STAT 542 Homework 10

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1. $P(Y_i = 1) = P(X_i > \mu) = 1 - F_X(\mu)$, $P(Y_i = 1) = P(X_i \leq \mu) = F_X(\mu)$. Thus $Y_i \sim Bernolli(p = 1 - F_X(\mu))$. As X_i's are iid, then Y_i's are iid. Thus

$$\sum_{i=1}^{n} Y_i \sim Binomial(n, p = 1 - F_X(\mu))$$

2. Let $V = \sqrt{\frac{S^2(n-1)}{\sigma^2}}$, then $V \sim \chi^2_{n-1}$.

$$\begin{aligned}
E(c\sqrt{S^2}) &= \frac{c\sigma}{\sqrt{n-1}} E\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right) \\
&= \frac{c\sigma}{\sqrt{n-1}} E(\sqrt{V}) \\
&= \frac{c\sigma}{\sqrt{n-1}} \int_0^\infty \sqrt{v} \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} v^{\frac{n-1}{2}-1} \mathrm{e}^{-\frac{v}{2}} \mathrm{d}v \\
&= \frac{c\sigma}{\sqrt{n-1}} \frac{\Gamma\frac{n}{2}2^{\frac{n}{2}}}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} \int_0^\infty \frac{1}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}} v^{\frac{n}{2}-1} \mathrm{e}^{-\frac{v}{2}} \mathrm{d}v \\
&= c\frac{\sqrt{2}\Gamma(\frac{n}{2})}{\sqrt{n-1}\Gamma(\frac{n-1}{2})}
\end{aligned}$$

Thus $E(c\sqrt{S^2}) = \sigma \Rightarrow c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$ and $g(S^2) = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}\sqrt{S^2}$.

3. (a) $X_1 - 1$, $\frac{X_2-2}{2}$, $\frac{X_3-3}{3}$ are iid $N(0,1)$. then

$$(X_1 - 1)^2 + \left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2 \sim \chi^2_3$$

(b) Let $Y_1 = X_1 - 1$, $Y_2 = \frac{X_2-2}{2}$, $Y_3 = \frac{X_3-3}{3}, \bar{Y_3} = \frac{Y_1+Y_2+Y_3}{3} = \frac{1}{3}(X_1 + X_2/2 + X_3/3) - 1, S = \sqrt{\frac{1}{2}\sum_{i=1}^3 (Y_i - \bar{Y_3})^2}$ Then

$$\frac{\sqrt{3}\bar{Y_3}}{S} \sim t_2$$

(c)

$$\frac{3\bar{Y_3}^2}{S^2} \sim F_{1,2}$$

4. $F_Z(z) = P(\min(X,Y) \le z) = 1 - P(\min(X,Y) > z) = 1 - P(X > z)P(Y > z) = 1 - (1 - \Phi(z))^2$, then

$$\begin{aligned} F_{Z^2}(z) &= P(Z^2 \le z) = P(-\sqrt{z} \le Z \le \sqrt{z}) \\ &= F_Z(\sqrt{z}) - F_Z(-\sqrt{z}) \\ &= 1 - (1 - \Phi(\sqrt{z}))^2 - 1 + (1 - \Phi(-\sqrt{z}))^2 \\ &= -(1 - \Phi(\sqrt{z}))^2 + (\Phi(\sqrt{z}))^2 \\ &= 2\Phi(\sqrt{z}) - 1 \end{aligned}$$

Thus

$$f_{Z^2}(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_{Z^2}(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z}{2}} z^{-\frac{1}{2}}$$

Hence $Z^2 \sim \chi_1^2$

5. $f_X(x) = 1/\theta$, $F_X(x) = x/\theta$, $0 < x < \theta$.

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u,v) &= \frac{n!}{0!(n-2)!0!} \frac{1}{\theta} \frac{1}{\theta} (\frac{u}{\theta})^0 (\frac{v-u}{\theta})^{n-2} (1 - \frac{v}{\theta})^0 \\ &= \frac{n(n-1)}{\theta^n} (v-u)^{n-2}, \quad 0 < u < v < \theta \end{aligned}$$

Let $Z = \frac{X_{(1)}}{X_{(n)}}$, $W = X_{(n)}$. Hence $X_{(1)} = ZW$, $X_{(n)} = W$, and

$$J = \begin{bmatrix} w & z \\ 0 & 1 \end{bmatrix}, |\det J| = |w|$$

Thus

$$\begin{aligned} f_{Z,W}(z,w) &= \frac{n(n-1)}{\theta^n} (w - zw)^{n-2} w \\ &= \frac{n(n-1)}{\theta^n} (1-z)^{n-2} w^{n-1}, \quad 0 < z < 1, 0 < w < \theta \end{aligned}$$

The support odf (Z,W) is a rectangle, and z and w can be seperated, so they are independent.

6. (a) X and Y are iid $N(0,1)$, thus $\begin{bmatrix} X \\ Y \end{bmatrix}$ is $MVN(\mathbf{0}, I)$. Thus $\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} X \\ X+Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$. And $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Hence $\begin{bmatrix} X \\ Z \end{bmatrix} \sim MVN(\mathbf{0}, \Sigma)$

(b) For $\begin{bmatrix} X - Y \\ X + Y \end{bmatrix}$ we have

$$\begin{bmatrix} X - Y \\ X + Y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Thus $\begin{bmatrix} X - Y \\ X + Y \end{bmatrix} \sim MVN(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $\Sigma_{12} = \Sigma_{21} = 0$, thus $X - Y$ and $X + Y$ are independent. Then $g(x,y) = x - y$.

(c) $F_{X|Z>0}(x) = \frac{P(X\leq x, Z>0)}{P(Z>0)} = 2\int_{-\infty}^{x}\int_{0}^{\infty} f(x,z)\mathrm{d}x\mathrm{d}z$. Then $f_{X|Z>0}(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_{X|Z>0}(x) = 2\int_{0}^{\infty} f(x,z)\mathrm{d}z$. $\begin{bmatrix} X & Z \end{bmatrix}^T$ is MVN, then

$$f(x,z) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(2x^2-2zx+z^2)}$$

Thus

$$\begin{aligned}
f_{X|Z>0}(x) &= 2\frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-\frac{1}{2}(2x^2-2zx+z^2)}\mathrm{d}z \\
&= \frac{\sqrt{2\pi}}{\pi}\mathrm{e}^{-x^2/2}\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(z-x)^2/2}\mathrm{d}z \\
&= \sqrt{\frac{2}{\pi}}\mathrm{e}^{-x^2/2}\int_{-x}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-t^2/2}\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\mathrm{e}^{-x^2/2}(1-\Phi(-x)) \\
&= \sqrt{\frac{2}{\pi}}\mathrm{e}^{-x^2/2}\Phi(x)
\end{aligned}$$

7. (a)

$$\begin{aligned}
E[X] &= E[E[X|Y]] = E[Y] = \mu \\
Var[X] &= E[Var[X|Y]] + Var[E[X|Y]] \\
&= E[\sigma^2] + Var[Y] \\
&= \sigma^2 + \tau^2 \\
E[XY] &= E[E[XY|Y]] \\
&= E[YE[X|Y]] \\
&= E[Y^2] \\
&= Var[Y] + (E[Y])^2 \\
&= \tau^2 + \mu^2 \\
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= \tau^2 + \mu^2 - \mu^2 = \tau^2
\end{aligned}$$

(b)

$$f(x,y) = f(x|y)f(y) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-y)^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\tau}\mathrm{e}^{-\frac{(y-\mu)^2}{2\tau^2}} = \frac{1}{2\pi\sigma\tau}\mathrm{e}^{-\frac{1}{2}((\frac{x-y}{\sigma})^2+(\frac{y-\mu}{\tau})^2)}$$

From the variance of $[X,Y]^T$, we have $\Sigma = \begin{bmatrix} \sigma^2+\tau^2 & \tau^2 \\ \tau^2 & \tau^2 \end{bmatrix}$, $\det\Sigma = (\sigma^2+\tau^2)\tau^2 - \tau^2\cdot\tau^2 = \sigma^2\tau^2$, and $\Sigma^{-1} = \begin{bmatrix} 1/\sigma^2 & -1/\sigma^2 \\ -1/\sigma^2 & 1/\sigma^2+1/\tau^2 \end{bmatrix}$. Hence if $\begin{bmatrix} X & Y \end{bmatrix}^T$ is MVN, then

$$\begin{aligned}
f(x,y) &= \left(\frac{1}{(2\pi)^2\det\Sigma}\right)^{1/2}\mathrm{e}^{-\frac{1}{2}\begin{bmatrix} x-\mu & y-\mu \end{bmatrix}\Sigma^{-1}\begin{bmatrix} x-\mu \\ y-\mu \end{bmatrix}} \\
&= \frac{1}{2\pi\sigma\tau}\mathrm{e}^{-\frac{1}{2}((\frac{x-\mu}{\sigma})^2 - \frac{(y-\mu)(x-\mu)}{\sigma^2} - \frac{(x-\mu)(y-\mu)}{\sigma^2} + (\frac{y-\mu}{\sigma})^2 + (\frac{y-\mu}{\tau})^2)} \\
&= \frac{1}{2\pi\sigma\tau}\mathrm{e}^{-\frac{1}{2}((\frac{x-y}{\sigma})^2+(\frac{y-\mu}{\tau})^2)}
\end{aligned}$$

It is the same as what we get from the conditional density, thus $\begin{bmatrix} X & Y \end{bmatrix}^T \sim MVN(\boldsymbol{\mu}, \Sigma)$.