Functions of Several Random Variables (Ch. 5.5)

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Outline

Functions of Several Random Variables

Expectations and variances of linear combinations

Approximating the Mean and Variance of a Function

The Central Limit Theorem

Functions of several random variables

functions of random variables
are still random variables

- We often consider functions of random variables of the form:

$$U = \underline{g(X, Y, \ldots, Z)}$$

where X, $Y, \ldots, Z$ are random variables.

- U is itself a random variable.

Example: connecting steel parts

- Suppose that a steel plate with nominal thickness .15 in. is to rest in a groove of nominal width .155 in., machined on the surface of a steel block.

Relative Frequency Distribution of Plate Thicknesses

Plate Thickness (in.)	Relative Frequency
.148	.4
.149	.3
.150	.3

Relative Frequency Distribution of Slot Widths

Slot Width (in.)	Relative Frequency
.153	.2
.154	.2
.155	.4
.156	.2

marginal dist.

- X = plate thickness
- Y = slot width
- $U = Y - X$, the "wiggle room" of the plate

The distributions of X, Y, and U

$Y-X$

$[.153-.150, .156-.148]$

$[0.03, 0.08]$

Marginal and Joint Probabilities for X and Y

$y \setminus x$	.148	.149	.150	$f_Y(y)$
.156	.08	.06	.06	.2
.155	.16	.12	.12	.4
.154	.08	.06	.06	.2
.153	.08	.06	.06	.2
$f_X(x)$	.4	.3	.3	

The Probability Function for the Clearance $U = Y - X$

u	$f(u)$
.003	.06
.004	$.12 = .06 + .06$
.005	$.26 = .08 + .06 + .12$
.006	$.26 = .08 + .12 + .06$
.007	$.22 = .16 + .06$
.008	.08

- Determining the distribution of U is difficult in the continuous case. $g(x,y) = y - x = u$.

$$f(u) = \sum_{y-x=u} f(x,y) = \boxed{\sum_x f(x, x+u)}$$

intuitively: $f(u) = \int_{-\infty}^{\infty} f(x, x+u)\,dx$

Outline

Expectations and variances of linear combinations

- $X_1, X_2, \ldots, X_n$ are independent random variables and

$$Y = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

then: also true if $X_1, \ldots, X_n$ are not ind.

$$E(Y) = E(a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n)$$
$$= a_0 + a_1 E(X_1) + a_2 E(X_2) + \cdots + a_n E(X_n)$$

is true only when $X_1, \ldots, X_n$ are ind.

$$Var(Y) = Var(\boxed{a_0} + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n)$$
$$= a_1^2 \cdot Var(X_1) + a_2^2 \cdot Var(X_2) + \cdots + a_n^2 \cdot Var(X_n)$$

Your turn: linear combinations

- Say we have two independent random variables X and Y with $E(X) = 3.3$, $Var(X) = 1.91$, $E(Y) = 25$, and $Var(Y) = 65$.
- Find:

$$\begin{aligned} E(3 + 2X - 3Y) \\ E(-4X + 3Y) \\ E(-4X - 6Y) \\ Var(3 + 2X - 3Y) \\ Var(2X - 5Y) \\ Var(-4X - 6Y) \end{aligned}$$

Answers: linear combinations

$$\begin{aligned} E(3+2X-3Y) &= 3+2E(X)-3E(Y) \\ &= 3+2\cdot 3.3-3\cdot 25 \\ &= -65.4 \end{aligned}$$

$$\begin{aligned} E(-4X+3Y) &= -4E(X)+3E(Y) \\ &= -4\cdot 3.3+3\cdot 25 \\ &= 61.8 \end{aligned}$$

$$\begin{aligned} E(-4X-6Y) &= -4\cdot E(X)-6\cdot E(Y) \\ &= -4\cdot 3.3-6\cdot 25 \\ &= -163.2 \end{aligned}$$

Answers: linear combinations

$Var(Y) \geq 0$

$$
\begin{aligned}
Var(3 + 2X - 3Y) &= 2^2 \cdot Var(X) + (-3)^2 Var(Y) \\
&= 4 \cdot 1.91 + 9 \cdot 65 \\
&= 592.64
\end{aligned}
$$

$$
\begin{aligned}
Var(2X - 5Y) &= 2^2 \cdot Var(X) + (-5)^2 Var(Y) \\
&= 4 \cdot 1.91 + 25 \cdot 65 \\
&= 1632.64
\end{aligned}
$$

$$
\begin{aligned}
Var(-4X - 6Y) &= (-4)^2 \cdot Var(X) + (-6)^2 Var(Y) \\
&= 16 \cdot 1.91 + 36 \cdot 65 \\
&= 2370.56
\end{aligned}
$$

Your turn: more linear combinations

- Say $X \sim \text{Binomial}(n = 10, p = 0.5)$ and $Y \sim \text{Poisson}(\lambda = 3)$. independent.
- Calculate:

$$E(5 + 2X - 7Y)$$
$$Var(5 + 2X - 7Y)$$

$E(X) = np = 5.$

$Var(X) = np(1-p) = 2.5$

$E(Y) = 3$

$Var(Y) = 3$

Answer: more linear combinations

- First, note that:

$$E(X) = np = 10 \cdot 0.5 = 5$$
$$E(Y) = \lambda = 3$$
$$Var(X) = np(1-p) = 10(0.5)(1-0.5) = 2.5$$
$$Var(Y) = \lambda = 3$$

Now, we can calculate:

$$\begin{aligned} E(5 + 2X - 7Y) &= 5 + 2E(X) - 7E(Y) \\ &= 5 + 2 \cdot 5 - 7 \cdot 3 \\ &= -6 \end{aligned}$$

$$\begin{aligned} Var(5 + 2X - 7Y) &= 2^2 \cdot Var(X) + (-7)^2 \cdot Var(Y) \\ &= 4 \cdot 2.5 + 49 \cdot 3 \\ &= 157 \end{aligned}$$

Outline

Functions of Several Random Variables

Expectations and variances of linear combinations

Approximating the Mean and Variance of a Function

The Central Limit Theorem

Approximating $E(U)$ and $Var(U)$ when determining $f_U(u)$ is too hard

g has first derivative.

- If $X, Y, \ldots, Z$ are independent, g is well-behaved, and the variances $\text{Var}(X)$, $\text{Var}(Y), \ldots$, $\text{Var}(Z)$ are small enough, then $U = g(X, Y, \ldots Z)$ has:

$$E(U) \approx g(E(X), E(Y), \ldots, E(Z))$$

$$\text{Var}(U) \approx \left(\frac{\partial g}{\partial x}\right)^2 Var(X) + \left(\frac{\partial g}{\partial y}\right)^2 Var(Y) + \cdots + \left(\frac{\partial g}{\partial z}\right)^2 Var(Z)$$

- These formulas are often called the **propagation of error formulas**.

approximate g with a linear function.

Taylor expansion:

$g(x, y, \dots z)$.

expand it around $(E(X), E(Y), \dots, E(Z))$.

$$g(x, y, \dots, z) = g(E(X), E(Y), \dots, E(Z))$$
$$+ \boxed{\left(\frac{\partial g}{\partial x}\right) \cdot (X - E(X))} + \left(\frac{\partial g}{\partial y}\right)(Y - E(Y)) + \dots + \left(\frac{\partial g}{\partial z}\right)(Z - E(Z))$$
$$+ \text{high order terms.} \leftarrow \text{ignored.}$$

$$E(g(X, Y, \dots, Z)) = g(E(X), E(Y), \dots, E(Z))$$
$$+ \underbrace{\left(\frac{\partial g}{\partial x}\right)(E(X) - E(X))}_{=0} + 0 \dots + 0 \dots$$

$$g(x, y, \ldots, z) \approx g(E(x), E(y), \ldots, E(z))$$

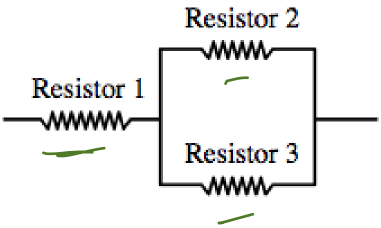
$$+ \left(\frac{\partial g}{\partial x}\right)(x - E(x)) + \left(\frac{\partial g}{\partial y}\right)(y - E(y)) +$$

$$\cdots + \left(\frac{\partial g}{\partial z}\right)(z - Ez) .$$

$$Var(g(x, y, \ldots, z)) = \left(\frac{\partial g}{\partial x}\right)^2 Var(x)$$

$$+ \left(\frac{\partial g}{\partial y}\right)^2 Var(y) + \cdots + \left(\frac{\partial g}{\partial z}\right)^2 Var(z)$$

Example: an electric circuit



- R is the total resistance of the circuit.
- R_1, R_2, and R_3 are the resistances of resistors 1, 2, and 3, respectively. SD
- $E(R_i) = 100$, $\cancel{\mathrm{Var}}(R_i) = 2$, $i = 1, 2, 3$.

$$R = g(R_1, R_2, R_3) = R_1 + \boxed{\frac{R_2 R_3}{R_2 + R_3}}$$

Example: an electric circuit

$$g(r_1, r_2, r_3) = \underline{r_1} + \boxed{\frac{r_2 r_3}{r_2 + r_3}}$$

$$E(R) \approx g(100, 100, 100) = 100 + \frac{(100)(100)}{100 + 100} = 150\Omega$$

$$\frac{\partial g}{\partial r_1} = 1 \quad = \frac{\partial r_1}{\partial r_1} = 1$$

$$\frac{\partial g}{\partial r_2} = \frac{(r_2 + r_3)r_3 - r_2 r_3}{(r_2 + r_3)^2} = \frac{r_3^2}{(r_2 + r_3)^2} \qquad \frac{(r_2 + r_3) r_3 - r_2 r_3}{(r_2 + r_3)^2}$$

$$\frac{\partial g}{\partial r_3} = \frac{(r_2 + r_3)r_2 - r_2 r_3}{(r_2 + r_3)^2} = \frac{r_2^2}{(r_2 + r_3)^2}$$

$$\text{Var(R)} \approx (1)^2(2)^2 + \left(\frac{(100)^2}{(100 + 100)^2}\right)^2 (2)^2 + \left(\frac{(100)^2}{(100 + 100)^2}\right)^2 (2)^2$$

$$= 4.5$$

$$\text{SD}(R) = \sqrt{4.5} \approx 2.12\Omega$$

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Outline

iid random variables.

- **Identically Distributed**: Random variables $X_1, X_2, \ldots, X_n$ are identically distributed if they have the same probability distribution.
- **“iid”**: Random variables $X_1, X_2, \ldots, X_n$ are iid if they are **I**ndependent and **I**dentically **D**istributed.

Averages of iid random variables

- $X_1, X_2, \ldots, X_n$ are iid with expectation μ and variance σ^2.
- Derive:

$$E(\overline{X})$$
$$Var(\overline{X})$$

where:

$$\overline{X} = \underbrace{\frac{X_1 + X_2 + \cdots + X_n}{n}}$$

the mean of the X_i's.

a function ⇒ the mean is a random variable.

Averages of iid random variables

$$
\begin{aligned}
E(\overline{X}) &= E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \cdots + \frac{1}{n}E(X_n) \\
&= \underbrace{\frac{1}{n}\mu + \frac{1}{n}\mu + \cdots + \frac{1}{n}\mu}_{n \text{ times}} \\
&= n \cdot \frac{1}{n}\mu \\
&= \boxed{\mu}
\end{aligned}
$$

- Remember $E(\overline{X}) = \mu$: it's an important result.

Answers: averages of iid random variables

$$
\begin{aligned}
Var(\overline{X}) &= Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \left(\frac{1}{n}\right)^2 Var(X_1) + \left(\frac{1}{n}\right)^2 Var(X_2) + \cdots + \left(\frac{1}{n}\right)^2 \cdot Var(X_n) \\
&= \underbrace{\frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \cdots + \frac{1}{n^2}\sigma^2}_{n \text{ times}} \\
&= n \cdot \frac{1}{n^2}\sigma^2 \\
&= \boxed{\frac{\sigma^2}{n}}
\end{aligned}
$$

$\downarrow$ as $n \uparrow$

- Remember $Var(\overline{X}) = \frac{\sigma^2}{n}$: it's another important result.

That's averaging can improve precision.

Example: length of seeds

- A botanist has collected a sample of 10 seeds and measures the length of each.
- The seed lengths $X_1, X_2, \ldots, X_{10}$ are supposed to be iid with mean $\mu = 5$ mm and variance $\sigma^2 = 2\ mm^2$.

$$E(\overline{X}) = \mu = 5$$
$$Var(\overline{X}) = \sigma^2/n = 2/10 = 0.2$$

The Central Limit Theorem

could be continuous or discrete.

- If $X_1, X_2, \ldots, X_n$ are *any* iid random variables with mean μ and variance $\sigma^2 < \infty$, and if $n \geq 25$,

$$\overline{X} \approx \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$$

- The Central Limit Theorem (CLT) one of the most important and useful results in statistics.

Example: tool serial numbers

- W_1 = last digit of the serial number observed next Monday at 9 AM
- W_2 = last digit of the serial number the Monday after at 9 AM
- W_1 and W_2 are independent with pmf:

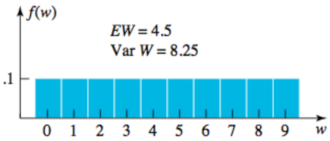
$$f(w) = \begin{cases} 0.1 & w = 0, 1, 2, \ldots, 9 \\ 0 & \text{otherwise} \end{cases}$$

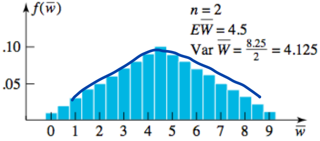
- $\overline{W} = \frac{1}{2}(W_1 + W_2)$ has the pmf:

The Probability Function for $\overline{W}$ for $n = 2$

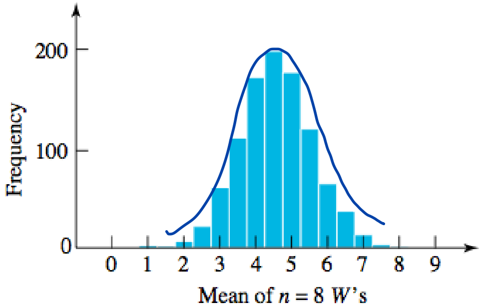
$\bar{w}$	$f(\bar{w})$	$\bar{w}$	$f(\bar{w})$	$\bar{w}$	$f(\bar{w})$	$\bar{w}$	$f(\bar{w})$	$\bar{w}$	$f(\bar{w})$
0.0	.01	2.0	.05	4.0	.09	6.0	.07	8.0	.03
0.5	.02	2.5	.06	4.5	.10	6.5	.06	8.5	.02
1.0	.03	3.0	.07	5.0	.09	7.0	.05	9.0	.01
1.5	.04	3.5	.08	5.5	.08	7.5	.04		

Example: tool serial numbers



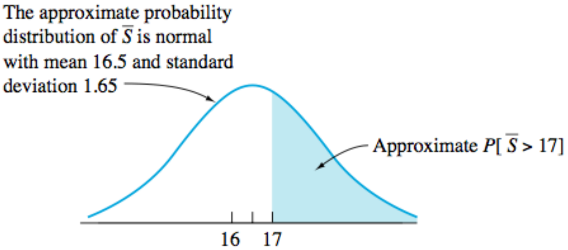


- What if $\overline{W} = \frac{1}{8}(W_1 + W_2 + \cdots + W_8)$, the average of 8 days of initial serial numbers?



Example: excess sale time

- $\overline{S}$ = sample mean excess sale time (over a 7.5 s threshold) for 100 stamp sales.
- Each individual excess sale time should have an Exp($\alpha = 16.5$ s) distribution. That means:
  - $E(\overline{S}) = \alpha = 16.5$ s
  - $SD(\overline{S}) = \sqrt{\text{Var}(\overline{S})} = \sqrt{\frac{\alpha^2}{100}} = 1.65$ s σ^2 n
  - By the Central Limit Theorem, $\overline{S} \approx N(16.5, 1.65^2)$
- We want to approximate $P(\overline{S} > 17)$.



Example: excess sale time

$\overline{S} \sim N(16.5, 1.65^2)$.

$$
\begin{aligned}
P(\overline{S} > 17) &= P\left(\frac{\overline{S} - 16.5}{1.65} > \boxed{\frac{17 - 16.5}{1.65}}\right) \\
&\approx P(Z > 0.303) \qquad (Z \sim N(0,1)) \\
&= 1 - P(Z \leq 0.303) \\
&= 1 - \Phi(0.303) \\
&= 1 - 0.62 \quad \text{from the standard normal table} \\
&= 0.38
\end{aligned}
$$

Example: net weight of baby food jars

- Individual jar weights are iid with unknown mean μ and standard deviation $\sigma = 1.6$ g
- $\overline{V}$ = sample mean weight of n jars $\approx N\left(\mu, \frac{1.6^2}{n}\right)$.
- We want to find μ. One way to hone in on μ is to find n such that:

$|\bar{V} - \mu| < 0.3$

$$P(\mu - 0.3 < \overline{V} < \mu + 0.3) = 0.8$$

the prob that difference is small is high.

That way, our measured value of $\overline{V}$ is likely to be close to μ.

Example: net weight of baby food jars

$\overline{V} \sim N(\mu, \frac{1.6^2}{n})$.

$$
\begin{aligned}
0.8 &= P(\mu - 0.3 < \overline{V} < \mu + 0.3) \\
&= P\left(\frac{-0.3}{1.6/\sqrt{n}} < \boxed{\frac{\overline{V} - \mu}{1.6/\sqrt{n}}} < \frac{0.3}{1.6/\sqrt{n}}\right) \\
&\approx P(-0.19\sqrt{n} < Z < 0.19\sqrt{n}) \quad \text{(by CLT)} \\
&= 1 - 2\Phi(-0.19\sqrt{n}) \quad \text{(look at the N(0,1) pdf)} \\
\Phi^{-1}(0.1) &= -0.19\sqrt{n} \\
n &= \frac{\Phi^{-1}(0.1)^2}{(-0.19)^2} \\
&= \frac{(-1.28)^2}{(-0.19)^2} \quad \text{(standard normal table)} \\
&= 46.10
\end{aligned}
$$

when $n \uparrow$. $\uparrow$.

round it up. to be conservative.

- Hence, we'll need a sample size of $n = 47$.

Example: cars

- Suppose a bunch of cars pass through certain stretch of road. Whenever a car comes, you look at your watch and record the time.
- Let X_i be the time (in hours) between when the i'th car comes and the $(i+1)$'th car comes, $i = 1, \ldots, 44$. Suppose you know:

$$X_1, X_2, \ldots, X_{44} \sim \text{ iid } f(x) = e^{-x} \quad x \geq 0$$

Exp(1).

- Find the probability that the average time gap between cars exceeds 1.05 hours.

Example: cars

$X_1 \sim Exp(\alpha = 1)$.

$E(X_1) = 1$. $Var(X_1) = 1$.

$$\mu = E(X_1)$$

$$= \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{0}^{\infty} x e^{-x} dx$$

$$= -e^{-x}(x+1)\Big|_0^{\infty} \quad \text{integration by parts}$$

$$= 1$$

Example: cars

$$
\begin{aligned}
E(X_1^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_0^{\infty} x^2 e^{-x} dx \\
&= -e^{-x}(x^2 + 2x + 2)\big|_0^{\infty} \qquad \text{integration by parts} \\
&= 2 \\
\sigma^2 &= Var(X_1) \\
&= E(X_1^2) - E^2(X_1) \\
&= 2 - 1^2 \\
&= 1
\end{aligned}
$$

Example: cars

$$\overline{X} \sim \text{ approx. } N(\mu, \sigma^2/n)$$
$$= N(1, 1/44)$$

Thus:

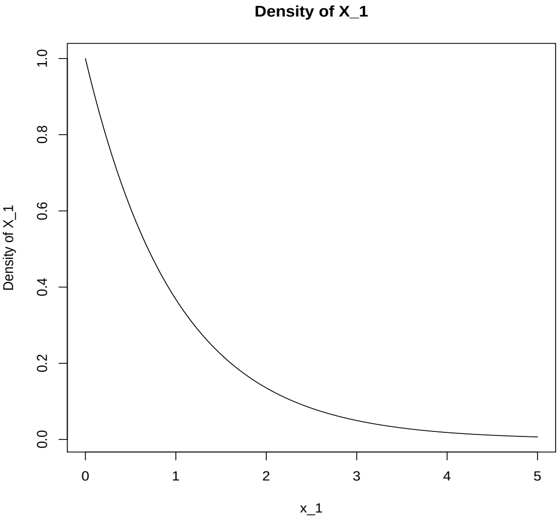
$$\frac{\overline{X} - 1}{\sqrt{1/44}} \sim N(0, 1)$$

Example: cars

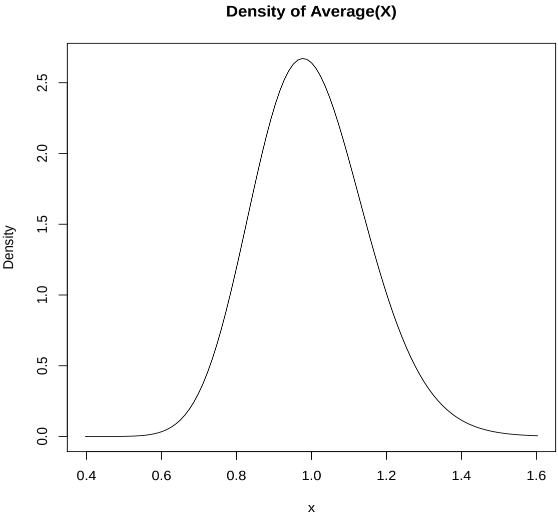
Now, we're ready to approximate:

$$
\begin{aligned}
P(\overline{X} > 1.05) &= P\left(\frac{\overline{X} - 1}{\sqrt{1/44}} > \frac{1.05 - 1}{\sqrt{1/44}}\right) \\
&= P\left(\frac{\overline{X} - 1}{\sqrt{1/44}} > 0.332\right) \\
&\approx P(Z > 0.332) \\
&= 1 - P(Z \leq 0.332) \\
&= 1 - \Phi(0.332) \\
&= 1 - 0.630 = 0.370
\end{aligned}
$$

Example: cars



Example: cars



Example: cars