STAT 542 Homework 9

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1. (a) X_1, $X_2 \sim Exponential(\theta)$, thus $M_{X_1}(t) = M_{X_2}(t) = \frac{1}{1-\theta t} \quad t < 1/\theta$. Hence,

$$M_{X_1,X_2}(t_1,t_2) = M_{X_2}(t_1)M_{X_2}(t_2) = \frac{1}{(1-\theta t_1)(1-\theta_2 t)}, \quad t_1, t_2 < 1/\theta$$

(b)

$$\begin{aligned} M_{X_1-X_2}(t) &= E(\mathrm{e}^{X_1-X_2}t) \\ &= E(\mathrm{e}^{tX_1+(-t)X_2}) \\ &= M_{X_1,X_2}(t,-t) \end{aligned}$$

(c) $Y = X_1 - X_2$, the support of Y is $(-\infty, \infty)$. We also have

$$\begin{aligned} M_{X_1-X_2}(t) &= M_{X_1,X_2}(t,-t) \\ &= \frac{1}{(1-\theta t)(1+\theta t)} \\ &= \frac{1}{2}[\frac{1}{1-\theta t} + \frac{1}{1-\theta(-t)}] \\ &= \int_0^\infty \frac{1}{2}\frac{1}{\theta}\mathrm{e}^{-y/\theta}\mathrm{e}^{ty}\mathrm{d}y + \int_0^\infty \frac{1}{2}\frac{1}{\theta}\mathrm{e}^{-y/\theta}\mathrm{e}^{-ty}\mathrm{d}y \\ &= \int_0^\infty \frac{1}{2}\frac{1}{\theta}\mathrm{e}^{-y/\theta}\mathrm{e}^{ty}\mathrm{d}y + \int_{-\infty}^0 \frac{1}{2}\frac{1}{\theta}\mathrm{e}^{y/\theta}\mathrm{e}^{ty}\mathrm{d}y \\ &= \int_{-\infty}^\infty f_Y(y)\mathrm{e}^{ty}\mathrm{d}y \end{aligned}$$

Hence, we have

$$f_Y(y) = \begin{cases} \frac{1}{2\theta}\mathrm{e}^{-y/\theta} & y \geq 0 \\ \frac{1}{2\theta}\mathrm{e}^{y/\theta} & y < 0 \end{cases}$$

2. (a) $Y|X = x \sim N(x, x^2)$, $X \sim Uniform(0,1)$, thus

$$\begin{aligned}
EY &= E[E[Y|X]] = E[X] = 1/2 \\
Var[Y] &= E[Var[Y|X]] + Var[E[Y|X]] \\
&= E[X^2] + Var[X] \\
&= Var[X] + (E[X])^2 + Var[Y] \\
&= \frac{1}{12} + \frac{1}{4} + \frac{1}{12} \\
&= \frac{5}{12} \\
E[XY] &= E[E[XY|X]] \\
&= E[XE[Y|X]] \\
&= E[X^2] = Var[X] + (E[X])^2 \\
&= \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \\
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{1}{12}
\end{aligned}$$

(b)

$$f(x,y) = f(y|x)f(x) = \frac{1}{\sqrt{2\pi}x} \mathrm{e}^{-\frac{(y-x)^2}{2x^2}}, \quad (x,y) \in (0,1) \times (-\infty, \infty)$$

Let

$$U = Y/X,\ V = X$$

Then the support of (U,V) is $(-\infty,\infty) \times (0,1)$. The transformation

$$\begin{cases} x = v \\ y = uv \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ v & u \end{bmatrix}$$

$$|\det J| = |-v| = v$$

$$\begin{aligned}
f(u,v) &= \frac{1}{\sqrt{2\pi}v} \mathrm{e}^{-\frac{(uv-v)^2}{2v^2}} \cdot v \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(u-1)^2}{2}} \\
&= f_U(u) \cdot f_V(v)
\end{aligned}$$

Hence U, V are independent, and we have $U \sim N(1,1)$ and $V \sim Uniform(0,1)$. Thus Y/X and X are independent.

3. $X_i \sim Uniform(0,1)$, Let $Y_i = -\log X_i$, then

$$P(Y_i \leq y) = P(-\log X_i \leq y) = P(X_i \geq \mathrm{e}^{-y}) = 1 - P(X_i < \mathrm{e}^{-y})$$

Hence $Y_i \sim \mathit{Exponential}(1) \sim \mathit{Gamma}(1,1)$.
Let $Y = -\log(\prod_{i=1}^n X_i)$, then

$$Y = -\log(\prod_{i=1}^n X_i) = \sum_{i=1}^n -\log X_i = \sum_{i=1}^n Y_i \sim \mathit{Gamma}(n,1)$$

Let $S = \prod_{i=1}^n X_i$, then the transformation $y = -\log(u)$. Thus

$$f_U(u) = f_Y(y(u))|y'(u)| = \frac{(-\log y)^{n-1}\mathrm{e}^{\log y}}{\Gamma(n)}\frac{1}{y} = \frac{(-\log y)^{n-1}}{\Gamma(n)}, \quad y \in (0,1)$$

4. (a) When $z < 0$, we have

$$\begin{aligned} P(Z \leq z) &= P(Z \leq z \cap XY > 0) + P(Z \leq z \cap XY < 0) \\ &= P(X \leq z \cap XY > 0) + P(-X \leq z \cap XY < 0) \\ &= P(X \leq z, Y < 0) + P(X \geq -z, Y < 0) \\ &= P(X \leq z, Y < 0) + P(X \leq z, Y > 0) \quad (P(X \geq -z, Y < 0) = P(X \leq z, Y > 0)) \\ &= P(X \leq z)(P(Y < 0) + P(Y > 0)) \\ &= F_X(z) \end{aligned}$$

When $z \geq 0$, we can also show that $P(Z > 0) = P(X > 0)$, thus $P(Z \leq z) = P(X \leq z) = F_X(z)$. Hence $Z \sim X \sim N(0,1)$.

(b) When $Y > 0$, $Z = X > 0$ when $X > 0$ and $Z = -X > 0$ when $X < 0$. When $Y < 0$, $Z = X < 0$ when $X < 0$ and $Z = -X < 0$ when $X > 0$. Hence Z and Y always have the same sign. The joint distribution cannot be multi-normal.

5. Let $(X_1, Y_1), (X_2, Y_2)$ be two points hit by bullets. And we know that X_1, X_2, Y_1, Y_2 are iid $N(0,1)$. Then we have $X_1 - X_2 \sim N(0,2)$ and $Y_1 - Y_2 \sim N(0,2)$. Thus $(\frac{X_1 - X_2}{\sqrt{2}})^2 + (\frac{Y_1 - Y_2}{\sqrt{2}})^2 \sim \chi_2^2$. The distance between this two points are $S = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$. Let $U = \frac{1}{2}(X_1 - X_2)^2 + \frac{1}{2}(Y_1 - Y_2)^2$, then $S = \sqrt{2U} \Rightarrow U = S^2/2$. $f_U(u) = \frac{1}{2}\mathrm{e}^{-u/2}, \quad u > 0$, then

$$f_S(s) = f_U(s^2/2)|s| = \frac{s}{2}\mathrm{e}^{-s^2/4}, \quad s > 0.$$

6. X_1, X_2, X_3 are iid $\mathit{Exponential}(\lambda)$. $X_{(3)} = \max X_1, X_2, X_3$, then by formula for the density of order statistics

$$f_{X_{(3)}}(x) = 3f_{X_1}(x)(F_{X-1}(x))^2 = \frac{3}{\lambda}\mathrm{e}^{-x/\lambda}(1 - \mathrm{e}^{-x/\lambda})^2,\ x > 0$$

7. (a) Define the transformation

$$\begin{cases} u = \frac{x}{x+y} \\ v = x \end{cases}$$

then

$$\begin{cases} x = v \\ y = \frac{v}{u} - v \end{cases}$$

Then $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u} - 1 \end{bmatrix}$ and $|J| = \left|\frac{v}{u^2}\right| = \frac{|v|}{u^2}$.

Hence

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v))|J| = \frac{1}{2\pi}\mathrm{e}^{-\frac{v^2}{2}}\mathrm{e}^{\frac{-v^2(\frac{1}{u}-1)^2}{2}}\frac{|v|}{u^2}$$

The marginal density of U,

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{-\frac{v^2}{2}}\mathrm{e}^{\frac{-v^2(\frac{1}{u}-1)^2}{2}}\frac{|v|}{u^2} \\
&= \frac{2}{2\pi u^2}\int_0^{\infty} \mathrm{e}^{\frac{(1/u-1)^2}{2}+\frac{1}{2}v^2} v\mathrm{d}v \\
&= \frac{1}{2\pi u^2}\int_0^{\infty} \mathrm{e}^{\frac{(1/u-1)^2}{2}+\frac{1}{2}t}\mathrm{d}t \\
&= \frac{1}{2\pi u^2}\frac{2}{1+(1/u-1)^2} \\
&= \frac{1}{\pi}\frac{1}{2u^2-2u+1} \\
&= \frac{1}{\pi\frac{1}{2}}\frac{1}{((u-\frac{1}{2})/\frac{1}{2})^2+1}
\end{aligned}$$

Hence $\frac{X}{X+y} = U \sim Cauchy(x_0 = \frac{1}{2}, \gamma = \frac{1}{2})$

(b) Define the transformation

$$\begin{cases} u = \frac{x}{|y|} \\ v = x \end{cases}$$

Then

$$\begin{cases} x_1 = v \\ y_1 = \frac{v}{u} \end{cases}$$

and

$$\begin{cases} x_2 = v \\ y_2 = -\frac{v}{u} \end{cases}$$

Thus $J_1 = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{bmatrix}$ and $J_2 = \begin{bmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{bmatrix}$. Thus we have $|J_1| = |J_2| = \frac{|v|}{u^2}$.

The joint density of (U,V), notice that U and V have the same sign, thus the support of $(U,V) \in (0,\infty)\times(0,\infty) \cup (-\infty,0)\times(-\infty,0)$.

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(v, \frac{v}{u})\frac{|v|}{u^2} + f_{X,Y}(v, -\frac{v}{u})\frac{|v|}{u^2} \\
&= \frac{1}{\pi}\mathrm{e}^{-v^2/2}\mathrm{e}^{-(\frac{v}{u})^2/2}\frac{|v|}{u^2} \\
&= \frac{1}{\pi u^2}\mathrm{e}^{-v^2(\frac{1}{2}+\frac{1}{2u^2})}\frac{|v|}{u^2}
\end{aligned}$$

When $u > 0$,

$$\begin{aligned} f_U(u) &= \int_0^\infty \frac{1}{\pi u^2} \mathrm{e}^{-v^2(\frac{1}{2}+\frac{1}{2u^2})} \frac{v}{u^2} \mathrm{d}v \\ &= \int_0^\infty \frac{1}{2\pi u^2} \mathrm{e}^{-t(\frac{1}{2}+\frac{1}{2u^2})} \mathrm{d}t \\ &= \frac{1}{2\pi u^2} \frac{1}{\frac{1}{2}+\frac{1}{2u^2}} \\ &= \frac{1}{\pi} \frac{1}{1+u^2} \end{aligned}$$

When $u < 0$,

$$\begin{aligned} f_U(u) &= \int_{-\infty}^0 \frac{1}{\pi u^2} \mathrm{e}^{-v^2(\frac{1}{2}+\frac{1}{2u^2})} \frac{-v}{u^2} \mathrm{d}v \\ &= \int_0^\infty \frac{1}{2\pi u^2} \mathrm{e}^{-t(\frac{1}{2}+\frac{1}{2u^2})} \mathrm{d}t \\ &= \frac{1}{2\pi u^2} \frac{1}{\frac{1}{2}+\frac{1}{2u^2}} \\ &= \frac{1}{\pi} \frac{1}{1+u^2} \end{aligned}$$

Hence $\frac{X}{|Y|} = U \sim Cauchy(x_0 = 0, \gamma = 1)$.

(c) A normally distributed r.v. divided by another normally distributed r.v. will be a Cauchy distribution r.v.

8. Let $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ Thus $\det \Sigma = 1 - \rho^2$. And $\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$. The distribution of X, Y with covariance matrix Σ and mean $\mu = (0,0)^T$ is like the density in the problem.

Thus $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{VarX}\sqrt{VarY}} = \frac{\rho}{1 \cdot 1} = \rho$.