STAT 543 Homework 3

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1. The posterior pdf is

$$f(\theta|\tilde{x}) \propto f(\tilde{x}|\theta)\pi(\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}$$

Thus

$$E(L(t,\theta)|\tilde{x}) \propto \int_0^1 \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}\frac{(t-\theta)^2}{\theta(1-\theta)}\mathrm{d}\theta = \int_0^1 \theta^{\sum_{i=1}^n x_i - 1}(1-\theta)^{n-\sum_{i=1}^n x_i - 1}(t-\theta)^2\mathrm{d}\theta$$

If we take the $(t-\theta)^2$ above as a loss function and $\theta^{\sum_{i=1}^n -1}(1-\theta)^{n-\sum_{i=1}^n -1}$ as a posterior, then to minimize the above equation, the t we chose is the expectation of a Beta distribution, where $\alpha = \sum_{i=1}^n x_i$ and $\beta = n - \sum_{i=1}^n x_i$. Thus

$$t = \frac{\alpha}{\alpha+\beta} = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + n - \sum_{i=1}^n x_i} = \frac{\sum_{i=1}^n x_i}{n}$$

The Bayes estimator $T_0 = \frac{\sum_{i=1}^n X_i}{n}$.

2. (a) As $X_1, X_2, \cdots, X_n$ are iid $N(\theta, 1)$, then $\bar{X}_n \sim N(\theta, 1/n)$, hence

$$\begin{aligned} E(T) &= E((\bar{X}_n)^2 - \frac{1}{n}) \\ &= E((\bar{X}_n)^2) - \frac{1}{n} \\ &= Var((\bar{X}_n)^2) + (E(\bar{X}_n))^2 - \frac{1}{n} \\ &= \frac{1}{n} + \theta^2 - \frac{1}{n} = \theta^2 \end{aligned}$$

Thus T is an unbiased estimator.
$\sqrt{n}(\bar{X}_n - \theta) = Y \Rightarrow \bar{X}_n = Y/\sqrt{n} + \theta \Rightarrow (\bar{X}_n)^2 = Y^2/n + 2\theta Y/\sqrt{n} + \theta^2$. Here we know $Y \sim N(0,1)$ and $Y^2 \sim \chi_1^2$. Thus $Var_\theta(Y^2) = 2$ and $Var_\theta(Y) = 1$.
Also, for $Y \sim N(0,1)$,

$$Cov_\theta(Y^2, Y) = E(Y^3) - E(Y^2)E(Y) = E(Y^3) = \int_{-\infty}^{\infty} y^3 \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y = 0$$

As the function in the integral is an odd function and the integral interval is symmetric about 0, thus the integral is 0.

Then

$$
\begin{aligned}
Var_\theta(T) &= Var_\theta((\bar{X}_n)^2 - \frac{1}{n}) \\
&= Var_\theta((\bar{X}_n)^2) \\
&= Var_\theta(Y^2/n + 2\theta Y/\sqrt{n} + \theta^2) \\
&= Var_\theta(Y^2/n + 2\theta Y/\sqrt{n}) \\
&= \frac{1}{n^2} Var_\theta(Y) + \frac{4\theta^2}{n} Var_\theta(Y) + \frac{4\theta}{n\sqrt{n}} Cov_\theta(Y^2, Y) \\
&= \frac{2}{n^2} + \frac{4\theta^2}{n}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\gamma(\theta) = \theta^2 \Rightarrow \gamma'(\theta) = 2\theta \\
&\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(X_1|\theta) \\
=&\frac{\mathrm{d}}{\mathrm{d}\theta} \log \left(\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(X_1-\theta)^2}{2}} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\log \frac{1}{\sqrt{2\pi}} - \frac{(X_1-\theta)^2}{2} \right) \\
&= X_1 - \theta
\end{aligned}
$$

Then

$$
I_1(\theta) = E_\theta((X_1-\theta)^2) = Var_\theta(X_1) = 1 \Rightarrow I_n(\theta) = nI_1(\theta) = n
$$

.

Thus

$$
CRLB = \frac{(\gamma'(\theta))^2}{I_n(\theta)} = \frac{4\theta^2}{n}
$$

(c)

$$
Var_\theta(T) - CRLB = \frac{4\theta^2}{n} + \frac{2}{n^2} - \frac{4\theta^2}{n} = \frac{2}{n^2} > 0, \forall \theta \in \Theta
$$

Hence $Var_\theta(T) > CRLB$ for all θ.

3. (a) $\bar{X}_n|\theta \sim N(\theta, \frac{9}{25})$,

$$
\begin{aligned}
f(\bar{x}, \theta) &= f(\bar{x}|\theta) f(\theta) \\
&= \frac{1}{\sqrt{2\pi}\frac{3}{5}} \mathrm{e}^{-\frac{(\bar{x}-\theta)^2}{2\cdot\frac{9}{25}}} \frac{1}{\sqrt{2\pi}4} \mathrm{e}^{-\frac{(\theta-10)^2}{2\cdot 16}} \\
&= \frac{5}{24\pi} \mathrm{e}^{-\frac{25(\bar{x}-\theta)^2}{18} - \frac{(\theta-10)^2}{32}},\ (\bar{x}, \theta) \in \mathbb{R}^2
\end{aligned}
$$

(b) For $X_i|\theta \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, we have

$$
m(\bar{x}) \sim N(\mu, \sigma_n^2 + \tau^2)
$$

Here $\sigma_n^2 = \frac{\sigma^2}{n}$.
For this question, $\sigma_n^2 = \frac{9}{25}, \tau^2 = 16, \mu = 10$, then $\sigma_n^2 + \tau^2 = \frac{9}{25} + 16 = \frac{409}{25}$. Then

$$
m(\bar{x}) \sim N(10, \frac{409}{25})
$$

(c) For $X_i|\theta \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, we have

$$\theta|\bar{x} \sim N\left(\frac{\tau^2}{\tau^2+\sigma_n^2}\bar{x} + \frac{\sigma_n^2}{\tau^2+\sigma_n^2}\mu, \frac{\tau^2\sigma_n^2}{\tau^2+\sigma_n^2}\right)$$

Then

$$E(\theta|\bar{x}=18) = \frac{\tau^2}{\tau^2+\sigma_n^2}\bar{x} + \frac{\sigma_n^2}{\tau^2+\sigma_n^2}\mu = \frac{16}{16+\frac{9}{25}}\cdot 18 + \frac{\frac{9}{25}}{16+\frac{9}{25}}\cdot 10 = \frac{7290}{409}$$
$$Var(\theta|\bar{x}=18) = \frac{\tau^2\sigma_n^2}{\tau^2+\sigma_n^2} = \frac{16\cdot\frac{9}{25}}{16+\frac{9}{25}} = \frac{144}{409}$$

4. (a)

$$\begin{aligned} f(\tilde{x}|\lambda) &= f(x_1|\lambda)f(x_2|\lambda)f(x_3|\lambda) \\ &= \frac{\lambda^{x_1}}{x_1!}\mathrm{e}^{-\lambda}\frac{\lambda^{x_2}}{x_2!}\mathrm{e}^{-\lambda}\frac{\lambda^{x_3}}{x_3!}\mathrm{e}^{-\lambda} \\ &= \frac{\lambda^{x_1+x_2+x_3}}{x_1!x_2!x_3!}\mathrm{e}^{-3\lambda} \end{aligned}$$

$$f(\lambda) = \lambda^{1-1}\frac{\mathrm{e}^{-\lambda/2}}{\Gamma(1)2} = \frac{1}{2}\mathrm{e}^{-\lambda/2}$$

$$\begin{aligned} f(\tilde{x},\lambda) &= f(\tilde{x}|\lambda)f(\lambda) \\ &= \frac{\lambda^{x_1+x_2+x_3}}{2x_1!x_2!x_3!}\mathrm{e}^{-\frac{7}{2}\lambda} \end{aligned}$$

$$\begin{aligned} m(\tilde{x}) &= \int_0^\infty \frac{\lambda^{x_1+x_2+x_3}}{2x_1!x_2!x_3!}\mathrm{e}^{-\frac{7}{2}\lambda}\mathrm{d}\lambda \\ &= \frac{1}{2x_1!x_2!x_3!}\int_0^\infty \lambda^{x_1+x_2+x_3}\mathrm{e}^{-\frac{7}{2}\lambda}\mathrm{d}\lambda \\ &= \frac{1}{2x_1!x_2!x_3!}\int_0^\infty (\frac{2}{7})^{x_1+x_2+x_3+1}t^{x_1+x_2+x_3-1}\mathrm{e}^{-t}\mathrm{d}t \\ &= \frac{2^{x_1+x_2+x_3}}{7^{x_1+x_2+x_3+1}x_1!x_2!x_3!}\Gamma(x_1+x_2+x_3+1) \end{aligned}$$

Thus

$$f(\lambda|\tilde{x}) = \frac{7^{x_1+x_2+x_3+1}x_1!x_2!x_3!}{2^{x_1+x_2+x_3}\Gamma(x_1+x_2+x_3+1)}\frac{\lambda^{x_1+x_2+x_3}}{2x_1!x_2!x_3!}\mathrm{e}^{-\frac{7}{2}\lambda} = \frac{1}{(2/7)^{x_1+x_2+x_3+1}}\lambda^{x_1+x_2+x_3+1-1}\mathrm{e}^{-\frac{\lambda}{2/7}}$$

Thus

$$\lambda|\tilde{x} \sim Gamma(x_1+x_2+x_3+1, \frac{2}{7})$$

(b)

$$E(\lambda|\tilde{x}) = \frac{2}{7}(x_1+x_2+x_3+1)$$
$$Var(\lambda|\tilde{x}) = \frac{4}{49}(x_1+x_2+x_3+1)$$

5. (a) For double exponential distribution, $E(X_i) = \mu$, then

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i) = \sum_{i=1}^{n} a_i \mu = \mu$$

Thus it is unbiased estimator.

(b)

$$\sum_{i=1}^{n}(a_i - \bar{a})^2 = \sum_{i=1}^{n}(a_i^2 - 2\bar{a}a_i + (\bar{a})^2) = \sum_{i=1}^{n} a_i^2 - 2\bar{a}\sum_{i=1}^{n} a_i + n(\bar{a})^2 = \sum_{i=1}^{n} a_i^2 - n(\bar{a})^2$$

Then

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n}(a_i - \bar{a})^2 + n(\bar{a})^2$$

$$\begin{aligned}
Var(\sum_{i=1}^{n} a_i X_i) &= \sum_{i=1}^{n} a_i^2 Var(X_i) = \sum_{i=1}^{n} a_i^2 2\sigma^2 \\
&= 2\sigma^2 \sum_{i=1}^{n} a_i^2 \\
&= 2\sigma^2 \left[\sum_{i=1}^{n}(a_i - \bar{a})^2 + n(\bar{a})^2\right] \\
&= 2\sigma^2 \left[\sum_{i=1}^{n}(a_i - \frac{1}{n})^2 + \frac{1}{n}\right] \\
&\geq \frac{2\sigma^2}{n}
\end{aligned}$$

The equation holds only when $a_i = \frac{1}{n}$. Thus for the case $n = 3$, among all the linear unbiased estimator $\sum_{i=1}^{3} \frac{1}{3} X_i$ has the smallest variance.

6. (a)

$$\begin{aligned}
\log L(\theta) &= |x| \log \frac{\theta}{2} + (1 - |x|) \log(1 - \theta) \\
\frac{\mathrm{d}}{\mathrm{d}\theta} \log L(\theta) &= \frac{1}{\theta}|x| - (1 - |x|)\frac{1}{1 - \theta} \\
&= \frac{|x|(1 - \theta) - (1 - |x|)\theta}{\theta(1 - \theta)} \\
&= \frac{|x| - \theta}{\theta(1 - \theta)}
\end{aligned}$$

Hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta} \log L(\theta)\right|_{\hat{\theta}} = 0 \Rightarrow \hat{\theta} = |x|$$

We also have

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log L(\theta) = -\frac{1}{\theta^2}|x| - (1 - |x|)\frac{1}{(1 - \theta)^2} < 0$$

Then $\hat{\theta} = |X|$.

(b)

$$E(T(X)) = 2P(X = 1) = 2\frac{\theta}{2} = \theta$$

Thus $T(X)$ is unbiased.

(c)

$$
\begin{aligned}
MSE_{T(X)} &= E((T(X)-\theta)^2) \\
&= (2-\theta)^2\frac{\theta}{2} + (0-\theta)^2(1-\frac{\theta}{2}) \\
&= (\theta^2-4\theta+4)\frac{\theta}{2} + \theta^2(1-\frac{\theta}{2}) \\
&= -\theta^2+2\theta
\end{aligned}
$$

Take another estimator $T_1(X) = |X|$. Then

$$
\begin{aligned}
MSE_{T_1(X)} &= E((T_1(X)-\theta)^2) \\
&= (1-\theta)^2\frac{\theta}{2} + \theta^2(1-\frac{\theta}{2}) \\
&= (\theta^2-2\theta+1)\frac{\theta}{2} + \theta^2 - \frac{\theta^3}{2} \\
&= \frac{\theta}{2}
\end{aligned}
$$

$$
MSE_{T_1(X)} - MSE_{T(X)} = \frac{\theta}{2} + \theta^2 - 2\theta = \theta^2 - \frac{3}{2}\theta = \theta(\theta - \frac{3}{2}) \leq 0, \, \forall\theta \in [0,1]
$$

And the less sign holds when $\theta \in (0,1]$. Hence $T_1(X) = |X|$ is a better estimator in terms of MSE.