Homework 7

Due February 13, 2020 at 11:59 PM

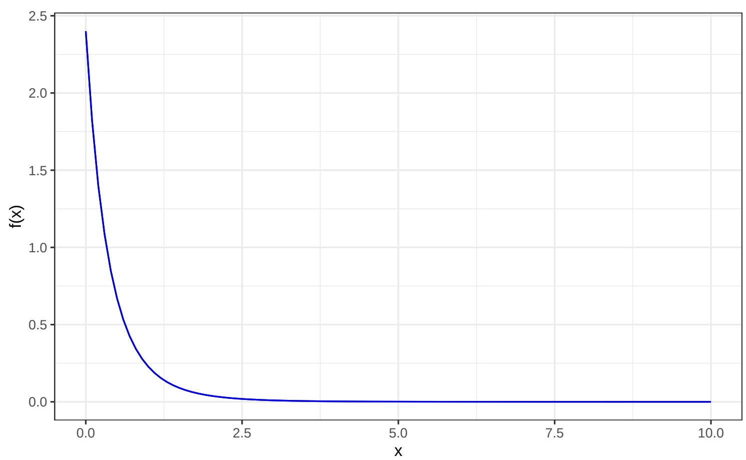
1. Let X be the total medical expenses incurred by a particular individual during a given year. Although X is a discrete random variable, suppose its distribution is quite well approximated by a continuous distribution with pdf $f(x) = k(1 + x/2.5)^{-7}$ for $x \geq 0$.

   (a) What is the value of k? (2 points)

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty k(1 + x/2.5)^{-7}dx = 2.5k(-\frac{1}{6})(1 + \frac{x}{2.5})^{-6}\Big|_0^\infty = \frac{k}{2.4} = 1$$

   Therefore $k = 2.4$.

   (b) Graph the pdf of X. (2 points)



   (c) What are the expected value and standard deviation of total mediacal expenses? (4 points)

$$E(X) = \int_0^\infty x \cdot 2.4(1 + x/2.5)^{-7}dx$$

   Let $1 + x/2.5 = u \Rightarrow x = 2.5(u - 1)$, $dx = 2.5du$, the limits are then 1 and ∞.

$$= \int_1^\infty 2.5(u-1) \cdot 2.4u^{-7}2.5du$$

$$= 15(-\frac{1}{5}u^{-5} + \frac{1}{6}u^{-6})\Big|_0^\infty = 0.5$$

$$E(X^2) = \int_0^\infty x^2 \cdot 2.4x(1 + x/2.5)^{-7}dx$$

Let u = 1 + x/2.5,

$$\begin{aligned}
&= \int_1^{\infty} (2.5(u-1))^2 \cdot 2.4u^{-7} \cdot 2.5du \\
&= \frac{75}{2}\left(-\frac{1}{4}u^{-4} + \frac{2}{5}u^{-5} + u^{-7}\right)\Big|_0^{\infty} = 0.625
\end{aligned}$$

Therefore

$$SD(X) = \sqrt{Var(X)} = \sqrt{E(X^2) - (E(X))^2} = \sqrt{0.625 - 0.5^2} = 0.612.$$

(d) This individual is covered by an insurance plan that entails a \$500 deductible provision (so the first \$500 worth of expenses are paid by the individual). Then the plan will pay 80% of any additional expenses exceeding \$500, and the maximum payment by the individual (including the deductible amount) is \$2500. Let Y denote the amount of this individual's medical expenses paid by the insurance company. What is the expected value of Y? [Hint: First figure out what value of X corresponds to the maximum out-of-pockect expenses of \$2500. Then write an expression for Y as a function of X (which involves several different pieces) and calculate the expected value of this function.] (4 points)

The maximum out-of-pocket expense occurs when $500+0.2(x-500) = 2500$. Solving this equation, we have $x = 10500$. Therefore, the amount paid by the insurance Y can be wriiten as a function of X, $Y = g(X)$, where

$$g(X) = \begin{cases} 0 & , X \leq 500 \\ 0.8(X-500) & , 500 < X \leq 10500 \\ X - 2500 & , X > 10500 \end{cases}$$

Then $E(Y) = E(g(X))$, we have

$$\begin{aligned}
E(g(X)) &= \int_0^{\infty} g(x)f(x)dx \\
&= \int_0^{500} 0dx + \int_{500}^{10500} 0.8(x-500) \cdot 2.4(1+x/2.5)^{-7}dx + \int_{10500}^{\infty} (x-2500) \cdot 2.4(1+x/2.5)dx \\
&= 1.22 \times 10^{-12} + 1.84 \times 10^{-18} \\
&\approx 1.22 \times 10^{-12}
\end{aligned}$$

Another answer: If X and Y are in 1000s dollars as in the original question, then

$$g(X) = \begin{cases} 0 & , X \leq 0.5 \\ 0.8(X-0.5) & , 0.5 < X \leq 10.5 \\ X - 2.5 & , X > 10.5 \end{cases}$$

Then

$$
\begin{aligned}
E(Y) &= E(g(X)) \\
&= \int_0^\infty g(x)f(x)dx \\
&= \int_0^{0.5} 0dx + \int_{0.5}^{10.5} 0.8(x-0.5)\cdot 2.4(1+x/2.5)^{-7}dx + \int_{10.5}^\infty (x-2.5)\cdot 2.4(1+x/2.5)^{-7}dx \\
&= 0.1608
\end{aligned}
$$

2. In a system with a large number of particles, the magnitude of velocity X of these particles can be described by the Maxwell distritbuion, the pdf is given by

$$
f(x) = \begin{cases} Ax^2 e^{-x^2/b} & , x > 0 \\ 0 & , \text{otherwise} \end{cases},
$$

where $b = m/(2kT)$. k is the Bolzmann's constant, T is the thermodynamic temperature, and m is the particle mass. Suppose b is known, express the constant A in terms of b.

(4 points)

We know the variance of the standard normal distribution is 1, so

$$
\int_{-\infty}^\infty z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 1 \Rightarrow \int_{-\infty}^\infty z^2 e^{-z^2/2} dz = \sqrt{2\pi}.
$$

And with the proerty of even function, we have

$$
\int_0^\infty z^2 e^{-z^2/2} dz = \frac{1}{2}\int_{-\infty}^\infty z^2 e^{-z^2/2} dz = \frac{\sqrt{2\pi}}{2}.
$$

Now let $x/\sqrt{b} = z/\sqrt{2} \Rightarrow x = \sqrt{\frac{b}{2}}z$.

So

$$
\begin{aligned}
\int_0^\infty x^2 e^{-x^2/b} dx &= \int_0^\infty \frac{b}{2} z^2 e^{-z^2/2} \sqrt{\frac{b}{2}} dz \\
&= \frac{b}{2}\sqrt{\frac{b}{2}} \frac{\sqrt{2\pi}}{2} \\
&= \frac{b\sqrt{b\pi}}{4}
\end{aligned}
$$

Since $\int_0^\infty f(x)dx = A\int_0^\infty x^2 e^{-x^2/b} dx = 1$, we have

$$
A\frac{b\sqrt{b\pi}}{4} = 1 \Rightarrow A = \frac{4}{b\sqrt{b\pi}}.
$$

3. P. 263: 2 (2×9 points)

   (a) $P(Z < -0.62) = \Phi(-0.62) = 0.2676$
   (b) $P(Z > 1.06) = 1 - P(Z \leq 1.06) = 1 - \Phi(1.06) = 1 - 0.8554 = 0.1446$
   (c) $P(-0.37 < Z < 0.51) = P(Z < 0.51) - P(Z \leq -0.37) = 0.6950 - 0.3557 = 0.3393$
   (d) $P(|Z| \leq 0.47) = P(-0.47 \leq Z \leq 0.47) = P(Z \leq 0.47) - P(Z < -0.47) = 0.6808 - 0.3192 = 0.3616$
   (e) $P(|Z| > 0.93) = P(Z < -.93) + P(Z > 0.93) = 2P(Z < -0.93) = 2(0.1762) = 0.3524$
   (f) $P(-3 < Z < 3) = P(Z < 3) - P(Z \leq -3) = 0.9987 - 0.0013 = 0.9974$
   (g) Looking up 0.90 in the body of the table, $\# \approx 1.28$
   (h) $P(|Z| < \#) = 0.90$ is equivalent to $P(Z < \#) = 0.95$ by symmetry. So $\# \approx 1.645$.
   (i) $P(|Z| > \#) = 0.03$ is equivalent to $P(Z < \#) = 1 - 0.03/2 = 0.985$ by symmetry. So $\# \approx 2.17$

4. P. 263: 3 (2×8 points)

   Probabilities involving X are just areas under the normal curve with $\mu = 43.0$ and $\sigma = 3.6$. Each of these areas has an equal corresponding area under the standard normal curve. Define $Z = \frac{X-43.0}{3.6}$. Then Z is a standard normal random variable. Re-express each of the problems below in terms of Z.

   (a) $P(X < 45.2) = P(Z < 0.61) = 0.7291$
   (b) $P(X \leq 41.7) = P(Z \leq -0.36) = 0.3594$
   (c) $P(43.8 < X < 47.0) = P(0.22 < Z < 1.11) = P(Z < 1.11) - P(Z \leq 0.22) = 0.8665 - 0.5871 = 0.2794$
   (d) $P(|X - 43.0| \leq 2.0) = P(-0.56 \leq Z \leq 0.56) = P(Z \leq 0.56) - P(Z < -0.56) = 0.7123 - 0.2877 = 0.4246$
   (e) $P(|X - 43.0| > 1.7) = 1 - P(|Z| \leq 0.47) = 1 - (P(Z \leq 0.47) - P(Z < -0.47)) = 1 - (0.6808 - 0.3192) = 0.6384$
   (f) $P(X < \#) = 0.95$ is equivalent to $P(Z < \frac{\#-43.0}{3.6}) = 0.95$. So

   $$\frac{\# - 43.0}{3.6} \approx 1.645 \Rightarrow \# \approx 48.922$$

   (g) $P(X \geq \#) = 0.3$ is equivalent to $P(X < \#) = 0.7$, which is equivalent to $P(Z < \frac{\#-43.0}{3.6}) = 0.70$. So

   $$\frac{\# - 43.0}{3.6} \approx 0.52 \Rightarrow \# \approx 44.872$$

   (h) $P(|X - 43.0| > \#) = 0.5$ is equivalent to $P(|\frac{X-43.0}{3.6}| \leq \frac{\#}{3.6}) = 0.95$. This is equivalent to $P(|Z| \leq \#/3.6) = 0.95 \Rightarrow P(Z \leq \#/3.6) = 0.975$. So

   $$\frac{\#}{3.6} = 1.96 \Rightarrow \# = 7.056.$$