STAT 510 Homework 4

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1. (a)

$$\begin{bmatrix} \hat{y} \\ y - y \end{bmatrix} = \begin{bmatrix} P_X y \\ (I - P_X)y \end{bmatrix} = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} y$$

We know $y \sim N(X\beta, \sigma^2 I)$, thus $\begin{bmatrix} \hat{y} \\ y - \hat{y} \end{bmatrix} \sim N(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} X\beta = \begin{bmatrix} P_X X\beta \\ (I - P_X)X\beta \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} \sigma^2 I \begin{bmatrix} P_X^T & (I - P_X)^T \end{bmatrix}$$

$$= \begin{bmatrix} P_X P_X^T & P_X(I - P_X)^T \\ (I - P_X)P_X^T & (I - P_X)(I - P_X)^T \end{bmatrix}$$

$$= \begin{bmatrix} P_X & 0 \\ 0 & I - P_X \end{bmatrix}$$

Hence

$$\begin{bmatrix} \hat{y} \\ y - \hat{y} \end{bmatrix} \sim N\left(\begin{bmatrix} X\beta \\ 0 \end{bmatrix}, \begin{bmatrix} P_X & 0 \\ 0 & I - P_X \end{bmatrix} \right)$$

(b) We have

$$\hat{y}^T \hat{y} = y^T P_X^T P_X y = y^T P_X y$$

So let $A = \frac{1}{\sigma^2} P_X$, we have $rank(A) = rank(P_X) = rank(X) = r$. Also, $\Sigma = \sigma^2 I$ is positive definite, and

$$A\Sigma A\Sigma = \frac{1}{\sigma^2} P_X \sigma^2 I \frac{1}{\sigma^2} P_X \sigma^2 I = \frac{1}{\sigma^2} P_X \sigma^2 I = A\Sigma$$

From $y \sim N(X\beta, \sigma^2 I)$ we have

$$y^T A y = \frac{1}{\sigma^2} y^T P_X y \sim \chi_r^2 \left(\beta^T X^T \frac{1}{\sigma^2} P_X X\beta/2 \right) = \chi_r^2 \left(\frac{1}{\sigma^2} \beta^2 X^T X\beta/2 \right)$$

Thus

$$\hat{y}^T \hat{y} \sim \sigma^2 \chi_r^2 \left(\frac{1}{\sigma^2} \beta^T X^T X\beta/2 \right)$$

2. (a)

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) By Gauss-Markov Theorem, BLUE here is OLSE, thus

$$\text{BLUE}(\beta_4) = \text{OLSE}(\beta_4) = \text{OLSE}(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \text{OLSE}(\beta_1 + \beta_2 + \beta_3) = 26.3 - 22.8 = 3.5$$

(c)

$$\hat{\text{Var}}(\beta_4) = \hat{\sigma}^2 \boldsymbol{C}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{C}^T = \boldsymbol{AX}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{X}^T\boldsymbol{A}^T = \boldsymbol{AP_X}\boldsymbol{A}^T$$

The matrix $\boldsymbol{A}$ we use here is $\begin{bmatrix} 0 & \cdots & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$, which is literally take the last row of a matrix and minus the last 5th row. And in this model, the estimated response variable is just sample means of each treatment. Then we have

$$\boldsymbol{AP_X} = \begin{bmatrix} 0 & \cdots & 0 - 1/2 & -1/2 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Then we have $\boldsymbol{AP_X}\boldsymbol{A}^T = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.

And

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{4} s_i^2(n_i - 1)}{\sum_{i=1}^{4} n_i - 4} = 3.428571$$

Then

$$se(\hat{\beta}_4) = \sqrt{\frac{3}{4}\hat{\sigma}^2} = 1.603576$$

3. (a)

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{bmatrix} \boldsymbol{1}^T \\ \boldsymbol{x}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{x} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{1}^T\boldsymbol{1} & \boldsymbol{1}^T\boldsymbol{x} \\ \boldsymbol{x}^T\boldsymbol{1} & \boldsymbol{x}^T\boldsymbol{x} \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

Take the inverse, we have

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} & \frac{-\sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \\ \frac{-\sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} & \frac{n}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \end{bmatrix}$$

And

$$\boldsymbol{X}^T\boldsymbol{y} = \begin{bmatrix} \boldsymbol{1}^T \\ \boldsymbol{x}^T \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} \boldsymbol{1}^T\boldsymbol{y} \\ \boldsymbol{x}^T\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Then

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^n x_i^2}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} & \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} \\ \frac{-\sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} & \frac{n}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i)-(\sum_{i=1}^n x_i)(\sum_{i=1}^n x_i y_i)}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} \\ \frac{n\sum_{i=1}^n x_i y_i-(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n\sum_{i=1}^n x_i^2-(\sum_{i=1}^n x_i)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\bar{x^2}\bar{y}-\bar{x}\bar{xy}}{\bar{x^2}-(\bar{x})^2} \\ \frac{\bar{xy}-\bar{x}\bar{y}}{\bar{x^2}-(\bar{x})^2} \end{bmatrix}$$

(b) $\begin{bmatrix} \mathbf{1} & \boldsymbol{x}-\bar{x}\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \boldsymbol{x} \end{bmatrix} \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix}$. Thus we choose $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix}$

(c) We have

$$\boldsymbol{W}^T\boldsymbol{W} = \begin{bmatrix} \mathbf{1}^T \\ (\boldsymbol{x}-\bar{x}\mathbf{1})^T \end{bmatrix} \begin{bmatrix} \mathbf{1} & \boldsymbol{x}-\bar{x}\mathbf{1} \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \sum_{i=1}^n(x_i-\bar{x}) \end{bmatrix}$$

Also,

$$\boldsymbol{W}^T\boldsymbol{y} = \begin{bmatrix} \mathbf{1}^T\boldsymbol{y} \\ (\boldsymbol{x}-\bar{x}\mathbf{1})^T\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n(x_i-\bar{x})y_i \end{bmatrix}$$

Hence

$$\hat{\boldsymbol{\alpha}} = (\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T\boldsymbol{y}$$

$$= \begin{bmatrix} 1/n & 0 \\ 0 & 1/\sum_{i=1}^n(x_i-\bar{x}) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n(x_i-\bar{x})y_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^n y_i}{n} \\ \frac{\sum_{i=1}^n(x_i-\bar{x})y_i}{\sum_{i=1}^n(x_i-\bar{x})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y} \\ \frac{\bar{xy}-\bar{x}\bar{y}}{\bar{x^2}-(\bar{x})^2} \end{bmatrix}$$

(d)

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sum_{i=1}^n y_i}{n} \\ \frac{\sum_{i=1}^n(x_i-\bar{x})y_i}{\sum_{i=1}^n(x_i-\bar{x})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sum_{i=1}^n y_i/n-\bar{x}\sum_{i=1}^n(x_i-\bar{x})y_i}{\sum_{i=1}^n(x_i-\bar{x})^2} \\ \frac{\sum_{i=1}^n(x_i-\bar{x})y_i}{\sum_{i=1}^n(x_i-\bar{x})^2} \end{bmatrix}$$

(e)

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & -\bar{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{y} \\ \frac{\bar{xy}-\bar{x}\bar{y}}{\bar{x^2}-(\bar{x})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\bar{x^2}\bar{y}-(\bar{x})^2\bar{y}-\bar{x}\bar{xy}+(\bar{x})^2\bar{y}}{\bar{x^2}-(\bar{x})^2} \\ \frac{\bar{xy}-\bar{x}\bar{y}}{\bar{x^2}-(\bar{x})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\bar{x^2}\bar{y}-\bar{x}\bar{xy}}{\bar{x^2}-(\bar{x})^2} \\ \frac{\bar{xy}-\bar{x}\bar{y}}{\bar{x^2}-(\bar{x})^2} \end{bmatrix}$$

It matches (a).

4. (a) The null hypothesis tested is the difference in score students gave for different perceived instructor when ruling out any possible impact the actually instructor gave is zero.

(b) The matrix $(\boldsymbol{X}_R^T \boldsymbol{X}_R)^-$ we got is

$$
\begin{bmatrix}
0.1 & -0.1 & -0.1 & 0.1 \\
-0.1 & 0.2 & 0.1 & -0.2 \\
-0.1 & 0.1 & 0.2 & -0.2 \\
0.1 & -0.2 & -0.2 & 0.377
\end{bmatrix}
$$

With we can have

$$\hat{\sigma}^2 = (se(\text{Intercept}))^2 = 0.0507$$

To calculate the variance of main effect of perceived instructor, we use $\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 & 1/2 \end{bmatrix}$
Hence

$$\hat{\text{Var}}(\text{main effect of perceived instructor}) = \hat{\sigma}^2 \boldsymbol{C}(\boldsymbol{X}_R^T \boldsymbol{X}_R)^- \boldsymbol{C}^T = 0.0047775$$

Thus the standard error is $\sqrt{0.0047775} = 0.0691$.

Also, we have the estimate of main effect of perceived instructor to be $\begin{bmatrix} 0 & 0 & 1 & 1/2 \end{bmatrix} \hat{\boldsymbol{\beta}} = 0.8700 - 0.1831/2 = 0.77845$. And we know $t_{33-4,0.975} = t_{29,0.975} = 2.04523$. Thus the 95% confidence interval

$$[0.77845 - 0.0691 \times 2.04523, 0.77845 + 0.0691 \times 2.04523] = [0.6371, 0.9198]$$