# STAT 305 FINAL EXAM VERSION A SOLUTIONS

**Exercise 3.1.**
Say we have a continuous random variable $X$ with the following pdf:

$$f(x) = \begin{cases} k & : 0 \leq x \leq 5 \\ 0 & : x \text{ otherwise.} \end{cases}$$

where $k$ is some real constant.

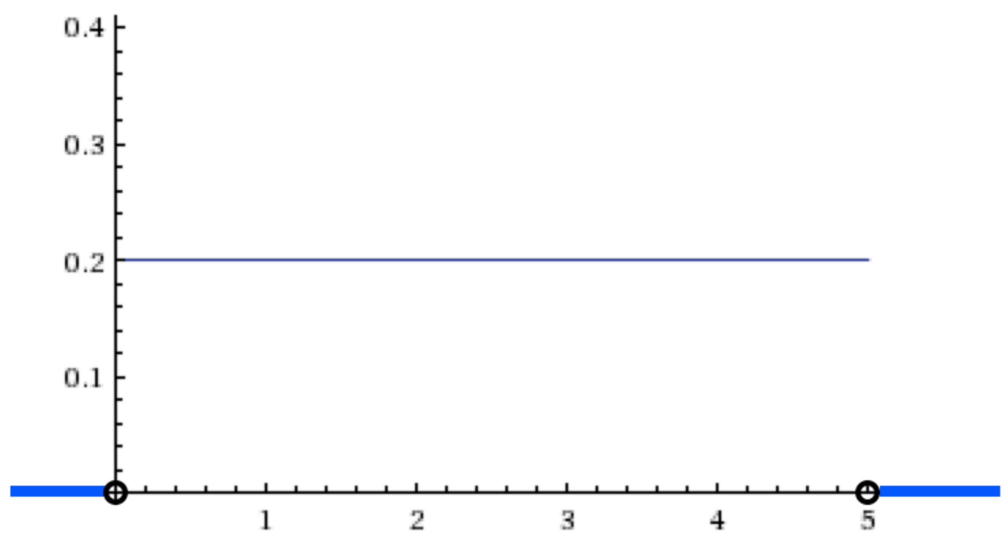
a. Find $k$ such that $f(x)$ is a valid pdf.

We know $\int_{-\infty}^{\infty} f(x)dx = 1$ if the pdf is valid.

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^5 kdx = [k \cdot x]_{x=0}^5 = 5k$$

Hence, $k = 1/5$.

b. Sketch a graph of $f(x)$ on the Cartesian plane.



c. Find the cdf $F(x)$ of $X$.

If $x < 0$, then:

$$0 \leq F(x) = \int_{-\infty}^{x} f(t)dt \leq \int_{-\infty}^{0} f(t)dt = \int_{-\infty}^{0} 0dt = 0$$

And hence, $F(x) = 0$. If $0 \leq x \leq 5$, then:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt = 0 + \int_{0}^{x} \frac{t}{5}dt = \frac{x}{5}$$

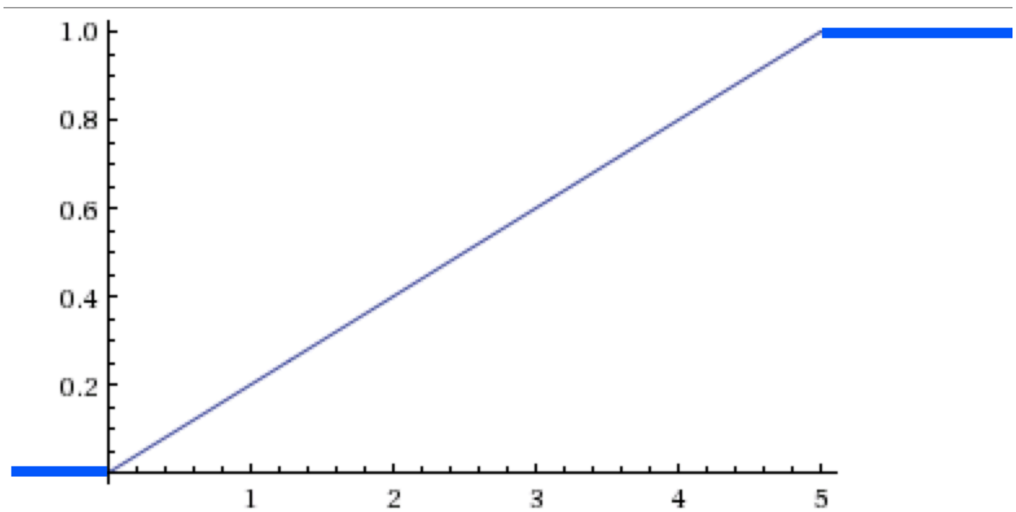
If $x > 5$, then:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{5} f(t)dt + \int_{5}^{x} f(t)dt \\ &= 0 + (\frac{x}{5})_{x=0}^{5} + 0 \\ &= 1 \end{aligned}$$

Thus,

$$F(x) = \begin{cases} 0 & : x < 0 \\ \frac{x}{5} & : 0 \leq x \leq 5 \\ 1 & : x > 5 \end{cases}$$

d. Sketch a graph of $F(x)$ on the Cartesian plane.




e. Find $P(0.2 \le X \le 2)$

$$P(0.2 \le X \le 2) = F(2) - F(0.2) = \frac{2}{5} - \frac{0.2}{5} = 0.36$$

f. Find $P(X \ge 3)$

$$P(X \ge 3) - 1 - P(X \le 3) = 1 - F(3) = 1 - 3/5 = 2/5$$

g. Find $P(X = 0.5)$

$P(X = 0.5) = P(X \le 0.5) - P(X < 0.5) = F(0.5) - F(0.5) = 0$

h. Find $E(X)$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^5 x \cdot \frac{1}{5} = \frac{1}{10} x^2 |_{x=0}^5 = 2.5$$

i. Find $Var(X)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^5 x^2 \cdot \frac{1}{5} = \frac{1}{15}x^3|_{x=0}^5 = \frac{125}{15} = \frac{25}{3}$$

Hence:

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{25}{3} - (2.5)^2 = 2.0833$$

**Exercise 3.2.**

a. $z_{0.8}$

$$z_{0.8} = Q(0.8) = 0.84$$

b. $z_{0.02}$

$$z_{0.02} = Q(0.02) = -2.05$$

c. $c$ such that $P(|Z| < c) = 0.9$, where $Z \sim N(0, 1)$

$$\begin{aligned} 0.9 &= P(|Z| < c) \\ &= P(-c < Z < c) \\ &= P(Z < c) - P(Z \leq -c) \\ &= P(Z \leq c) - P(Z \geq c) \quad \text{by symmetry} \\ &= P(Z \leq c) - (1 - P(Z \leq c)) \\ &= 2P(Z \leq c) - 1 \\ &= 2\Phi(c) - 1 \end{aligned}$$

Hence:

$$\begin{aligned}
&0.9 = 2\Phi(c) - 1 \\
\Rightarrow &\Phi(c) = 0.95 \\
\Rightarrow &c = \Phi^{-1}(0.95) = \boxed{1.64}
\end{aligned}$$

d. $c$ such that $P(|X| > c) = 0.05$, where $X \sim N(0, 8)$

$$\begin{aligned}
0.05 &= P(|X| > c) \\
&= P(X > c \text{ or } X < -c) \\
&= P(X > c) + P(X < -c) \\
&= P(\frac{X-0}{\sqrt{8}} > \frac{c-0}{\sqrt{8}}) + P(\frac{X-0}{\sqrt{8}} < \frac{-c-0}{\sqrt{8}}) \quad = P(Z > \frac{c}{\sqrt{8}}) + P(Z < \frac{-c}{\sqrt{8}}) \\
&= P(Z < \frac{-c}{\sqrt{8}}) + P(Z < \frac{-c}{\sqrt{8}}) \quad \text{by symmetry} \\
&= 2P(Z < \frac{-c}{\sqrt{8}}) \\
&= 2P(Z \leq \frac{-c}{\sqrt{8}}) \\
&= 2\Phi(\frac{-c}{\sqrt{8}})
\end{aligned}$$

Hence:

$$\begin{aligned}
0.05 &= 2\Phi(\frac{-c}{\sqrt{8}}) \\
0.025 &= \Phi(\frac{-c}{\sqrt{8}}) \\
\Phi^{-1}(0.025) &= \frac{-c}{\sqrt{8}} \\
-1.96 &= \frac{-c}{\sqrt{8}} \\
c &= (-1.96)(\sqrt{8}) \\
&= 5.54
\end{aligned}$$

e. $c$ such that $P(|T| < c) = 0.9$, where $T \sim t_4$

$$\begin{aligned}
0.9 &= P(|T| < c) \\
&= P(-c < T < c) \\
&= P(T < c) - P(T < -c) \\
&= P(T < c) - P(T > c) \quad \text{by symmetry} \\
&= P(T \le c) - (1 - P(T \le c)) \\
&= 2P(T \le c) - 1
\end{aligned}$$

Hence:

$$\begin{aligned}
0.9 &= 2P(T \le c) - 1 \\
0.95 &= P(T \le c)
\end{aligned}$$

Hence:

$$c = t_{4,0.95} = \boxed{2.1318}$$

f. $c$ such that $P(|T| > c) = 0.1$, where $T \sim t_9$

$$\begin{aligned}
0.1 &= P(T > c \text{ or } T < -c) \\
&= P(T > c) + P(T < -c) \\
&= P(T < -c) + P(T < -c) \quad \text{by symmetry} \\
&= 2P(T \le -c)
\end{aligned}$$

Hence:

$$\begin{aligned}
0.1 &= 2P(T \le -c) \\
0.05 &= P(T \le -c)
\end{aligned}$$

Hence:

$$\begin{aligned}
-c &= t_{9,0.05} \\
&= -1.833 \\
c &= 1.833
\end{aligned}$$

g. $t_{3,0.9}$

$$t_{3,0.9} = 1.638$$

h. $t_{1,0.99}$

$$t_{1,0.99} = 31.821$$

**Exercise 3.3.**
Find the following:

a. $E(X + 2Y)$, where $X \sim$ Weibull with mean 2.1 and variance 0.63 and is independent of $Y \sim$ Maxwell with mean 3.33 and variance 7.78

$$\begin{aligned} E(X + 2Y) &= E(X) + 2E(Y) \\ &= 2.1 + 2 \cdot 3.33 \\ &= 8.76 \end{aligned}$$

b. $Var(X + 2Y)$, where $X \sim$ Weibull with mean 2.1 and variance 0.63 and is independent of $Y \sim$ Maxwell with mean 3.33 and variance 7.78

$$\begin{aligned} Var(X + 2Y) &= Var(X) + 2^2 E(Y) \\ &= 0.63 + 4 \cdot 7.78 \\ &= 31.75 \end{aligned}$$

c. $E(3X + 5Y - 8Z)$, where $X, Y,$ and $Z$ are independent, $E(X) = 4$, $E(Y) = 3$, $E(Z) = 10$

$$\begin{aligned} E(3X + 5Y - 8Z) &= 3E(X) + 5E(Y) - 8E(Z) \\ &= 3 \cdot 4 + 5 \cdot 3 - 8 \cdot 10 \\ &= -53 \end{aligned}$$

d. $Var(3X + 5Y - 8Z)$, where $X, Y,$ and $Z$ are independent, $Var(X) = 9$, $Var(Y) = 16$, $Var(Z) = 2$

$$\begin{aligned} Var(3X + 5Y - 8Z) &= 3^2 Var(X) + 5^2 Var(Y) + 8^2 Var(Z) \\ &= 9 \cdot 9 + 25 \cdot 16 + 64 \cdot 2 \\ &= 609 \end{aligned}$$

e. $E(\overline{X})$, where $X_1, X_2, \ldots, X_n$ are iid with mean 0 and variance 2.

$$\begin{aligned}
E(\overline{X}) &= E(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n) \\
&= \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \cdots + \frac{1}{n}E(X_n) \\
&= \frac{1}{n}0 + \frac{1}{n}0 + \cdots + \frac{1}{n}0 \\
&= 0
\end{aligned}$$

**Exercise 3.4.**
Use the Central Limit Theorem to approximate the following:

a. $P(|\overline{X} - 1| < 2)$, where $X_1, X_2, \ldots, X_{41}$ are iid Exponential(4), each with mean 0.25 and variance 0.0625.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(0.25, 0.0625/41) = N(0.25, 0.0015)$. Hence:

$$\begin{aligned}
P(|\overline{X} - 1| < 2) &= P(-2 < \overline{X} - 1 < 2) \\
&= P(\frac{-2 + 0.75}{\sqrt{0.0015}} < \frac{\overline{x} - 0.25}{\sqrt{0.0015}} < \frac{2 + 0.75}{\sqrt{0.0015}}) \\
&\approx P(-32.27 < Z < 71.00) \\
&= P(Z < 71.00) - P(Z < -32.27) \\
&\approx 1 - 0 \\
&= 1
\end{aligned}$$

b. The number $c$ such that $P(\overline{X} > c) = 0.95$, where $X_1, X_2, \ldots, X_{26}$ are iid Gamma(1,2), each with mean 2 and variance 4.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(2, 4/26) = N(2, 0.154)$.

$$\begin{aligned}
0.95 &= P(\overline{X} > c) \\
&= P(\frac{\overline{X} - 2}{\sqrt{0.154}} > \frac{c - 2}{\sqrt{0.154}}) \\
&\approx P(Z > \frac{c - 2}{\sqrt{0.154}})
\end{aligned}$$

Hence:

$$
\begin{aligned}
0.95 &= P(Z > \frac{c-2}{\sqrt{0.154}}) \\
0.95 &= 1 - P(Z \leq \frac{c-2}{\sqrt{0.154}}) \\
0.05 &= P(Z \leq \frac{c-2}{\sqrt{0.154}}) \\
0.05 &= \Phi(\frac{c-2}{\sqrt{0.154}}) \\
\Phi^{-1}(0.05) &= \frac{c-2}{\sqrt{0.154}} \\
-1.64 &= \frac{c-2}{\sqrt{0.154}} \\
1.36 &= c
\end{aligned}
$$

c. $P(|\overline{X} - 5| > 1)$, where $X_1, X_2, \ldots, X_{38} \sim$ are iid $\chi^2_5$, each with mean 5 and variance 10.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(5, 10/38) = N(5, 0.263)$

$$
\begin{aligned}
P(|\overline{X} - 5| > 1) &= P(\overline{X} - 5 > 1) + P(\overline{X} - 5 < -1) \\
&= P\left(\frac{\overline{X} - 5}{\sqrt{0.263}} > \frac{1}{\sqrt{0.263}}) + P(\frac{\overline{X} - 5}{\sqrt{0.263}} < \frac{-1}{\sqrt{0.263}}\right) \\
&\approx P(Z > 1.95) + P(Z < -1.95) \\
&= 2P(Z \leq -1.95) \\
&= 0.0512
\end{aligned}
$$

**Exercise 3.5.**
David Bowie wanted to measure the quality of his music. So on $n$ =123 separate occasions, he flew to Tibet to perform for the Dalai Lama. On each visit, after meditation, tea, and a contemplative friendly discussion, Bowie performed a new piece, and the wise Lama rated the morality of Bowie's music on a scale from 0 to 100 (100 = perfectly virtuous, 0 = nauseatingly

abominable). (Disclaimer: this question is complete fiction.)

The sample mean of the morality ratings was $\overline{x}$ =77.89, which is an estimate of the true mean morality rating $\mu$ of David Bowie's music. The sample standard deviation of the ratings was $s = 10$. Use the above information to answer the following:

a. In his benevolence and jest, the Dalai Lama proclaims a musician "enlightened" if his true mean morality rating is above 75. Is there enough evidence at significance level $\alpha = 0.1$ to suggest that David Bowie is enlightened?

Use a 5-step hypothesis test:

1. **State the hypotheses and the significance value:**

$$\begin{aligned} H_0 &: \mu = 75 \\ H_a &: \mu > 75 \\ \alpha &= 0.1 \end{aligned}$$

2. **Find the test statistic and state the distribution it came from:**

Since $n = 123 \geq 40$, the test statistic is:

$$z^* = \frac{\overline{x - 75}}{s/\sqrt{n}} = \frac{77.89 - 75}{10/\sqrt{123}} = 3.21$$

which came from:

$$Z^* \sim \text{ approx. } N(0,1)$$

3. **Compute the p-value, or at least compare it to $\alpha$:**

Since we have a one-sided upper alternative hypothesis and a test statistic from an approximate standard normal distribution, the p-value is:

$P(Z^* > z^*) = P(Z^* > 3.21) = 1 - P(Z^* \leq 3.21) = 1 - 0.9993 = 0.0007$

4. **Make a decision in terms of the hypotheses:**

   Since the p-value is $< \alpha$, we reject $H_0$ and conclude $H_a$.

5. **State a conclusion in the context of the problem:**

   We have enough evidence to claim that David Bowie's true mean morality rating is above 75: i.e., that David Bowie is enlightened.

b. Calculate and interpret a 2-sided 95% confidence interval for David Bowie's true mean morality rating.

Since $n = 123 \geq 40$, $1 - \alpha = 0.95$, and $\sigma$ is unknown, our confidence interval is:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}},\ \overline{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

$$\left(\overline{x} - z_{0.975}\frac{s}{\sqrt{n}},\ \overline{x} + z_{0.975}\frac{s}{\sqrt{n}}\right)$$

$$\left(77.89 - 1.96 \cdot \frac{10}{\sqrt{123}},\ 77.89 + 1.96\frac{10}{\sqrt{123}}\right)$$

$$(76.12, 79.66)$$

We are 95% confident that David Bowie's true morality rating $\mu$ lies in the interval, ( 76.12, 79.66).

c. In another inside jest, the Dalai Lama proclaims a musician to be in the "Sage Range" if his true mean morality rating is greater than 60 and less than 80. Based on the confidence interval in part (b), are you at least 95% confident that David Bowie is in the Sage Range?

We're 95% confident that David Bowie's true morality rating is inside (

76.12, 79.66), which in turn, is inside the Sage Range of (60, 80). Hence, we're at least 95% confident that David Bowie is in the Sage Range.

**Exercise 3.6.**
You are a manufacturer of the Fidget Widget©: a little device that, when placed on one's head, massages the subjects temples to curb fidgeting during class (again, fiction). You need to make sure that this device's true mean mass is close to 5 kg. Too high and the subject's neck will strain, but too low and the device is either flimsy or missing some components.

Suppose you take a sample of 41 newly-made Fidget Widgets and calculate their sample mean mass to be 6 kg. Suppose the standard deviation of the sample masses is 0.005 kg.

a. Is there enough evidence to conclude that the true mean mass of the Fidget Widgets is different from the target of 5 kg? Do a hypothesis test at significance level $\alpha = 0.01$ to justify your answer.

   Use a 5-step hypothesis test:

   1. **State the hypotheses and the significance value:**

   $$\begin{aligned} H_0 &: \mu = 5 \\ H_a &: \mu \neq 5 \\ \alpha &= 0.01 \end{aligned}$$

   2. **Find the test statistic and state the distribution it came from:**

      Since $n = 41 \geq 40$, the test statistic is:

   $$z^* = \frac{\overline{x - 5}}{s/\sqrt{n}} = \frac{6 - 5}{0.005/\sqrt{41}} = 1280.625$$

      which came from:

   $$Z^* \sim \text{ approx. } N(0, 1)$$

3. **Compute the p-value, or at least compare it to $\alpha$:**

   Since we have a one-sided upper alternative hypothesis and a test statistic from an approximate standard normal distribution, the p-value is:

   $P(|Z^*| > |z^*|) = P(Z^* > 1280.625) + P(Z^* < -1280.625) \approx 0 + 0 = 0$

4. **Make a decision in terms of the hypotheses:**

   Since the p-value is $\approx 0 < \alpha$, we reject $H_0$ and conclude $H_a$.

5. **State a conclusion in the context of the problem:**

   We have enough evidence to claim that the true mean weight of the Fidget Widgets produced is different from the target weight of 5 kg.

b. Calculate a two-sided 95% confidence interval for the true mean mass. Based on the confidence interval, is 5 kg a plausible guess for the true mean mass of the Fidget Widgets?

Since $n = 41 \geq 40$, $1 - \alpha = 0.95$, and $\sigma$ is unknown, our confidence interval is:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}},\ \overline{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

$$\left(\overline{x} - z_{0.975}\frac{s}{\sqrt{n}},\ \overline{x} + z_{0.975}\frac{s}{\sqrt{n}}\right)$$

$$\left(6 - 1.96 \cdot \frac{0.005}{\sqrt{41}},\ 6 + 1.96\frac{0.005}{\sqrt{41}}\right)$$

$$(5.998, 6.002)$$

We are 95% confident that the true mean weight of all the Fidget Widgets produced is in the interval, ( 5.998, 6.002). Since 5 kg is not in this interval, 5 kg is not a plausible guess for the true mean mass of the Fidget Widgets.