Spring 2020 **Stat 305 (Section 4)** **Quiz 3**

Name:
Solution

Total points for the exam is 50. Points for individual questions are given at the beginning of each problem. Show all your calculations clearly. Put final answers in the box at the right (except for the diagrams!).

Some quantiles might be used: $z_{0.975} = 1.96$, $z_{0.95} = 1.64$.

1. [5+5+5=15 points]

Suppose that a factory manufactures a rod-shaped part. The lengths of this kind of part are normally distributed with mean 7.50 cm and standard deviation 0.015 cm (*round to 3 decimal places*).

(a) Evaluate the probability that the length of the next manufactured part is between 7.49 cm and 7.51 cm.

Let X be the length of this kind of part.

probability= 0.495

then $X \sim N(7.5, 0.015^2)$.

$\Rightarrow Z = \frac{X-7.5}{0.015} \sim N(0,1)$.

$$\Rightarrow P(7.49 < X < 7.51) = P\left(\frac{7.49-7.5}{0.015} < \frac{X-7.5}{0.015} < \frac{7.51-7.5}{0.015}\right)$$

$$= P(-0.667 < Z < 0.667) = P(Z < 0.667) - P(Z < -0.667)$$

$$= \Phi(0.667) - \Phi(-0.667) = 2\Phi(0.667) - 1 = 0.495$$

(b) Evaluate the probability that a the mean length of the next 30 parts is between 7.49 cm and 7.51 cm (*round to 4 decimal places*).

$X_1, X_2, \ldots, X_{30} \sim N(7.5, 0.015^2)$

probability= 0.9997

$\Rightarrow \bar{X} \sim N\left(7.5, \frac{0.015^2}{30}\right)$

$$\Rightarrow P(7.49 < \bar{X} < 7.51) = P\left(\frac{7.49-7.5}{0.015/\sqrt{30}} < \frac{\bar{X}-7.5}{0.015/\sqrt{30}} < \frac{7.51-7.5}{0.015/\sqrt{30}}\right)$$

$$= P(-3.651 < Z < 3.651)$$

$$= \Phi(3.651) - \Phi(-3.651)$$

$$= 2\Phi(3.651) - 1 = 0.9997$$

(c) If the factory want to control that 95% of the parts' lengths are between 7.49 cm and 7.51 cm, what manufaturing precision (as measured by standard deviation σ) is required if the mean length is still 7.5 cm (*round to 4 decimal places*)?

$X \sim N(7.5, \sigma^2)$

s.d.= 0.0051

$$\Rightarrow P(7.49 < X < 7.51)$$
$$= P\left(\frac{7.49 - 7.5}{\sigma} < \frac{X - 7.5}{\sigma} < \frac{7.51 - 7.5}{\sigma}\right)$$
$$= P\left(\frac{-0.01}{\sigma} < Z < \frac{0.01}{\sigma}\right)$$
$$= \Phi\left(\frac{0.01}{\sigma}\right) - \Phi\left(-\frac{0.01}{\sigma}\right)$$
$$= 2\Phi\left(\frac{0.01}{\sigma}\right) - 1 = 0.95.$$
$$\Rightarrow \Phi\left(\frac{0.01}{\sigma}\right) = 0.975$$
$$\Rightarrow \frac{0.01}{\sigma} = 1.96 \Rightarrow \sigma = 0.0051.$$

2. [5 points]

Suppose that X, Y, and W are independent random variables each with mean 5 and standard deviation 1. Find the mean and standard deviation of $(X + 3Y - \sqrt{2}W)$ (*round to 4 decimal places*).

mean= 12.9289

s.d.= 3.4641

$$E(X + 3Y - \sqrt{2}W)$$
$$= E(X) + 3E(Y) - \sqrt{2}E(W)$$
$$= 5 + 3 \times 5 - \sqrt{2} \times 5 = 20 - 5\sqrt{2} = 12.9289$$

$$Var(X + 3Y - \sqrt{2}W)$$
$$= Var(X) + 3^2 Var(Y) + (-\sqrt{2})^2 Var(W)$$
$$= 1 + 9 + 2 = 12$$

$$SD(X + 3Y - \sqrt{2}W) = \sqrt{12} \approx 3.4641$$

3. [4 + 6 points]

Suppose that the weights for each bottle of the bottled water from a factory are normally distributed with mean 500 g and standard deviation 5 g. 40 bottles of water will be packed in a box.

(a) Find the mean and variance for the weight of a box of bottled water (ignore the weight of the box).

$X_1, X_2, \ldots, X_{40} \sim N(500, 5^2)$

mean= 20000

variance= 1000

$E\left(\sum_{i=1}^{40} X_i\right) = \sum_{i=1}^{40} E(X_i)$

$= 40 \times 500 = 20000$

$Var\left(\sum_{i=1}^{40} X_i\right) = \sum_{i=1}^{40} Var(X_i) = 40 \times 5^2 = 1000$

(b) Find the probability that the weight of a box of bottled water is less than 19950 g (ignore the weight of the box) (*round to 4 decimal places*).

probability= 0.0569

$\sum_{i=1}^{40} X_i \sim N(20000, 1000)$

$P\left(\sum_{i=1}^{40} X_i < 19950\right)$

$= P\left(\frac{\sum_{i=1}^{40} X_i - 20000}{\sqrt{1000}} < \frac{19950 - 20000}{\sqrt{1000}}\right)$

$= P(Z < -1.5811)$

$= 0.0569$

4. [5+5+10=20 points]

Some students measured the heights of 405 steel punches of a particular type. These were all from a single manufacturer and were supposed to have heights of 0.500 in. (The stamping machine in which these are used is designed to use 0.500 in. punches.) The students measurements had $\bar{x} = 0.5004$ in. and $s = 0.0026$ in.

(a) Give a 95% two-sided confidence interval for the mean height of the steel punch of this particular type (*round to 5 decimal places*).

The sample size is large, so the quantile used is $z_{1-\alpha/2} = z_{0.975}$

$C.I = (\ 0.50015 \ , \ 0.50065 \ )$

$$\bar{X} \pm z_{0.975} \cdot \frac{s}{\sqrt{n}}$$

$$= 0.5004 \pm 1.96 \cdot \frac{0.0026}{\sqrt{405}}$$

$$\simeq 0.5004 \pm 0.00025$$

$$= (0.50015 \ , \ 0.50065)$$

(b) Give a 95% upper confidence bound for the mean height of the steel punch of this particular type (*round to 5 decimal places*).

We use $z_{1-\alpha} = z_{0.95}$

upper conf. bd = 0.50061

$$\bar{x} + z_{1-\alpha} \cdot \frac{s}{\sqrt{n}}$$

$$= 0.5004 + 1.64 \cdot \frac{0.0026}{\sqrt{405}}$$

$$= 0.50061$$

(c) Test if the mean height of the steel punch of this particular type differs from the desired height of 0.500 in. with **p-value**. **Show all the steps of the testing procedure.**

1. $H_0: \mu = 0.500$, $H_a: \mu \neq 0.500$

2. Test statistic:

$$Z = \frac{\bar{X} - 0.5}{s / \sqrt{n}}$$

Assuming $X_1, X_2, \ldots X_{405}$ are iid. then under H_0,

$Z \sim N(0,1)$.

3. The observed test statistic:

$$z = \frac{0.5004 - 0.5}{0.0026/\sqrt{405}} = 3.096094$$

This is a two-sided test, so

$$\text{p-value} = P(|Z| > |z|) = P(|Z| > 3.096094)$$

$$= 2P(Z < -3.096094)$$

$$= 0.00196$$

4. The pvalue is very small, so we reject H_0.

5. There is significant evidence the the mean height of the steel punch of this particular type differs from the desired height of 0.500 in.