Special Discrete Random Variables (Ch. 5.1)

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Outline

The Bernolli Distribution

- $X \sim \text{Bernolli}(p)$ – i.e., X is distributed as a bernolli random variable with parameter p $(0 < p < 1)$ if:

$$f_X(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = p$
- $Var(X) = p(1-p)$
- A bernolli random variable indicate success (encoded as 1) or failure (encoded as 0) in one success-failure trial.
- Examples:
  - A hexamine pellet made from a pelletizing machine conforms the specification (success, 1) or not (failure, 0).
  - A run of the chemical process has a percent yield above 80% (success, 1) or not (failure, 0).

X, support : $\{0, 1\}$.

$f(x)$: $x = 1$. $f(1) = p$

$x = 0$. $f(0) = 1-p$.

$f(x) = p^x (1-p)^{1-x}$

$$E(X) = \sum_{x=0}^{1} x \cdot f(x) = 0 \cdot (1-p) + 1 \cdot p = p.$$

$$E(X^2) = \sum_{x=0}^{1} x^2 f(x) = 0 \cdot (1-p) + 1 \cdot p = p.$$

$$Var(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$$

Outline

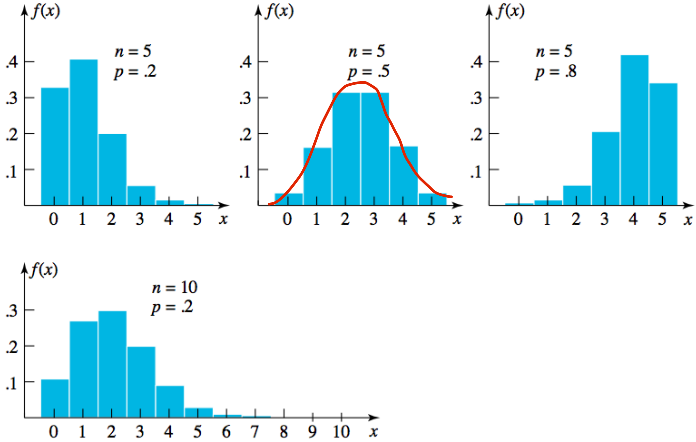
The Binomial Distribution

- $X \sim \text{Binomial}(n,\, p)$ – i.e., X is distributed as a binomial random variable with parameters n and p $(0 < p < 1)$ if:

$$f_X(x) = \begin{cases} \boxed{\binom{n}{x}} p^x (1-p)^{n-x} & x = 0, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

  where:
  - $\binom{n}{x} = \boxed{\frac{n!}{x!(n-x)!}}$, read "$n$ choose x"
  - $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$, the factorial function.
- $E(X) = np$
- $Var(X) = np(1-p)$

The Binomial Distribution



X : # of successes out of n trials.

$\square \ \square \ \square \ \cdots \ \square \leftarrow$ n success-failure trials.

independent.

for each one:

success prob: p.

support: $\{0, 1, \ldots, n\}$.

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E(X) = \sum_{x=0}^{n} x \cdot f(x) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} \cdot p^x (1-p)^{n-x}, \qquad x-1 \to x$$

$$= \sum_{x=0}^{n-1} \frac{n!}{x!\,(n-1-x)!} p^{x+1} (1-p)^{n-x-1}$$

$$= \sum_{x=0}^{n-1} \underbrace{\frac{(n-1)!}{x!\,(n-1-x)!}}_{\binom{n-1}{x}} \cdot n \cdot p^x \cdot p \, (1-p)^{n-1-x}$$

$$= np \sum_{x=0}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x} = np$$

$$\underline{\underline{E(X^2)}} = E((X-1)X) + E(X)$$

$$E(X(X-1))$$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \underline{x(x-1)} \frac{n!}{\underline{x!}\,(n-p)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} p^x (1-p)^{n-x}, \qquad x-2 \to x$$

$$= \sum_{x=0}^{n-2} \boxed{\frac{(n-2)!}{x!\,(n-2-x)!}}\, n(n-1)\, p^2 \cdot \boxed{p^x} \cdot \boxed{(1-p)^{n-2-x}}$$

$$= n(n-1)p^2$$

$$E(X(X-1)) = n(n-1)p^2$$

$$E(X^2) = n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np - np^2 = np(1-p)$$

Purpose of the binomial random variable

- A Bin(n, p) random variable counts the number of successes in n success-failure trials that:
  - are independent of one another.
  - each succeed with probability p.
- Examples:
  - Number of conforming hexamine pellets in a batch of $n = 50$ total pellets made from a pelletizing machine.
  - Number of runs of the same chemical process with percent yield above 80%, given that you run the process a total of $n = 1000$ times.
  - Number of rivets that fail in a boiler of $n = 25$ rivets within 3 years of operation. (Note; "success" doesn't always have to be good.)

Relationship between Bernolli and Binomail random variables

- Bernolli(p) = Binomial(1, p).
- Let $X_1, X_2, \ldots, X_n$ be i.i.d. (independent and identically distributed) Bernolli(p) random variables, then

$$\sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$$

  - Informally, $X_1, X_2, \ldots, X_n$ are independent means their outcomes do not affect each other. Formal definition of independence will be given in the lecture of "joint distribution and independence".

- $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$ (true for non independent too).
- $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$ (only true for independent variables)

$$E(X) = \sum_{i=1}^{n} E(X_i) = np.$$

$$Var(X) = \sum_{i=1}^{n} Var(X_i) = np(1-p).$$

Example: machine with 10 components

- Suppose you have a machine with 10 **independent** components in series. The machine only works if all the components work.
- Each component succeeds with probability $p = 0.95$ and fails with probability $1 - p = 0.05$.
- Let Y be the number of components that succeed in a given run of the machine. Then:

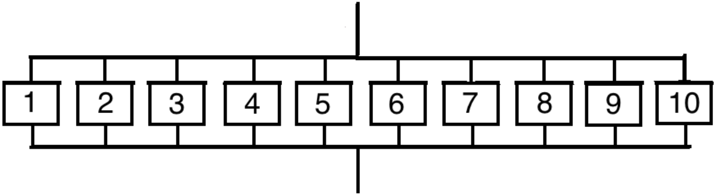
$$Y \sim \text{Binomial}(n = 10, p = 0.95)$$

Example: machine with 10 components

$$
\begin{aligned}
P(\text{machine succeeds}) &= P(Y = 10) \\
&= \binom{10}{10} p^{10} (1-p)^{10-10} \\
&= p^{10} \\
&= 0.95^{10} \\
&= 0.5987
\end{aligned}
$$

- This machine isn't very reliable.

Example: machine with 10 components



- What if I arrange these 10 components in parallel? This machine succeeds if at least 9 of the components succeed.
- What is the probability that the new machine succeeds?

Example: machine with 10 components

$$
\begin{aligned}
&P(\text{improved machine succeeds}) \\
&\qquad = P(Y \geq 9) \\
&\qquad = P(Y = 9) + P(Y = 10) \\
&\qquad = \binom{10}{9} p^9 (1-p) + \binom{10}{10} p^{10} (1-p)^{10-10} \\
&\qquad = (10) \cdot 0.95^9 \cdot 0.05 + (1) \cdot 0.95^{10} \\
&\qquad = 0.9139
\end{aligned}
$$

- By allowing just one component to fail, we made this machine far more reliable.

Example: machine with 10 components

- If we allow up to 2 components to fail:

$$
\begin{aligned}
&P(\text{improved machine succeeds}) \\
&\quad = P(Y \geq 8) \\
&\quad = P(Y = 8) + P(Y = 9) + P(Y = 10) \\
&\quad = \binom{10}{8} p^8 (1-p)^{10-8} + \binom{10}{9} p^9 (1-p) + \binom{10}{10} p^{10} (1-p)^{10-10} \\
&\quad = \frac{10!}{(10-8)!8!} \cdot 0.95^8 \cdot 0.05^2 + (10) \cdot 0.95^9 \cdot 0.05 + (1) \cdot 0.95^{10} \\
&\quad = 0.9885
\end{aligned}
$$

Example: machine with 10 components

- $E(Y) = np = 10 \cdot 0.95 = 9.5$. So the number of components to fail per run on average is 9.5.
- $Var(Y) = np(1-p) = 10 \cdot 0.95 \cdot (1 - 0.95) = 0.475$.
- $SD(Y) = \sqrt{Var(Y)} = \sqrt{np(1-p)} = 0.689$.

Outline

Bernolli Distribution

Binomial Distribution

Geometric Distribution

Poisson Distribution

Geometric random variables

- $X \sim \text{Geometric}(p)$ – that is, X has a geometric distribution with parameter p $(0 < p < 1)$ – if its pmf is:

no zero!

$$f_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

and its cdf is:

$$F_X(x) = \begin{cases} 1-(1-p)^{x} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = \frac{1}{p}$
- $Var(X) = \frac{1-p}{p^2}$

X : # of trials taken to get a first success.

$\{1, 2, \dots\}$.

$f(x) = P(X = x)$.

$x = 1$. $P(X=1) = p$.

$x=2$: FS. $\longrightarrow P(X=2) = (1-p)p$

$x=3$: $FFS \longrightarrow P(X=3) = (1-p)^2 p$.

X : $\underbrace{FF\cdots F}_{x-1}S$ $\quad P(X=x) = (1-p)^{x-1} p$

$$E(X) = \sum_{x=1}^{\infty} x\,(1-p)^{x-1} \cdot p$$

$$\sum_{x=0}^{\infty} (1-p)^{x} = \frac{1}{p}, \qquad \sum_{x=1}^{\infty} (1-p)^{x-1} p = 1$$

$$\sum_{x=1}^{\infty} (1-p)^{x} = \frac{1}{p} - 1,$$

$$\sum_{x=1}^{\infty} -x\,(1-p)^{x-1} = -\frac{1}{p^2}$$

$$\sum_{x=1}^{\infty} x\,p\,(1-p)^{x-1} = \frac{1}{p} = E(X)$$

$$\sum_{x=1}^{\infty} x(1-p)^{x-1} = \frac{1}{p^2}$$

$$\sum_{x=1}^{\infty} x(1-p)^{x} = \frac{1-p}{p^2} = \frac{1}{p^2} - \frac{1}{p}$$

$$\sum_{x=1}^{\infty} x \, x(1-p)^{x-1} \cdot (-1) = -\frac{2}{p^3} + \frac{1}{p^2}$$

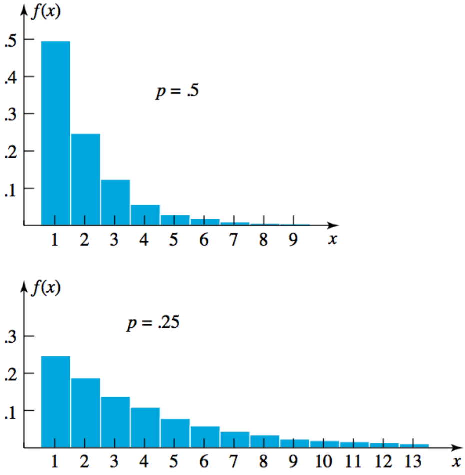
$$\Rightarrow \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} = \frac{2}{p^3} - \frac{1}{p^2}$$

$$\Rightarrow \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1} = \frac{2}{p^2} - \frac{1}{p} = E(X^2)$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

A look at the Geom(p) distribution



Uses of the $X \sim \text{Geom}(p)$

- For an indefinitely-long sequence of independent, success-failure trials, each with $\text{P(success)} = p$, X is the number of trials it takes to get a success.
- Examples:
  - Number of rolls of a fair die until you land a 5.
  - Number of shipments of raw material you get until you get a defective one.
  - The number of enemy aircraft that fly close before one flies into friendly airspace.
  - Number hexamine pellets you make before you make one that does not conform.
  - Number of buses that come before yours.

Example: shorts in NiCad batteries

- An experimental program was successful in reducing the percentage of manufactured NiCad cells with internal shorts to around 1%.
- Let T be the test number at which the first short is discovered. Then, $T \sim \text{Geom}(p)$.

$$\begin{aligned}
P(\text{1st or 2nd cell tested is has the 1st short}) &= P(T = 1 \text{ or } T = 2) \\
&= f(1) + f(2) \\
&= p + p(1-p) \\
&= 0.01 + 0.01(1 - 0.01) \\
&= 0.02
\end{aligned}$$

$$\begin{aligned}
P(\text{at least 50 cells tested w/o finding a short}) &= P(T > 50) \\
&= 1 - P(T \leq 50) \\
&= 1 - F(50) \\
&= 1 - (1 - (1-p)^x) \\
&= (1-p)^x \\
&= (1 - 0.01)^{50} \\
&= 0.61
\end{aligned}$$

Example: shorts in NiCad batteries

$$E(T) = \frac{1}{p} = \frac{1}{0.01}$$
$$= 100 \text{ tests for the first short to appear, on avg.}$$

$$SD(T) = \sqrt{Var(T)} = \sqrt{\frac{1-p}{p^2}}$$
$$= \sqrt{\frac{1-0.01}{0.01^2}} = 99.5 \text{ tested batteries}$$

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Bernolli Distribution

Binomial Distribution

Geometric Distribution

Poisson Distribution

Poisson random variables

- $X \sim \text{Poisson}(\lambda)$ – that is, X has a ~~geometric~~ Poisson distribution with parameter $\lambda > 0$ – if its pmf is:

$$f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = \lambda$
- $Var(X) = \lambda$

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \ldots$$

$$f(x) \geq 0.$$

$$\sum_{x=0}^{\infty} f(x) = 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

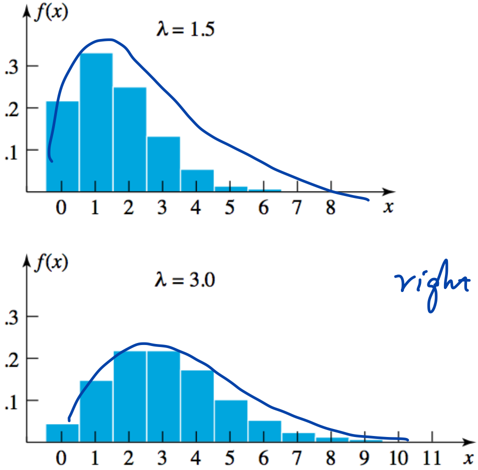
$x \to \lambda$

$n \to x.$

$$e^\lambda = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$\Rightarrow \quad 1 = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!}$$

A look at the Poisson distribution



right skew.

Meaning of the Poisson distribution

- A Poisson(λ) random variable counts the number of occurrences that happen over a fixed interval of time or space.
- These occurrences must:
  - be independent
  - be sequential in time (no two occurrences at once)
  - occur at the same constant rate, λ.
- λ, the **rate parameter**, is the expected number of occurrences in the specified interval of time or space.

Examples

- Y is the number of shark attacks off the coast of CA next year. $\lambda = 100$ attacks per year.
- Z is the number of shark attacks off the coast of CA next month. $\lambda = 100/12 = 8.3333333$ attacks per month
- N is the number of β particles emitted from a small bar of plutonium, registered by a Geiger counter, in a minute. $\lambda = 459.21$ particles/minute.
- J is the number of particles per three minutes. $\lambda = ?$

$$\lambda = \frac{459.21 \text{ (units particle)}}{1 \text{ (unit minute)}} \cdot \frac{3 \text{ (units minute)}}{1 \text{ (unit of 3 minutes)}}$$

$$= \frac{1377.63 \text{ (units particle)}}{1 \text{ (unit of 3 minutes)}} = 1377.62 \text{ particles per 3 minutes}$$

Example: Rutherford/Geiger experiment

- Rutherford and Geiger measured the number of α particles detected near a small bar of plutonium for 8-minute periods.
- The average number of particles per 8 minutes was $\lambda = 3.87$ particles / 8 min.
- Let $S \sim$ Poisson(λ), the number of particles detected in the next $\boxed{8}$ minutes.

$$f(s) = \begin{cases} \frac{e^{-3.87}(3.87)^s}{s!} & s = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
&P(\text{at least 4 particles recorded}) \\
&\quad = P(S \geq 4) \qquad = 1 - P(S < 4). \\
&\quad = f(4) + f(5) + f(6) + \cdots \\
&\quad = 1 - f(0) - f(1) - f(2) - f(3) \\
&\quad = 1 - \frac{e^{-3.87}(3.87)^0}{0!} - \frac{e^{-3.87}(3.87)^1}{1!} \\
&\qquad - \frac{e^{-3.87}(3.87)^2}{2!} - \frac{e^{-3.87}(3.87)^3}{3!} \\
&\quad = 0.54
\end{aligned}$$

Example: arrival at a university library

10 min

- Some students' data indicate that between 12:00 and 12:10 P.M. on Monday through Wednesday, an average of around 125 students entered a library at Iowa State University library.
- Let M be the number of students entering the ISU library between 12:00 and 12:01 PM next Tuesday. → 1 min
- Model $M \sim \text{Poisson}(\lambda)$.
- Having observed 125 students enter between 12:00 and 12:10 PM last Tuesday, we might choose:

$$\lambda = \frac{125 \text{ (units of student)}}{1 \text{ (unit of 10 minutes)}} \cdot \frac{1 \text{ (unit of 10 minutes)}}{10 \text{ (units of minute)}}$$

$$= \frac{12.5 \text{ (units of student)}}{1 \text{ (unit minute)}} = 12.5 \text{ students per minute}$$

Example: arrival at a university library

- Under this model, the probability that between 10 and 15 students arrive at the library between 12:00 and 12:01 PM is:

$$
\begin{aligned}
P(10 \leq M \leq 15) &= f(10) + f(11) + f(12) + f(13) + f(14) + f(15) \\
&= \frac{e^{-12.5}(12.5)^{10}}{10!} + \frac{e^{-12.5}(12.5)^{11}}{11!} + \frac{e^{-12.5}(12.5)^{12}}{12!} \\
&+ \frac{e^{-12.5}(12.5)^{13}}{13!} + \frac{e^{-12.5}(12.5)^{14}}{14!} + \frac{e^{-12.5}(12.5)^{15}}{15!} \\
&= \underline{0.60}
\end{aligned}
$$

Example: shark attacks

- Let X be the number of unprovoked shark attacks that will occur off the coast of Florida next year.
- Model $X \sim \text{Poisson}(\lambda)$.
- From the shark data at http://www.flmnh.ufl.edu/fish/sharks/statistics/FLactivity.htm, 246 unprovoked shark attacks occurred from 2000 to 2009.
- Hence, I calculate:

10 yr

$$\lambda = \frac{246 \text{ (units attack)}}{1 \text{ (unit of 10 years)}} \cdot \frac{1 \text{ (unit of 10 years)}}{10 \text{ (units year)}}$$
$$= \frac{24.6 \text{ (units attack)}}{1 \text{(unit year)}} = 24.6 \text{ attacks per year}$$

Example: shark attacks

$X \sim \text{Poisson}(24.6)$

$$P(\text{no attacks next year}) = f(0) = e^{-24.6} \cdot \frac{24.6^0}{0!}$$

$$\approx 2.07 \times 10^{-11}$$

$P(X \geq 5) = 1 - P(X < 5)$

$$P(\text{at least 5 attacks}) = 1 - P(\text{at most 4 attacks})$$

$$= 1 - F(4)$$

$$= 1 - f(0) - f(1) - f(2) - f(3) - f(4)$$

$$= 1 - e^{-24.6}\frac{24.6^0}{0!} - e^{-24.6}\frac{24.6^1}{1!} - e^{-24.6}\frac{24.6^2}{2!}$$

$$- e^{-24.6}\frac{24.6^3}{3!} - e^{-24.6}\frac{24.6^4}{4!}$$

$$\approx 0.9999996$$

$$P(\text{more than 30 attacks}) = 1 - P(\text{at least 30 attacks})$$

$$= 1 - e^{-24.6}\sum_{i=0}^{30}\frac{24.6^x}{x!} = 1 - e^{-24.6} \cdot 4.251 \times 10^{10}$$

$$\approx 0.1193$$

Example: shark attacks

- Now, let Y be the total number of shark attacks in Florida during the next 4 months.
- Let $Y \sim \text{Poisson}(\theta)$, where θ is the true shark attack rate per 4 months:

$$\theta = \frac{24.6 \text{ (units attack)}}{1 \text{ (unit year)}} \cdot \frac{1/3 \text{ (unit year)}}{1 \text{ (unit of 4 months)}}$$
$$= \frac{8.2 \text{ (units attack)}}{1 \text{ (unit of 4 months)}} = 8.2 \text{ attacks per 4 months}$$

Example: shark attacks

$Y \sim \text{Poisson}(\theta = 8.2)$.

$$P(\text{no attacks } \overset{\text{next 4 months}}{\text{ne}\cancel{\text{xt year}}\text{r}}) = f(0) = e^{-8.2} \cdot \frac{8.2^0}{0!}$$

$$\approx 0.000275$$

$$P(\text{at least 5 attacks}) = 1 - P(\text{at most 4 attacks})$$

$$= 1 - F(4)$$

$$= 1 - f(0) - f(1) - f(2) - f(3) - f(4)$$

$$= 1 - e^{-8.2}\frac{8.2^0}{0!} - e^{-8.2}\frac{8.2^1}{1!} - e^{-8.2}\frac{8.2^2}{2!}$$

$$- e^{-8.2}\frac{8.2^3}{3!} - e^{-8.2}\frac{8.2^4}{4!}$$

$$\approx 0.9113$$

$$P(\text{more than 30 attacks}) = 1 - P(\text{at least 30 attacks})$$

$$= 1 - e^{-8.2}\sum_{i=0}^{30}\frac{8.2^x}{x!} = 1 - e^{-8.2} \cdot 4.251 \times 10^{10}$$

$$\approx 9.53 \times 10^{-10}$$

$$f(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x}{x!} \cdot e^{-\lambda}$$

$x-1 \to x$

$$= \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} e^{-\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{x!} e^{-\lambda}$$

$$= \lambda \boxed{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda}} = 1$$

$$= \lambda$$

$$E(X(X-1)) = E(X^2) - E(X).$$

$$\| $$

$$\sum_{x=0}^{\infty} \underline{x(x-1)} \frac{\lambda^x}{\underline{x!}} e^{-\lambda}$$

$$= \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} e^{-\lambda}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^{x+2}}{x!} e^{-\lambda} = \lambda^2 \underbrace{\boxed{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda}}}_{} = \lambda^2$$

$$\boxed{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda}} = 1$$

$$E(X^2) = \lambda^2 + \lambda.$$

$$Var(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$