STAT 510 Homework 10

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1. $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{b} + \boldsymbol{\epsilon}$, here we have

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, Var(\boldsymbol{\epsilon}) = \boldsymbol{\Sigma} = \sigma^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

Then we have $rank(\boldsymbol{X}) = 2, n = 3, n - rank(\boldsymbol{X}) = 1$. Let $\boldsymbol{A}^T = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, then we have

$$w = \boldsymbol{A}^T \boldsymbol{y} = y_1 - y_2,\ E(w) = 0, Var(w) = \boldsymbol{A}^T \boldsymbol{\Sigma} \boldsymbol{A} = \sigma^2$$

Then the log likelihood is

$$\ell(\boldsymbol{\sigma}^2) = \log f(w|\sigma^2) = \log \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{w^2}{2\sigma^2}} = -\frac{w^2}{2\sigma^2} - \log \sigma - \frac{1}{2}\log(2\pi)$$

$$\frac{\partial}{\partial \sigma^2}\ell(\sigma^2) = 0 \Rightarrow \frac{w^2 - \sigma^2}{2\sigma^4} = 0 \Rightarrow \sigma^2 = w^2$$

We also have

$$\left.\frac{\partial^2}{\partial(\sigma^2)^2}\ell(\sigma^2)\right|_{\sigma^2 = w^2} = -\frac{1}{w^4} < 0$$

Hence $\sigma^2 = w^2 = (y_1 - y_2)^2$ is MLE and is REML.

2. (a)

$$L(\lambda) = \prod_{i=1}^n f(y_i|\lambda) = \left(\prod_{i=1}^n \frac{1}{y_i!}\right) \exp(-n\lambda)\lambda^{\sum_{i=1}^n y_i}$$

(b)

$$\begin{aligned} \ell(\lambda) &= \log \prod_{i=1}^n f(y_i|\lambda) \\ &= \sum_{i=1}^n \log f(y_i|\lambda) \\ &= \log \lambda \sum_{i=1}^n y_i - n\lambda - \sum_{i=1}^n y_i! \end{aligned}$$

Thus score eqation is

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \sum_{i=1}^n y_i - n = 0$$

(c)

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = 0 \Rightarrow \lambda = \frac{\sum_{i=1}^n y_i}{n}$$

(d)

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^n y_i}{\lambda^2}$$

Thus

$$\left.\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right|_{\lambda = \frac{\sum_{i=1}^n y_i}{n}} = -\frac{1}{\sum_{i=1}^n y_i} < 0$$

(e) $y_i \overset{\text{iid}}{\sim} \text{Poisson}(\lambda) \Rightarrow \sum_{i=1}^n y_i \sim \text{Poisson(n}\lambda)$

$$I(\lambda) = -E\left[\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right] = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

(f)

$$I(\lambda)^{-1} = \frac{\lambda}{n}$$

(g)

$$\begin{aligned} I(\lambda)^{-1} &= \frac{\lambda}{n} \\ Var(\hat{\lambda}) &= Var\left(\sum_{i=1}^n y_i/n\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^n y_i\right) = \frac{1}{n^2} n\lambda = \frac{\lambda}{n} \end{aligned}$$

They are equal.

(h)

$$\hat{Var}(\hat{\lambda}) = \frac{\hat{\lambda}}{n} = \frac{\sum_{i=1}^n y_i}{n^2}$$

3. (a)

$$\begin{aligned} \ell(\boldsymbol{\lambda}) &= \log \prod_{i=1}^2 \prod_{j=1}^7 f(y_{ij}|\boldsymbol{\lambda}) \\ &= \sum_{i=1}^2 \sum_{j=1}^7 \log f(y_{ij}|\boldsymbol{\lambda}) \\ &= \sum_{i=1}^2 \sum_{j=1}^7 (y_{ij} \log \lambda_i - \lambda_i - \log y_{ij}!) \end{aligned}$$

Then we have

$$\frac{\partial \ell(\boldsymbol{\lambda})}{\partial \lambda_i} = \frac{\sum_{j=1}^7 y_{ij}}{\lambda_i} - 7 = 0 \Rightarrow \lambda_i = \frac{\sum_{j=1}^7 y_{ij}}{7}$$

Thue $\hat{\boldsymbol{\lambda}} = \left(\frac{\sum_{j=1}^7 y_{1j}}{7}, \frac{\sum_{j=1}^7 y_{2j}}{7}\right)$.

Also

$$\left.\frac{\partial^2 \ell(\boldsymbol{\lambda})}{\partial \lambda_i}\right|_{\hat{\boldsymbol{\lambda}}} = -\frac{1}{\hat{\lambda_i}} < 0, \left.\frac{\partial^2 \ell(\boldsymbol{\lambda})}{\partial \lambda_1 \partial \lambda_2}\right|_{\hat{\boldsymbol{\lambda}}} = 0$$

Thus $H = \begin{bmatrix} -\frac{1}{\hat{\lambda}_1} & 0 \\ 0 & -\frac{1}{\hat{\lambda}_2} \end{bmatrix}$ is non-positive definite. Hence $\hat{\boldsymbol{\lambda}} = \left(\sum_{j=1}^{7} y_{1j}/7, \sum_{j=1}^{7} y_{2j}/7\right)$ is MLE.

Then

$$\ell(\hat{\boldsymbol{\lambda}}) = \sum_{i=1}^{2}\sum_{j=1}^{7}(y_{ij}\log\bar{y}_{i\cdot} - \bar{y}_{i\cdot} - \log y_{ij}!) = -37.10781,\ k = 2, n = 14$$

Hence

$$\text{AIC} = -2\ell(\hat{\boldsymbol{\lambda}}) + 2k = 78.21563$$

(b)

$$\text{BIC} = -2\ell(\hat{\boldsymbol{\lambda}}) + k\log(n) = 79.49374$$

(c) Like wise, we have

$$\ell(\hat{\lambda}) = \sum_{i=1}^{2}\sum_{j=1}^{7}(y_{ij}\log\bar{y}_{\cdot\cdot} - \bar{y}_{\cdot\cdot} - \log y_{ij}!) = -40.30237, k = 1, n = 14$$

Hence

$$\text{AIC} = -2\ell(\hat{\lambda}) + 2k = 82.60474$$

(d)

$$\text{BIC} = -2\ell(\hat{\lambda}) + k\log(n) = 83.2438$$

(e) Model (1).

(f) Model (1).

(g)

$$-2\log\Lambda = 2(\ell(\hat{\boldsymbol{\lambda}}) - \ell(\hat{\lambda})) = 6.389118$$

(h)

$$p = 1 - P(\chi_1^2 \leq 6.389118) = 0.01148221$$

(i)

$$\hat{Var}(\hat{\lambda}_1 - \hat{\lambda}_2) = \hat{Var}\left(\frac{\hat{\lambda}_1 - \hat{\lambda}_2}{7}\right) = \frac{\hat{\lambda}_1 + \hat{\lambda}_2}{7}$$

Then

$$\text{Wald Statistic} = \frac{(\hat{\lambda}_1 - \hat{\lambda}_2)^2}{(\hat{\lambda}_1 + \hat{\lambda}_2)/7} = \frac{7(\bar{y}_{1\cdot} - \bar{y}_{2\cdot})^2}{\bar{y}_{1\cdot} + \bar{y}_{2\cdot}} = 6.351648$$

(j)

$$p = P(\chi_1^2 \geq 6.351648) = 0.01172723$$