STAT 542 Homework 1

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1. Approximately one-third of all human twins are identical (one-egg) and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, one-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth. Define the following events:

$$A = \{\text{a U.S. birth results in twin females}\}$$
$$B = \{\text{a U.S. birth results in identical twins}\}$$
$$C = \{\text{a U.S. birth results in twins}\}$$

Find $P(A \cap B \cap C)$.

$P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C) = \frac{1}{2}\frac{1}{3}\frac{1}{90} = \frac{1}{540}$

2. Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two grayhaired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

(a) What is the probability that the animal chosen is brown-haired?

$P(\{\text{brown-haired}\}) = P(\{\text{brown-haired}\}|\{\text{litter 1}\})P(\{\text{litter 1}\}) +$
$P(\{\text{brown-haired}\}|\{\text{litter 2}\})P(\{\text{litter 2}\}) = \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{19}{30}$

(b) Given that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?

$P(\{\text{litter 1}\}|\{\text{brown-haired}\}) = \frac{P(\{\text{brown-haired}\} \cap \{\text{litter 1}\})}{P(\{\text{brown-haired}\})} = \frac{P(\{\text{brown-haired}\}|\{\text{litter 1}\})P(\{litter 1\})}{P(\{\text{brown-haired}\})} =$
$\frac{2/3 \cdot 1/2}{19/30} = \frac{10}{19}$

3. Prove each of the following statements. (Assume that any conditioning event has positive probability.)

(a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A.

$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A \cap B)$. Also, we have $P(A) = P(A \cap B) + P(A \cap B^c)$. $P(B^c) = 1 - P(B) = 0$, $A \cap B^c \subset B^c \Rightarrow P(A \cap B^c) \leq P(B^c) = 0$. Thus $P(A \cap B^c) = 0$ and $P(A) = P(A \cap B)$. Therefore $P(A|B) = P(A \cap B) = P(A)$.

(b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.
$A \subset B \Rightarrow A \cap B = A$.
$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$.
$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$

(c) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

$A \subset A \cup B \Rightarrow A \cap (A \cup B) = A$, $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.
$P(A|A \cup B) = \frac{A \cap (A \cup B)}{P(A \cup B)} = \frac{P(A)}{P(A)+P(B)}$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$
$P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$

4. A pair of events A and B cannot be simultaneously *mutually exclusive* and *independent*. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

(a) If A and B are mutually exclusive, they cannot be independent.
$A \cap B = \emptyset \Rightarrow P(A \cap B) = 0$. Therefore, $P(A|B) = \frac{P(A \cap B)}{P(B)} = 0 \neq P(A)$, as $P(A) > 0$. Thus A and B are not independent.

(b) If A and B are independent, they cannot be mutually exclusive.
$P(A|B) = P(A) \Rightarrow P(A \cap B) = P(A)P(B) > 0$. Thus $A \cap B$ cannot be an empty set, A and B are not mutually exclusive.

5. As in Example 1.3.6, consider telegraph signals "dot"and "dash" sent in the proportion 3:4, where erratic transmissions cause a dot to become a dash with probability $\frac{1}{4}$ and a dash to become a dot with probability $\frac{1}{3}$.

(a) If a dash is received, what is the probability that a dash has been sent?
$P(\{\text{dash sent}\}|\{\text{dash received}\}) = \frac{P(\{\text{dash sent}\} \cap \{\text{dash received}\})}{P(\{\text{dash received}\})}$.
$P(\{\text{dash received}\}) = P(\{\text{dash received}|\text{dash sent}\})P(\{\text{dash sent}\}) +$
$P(\{\text{dash received}|\text{dot sent}\})P(\{\text{dot sent}\}) = \frac{2}{3} \cdot \frac{4}{7} + \frac{1}{4} \cdot \frac{3}{7} = \frac{41}{84}$.
$P(\{\text{dash sent}\} \cap \{\text{dash received}\}) = P(\{\text{dash received}\}|\{\text{dash sent}\})P(\{\text{dash sent}\}) =$
$\frac{2}{3} \cdot \frac{4}{7} = \frac{8}{21}$.
Therefore, $P(\{\text{dash sent}\}|\{\text{dash received}\}) = \frac{8/21}{41/84} = \frac{32}{41}$.

(b) Assuming independence between signals, if the message dot-dot was received, what is the probability distribution of the four possible messages that could have been sent?
$P(\{\text{dot received}\} \cap \{\text{dot sent}\}) = P(\{\text{dot received}\}|\{\text{dot sent}\})P(\{\text{dot sent}\}) =$
$\frac{3}{4} \cdot \frac{3}{7} = \frac{9}{28}$.
$P(\{\text{dot received}\}) = 1 - P(\{\text{dash received}\}) = \frac{43}{84}$.
Thus, $P(\{\text{dot sent}\}|\{\text{dot received}\}) = \frac{(9/28)}{43/84} = \frac{27}{43}$, $P(\{\text{dash sent}\}|\{\text{dot received}\}) =$
$\frac{16}{43}$. By indepence, the distribution of four messages could have been sent is:

Signal sent	Probability
dot-dot	$(27/43)^2$
dot-dash	$(27/43) \cdot (16/43)$
dash-dot	$(16/43) \cdot (27/43)$
dash-dash	$(16/43)^2$

6. Prove that the following functions are cdfs.

(c) $\mathrm{e}^{-\mathrm{e}^{-x}}$, $x \in (-\infty, \infty)$

$$F(x) = \mathrm{e}^{-\mathrm{e}^{-x}},\ x \in (-\infty, \infty)$$

$F(x)$ is right continuous. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$. $F'(x) = \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{e}^{-x}} > 0$. Thus $F(x)$ is a cdf.

(d) $1 - \mathrm{e}^{-x}$, $x \in (0, \infty)$

$$F(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 - \mathrm{e}^{-x} & x \in (0, \infty) \end{cases}$$

$\lim_{x \to 0^+} F(x) = F(0) = 0$, $F(x)$ is right continuous. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$. $F'(x) = \mathrm{e}^{-x} > 0$. Thus $F(x)$ is a cdf.

7. An appliance store receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 ovens at random, without replacement, and tests to see if they are defective. Let $X =$ number of defectives found. Calculate the pmf and cdf of X and plot the cdf.

$$P(X = 0) = \frac{\binom{25}{4}}{\binom{30}{4}} = 0.4616$$

$$P(X = 1) = \frac{\binom{5}{1}\binom{25}{3}}{\binom{30}{4}} = 0.4196$$

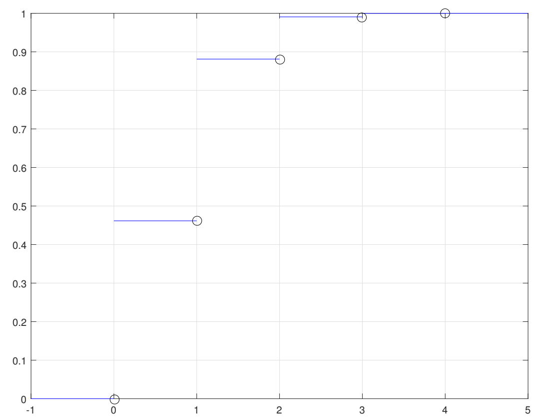
$$P(X = 2) = \frac{\binom{5}{2}\binom{25}{2}}{\binom{30}{4}} = 0.1095$$

$$P(X = 3) = \frac{\binom{5}{3}\binom{25}{2}}{\binom{30}{4}} = 0.0091$$

$$P(X = 4) = \frac{\binom{5}{4}\binom{25}{1}}{\binom{30}{4}} = 0.0002$$

$F(x)$ is the cdf corresponding to the pmf above.

$$F(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ 0.4616 & x \in [0, 1) \\ 0.8812 & x \in [1, 2) \\ 0.9907 & x \in [2.3) \\ 0.9998 & x \in [3.4) \\ 1 & x \in [4, \infty) \end{cases}$$



$P(3 < X \le 4.7) = P(X = 4) = 0.0002$, $P(3 \le X < 4.7) = P(X = 3) + P(X = 4) = 0.0093$.

8. For each o f the following, determine the value of c that makes $f(x)$ a pdf.

(a) $f(x) = c\sin(x)\mathrm{d}x$, $0 < x < \pi/2$
$\int_0^{\pi/2} c\sin x = c = 1 \Rightarrow c = 1$.

(b) $f(x) = c\mathrm{e}^{-|x|}$, $-\infty < x < \infty$
$\int_{-\infty}^{\infty} c\mathrm{e}^{-|x|}\mathrm{d}x = 2c\int_0^{\infty} \mathrm{e}^{-x} = 2c = 1 \Rightarrow c = 1/2$.

9. From the axioms of probability, it follows that probability functions $P(\cdot)$ exhibit "monotone continuity from above (mcfa)", meaning that for any decreasing sequence of sets/events $A_1 \supset A_2 \supset A_3 \supset \cdots$,

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{i=1}^{\infty} A_i\right).$$

Using the mcfa property, show that the cdf F of a random variable X must be right continuous: for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F(x + n^{-1}) = F(x)$$

holds.

$F(x) = P_X(X \in (-\infty, x])$. Let $A_n = (-\infty, x + \frac{1}{n}]$, thus $A_1 \supset A_2 \supset A_3 \supset \cdots$, $\bigcap_{i=1}^{\infty} A_i = (-\infty, x]$.

By mcfa, we have $\lim_{n\to\infty} F(x + n^{-1}) = \lim_{n\to\infty} P_X(A_n) = P_X(\bigcap_{i=1}^{\infty} A_i) = P_X((-\infty, x]) = F(x)$.

10. Statistical reliability involves studying the time to failure of manufactured units. In many reliability textbooks. one can find the exponential distribution

$$f(x) = \begin{cases} \frac{1}{\theta}\mathrm{e}^{-x/\theta} & x > 0 \\ 0 & x \le 0 \end{cases}$$

where $\theta > 0$ is a fixed value, for modeling the time X that a random unit runs until failure (i.e. X is a survival time). While a useful distribution gernerlly, the exponential distribution is not typically realistic for modeling failure times. Show that if X has an exponential distribution as above, then

$$P(X > s + t | X > t) = P(X > s)$$

for any calues $t, s > 0$; this feature is called teh "memoryless" property of the exponential distribution. Explain intuitively why the "memoryless" property might make the exponential distribution an unappealing model for the survival time of a randomly selected, manufactured unit.

$P(X > t) = 1 - P(X \le t) = 1 - F(t) = 1 - \int_{-\infty}^{t} f(x)\mathrm{d}x = 1 - \int_{0}^{t} f(x)\mathrm{d}x = 1 - (1 - \mathrm{e}^{-t/\theta}) = \mathrm{e}^{-t/\theta}$.

Therefore, $P(X > s + t | X > t) = \frac{P(X \in (s+t,\infty) \cap (t,\infty))}{P(X>t)} = \frac{P(X>s+t)}{P(X>t)} = \frac{\mathrm{e}^{-(s+t)/\theta}}{\mathrm{e}^{-t/\theta}} = \mathrm{e}^{-s/\theta} = P(X > s)$

The chance to survive a period of time s for the manufactured unit should go down as the time the manufactured unit has already survived get longer, so the "memoryless" property of exponential distribution makes it an unappealing model.