# Discrete Random Variables (Ch. 5.1)

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# Outline

# What is a random variable?

- **Random variable**; a quantity that can be thought of as dependent on chance phenomena.
  - $X$ = the value of a coin toss (heads or tails).
  - $Z$ = the amount of torque required to loosen the next bolt.
  - $T$ = the time you'll have to wait for the next bus home.
  - $N$ = the number of defective widgets in manufacturing process in a day.
  - $S$ = the number of provoked shark attacks off the coast of Florida next year.
- Two types:
  - **Discrete random variable**: one that can only take on a set of isolated points ($X$, $N$, and $S$).
  - **Continuous random variable**: one that can fall in an interval of real numbers ($T$ and $Z$).

# Discrete random variables

- A discrete random variable has a list of possible values:
  - $X$ = roll of a 6-sided fair die = 1, 2, 3, 4, 5, or 6.
  - $Y$ = roll of a 6-sided *unfair* die = 1, 2, 3, 4, 5, or 6.
- But how do you distinguish between $X$ and $Y$?

# Outline

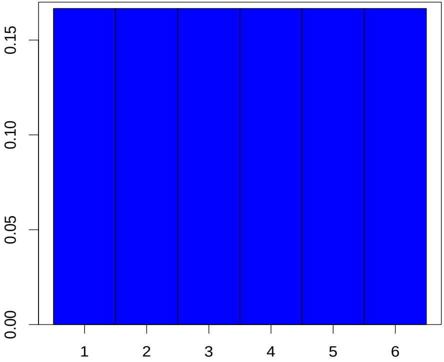
# Probability



- $P(X = x)$, the **probability** that $X$ equals $x$, is the fraction of times that $X$ will land on $x$
  1. We expect a fair die to land the number 3 roughly one out of every 6 tosses. Thus, $P(X = 3) = 1/6$
  2. Suppose the unfair die is weighted so that the number 3 only lands one out of every 22 tosses. Then, $P(Y = 3) = 1/22$.

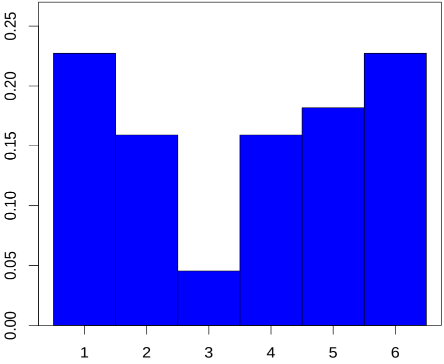
- $X$ has the following probabilities:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ |



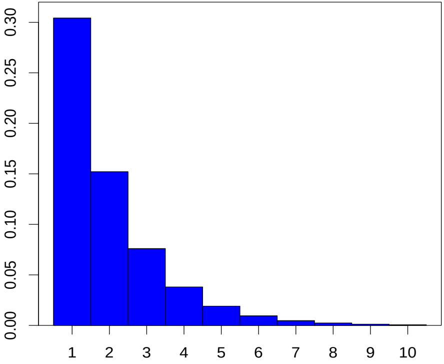
- Say $Y$ has the probabilities:

| $y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(Y=y)$ | $5/22$ | $7/44$ | $1/22$ | $7/44$ | $2/11$ | $5/22$ |



- $S$, the number of provoked shark attacks off FL next year, has infinite number of possible values. Here is one possible (made up) distribution:

| $s$ | 1 | 2 | 3 | $\cdots$ | $k$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $P(S=s)$ | $\frac{6}{\pi^2}$ | $\frac{1}{2}\frac{6}{\pi^2}$ | $\frac{1}{4}\frac{6}{\pi^2}$ | $\cdots$ | $\frac{1}{2^k}\frac{6}{\pi^2}$ | $\cdots$ |



# Outline

# Probability mass functions (pmf)

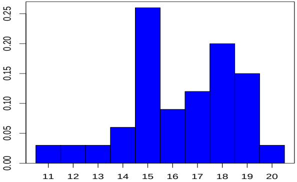
- The **probability mass function (pmf)** $f(x)$ of a random variable $X$ is just $P(X = x)$
  - $X$ has $f(x) = 1/6$
  - $S$ has $f(s) = \frac{1}{2^s}\frac{6}{\pi^2}$.
- We could also write $f_X$ for the pmf of $X$ and $f_S$ for the pmf of $S$.
- Rules of the pmf $f$:
  - $f(x) \geq 0$ for all $x$.
  - $\sum_x f(x) = 1$.

# Your turn: calculating probabilities

- Let $Z$ = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

| $z$ | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| $f(z) = P(Z = z)$ | 0.03 | 0.03 | 0.03 | 0.06 | 0.26 |

| $z$ | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|
| $f(z) = P(Z = z)$ | 0.09 | 0.12 | 0.20 | 0.15 | 0.03 |




- Calculate:
  1. $P(Z \leq 14)$
  2. $P(Z > 16)$
  3. $P(Z$ is an even number$)$
  4. $P(Z$ in $\{15, 16, 18\})$

# Answers: calculating probabilities

1.

$$
\begin{aligned}
P(Z \leq 14) &= P(Z = 11 \text{ or } Z = 12 \text{ or } Z = 13 \text{ or } Z = 14) \\
&= P(Z = 11) + P(Z = 12) + P(Z = 13) + P(Z = 14) \\
&= f(11) + f(12) + f(13) + f(14) \\
&= 0.03 + 0.03 + 0.03 + 0.06 \\
&= 0.15
\end{aligned}
$$

2.

$$
\begin{aligned}
P(Z > 16) &= P(Z = 17 \text{ or } Z = 18 \text{ or } Z = 19 \text{ or } Z = 20) \\
&= P(Z = 17) + P(Z = 18) + P(Z = 19) + P(Z = 20) \\
&= f(17) + f(18) + f(19) + f(20) \\
&= 0.12 + 0.20 + 0.15 + 0.03 \\
&= 0.5
\end{aligned}
$$

# Answers: calculating probabilities



3.

$$\begin{aligned} P(Z \text{ even}) &= P(Z=12 \text{ or } Z=14 \text{ or } Z=16 \text{ or } Z=18 \text{ or } Z=20) \\ &= P(Z=12)+P(Z=14)+P(Z=16)+P(Z=18) \\ &\quad +P(Z=20) \\ &= f(12)+f(14)+f(16)+f(18)+f(20) \\ &= 0.03+0.06+0.09+0.20+0.03 \\ &= 0.41 \end{aligned}$$

4.

$$\begin{aligned} P(Z \text{ in } \{15,16,18\}) &= P(Z=15 \text{ or } Z=16 \text{ or } Z=18) \\ &= P(Z=15)+P(Z=16)+P(Z=18) \\ &= f(15)+f(16)+f(18) \\ &= 0.26+0.09+0.02 \\ &= 0.37 \end{aligned}$$

# Outline

# The cumulative distribution function (cdf)

- **Cumulative distribution function (cdf)**: a function, $F$, defined by:

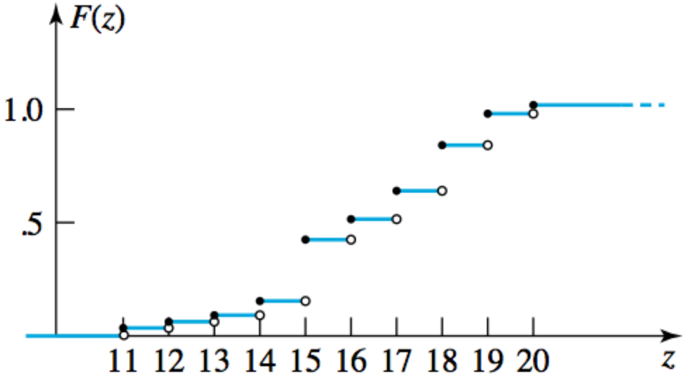
$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \sum_{z \leq x} f(z) \end{aligned}$$

- $F$ has the following properties:
  - $F(x) \geq 0$ for all real numbers $x$.
  - $F$ is monotonically increasing.
  - $\lim_{x \to -\infty} F(x) = 0$
  - $\lim_{x \to \infty} F(x) = 1$

# Example: torque random variable, $Z$

| $z$, Torque | $f(z) = P[Z = z]$ | $F(z) = P[Z \leq z]$ |
|---|---|---|
| 11 | .03 | .03 |
| 12 | .03 | .06 |
| 13 | .03 | .09 |
| 14 | .06 | .15 |
| 15 | .26 | .41 |
| 16 | .09 | .50 |
| 17 | .12 | .62 |
| 18 | .20 | .82 |
| 19 | .15 | .97 |
| 20 | .03 | 1.00 |

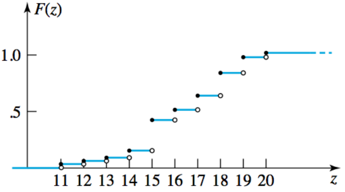
# Example: torque random variable, $Z$



# Your turn: calculating probabilities

| $z$ | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| $F(z) = P(Z \leq z)$ | 0.03 | 0.06 | 0.09 | 0.15 | 0.41 |

| $z$ | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|
| $F(z) = P(Z \leq z)$ | 0.50 | 0.62 | 0.82 | 0.97 | 1 |




- Using the cdf only, calculate:
  1. F(10.7)
  2. $P(Z \leq 15.5)$
  3. $P(12.1 < Z \leq 14)$
  4. $P(15 \leq Z < 18)$

# Answers: calculating probabilities

1. $F(10.7) = P(Z \le 10.7) = 0$
2. $P(Z \le 15.5) = P(Z \le 15) = 0.41$
3. 

$$\begin{aligned}
P(12.1 < Z \le 14) &= P(Z = 13 \text{ or } 14) \\
&= f(14) + f(13) \\
&= [f(14) + f(13) + f(12) + f(11)] \\
&\quad - [f(12) + f(11)] \\
&= P(Z \le 14) - P(Z \le 12) \\
&= F(14) - F(12) \\
&= 0.15 - 0.06 \\
&= 0.09
\end{aligned}$$

# Answers: calculating probabilities

4.

$$\begin{aligned} P(15 \leq Z < 18) &= P(Z = 15,\ 16,\ \text{or } 17) \\ &= P(Z \leq 17) - P(Z \leq 14) \\ &= F(17) - F(14) \\ &= 0.62 - 0.15 \\ &= 0.47 \end{aligned}$$

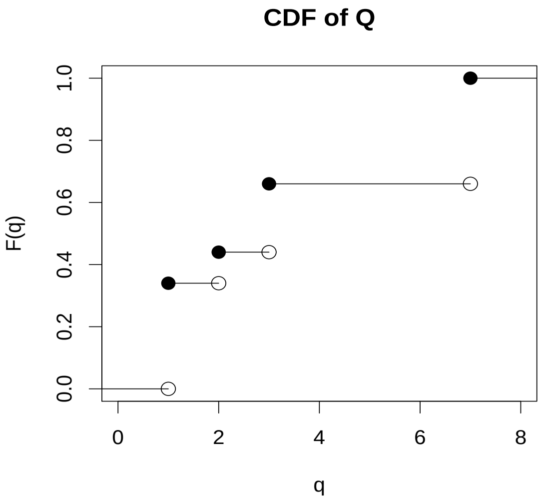
# Your turn: drawing the cdf

- Say we have a random variable $Q$ with pmf:

| $q$ | 1 | 2 | 3 | 7 |
|---|---|---|---|---|
| $f(q)$ | 0.34 | 0.1 | 0.22 | 0.34 |

- Draw the cdf.

# Answer: drawing the cdf



# Outline

What is a random variable?

Probability

Probability Mass Functions (pmf)

Cumulative Distribution Functions (cdf)

Expected Value

Variance and Standard Deviation

# Expected Value

- The **expected value** $E(X)$ (also called $\mu$) of a random variable $X$ is given by:

$$\sum_x x \cdot f(x)$$

- When $X$ is the roll of a fair die,

$$\begin{aligned} E(X) &= 1f(1) + 2f(2) + 3f(3) + 4f(4) + 5f(5) + 6f(6) \\ &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) \\ &= \frac{1+2+3+4+5+6}{6} \\ &= 3.5 \end{aligned}$$

- $E(X)$ is a *weighted average* of the possible values of $X$, weighted by their probabilities.
- $E(X)$ is the **mean of the distribution** of $X$

# Property of Expected Value

- Expected value of the function of a discrete random variable:

$$E(g(X)) = \sum_x g(x) \cdot f(x)$$

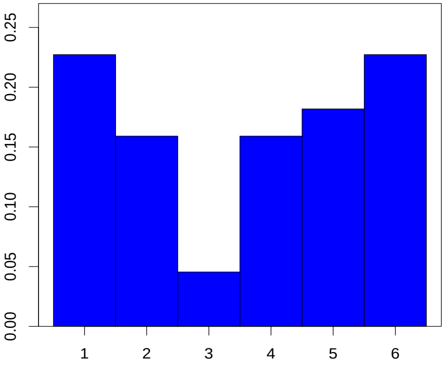
- Linearity (true for both discrete and continuous random variables):

$$E(X + Y) = E(X) + E(Y)$$

$$E(aX + b) = aE(X) + b$$

# Your turn: expected value

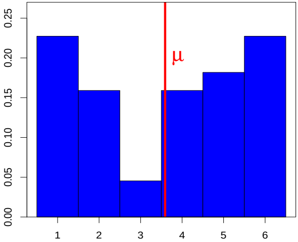
| $y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(Y=y)$ | $5/22$ | $7/44$ | $1/22$ | $7/44$ | $2/11$ | $5/22$ |




- Calculate $E(Y)$, the expected value of a toss of the *unfair* die.

# Answer: expected value

- $$\begin{aligned} E(Y) &= 1(5/22) + 2(7/44) + 3(1/22) \\ &\quad + 4(7/44) + 5(2/11) + 6(5/22) \\ &= 3.5909091 \end{aligned}$$
- The average roll of the unfair die is 3.5909.
- $E(Y)$ is the mean of the distribution of $Y$.



# Your turn: expected value

- Calculate $E(2X+Y)$, the expected value of the sum of two tosses of a fair die and a toss of the unfair die.

# Answer: expected value

$$E(2X + Y) = 2E(X) + Y = 2 \times 3.5 + 3.5909 = 10.5909$$

- The average of the sum of two tosses of a fair die and a toss of the unfair die is 10.5909.

# Your turn: expected value

| $z$ | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| $f(z) = P(Z = z)$ | 0.03 | 0.03 | 0.03 | 0.06 | 0.26 |

| $z$ | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|
| $f(z) = P(Z = z)$ | 0.09 | 0.12 | 0.20 | 0.15 | 0.03 |

- Calculate $E(Z)$, the expected value of the torque required to loosen the next bolt.

# Answer: expected value

$$
\begin{aligned}
E(Z) &= 11(0.03) + 12(0.03) + 13(0.03) + 14(0.06) + 15(0.26) \\
&= 16(0.09) + 17(0.12) + 18(0.20) + 19(0.15) + 20(0.03) \\
&= 16.35
\end{aligned}
$$

- The average torque required to loosen the next bolt is 16.35 units.

# Outline

# Variance

- **Variance**: the variance $Var(X)$ (also called $\sigma^2$) of a random variable $X$ is given by:

$$\begin{aligned} \text{Var}(X) &= E((X - E(X))^2) \\ &= \sum_x (x - E(X))^2 f(x) \end{aligned}$$

- Shortcut formulas:

$$\begin{aligned} \text{Var}(X) &= \left[\sum_x x^2 f(x)\right] - (E(X))^2 \\ &= E(X^2) - E^2(X) \end{aligned}$$

- The variance is the average squared deviation of random variable from its mean

# Example: calculating the variance

| $q$ | 1 | 2 | 3 | 7 |
|---|---|---|---|---|
| $f(q)$ | 0.34 | 0.1 | 0.22 | 0.34 |

- Long way:

$$
\begin{aligned}
E(Q) &= 1(0.34) + 2(0.1) + 3(0.22) + 7(0.34) \\
&= 3.58 \\
Var(Q) &= (1-3.58)^2 0.34 + (2-3.58)^2 0.1 \\
&\quad + (3-3.58)^2 0.22 + (7-3.58)^2 0.34 \\
&= 6.56
\end{aligned}
$$

- Short way:

$$
\begin{aligned}
E(Q^2) &= \sum_q q^2 f(q) \\
&= 1(0.34) + 4(0.1) + 9(0.22) + 49(0.34) \\
&= 19.38 \\
Var(Q) &= E(Q^2) - E^2(Q) \\
&= 19.38 - 3.58^2 \\
&= 6.56
\end{aligned}
$$

# Your turn: calculating the variance

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(x)$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ |

- Calculate $Var(X)$
- Calculate $SD(X)$

# Your turn: answers



- $$\begin{aligned} E(X) &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) \\ &= 3.5 \\ E(X^2) &= \sum_{x=1}^{6} x^2 f(x) \\ &= 1^2(1/6) + 2^2(1/6) + 3^2(1/6) + 4^2(1/6) + 5^2(1/6) + 6^2(1/6) \\ &= 15.17 \\ Var(X) &= E(X^2) - E^2(X) \\ &= 15.17 - 3.5^2 \\ &= 2.92 \end{aligned}$$
- $SD(X) = \sqrt{2.92} = 1.7088007$