STAT 500 Homework 5

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1. (a) Four components in $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{bmatrix}$$

(b) Calculate $\boldsymbol{\beta}$ using $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} Y_{11} + Y_{12} + Y_{13} \\ Y_{21} + Y_{22} + Y_{23} \\ Y_{31} + Y_{32} + Y_{33} \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} Y_{11} + Y_{12} + Y_{13} \\ Y_{21} + Y_{22} + Y_{23} \\ Y_{31} + Y_{32} + Y_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(Y_{11} + Y_{12} + Y_{13}) \\ \frac{1}{3}(Y_{21} + Y_{22} + Y_{23}) \\ \frac{1}{3}(Y_{31} + Y_{32} + Y_{33}) \end{bmatrix} = \begin{bmatrix} \bar{Y}_{1.} \\ \bar{Y}_{2.} \\ \bar{Y}_{3.} \end{bmatrix}$$

(c) Calculate the projection matrix $P_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$.

$$P_{\boldsymbol{X}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

2. (a) Four components in $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{bmatrix}$$

(b) Calculate $\boldsymbol{\beta}$ using $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^3 Y_{1j} + \sum_{j=1}^3 Y_{2j} + \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{1j} \\ \sum_{j=1}^3 Y_{2j} \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \sum_{j=1}^3 Y_{1j} + \sum_{j=1}^3 Y_{2j} + \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{1j} \\ \sum_{j=1}^3 Y_{2j} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}\sum_{j=1}^3 Y_{3j} \\ \frac{1}{3}\sum_{j=1}^3 Y_{1j} - \frac{1}{3}\sum_{j=1}^3 Y_{3j} \\ \frac{1}{3}\sum_{j=1}^3 Y_{2j} - \frac{1}{3}\sum_{j=1}^3 Y_{3j} \end{bmatrix} = \begin{bmatrix} \bar{Y}_{3.} \\ \bar{Y}_{1.} - \bar{Y}_{3.} \\ \bar{Y}_{1.} - \bar{Y}_{3.} \end{bmatrix}$$

(c) Calculate the projection matrix $P_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$.

$$
\begin{aligned}
P_{\boldsymbol{X}} &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & -1/3 & -1/3 & -1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -1/3 & -1/3 & -1/3 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}
\end{aligned}
$$

3. (a) Four components in $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{33} \end{bmatrix}$$

(b) Calculate $\boldsymbol{\beta}$ using $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 2/9 & -1/9 \\ 0 & -1/9 & 2/9 \end{bmatrix}$$

$$\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{31} \\ Y_{32} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^3 Y_{1j} + \sum_{j=1}^3 Y_{2j} + \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{1j} - \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{2j} - \sum_{j=1}^3 Y_{3j} \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 2/9 & -1/9 \\ 0 & -1/9 & 2/9 \end{bmatrix} \begin{bmatrix} \sum_{j=1}^3 Y_{1j} + \sum_{j=1}^3 Y_{2j} + \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{1j} - \sum_{j=1}^3 Y_{3j} \\ \sum_{j=1}^3 Y_{2j} - \sum_{j=1}^3 Y_{3j} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9}\sum_{i=1}^3\sum_{j=1}^3 Y_{ij} \\ \frac{2}{9}\sum_{j=1}^3 Y_{1j} - \frac{1}{9}\sum_{j=1}^3 Y_{2j} - \frac{1}{9}\sum_{j=1}^3 Y_{3j} \\ \frac{2}{9}\sum_{j=1}^3 Y_{2j} - \frac{1}{9}\sum_{j=1}^3 Y_{1j} - \frac{1}{9}\sum_{j=1}^3 Y_{3j} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sum_{i=1}^3 \bar{Y}_{i.} \\ \bar{Y}_{1.} - \frac{1}{3}(\bar{Y}_{2.} + \bar{Y}_{3.}) \\ \bar{Y}_{2.} - \frac{1}{3}(\bar{Y}_{1.} + \bar{Y}_{3.}) \end{bmatrix}$$

(c) Calculate the projection matrix $P_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$.

$$P_{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 2/9 & -1/9 \\ 0 & -1/9 & 2/9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 2/9 & 2/9 & 2/9 & -1/9 & -1/9 & -1/9 & -1/9 & -1/9 & -1/9 \\ -1/9 & -1/9 & -1/9 & 2/9 & 2/9 & 2/9 & -1/9 & -1/9 & -1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

4. In this problem, $P_{\boldsymbol{1}} = \begin{bmatrix} 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \end{bmatrix}$, then

$$(\boldsymbol{I} - P_{\boldsymbol{1}})\boldsymbol{Y} = \boldsymbol{Y} - P_{\boldsymbol{1}}\boldsymbol{Y} = \begin{bmatrix} Y_{11} - \bar{Y}_{..} \\ Y_{12} - \bar{Y}_{..} \\ Y_{13} - \bar{Y}_{..} \\ Y_{21} - \bar{Y}_{..} \\ Y_{22} - \bar{Y}_{..} \\ Y_{23} - \bar{Y}_{..} \\ Y_{31} - \bar{Y}_{..} \\ Y_{32} - \bar{Y}_{..} \\ Y_{33} - \bar{Y}_{..} \end{bmatrix}$$

$\Rightarrow$

$$
\begin{aligned}
SS_{total} &= \boldsymbol{Y}^T(\boldsymbol{I} - P_{\mathbf{1}})\boldsymbol{Y} \\
&= \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{21} & Y_{22} & Y_{23} & Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} Y_{11} - \bar{Y}_{..} \\ Y_{12} - \bar{Y}_{..} \\ Y_{13} - \bar{Y}_{..} \\ Y_{21} - \bar{Y}_{..} \\ Y_{22} - \bar{Y}_{..} \\ Y_{23} - \bar{Y}_{..} \\ Y_{31} - \bar{Y}_{..} \\ Y_{32} - \bar{Y}_{..} \\ Y_{33} - \bar{Y}_{..} \end{bmatrix} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} Y_{ij}(Y_{ij} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{..})(Y_{ij} - \bar{Y}_{..}) + \bar{Y}_{..}(Y_{ij} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{..})^2 + \bar{Y}_{..}\sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{..})^2 + \bar{Y}_{..}(9\bar{Y}_{..} - 9\bar{Y}_{..}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{..})^2
\end{aligned}
$$

$$(P_{\boldsymbol{X}} - P_{\mathbf{1}})\boldsymbol{Y} = P_{\boldsymbol{X}}\boldsymbol{Y} - P_{\mathbf{1}}\boldsymbol{Y} = \begin{bmatrix} \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned}
SS_{model} &= \boldsymbol{Y}^T(\boldsymbol{P_X} - P_{\mathbf{1}})\boldsymbol{Y} \\
&= \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{21} & Y_{22} & Y_{23} & Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{1.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{2.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \\ \bar{Y}_{3.} - \bar{Y}_{..} \end{bmatrix} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} Y_{ij}(\bar{Y}_{i.} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3}(\bar{Y}_{i.} - \bar{Y}_{..})\sum_{j=1}^{3} Y_{ij} \\
&= \sum_{i=1}^{3}(\bar{Y}_{i.} - \bar{Y}_{..}) \cdot 3\bar{Y}_{i.} \\
&= \sum_{i=1}^{3} 3(\bar{Y}_{i.} - \bar{Y}_{..})(\bar{Y}_{i.} - \bar{Y}_{..}) + \bar{Y}_{..}(\bar{Y}_{i.} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3} 3(\bar{Y}_{i.} - \bar{Y}_{..})^2 + \bar{Y}_{..}\sum_{i=1}^{3}(\bar{Y}_{i.} - \bar{Y}_{..}) \\
&= \sum_{i=1}^{3} 3(\bar{Y}_{i.} - \bar{Y}_{..})^2 + \bar{Y}_{..}(3\bar{Y}_{..} - 3\bar{Y}_{..}) \\
&= \sum_{i=1}^{3} 3(\bar{Y}_{i.} - \bar{Y}_{..})^2
\end{aligned}$$

$$(\boldsymbol{I} - P_{\boldsymbol{X}})\boldsymbol{Y} = \boldsymbol{Y} - P_{\boldsymbol{X}}\boldsymbol{Y} = \begin{bmatrix} Y_{11} - \bar{Y}_{1.} \\ Y_{12} - \bar{Y}_{1.} \\ Y_{13} - \bar{Y}_{1.} \\ Y_{21} - \bar{Y}_{2.} \\ Y_{22} - \bar{Y}_{2.} \\ Y_{23} - \bar{Y}_{2.} \\ Y_{31} - \bar{Y}_{3.} \\ Y_{32} - \bar{Y}_{3.} \\ Y_{33} - \bar{Y}_{3.} \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned}
SS_{error} &= \boldsymbol{Y}^T(P_{\boldsymbol{X}} - P_{\boldsymbol{1}})\boldsymbol{Y} \\
&= \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{21} & Y_{22} & Y_{23} & Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} Y_{11} - \bar{Y}_{1.} \\ Y_{12} - \bar{Y}_{1.} \\ Y_{13} - \bar{Y}_{1.} \\ Y_{21} - \bar{Y}_{2.} \\ Y_{22} - \bar{Y}_{2.} \\ Y_{23} - \bar{Y}_{2.} \\ Y_{31} - \bar{Y}_{3.} \\ Y_{32} - \bar{Y}_{3.} \\ Y_{33} - \bar{Y}_{3.} \end{bmatrix} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} Y_{ij}(Y_{ij} - \bar{Y}_{i.}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.})(Y_{ij} - \bar{Y}_{i.}) + \bar{Y}_{i.}(Y_{ij} - \bar{Y}_{i.}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.})^2 + \sum_{i=1}^{3} \bar{Y}_{i.} \sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.})^2 + \sum_{i=1}^{3} \bar{Y}_{i.}(3\bar{Y}_{i.} - 3\bar{Y}_{i.}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (Y_{ij} - \bar{Y}_{i.})^2
\end{aligned}$$