STAT 601 Exam 1

Yifan Zhu

April 2, 2018

For each of the 4 models, we calculated the marginal expectation, marginal variance and autocorrelation. Here we use autocorrelation to examine the dependence of variables and see if it is decreasing as a function of time separation.

1. Model 1.

   For Model 1, we have

   $$\mu(t) = \lambda + \gamma[\mu(t-1) - \lambda]$$
   $$W(t) = \epsilon_t$$

   and $\mu(0) \sim N(\lambda, \tau^2)$, $\epsilon_t \sim N(0, \sigma^2)$. Let $\alpha(t) = \mu(t) - \lambda$, then we have

   $$\alpha(t) = \gamma\alpha(t-1)$$

   and $\alpha(0) \sim N(0, \tau^2)$. Then

   $$\begin{aligned}
   \mathrm{E}(Y(t)) &= \mathrm{E}(\mu(t) + W(t)) \\
   &= \mathrm{E}(\mu(t)) = \mathrm{E}(\alpha(t) + \lambda) \\
   &= \lambda + \mathrm{E}(\alpha(t)) \\
   &= \lambda + \mathrm{E}(\gamma\alpha(t-1)) \\
   &= \lambda + \gamma\,\mathrm{E}(\alpha(t-1)) \\
   &= \cdots \\
   &= \lambda + \gamma^t\,\mathrm{E}(\alpha(0)) \\
   \lambda + \gamma^t \cdot 0 &= \lambda
   \end{aligned}$$

   Thus $Y(t)$ has constant marginal expectation. For variance,

   $$\mathrm{Var}(Y(t)) = \mathrm{Var}(\mu(t) + W(t)) = \mathrm{Var}(\mu(t)) + \mathrm{Var}(W(t)) = \mathrm{Var}(\alpha(t)) + \sigma^2$$

   We have

   $$\alpha(t)\alpha(t) = \gamma\alpha(t)\alpha(t-1)$$
   $$\alpha(t)\alpha(t-1) = \gamma\alpha(t-1)\alpha(t-1)$$

   Taking expectation on both sides, since $\mathrm{E}(\alpha(t)) = 0$, we have

   $$\mathrm{Var}(\alpha(t)) = \gamma\,\mathrm{Cov}(\alpha(t), \alpha(t-1))$$
   $$\mathrm{Cov}(\alpha(t), \alpha(t-1)) = \gamma\,\mathrm{Var}(\alpha(t-1))$$

   Thus

   $$\mathrm{Var}(\alpha(t)) = \gamma^2\,\mathrm{Var}(\alpha(t-1)) = \cdots = \gamma^{2t}\,\mathrm{Var}(\alpha(0)) = \gamma^{2t}\tau^2$$

Hence
$$\text{Var}(Y(t)) = \gamma^{2t}\tau^2 + \sigma^2$$

Next we calculated the autocorrelation.

$$\text{Cov}(Y(t), Y(t-k)) = \text{Cov}(\mu(t)+W(t), \mu(t-k)+W(t-k)) = \text{Cov}(\mu(t), \mu(t-k)), \ k = 1, 2, \ldots, t$$

We also have

$$\text{Cov}(\mu(t), \mu(t-k)) = \text{Cov}(\alpha(t) + \lambda, \alpha(t-k) + \lambda) = \text{Cov}(\alpha(t), \alpha(t-k))$$

Then

$$\alpha(t)\alpha(t-k) = \gamma\alpha(t-1)\alpha(t-k)$$

Taking expectation on both sides,

$$\begin{aligned}
\text{Cov}(\alpha(t), \alpha(t-k)) &= \gamma \, \text{Cov}(\alpha(t-1), \alpha(t-k)) \\
&= \gamma^2 \, \text{Cov}(\alpha(t-2), \alpha(t-k)) \\
&= \cdots \\
&= \gamma^k \, \text{Var}(\alpha(t-k), \alpha(t-k)) \\
&= \gamma^k \gamma^{2(t-k)}\tau^2 \\
&= \gamma^{2t-k}\tau^2
\end{aligned}$$

Hence the autocorrelation is

$$\begin{aligned}
\text{Corr}(Y(t), Y(t-k)) &= \frac{\text{Cov}(Y(t), Y(t-k))}{\sqrt{\text{Var}(Y(t))}\sqrt{\text{Var}(Y(t-k))}} \\
&= \frac{\gamma^{2t-k}\tau^2}{\sqrt{\gamma^{2t}\tau^2 + \sigma^2}\sqrt{\gamma^{2(t-k)}\tau^2 + \sigma^2}} \\
&= \left(1 + \frac{\sigma^2}{\gamma^{2t}\tau^2}\right)^{-1/2} \left(1 + \frac{\sigma^2}{\gamma^{2(t-k)}\tau^2}\right)^{-1/2}
\end{aligned}$$

Alternatively, let $t = t_2$, $0 \le t - k = t_1 < t_2$, then

$$\text{Corr}(Y(t_2), Y(t_1)) = \left(1 + \frac{\sigma^2}{\gamma^{2t_2}\tau^2}\right)^{-1/2} \left(1 + \frac{\sigma^2}{\gamma^{2t_1}\tau^2}\right)^{-1/2}$$

Here we can see autocorrelation goes down as t_1 and t_2 goes far away from 0. However, the autocorrelation does not necessarily decrease as the separation increases. (When t_1 is fixed and t_2 goes up, the separation goes up and the autocorrelation goes down; when t_2 is fixed and t_1 goes down, the separation goes up but the autocorrelation goes up.)

2. Model 2.

   For Model 2, we have

   $$\begin{aligned}
   \mu(t) &= \lambda + \gamma[\mu(t-1) - \lambda] + v_t \\
   W(t) &= \epsilon_t
   \end{aligned}$$

   and $\mu(0) \sim N(\lambda, \tau^2/(1 - \gamma^2))$, $\epsilon_t \sim N(0, \sigma^2)$, $v_t \sim N(0, \tau^2)$. Let $\alpha(t) = \mu(t) - \lambda$, then we have

   $$\alpha(t) = \gamma\alpha(t-1) + v_t$$

and $\alpha(0) \sim N(0, \tau^2/(1 - \gamma^2))$. Then

$$
\begin{aligned}
\mathrm{E}(Y(t)) &= \mathrm{E}(\mu(t) + W(t)) \\
&= \mathrm{E}(\mu(t)) \\
&= \mathrm{E}(\alpha(t) + \lambda) \\
&= \lambda + \mathrm{E}(\alpha(t)) \\
&= \lambda + \mathrm{E}(\gamma\alpha(t-1) + v_t) \\
&= \lambda + \gamma\,\mathrm{E}(\alpha(t-1)) \\
&= \cdots \\
&= \lambda + \gamma^t\,\mathrm{E}(\alpha(0)) \\
&= \lambda
\end{aligned}
$$

Thus $Y(t)$ has constant marginal expectation.

For variance, like Model 1 we have

$$
\mathrm{Var}(Y(t)) = \mathrm{Var}(\alpha(t)) + \sigma^2
$$

And

$$
\mathrm{E}(v_t\alpha(t)) = \mathrm{E}(v_t \cdot (\gamma\alpha(t-1) + v_t)) = \mathrm{Var}(v_t) = \tau^2
$$

We have

$$
\begin{aligned}
\alpha(t)\alpha(t) &= \gamma\alpha(t)\alpha(t-1) + v_t\alpha(t) \\
\alpha(t)\alpha(t-1) &= (\gamma\alpha(t-1) + v_t)\alpha(t-1) = \gamma\alpha(t-1)\alpha(t-1) + v_t\alpha(t-1)
\end{aligned}
$$

Taking expectation on both sides, we have

$$
\begin{aligned}
\mathrm{Var}(\alpha(t)) &= \gamma\,\mathrm{Cov}(\alpha(t), \alpha(t-1)) + \tau^2 \\
\mathrm{Cov}(\alpha(t), \alpha(t-1)) &= \gamma\,\mathrm{Var}(\alpha(t-1))
\end{aligned}
$$

Hence

$$
\mathrm{Var}(\alpha(t)) = \gamma^2\,\mathrm{Var}(\alpha(t-1)) + \tau^2 = T\left(\mathrm{Var}(\alpha(t-1))\right)
$$

where $T(x) = \gamma^2 x + \tau^2$. Since $\tau^2/(1-\gamma^2)$ is a fixed point of T. $(T\left(\tau^2/(1 - \gamma^2)\right) = \tau^2/(1-\gamma^2))$. And $\mathrm{Var}(\alpha(0)) = \tau^2/(1 - \gamma^2)$. Thus

$$
\mathrm{Var}(\alpha(t)) = \mathrm{Var}(\alpha(0)) = \frac{\tau^2}{1 - \gamma^2}
$$

And thus

$$
\mathrm{Var}(Y(t)) = \frac{\tau^2}{1 - \gamma^2} + \sigma^2
$$

The variance is constant.

For autocorrelation, like in Model 1, we have the autocovariance

$$
\mathrm{Cov}(Y(t), Y(t-k)) = \mathrm{Cov}(\mu(t), \mu(t-k)) = \mathrm{Cov}(\alpha(t), \alpha(t-k))
$$

Then

$$
\alpha(t)\alpha(t-k) = (\gamma\alpha(t-1) + v_t)\alpha(t-k) = \gamma\alpha(t-1)\alpha(t-k) + v_t\alpha(t-k)
$$

Taking expectation on both sides,

$$
\begin{aligned}
\mathrm{Cov}(\alpha(t), \alpha(t-k)) &= \gamma\,\mathrm{Cov}(\alpha(t-1), \alpha(t-k)) \\
&= \gamma^2\,\mathrm{Cov}(\alpha(t-2), \alpha(t-k)) \\
&= \cdots \\
&= \gamma^k\,\mathrm{Var}(\alpha(t-k)) \\
&= \gamma^k \frac{\tau^2}{1-\gamma^2}
\end{aligned}
$$

Then the autocorrelation is

$$
\mathrm{Corr}(Y(t), Y(t-k)) = \frac{\mathrm{Cov}(Y(t), Y(t-k))}{\sqrt{\mathrm{Var}(Y(t))}\sqrt{\mathrm{Var}(\alpha(t-k))}} = \frac{\gamma^k \frac{\tau^2}{1-\gamma^2}}{\frac{\tau^2}{1-\gamma^2} + \sigma^2} = \gamma^k \frac{\tau^2}{\tau^2 + \sigma^2(1-\gamma^2)}
$$

Alternatively, let $t = t_2$, $t - k = t_1$, we can write

$$
\mathrm{Corr}(Y(t_2), Y(t_1)) = \gamma^{t_2 - t_1} \frac{\tau^2}{\tau^2 + \sigma^2(1-\gamma^2)}
$$

As the separation increases, the autocorrelation decreases. So the for Model 2 we can say the dependence decreases as the time separation increases.

3. Model 3.

For Model 3, we have

$$
\mu(t) = \mu(t-1) + v_t
$$
$$
W(t) = \epsilon_t
$$

and $\mu(0) \sim N(\lambda, \tau^2), v_t \sim N(0, \tau^2), \epsilon_t \sim N(0, \sigma^2)$. Let $\alpha(t) = \mu - \lambda$, then

$$
\alpha(t) = \alpha(t-1) + v_t
$$

and $\alpha(0) \sim N(0, \tau^2)$. Then

$$
\begin{aligned}
\mathrm{E}(Y(t)) &= \mathrm{E}(\mu(t) + W(t)) \\
&= \mathrm{E}(\mu(t)) \\
&= \mathrm{E}(\alpha(t) + \lambda) \\
&= \lambda + \mathrm{E}(\alpha(t)) \\
&= \lambda + \mathrm{E}(\alpha(t-1) + v_t) \\
&= \lambda + \mathrm{E}(\alpha(t-1)) \\
&= \cdots \\
&= \lambda + \mathrm{E}(\alpha(0)) \\
&= \lambda
\end{aligned}
$$

Thus $Y(t)$ has constant marginal expectation.

For variance, like previous models, we have

$$
\mathrm{Var}(Y(t)) = \mathrm{Var}(\alpha(t)) + \sigma^2
$$

And

$$
\mathrm{E}(v_t \alpha(t)) = \mathrm{E}(v_t(\alpha(t-1+v_t))) = \mathrm{Var}(v_t) = \tau^2
$$

4

We have

$$\alpha(t)\alpha(t) = \alpha(t)\alpha(t-1) + v_t\alpha(t)$$
$$\alpha(t)\alpha(t-1) = \alpha(t-1)\alpha(t-1) + v_t\alpha(t-1)$$

Taking expectation on both sides, we have

$$\text{Var}(\alpha(t)) = \text{Cov}(\alpha(t), \alpha(t-1)) + \tau^2$$
$$\text{Cov}(\alpha(t), \alpha(t-1)) = \text{Var}(\alpha(t-1))$$

Thus

$$\text{Var}(\alpha(t)) = \text{Var}(\alpha(t-1)) + \tau^2 = \text{Var}(\alpha(t-2)) + 2\tau^2 = \cdots = \text{Var}(\alpha(0)) + t\tau^2 = (t+1)\tau^2$$

and

$$\text{Var}(Y(t)) = (t+1)\tau^2 + \sigma^2$$

Then we calculate the autocorrelation. Like previous models, we have

$$\text{Cov}(Y(t), Y(t-k)) = \text{Cov}(\alpha(t), \alpha(t-k)), \ k = 1, 2, \ldots, k$$

Also we have

$$\alpha(t)\alpha(t-k) = \alpha(t-1)\alpha(t-k) + v_t\alpha(t-k)$$

Then

$$\text{Cov}(\alpha(t), \alpha(t-k)) = \text{Cov}(\alpha(t-1), \alpha(t-k)) = \cdots = \text{Cov}(\alpha(t-k), \alpha(t-k)) = \text{Var}(\alpha(t-k)) = (t-k+1)\tau^2$$

Then the autocorrelation is

$$
\begin{aligned}
\text{Corr}(Y(t), Y(t-k)) &= \frac{\text{Cov}(Y(t), Y(t-k))}{\sqrt{\text{Var}(Y(t))}\sqrt{\text{Var}(Y(t-k))}} \\
&= \frac{(t-k+1)\tau^2}{\sqrt{(t+1)\tau^2 + \sigma^2}\sqrt{(t-k+1)\tau^2 + \sigma^2}} \\
&= \left((t+1)\tau^2 + \sigma^2\right)^{-1}\left(\frac{1}{(t-k+1)\tau^2} + \frac{\sigma^2}{(t-k+1)^2\tau^4}\right)^{-1/2}
\end{aligned}
$$

Alternatively, let $t = t_2$, $t - k = t_1$, $0 \le t_1 < t_2$, then

$$\text{Corr}(Y(t_2), Y(t_1)) = \left((t_2+1)\tau^2 + \sigma^2\right)^{-1}\left(\frac{1}{(t_1+1)\tau^2} + \frac{\sigma^2}{(t_1+1)^2\tau^4}\right)^{-1/2}$$

Here we can see autocorrelation for Model 3 increase as t_1 increases and decreases as t_2 increases. So when we fix t_1, as the separation goes up, t_2 goes up and the autocorrelation goes down. When we fix t_2, as the separation increases, t_2 decreases and then autocorrelation goes down. Hence for Model 3 we can say as time separation increases, the dependence decreases.

4. Model 4.

For Model 4, we have

$$\mu(t) = \mu$$
$$W(t) = \gamma W(t-1) + \epsilon_t$$

and $W(0) \sim N(0, \sigma^2/(1 - \gamma^2))$, $\epsilon_t \sim N(0, \sigma^2)$. We have

$$\text{E}(W(t)) = \text{E}(\gamma W(t-1) + \epsilon_t) = \gamma\,\text{E}(W(t-1)) = \cdots = \gamma^t\,\text{E}(W(0)) = 0$$

Thus
$$\mathrm{E}(Y(t)) = \mathrm{E}(\mu(t) + W(t)) = \mathrm{E}(\mu(t)) + \mathrm{E}(W(t)) = \mu$$

Thus $Y(t)$ has constant marginal expectation.

For the marginal variance,
$$\mathrm{Var}(Y(t)) = \mathrm{Var}(\mu + W(t)) = \mathrm{Var}(W(t))$$

We have
$$\mathrm{E}(W(t)\epsilon_t) = \mathrm{E}(\gamma W(t-1)\epsilon_t + \epsilon_t\epsilon_t) = \mathrm{Var}(\epsilon_t) = \sigma^2$$

And
$$W(t)W(t) = \gamma W(t-1)W(t) + \epsilon_t W(t)$$
$$W(t)W(t-1) = \gamma W(t-1)W(t-1) + \epsilon_t W(t-1)$$

Taking expectation on both sides, we have
$$\mathrm{Var}(W(t)) = \gamma \, \mathrm{Cov}(W(t-1), W(t)) + \sigma^2$$
$$\mathrm{Cov}(W(t), W(t-1)) = \gamma \, \mathrm{Var}(W(t-1))$$

Thus
$$\mathrm{Var}(W(t)) = \gamma^2 \, \mathrm{Var}(W(t-1)) + \sigma^2 = T(\mathrm{Var}(W(t-1)))$$

where $T(x) = \gamma^2 x + \sigma^2$, and $\sigma^2/(1-\gamma^2)$ is fixed point of T. So since $\mathrm{Var}(W(0)) = \sigma^2/(1-\gamma^2)$, then
$$\mathrm{Var}(W(t)) = \mathrm{Var}(W(0)) = \sigma^2/(1-\gamma^2)$$

The marginal variance is constant.

Next we calculate the autocorrelation. We have
$$W(t)W(t-k) = \gamma W(t-1)W(t-k) + \epsilon_t W(t-k)$$

Then taking expectation on both sides,
$$\begin{aligned}
\mathrm{Cov}(Y(t), Y(t-k)) &= \mathrm{Cov}(W(t), W(t-k)) \\
&= \gamma \, \mathrm{Cov}(W(t-1), W(t-k)) \\
&= \cdots \\
&= \gamma^k \, \mathrm{Cov}(W(t-k), W(t-k)) \\
&= \gamma^k \, \mathrm{Var}(W(t-k)) \\
&= \gamma^k \frac{\sigma^2}{1-\gamma^2}
\end{aligned}$$

And then
$$\mathrm{Corr}(Y(t), Y(t-k)) = \frac{\mathrm{Cov}(Y(t), Y(t-k))}{\sqrt{\mathrm{Var}(Y(t))}\sqrt{\mathrm{Var}(Y(t-k))}} = \frac{\gamma^k \frac{\sigma^2}{1-\gamma^2}}{\frac{\sigma^2}{1-\gamma^2}} = \gamma^k$$

Alternatively, let $t = t_2$, $t - k = t_1$, $0 \leq t_1 < t_2$, then
$$\mathrm{Corr}(Y(t_2), Y(t_1)) = \gamma^{t_2-t_1}$$

Here we can see autocorrelation decreases as time separation increases. So We can say the dependence decreases as time separation increases.

Now we have the marginal expectation and variance of $Y(t)$, autocovariance and autocorrelation of these 4 models. These 4 models all have constant marginal expectation and thus are all first order stationary. But from the autocovariance we can see only model 2 and model 4 are second order stationary. Also, model 1 and model 3 does not have constant marginal variance. For model 1, the marginal variance will shrink to a constant in a exponential manner as time goes large; and for model 3, the marginal variance will increase linearly as time goes large. Model 2 and model 4 both have constant marginal variance. Since $Y(t)$ is from a normal distribution, with proper parameters, model 2 and model 4 can produce the same first and second moment for $Y(t)$, and thus the same distribution because of normality. With those parameters, the autocovariances are also going to same for these two models, which shrinks exponentially as time separation increases. So marginality model 2 and model 4 can be taken as the same model. For model 1 and 3, the autocovariacne behaves in a different way than model 2 and model 4.

For each model, we simulated 6 series and we also computed the sample autocovariance function. Then we visualize the simulation to illustrate what we have identified for these models.

In Figure 1, we can see model 2 and model 4 behave pretty much the same and $Y(t)$ oscillate around the mean with constant variance. The variance of model 1 also shrinks very quickly and then $Y(t)$ oscillate around the mean. And in model 3 we can see the variance is getting larger as time goes larger.
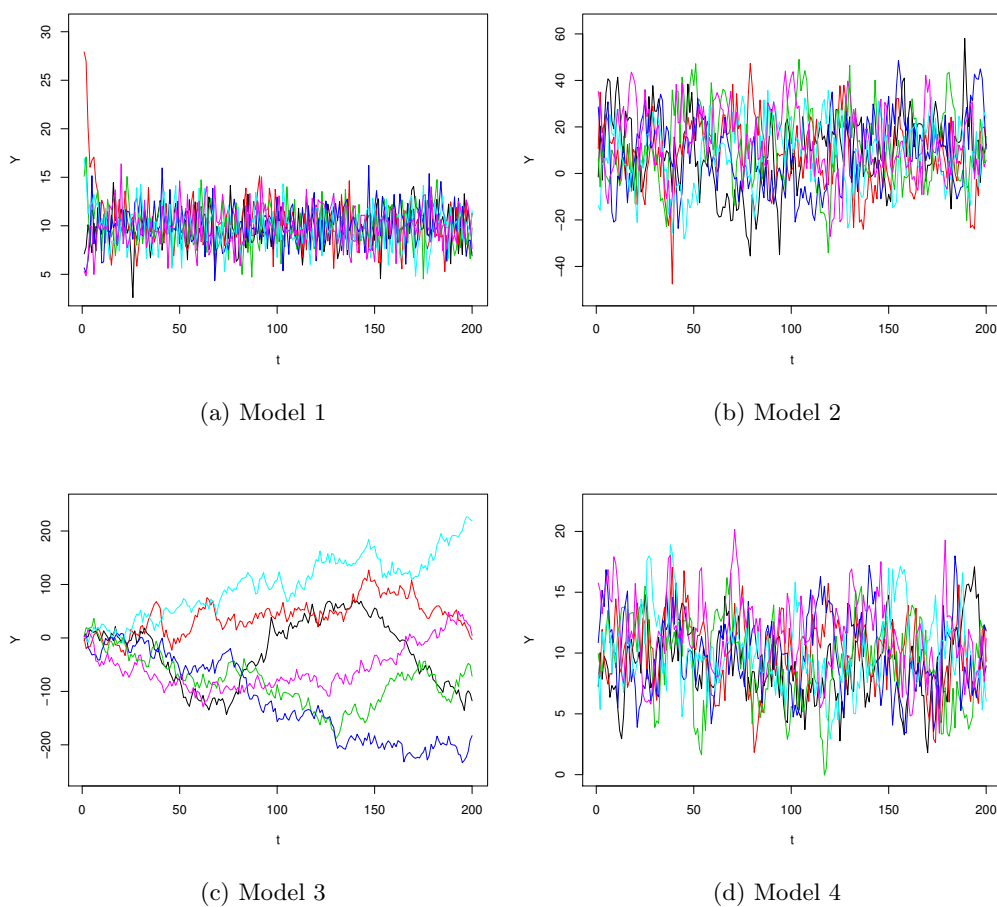

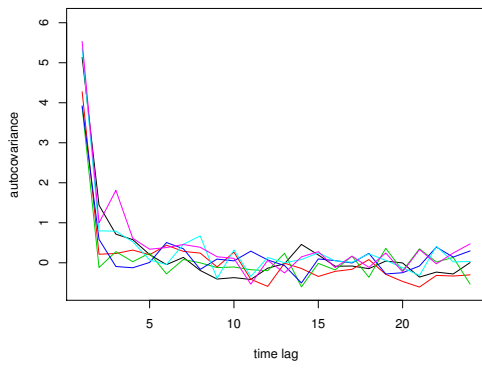(a) Model 1   (b) Model 2

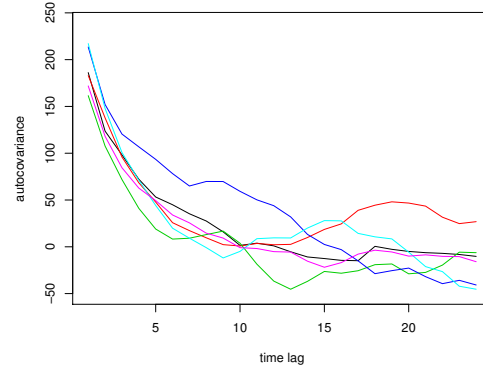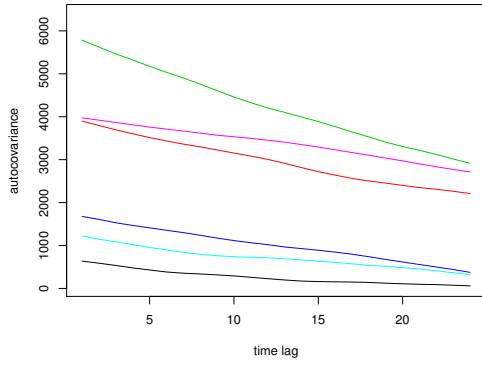(c) Model 3   (d) Model 4

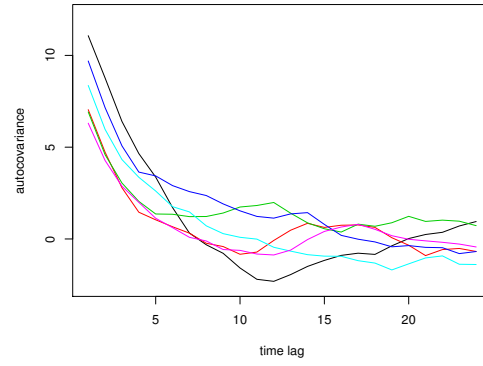Figure 1: $Y(t)$ for 4 models

(a) Model 1

(b) Model 2

(c) Model 3

(d) Model 4

Figure 2: Sample autocovariance function for 4 models

In Figure 2 we can see the sample autocovariance also behave pretty much the same for model 2 and model 4 and decreases exponentially. For model 3 the sample autocovariance seems to be decreasing linearly; and for model 1 the sample autocovariance keeps going up and down.