STAT 305 D Homework 6

Due March 7, 2013 at 12:40 PM in class

Remember: μ denotes $E(X)$, σ^2 denotes Var(X),m and σ denotes SD(X).

1.

The article "Modeling Sediment and Water Column Interactions for Hydrophobic Pollutants" (*Water Research,* 1984: 1169–1174) suggests the uniform distribution on the interval (7.5, 20) as a model for depth (cm) of the bioturbation layer in sediment in a certain region.

a. What are the mean and variance of depth?

b. What is the cdf of depth?

c. What is the probability that observed depth is at most 10? Between 10 and 15?

d. What is the probability that the observed depth is within 1 standard deviation of the mean value? Within 2 standard deviations?

a. If X is uniformly distributed on the interval from A to B, then

$$E(X) = \int_A^B x \cdot \frac{1}{B-A}\,dx = \frac{A+B}{2}, E(X^2) = \frac{A^2 + AB + B^2}{3}$$

V(X) = E(X^2) – [E(X)]^2 = $\frac{(B-A)^2}{2}$.

With A = 7.5 and B = 20, E(X) = 13.75, V(X) = 13.02

b. F(x) = $\begin{cases} 0 & x < 7.5 \\ \frac{x-7.5}{12.5} & 7.5 \le x < 20 \\ 1 & x \ge 20 \end{cases}$

c. P(X ≤ 10) = F(10) = .200; P(10 ≤ X ≤ 15) = F(15) – F(10) = .4

d. σ = 3.61, so $\mu \pm \sigma$ = (10.14, 17.36)
Thus, P($\mu - \sigma \le X \le \mu + \sigma$) = F(17.36) – F(10.14) = .5776
Similarly, P($\mu - 2\sigma \le X \le \mu + 2\sigma$) = P(6.53 ≤ X ≤ 20.97) = 1

2.

Let X be the total medical expenses (in 1000s of dollars) incurred by a particular individual during a given year. Although X is a discrete random variable, suppose its distribution is quite well approximated by a continuous distribution with pdf $f(x) = k(1 + x/2.5)^{-7}$ for $x \geq 0$.

a. What is the value of k?

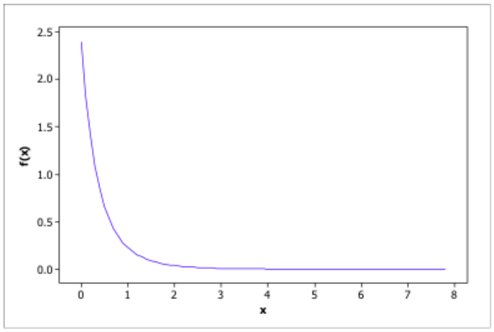
b. Graph the pdf of X.

c. What are the expected value and standard deviation of total medical expenses?

d. This individual is covered by an insurance plan that entails a \$500 deductible provision (so the first \$500 worth of expenses are paid by the individual). Then the plan will pay 80% of any additional expenses exceeding \$500, and the maximum payment by the individual (including the deductible amount) is \$2500. Let Y denote the amount of this individual's medical expenses paid by the insurance company. What is the expected value of Y?
[*Hint:* First figure out what value of X corresponds to the maximum out-of-pocket expense of \$2500. Then write an expression for Y as a function of X (which involves several different pieces) and calculate the expected value of this function.]

a. $1 = \int_0^\infty k(1+x/2.5)^{-7}\,dx = \frac{2.5k}{-6}(1+x/2.5)^{-6}\Big|_0^\infty = \frac{k}{2.4} \rightarrow k = 2.4$

b.



c. E(X) = $\int_0^\infty x \cdot 2.4(1+x/2.5)^{-7}\,dx = \int_1^\infty 2.5(u-1)\cdot 2.4u^{-7}\cdot 2.5du = 0.5$, or \$500. Similarly, $E(X^2) = \int_0^\infty x^2 \cdot 2.4(1+x/2.5)^{-7}\,dx = \int_1^\infty (2.5(u-1))^2 \cdot 2.4u^{-7}\cdot 2.5du = 0.625$, so V(X) = $0.625 - (0.5)^2 = 0.375$, and $\sigma_X = \sqrt{0.375} = 0.612$, or \$612.

d. The maximum out-of-pocket expense, \$2500, occurs when \$500 + 20%(X – \$500) equals \$2500; this accounts for the \$500 deductible and the 20% of costs above \$500 not paid by the insurance plan. Solve: \$2,500 = \$500 + 20%(X – \$500) → X = \$10,500. At that point, the insurance plan has already paid \$8,000, and the plan will pay all expenses thereafter.

A more thorough explanation of part (d):
Let $Y = g(X)$. Then:

$$g(X) = \begin{cases} 0 & X \le 500 \\ 0.8(x-500) & 500 < X \le m \\ X - 2500 & X > m \end{cases}$$

where m is the value of X corresponding to the maximum out-of-pocket expense of \$2500. This maximum individual out-of-pocket expense is equal to the \$500 deductible plus the 20% of the rest not paid by insurance:

$$2500 = 500 + 0.2(m - 500)$$

Hence:

$$m = 10500$$

and Y (a.k.a., $g(X)$) becomes:

$$g(X) = \begin{cases} 0 & X \le 500 \\ 0.8(x-500) & 500 < X \le 10500 \\ X - 2500 & X > 10500 \end{cases}$$

The question asks for $E(Y)$. This is $E(g(X))$:

$$
\begin{aligned}
E(g(X)) &= \int_{-\infty}^{\infty} g(x)f(x)dx \\
&= \int_{-\infty}^{0} 0 \cdot 0dx + \int_{0}^{500} 0 \cdot 2.4(1+x/2.5)^{-7}dx \\
&\quad + \int_{500}^{10500} 0.8(x-500) \cdot 2.4(1+x/2.5)^{-7}dx \\
&\quad + \int_{10500}^{\infty} (x-2500) \cdot 2.4(1+x/2.5)^{-7}dx \\
&= \int_{500}^{10500} 1.92(x-500)(1+x/2.5)^{-7}dx + \int_{10500}^{\infty} 2.4(x-2500)(1+x/2.5)^{-7}dx \\
&\approx 1.22 \times 10^{-12} + 1.84 \times 10^{-18} \\
&= 1.22 \times 10^{-12}
\end{aligned}
$$

Those greedy insurance companies...

3.

For X with a continuous distribution specified by the probability density

$$
f(x) = \begin{cases} .5x & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}
$$

find $P[X < 1.0]$ and find the mean, EX.

$$P(X < 1.0) = \int_0^1 .5x\,dx = .25.$$

using equation (5-18),

$$EX = \int_0^2 .5x^2\,dx = \frac{4}{3}.$$

4.

Suppose that Z is a standard normal random variable. Evaluate the following probabilities involving Z:

(a) $P[Z < -.62]$ (b) $P[Z > 1.06]$
(c) $P[-.37 < Z < .51]$ (d) $P[|Z| \leq .47]$
(e) $P[|Z| > .93]$ (f) $P[-3.0 < Z < 3.0]$

Now find numbers # such that the following statements involving Z are true:

(g) $P[Z \leq \#] = .90$ (h) $P[|Z| < \#] = .90$
(i) $P[|Z| > \#] = .03$

The values of $\Phi(z) = P(Z \leq z)$ are given in Table B-3. All of these probabilities correspond to areas under the standard normal curve.

(a) $P(Z < -.62) = \Phi(-.62) = .2676$.

(b) $P(Z > 1.06) = 1 - P(Z \leq 1.06) = 1 - \Phi(1.06) = 1 - .8554 = .1446$.

(c) $P(-.37 < Z < .51) = P(Z < .51) - P(Z \leq -.37) = .6950 - .3557 = .3393$.

(d) $P(|Z| \leq .47) = P(-.47 \leq Z \leq .47) = P(Z \leq .47) - P(Z < -.47) = .6808 - .3192 = .3616$.

(e) $P(|Z| > .93) = P(Z < -.93) + P(Z > .93) = 2(P(Z < -.93)) = 2(.1762) = .3524$.

(f) $P(-3.0 < Z < 3.0) = P(Z < 3.0) - P(Z \leq -3.0) = .9987 - .0013 = .9974$.

(g) Looking up .90 in the body of the table, $\# \approx 1.28$.

(h) $P(|Z| < \#) = .90$ is equivalent to $P(Z < \#) = .95$ (by symmetry). Looking up .95 in the body of the table, $\# \approx 1.645$.

(i) $P(|Z| > \#) = .03$ is equivalent to $P(Z < \#) = .985$ (by symmetry). Looking up .985 in the body of the table, $\# \approx 2.17$.

5. Find the following:

a.

a. $P(X \leq 2)$, where $X \sim N(5, 2)$

$$
\begin{aligned}
P(X \leq 2) &= P(\frac{X-5}{\sqrt{2}} \leq \frac{2-5}{\sqrt{2}}) \\
&= P(Z \leq -2.12) \\
&= \Phi(-2.12) \\
&= 0.0170
\end{aligned}
$$

b.

b. $P(|X| \leq 5)$, where $X \sim N(0, 15)$

$$
\begin{aligned}
P(|X| \leq 5) &= P(-5 \leq X \leq 5) \\
&= P(\frac{-5-0}{\sqrt{15}} \leq \frac{X-0}{\sqrt{15}} \leq \frac{5-0}{\sqrt{15}}) \\
&= P(-1.29 \leq Z \leq 1.29) \\
&= P(Z \leq 1.29) - P(Z \leq -1.29) \\
&= \Phi(1.29) - \Phi(-1.29) \\
&= 0.901 - 0.095 \\
&= 0.803
\end{aligned}
$$

c.

c. $P(T \geq 1.9432)$, where $T \sim t_6$

$$\begin{aligned} P(T \geq 1.9432) &= 1 - P(T < 1.9432) \\ &= 1 - P(T \leq 1.9432) \\ &= 1 - P(T \leq t_{6,0.95}) \\ &= 1 - 0.95 \\ &= 0.05 \end{aligned}$$

d.

$$P(|T| \leq 1.372), \text{ where } T \sim t_{10}.$$

$$\begin{aligned} P(|T| \leq 1.372) &= P(-1.372 \leq T \leq 1.372) \\ &= P(T \leq 1.372) - P(T \leq -1.372) \\ &= P(T \leq 1.372) - P(T \geq 1.3724) \quad \text{by symmetry} \\ &= P(T \leq 1.372) - (1 - P(T \leq 1.372)) \\ &= 2P(T \leq 1.372) - 1 \\ &= 2P(T \leq t_{10,0.9}) - 1 \\ &= 2 \cdot 0.9 - 1 \\ &= 0.8 \end{aligned}$$

e.

$$P(X > 7.779), \text{ where } X \sim \chi^2_4$$

$$
\begin{aligned}
P(X > 7.779) &= 1 - P(X \leq 7.779) \\
&= 1 - P(X \leq \chi^2_{4,0.9}) \\
&= 1 - 0.9 \\
&= 0.1
\end{aligned}
$$

f.

$$P(X \leq 11.3449), \text{ where } X \sim \chi^2_3$$

$$
\begin{aligned}
P(X \leq 11.3449) &= P(X \leq \chi^2_{3,0.99}) \\
&= 0.99
\end{aligned}
$$

g.

$$P(F \leq 30.8165), \text{ where } F \sim F_{2,3}$$

$$
\begin{aligned}
P(F \leq 30.8165) &= P(F \leq F_{2,3,0.99} \\
&= 0.99
\end{aligned}
$$

h.

$$P(F > 4.8759), \text{ where } F \sim F_{7,5}$$

$$\begin{aligned} P(F > 4.8759) &= 1 - P(F \leq 4.8759) \\ &= 1 - P(F \leq F_{7,5,0.95} \\ &= 1 - 0.95 \\ &= 0.05 \end{aligned}$$

6. Review the notation explained on the last slide of the Feb 28 lecture. Then, find the following.

    a. $z_{0.8}$ = Q(0.8) = 0.84

    b. $t_{3,0.9} = 1.638$

    c. $\chi^2_{2,0.95} = 5.991$

    d. $F_{4,2,0.99} = 99.249$

7. Let $S \sim \text{SquigglyJoe}(5, 3, \xi, \xi_\xi, \xi^{\xi^\xi}_{\xi_\xi}, \xi^{\xi^{\xi_\xi}}_{\xi_{\xi_\xi}})$, where $E(S) = 42$ and $Var(S) = 101$. Find the following:

    a. E(5 S + 7) = 5 E(S) + 7 $= 5 \cdot 42 + 7 = 217$

    b. Var(10S -2) $= 10^2$ Var(S) $= 10^2 \cdot 101 = 10100$

8. Weekly feedback. You get full credit as long as you write something.

    a. Is there any aspect of the subject matter that you currently struggle with? If so, what specifically do you find difficult or confusing? The more detailed you are, the better I can help you.

       You got full credit as long as you wrote something.

    b. Do you have any questions or concerns about the material, class logistics, or anything else? If so, fire away. You got full credit as long as you wrote something.