{1}\{\theta \leq y_1 < y_n \leq \theta + 1\}}{\mathbf{1}\{0 \leq y_1 < y_n \leq 1\}}$$

(d) $k = 1 - 0.1^{1/n} < 1 < \theta$, hence $\Pi(\theta) = 1$. Thus all $n = 1, 2, 3, \ldots$ satisfy.