Homework 6

Due February 27, 2020 at 11:59 PM

1. Find mean (expected value), median and variance of $X \sim \text{Exp}(\alpha)$. The pdf of $\text{Exp}(\alpha)$ is given by

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

- (3 points) Mean:

$$\begin{aligned} E(X) &= \int_0^\infty x \cdot \frac{1}{\alpha} e^{-x/\alpha} dx \\ &= \int_0^\infty x (-e^{-x/\alpha})' dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} &= -x e^{-x/\alpha} \Big|_0^\infty + \int_0^\infty e^{-x/\alpha} dx \\ &= 0 + (-\alpha e^{-x/\alpha}) \Big|_0^\infty \\ &= \alpha \end{aligned}$$

- (3 points) Variance:

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 \cdot \frac{1}{\alpha} e^{-x/\alpha} dx \\ &= \int_0^\infty x^2 (-e^{-x/\alpha})' dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} &= -x^2 e^{-x/\alpha} \Big|_0^\infty + \int_0^\infty e^{-x/\alpha} \cdot 2x dx \\ &= 0 + 2\alpha \int_0^\infty \frac{1}{\alpha} e^{-x/\alpha} dx \\ &= 2\alpha^2 \end{aligned}$$

Therefore

$$Var(X) = E(X^2) - (E(X))^2 = 2\alpha^2 - \alpha^2 = \alpha^2$$

- (3 points) Median:

$$\begin{aligned}
&\int_0^{Q(0.5)} \frac{1}{\alpha} e^{-x/\alpha} dx = 0.5 \\
\Rightarrow &(-e^{-x/\alpha})\Big|_0^{Q(0.5)} = 1 - e^{-Q(0.5)/\alpha} = 0.5 \\
\Rightarrow &\ln(0.5) = -Q(0.5)/\alpha \\
\Rightarrow &Q(0.5) = \alpha \ln 2
\end{aligned}$$

Since $\ln 2 < 1$, we can see the median for exponential distribution is always smaller than the mean.

2. P. 263: 5

   (a) (4 points)
   The mean is $\alpha = E(X) = 1000$. The cdf is given by

   $$F(x) = 1 - e^{-x/1000}.$$

   So the probability that a vehicle of this type gives less than 500 miles of service before first failure is

   $$P(X < 500) = F(500) = 1 - e^{-500/1000} = 0.3934.$$

   The probability that it gives at least 2000 miles of serive before first failure is

   $$P(X \geq 2000) = 1 - F(2000) = e^{-2000/1000} = 0.1353$$

   (b) (4 points)
   The q quantile $Q(q)$ satisfies

   $$\int_0^{Q(q)} f(x)dx = F(Q(q)) = 1 - e^{-Q(q)/1000} = q.$$

   Therefore

   $$Q(q) = -1000 \ln(1-q).$$

   Let $q = 0.05$, $Q(0.05) = -1000 \ln(0.95) = 51.29$.
   Let $q = 0.9$, $Q(0.9) = -1000 \ln(0.1) = 2302.58$.

3. P. 329: 31 (3×4 points)

   (a) (3 points)
   $P(X \leq 0.32) = F(0.32) = \sin(0.32) = 0.3146$.

   (b) (3 points)
   $f(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \sin(x) = \cos(x)$ for $0 < x \leq \pi/2$. For $x \leq 0$ or $x > \pi/2$, since the derivative is 0 for a constant, we have $f(x) = \frac{d}{dx} F(x) = 0$.
   Therefore

   $$f(x) = \begin{cases} \cos(x) & , 0 < x \leq \pi/2 \\ 0 & , \text{otherwise} \end{cases}$$

(c) (3 points)

$$\begin{aligned} E(X) &= \int_0^{\pi/2} x\cos(x)dx \\ &= \int_0^{\pi/2} x(\sin(x))'dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} &= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)dx \\ &= \pi/2 - (-\cos(x))\Big|_0^{\pi/2} \\ &= \pi/2 - (0-(-1)) \\ &= \pi/2 - 1 = 0.5708 \end{aligned}$$

(d) (3 points)

$$\begin{aligned} E(X^2) &= \int_0^{\pi/2} x^2\cos(x)dx \\ &= \int_0^{\pi/2} x^2(\sin(x))'dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} &= x^2\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)\cdot 2xdx \\ &= \pi^2/4 - 2\int_0^{\pi/2} x(-\cos(x))'dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} &= \pi^2/4 - 2(-\cos(x)x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos(x)dx) \\ &= \pi^2/4 - 2(0+\sin(x)\Big|_0^{\pi/2}) \\ &= \pi^2/4 - 2 \end{aligned}$$

Therfore

$$Var(X) = E(X^2) - (E(X))^2 = \pi^2/4 - 2 - (\pi/2-1)^2 = \pi - 3 = 0.1416$$

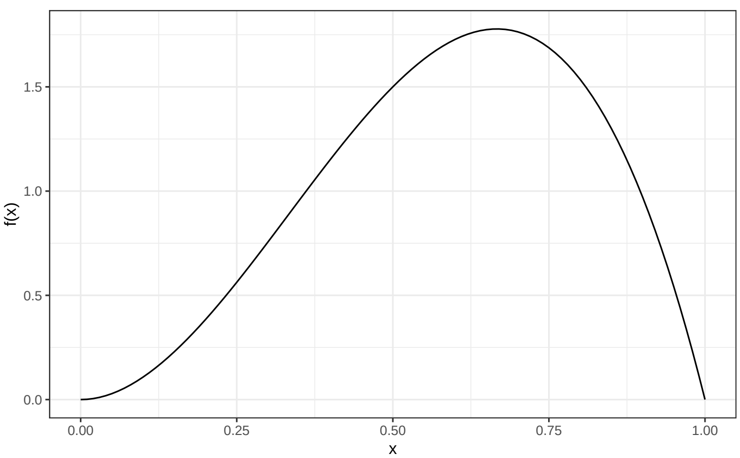
4. P. 332: 41 (6 + 4 + 4 + 7 points)

(a) (6 points)

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 k(x^2(1-x))dx = 1$$
$$\Rightarrow k\int_0^1 (x^2 - x^3)dx = 1$$
$$\Rightarrow k(\frac{1}{3}x^3 - \frac{1}{4}x^4)\big|_0^1 = 1$$
$$\Rightarrow k(\frac{1}{3} - \frac{1}{4}) = k\frac{1}{12} = 1$$
$$\Rightarrow k = 12$$

So the pdf is

$$f(x) = \begin{cases} 12x^2(1-x) & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$



(b) (4 points) $P(X \leq 0.25) = \int_0^{0.25} 12x^2(1-x)dx = (4x^3 - 3x^4)\big|_0^{0.25} = (4(0.25)^3 - 3(0.25)^4) - 0 =$ 0.0508.
$P(X \leq 0.75) = \int_0^{0.25} 12x^2(1-x)dx = (4x^3 - 3x^4)\big|_0^{0.75} = (4(0.75)^3 - 3(0.75)^4) - 0 = 0.7383$
$P(0.25 < X \leq 0.75) = P(X \leq 0.75) - P(X \leq 0.25) = 0.7383 - 0.0508 = 0.6875$
$P(|X - 0.5| > 0.1) = P(X > 0.5 + 1 \text{or} X < 0.5 - 0.1) = P(X > 0.6) + P(X < 0.4) = \int_0^{0.4} f(x)dx + \int_{0.6}^1 f(x)dx = 0.7040$

(c) (4 points)

$$\begin{aligned}
E(X) &= \int_0^1 x \cdot 12x^2(1-x)dx \\
&=(3x^4 - \frac{12}{5}x^5)\Big|_0^1 \\
&=(3 - \frac{12}{5}) - 0 \\
&=\frac{3}{5} = 0.6
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 \cdot 12x^2(1-x)dx \\
&=(\frac{12}{5}x^5 - 2x^6)\Big|_0^1 \\
&=(\frac{12}{5} - 2) - 0 \\
&=\frac{2}{5} = 0.4
\end{aligned}$$

So

$$Var(X) = E(X^2) - (E(X))^2 = 0.4 - 0.6^2 = 0.04 \Rightarrow SD(X) = \sqrt{0.04} = 0.2.$$

(d) (7 points)

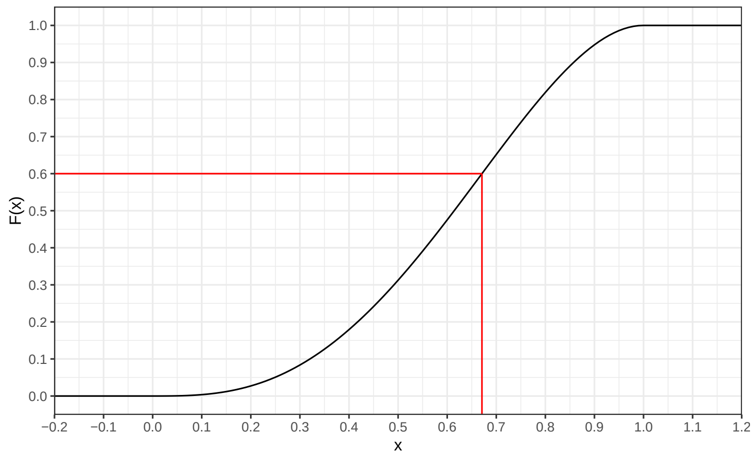
For $x \leq 0$, $F(x) = \int_{-\infty}^{x} 0dt = 0$.

For $x \geq 1$, $F(x) = \int_{-\infty}^{0} 0dt + \int_0^1 12t^2(1-t)dt + \int_1^x 0dt = 1$.

For $0 < x < 1$, $F(x) = \int_0^x 12t^2(1-t)dt = 4x^3 - 3x^4$.

So the cdf is

$$F(x) = \begin{cases} 0 & , x \leq 0 \\ 4x^3 - 3x^4 & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$



From the plot, we find the 0.6 quantile is around 0.6708.