

Game-theoretic Foundations of Multi-agent Systems

Lecture 4: Computing Solution Concepts of Normal-form Games

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Outline

1. Brief Overview of (Mixed Integer) Linear Programming
2. Dominated Strategies
3. Minmax and Maxmin Strategies
4. Nash Equilibrium
5. Correlated NE



Example: Reproduction of Two Paintings



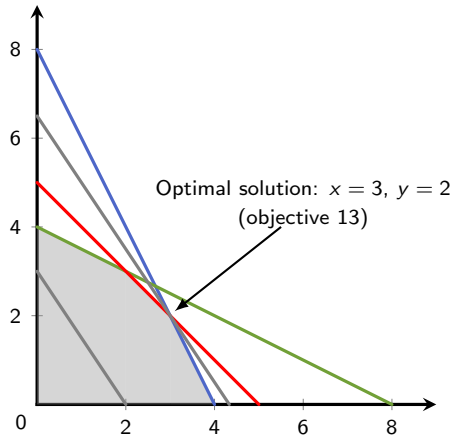
- Painting 1 sells for \$30
- Painting 2 sells for \$20
- We have 16 units of blue, 8 green, 5 red
- Painting 1 requires 4 blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \leq 16 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$



Solving Linear Program Graphically

$$\begin{array}{ll}\text{max.} & 3x + 2y \\ \text{s.t.} & 4x + 2y \leq 16 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \geq 0 \\ & y \geq 0\end{array}$$



Modified LP

$$\text{max. } 3x + 2y$$

$$\text{s.t. } 4x + 2y \leq 16$$

$$\text{s.t. } 4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

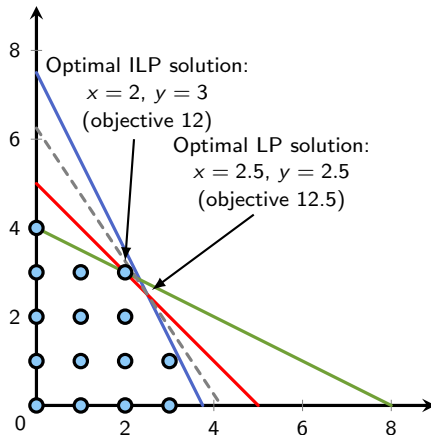
$$y \geq 0$$

- Optimal solution: $x = 2.5, y = 2.5$
- Objective = $7.5 + 5 = 12.5$
- Can we sell half paintings?



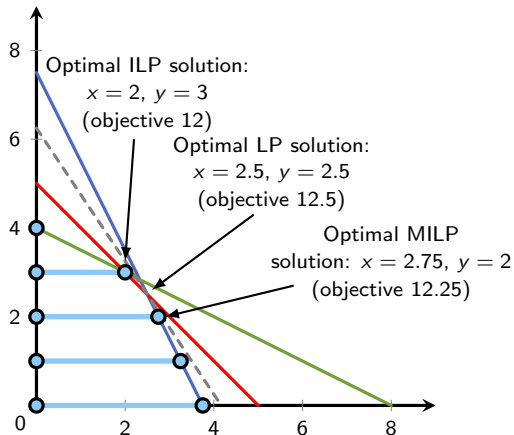
Integer Linear Program

$$\begin{array}{ll}\max. & 3x + 2y \\ \text{s.t.} & 4x + 2y \leq 15 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \in \mathbb{N}_0 \\ & y \in \mathbb{N}_0\end{array}$$



Mixed Integer Linear Program

$$\begin{array}{ll}\max. & 3x + 2y \\ \text{s.t.} & 4x + 2y \leq 15 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \geq 0 \\ & y \in \mathbb{N}_0\end{array}$$



Solving Mixed Linear/Integer Programs

- Linear programs can be solved efficiently
 - Simplex, ellipsoid, interior point methods, etc.
- (Mixed) integer programs are **NP-hard** to solve
 - Many standard NP-complete problems can be modeled as MILP
 - Search type algorithms such as branch and bound
- Standard packages for solving these
 - Gurobi, MOSEK, GNU Linear Programming Kit, CPLEX, CVXOPT, etc.
- LP relaxation of (M)ILP: remove integrality constraints
 - Gives upper bound on MILP (\sim admissible heuristic)



Exercise I: Knapsack-type Problem

- We arrive in room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for \$11 (3 units available)
- Unit of object B: 4kg, 4 liters, sells for \$4 (4 units available)
- Unit of object C: 6kg, 3 liters, sells for \$9 (1 unit available)
- What should we take?



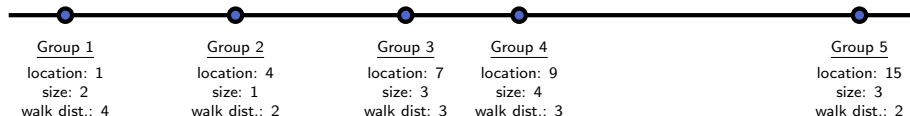
Exercise II: Cellphones (Set Cover)

- We want to have a working phone in every continent (besides Antarctica)
- But we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E



Exercise III: Hot-dog Stands

- We have two hot-dog stands to be placed in somewhere along beach
- We know where groups of people who like hot dogs are
- We also know how far each group is willing to walk
- Where do we put our stands to maximize # hot dogs sold? (price is fixed)



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Dominance by Pure Strategy

Algorithm 1: Determine whether s_i is strictly dominated by any pure strategy

```
for all  $a_i \in A_i$  where  $a_i \neq s_i$  do  
   $dom \leftarrow true$ ;  
  forall  $a_{-i} \in A_{-i}$  do  
    if  $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$  then  
       $dom \leftarrow false$ ;  
      break;  
  if  $dom = true$  then  
    return  $true$ ;  
return  $false$ ;
```



Recall: Strict Dominance

a_i **strictly dominates** s_i if $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$



Dominance by Pure Strategy: Discussion

- Complexity of the Algorithm is $O(|A|)$, linear in the size of normal-form game
- Recall: a_i **strictly dominates** s_i if $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$
- This definition refers to mixed-strategy profile of other agents
- In Alg. (1), we do not check every mixed-strategy profile of others, why?
 - Suppose a_i does not strictly dominate s_i for all a_{-i}
 - Then, there is no s_{-i} for which a_i strictly dominates s_i
 - This holds because of the linearity of expectation



Weak Dominance by Mixed Strategy

- Checking if strategy s_i is weakly dominated by any mixed strategy

$$\begin{aligned} \text{max.} \quad & \sum_{a_{-i} \in A_{-i}} \left[\left(\sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \right) - u_i(s_i, a_{-i}) \right] \\ \text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\ & \sum_{a_i \in A_i} p_{a_i} = 1 \\ & p_{a_i} \geq 0, && \forall a_i \in A_i \end{aligned}$$

- If optimal solution is strictly positive, then s_i is weakly dominated by $\{p_{a_i}\}$



Strict Dominance by Mixed Strategies

- Checking if strategy s_i is strictly dominated by any mixed strategy

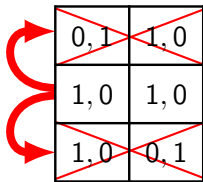
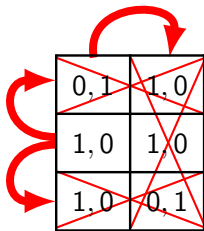
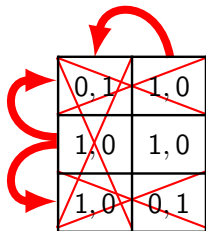
$$\begin{aligned} \max. \quad & \epsilon \\ \text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) + \epsilon \quad \forall a_{-i} \in A_{-i} \\ & \sum_{a_i \in A_i} p_{a_i} = 1 \\ & p_{a_i} \geq 0, \quad \forall a_i \in A_i \end{aligned}$$

- If optimal solution is strictly positive, then s_i is strictly dominated by $\{p_{a_i}\}$



Path Dependency of Iterated Dominance

- Iterated weak dominance is **path-dependent**:
 - Sequence of eliminations may determine which solution we get (if any)



- Iterated strict dominance is **path-independent**:
 - Elimination process will always terminate at the same point

Computational Questions for Iterated Dominance

- Is there some elimination path under which s_i is eliminated?
- Is there maximally reduced game where each agent has exactly 1 action?
- For **strict dominance**, both can be solved in polynomial time
 - Due to path-independence
 - Check if any strategy is dominated, remove it, repeat
 - With or without dominance by mixed strategies
- For **weak dominance**, both questions are NP-hard¹
 - Even when all utilities are 0 or 1
 - With or without dominance by mixed strategies

¹[Conitzer, Sandholm 05] and weaker version proved by [Gilboa, Kalai, Zemel 93]



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Recall: Minmax and Maxmin

- **Maxmin** strategy for agent i (maxmin value for agent i)

$$\operatorname{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- **Minmax** strategy against agent i (minmax value for agent i)

$$\operatorname{argmin}_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$



Maxmin Strategy and Value

- Finding maxmin strategy of agent i

$$\begin{aligned} \max. \quad & U_i \\ \text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq U_i, \quad \forall a_{-i} \in A_{-i} \\ & \sum_{a_i \in A_i} p_{a_i} = 1 \\ & p_{a_i} \geq 0, \quad \forall a_i \in A_i \end{aligned}$$

- Given p_{a_i} , first constraint ensures that U_i is less than any achievable expected utility for any pure strategies of opponents
- Objective of this LP, U_i , is **maxmin value** of agent i



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NE of Two-player, Zero-sum Games

- Maximin value for agent 1

$$\begin{aligned} \max. \quad & U_1 \\ \text{s.t.} \quad & \sum_{a_1 \in A_1} p_{a_1} u_1(a_1, a_2) \geq U_1 \quad \forall a_2 \in A_2 \\ & \sum_{a_1 \in A_1} p_{a_1} = 1 \\ & p_{a_1} \geq 0, \quad \forall a_1 \in A_1 \end{aligned}$$

- Minmax value for agent 1

$$\begin{aligned} \min. \quad & U_1 \\ \text{s.t.} \quad & \sum_{a_2 \in A_2} p_{a_2} u_1(a_1, a_2) \leq U_1 \quad \forall a_1 \in A_1 \\ & \sum_{a_2 \in A_2} p_{a_2} = 1 \\ & p_{a_2} \geq 0, \quad \forall a_2 \in A_2 \end{aligned}$$

- NE is expressed as LP \Rightarrow NE can be computed in polynomial time



Maximin Strategy for General-sum Games

- Agents could still play minimax strategy in general-sum games
 - I.e., pretend that the opponent is only trying to hurt you
- But this might not be rational:

		Agent 2	
		Left	Right
Agent 1	Up	0, 0	3, 1
	Down	1, 0	2, 1

- If A2 was trying to hurt A1, she would play Left, so A1 should play Down
- In reality, A2 will play Right (strictly dominant), so A1 should play Up



Hardness of Computing NE for General-sum Games

- Complexity was open for long time
 - “together with factoring [...] the most important concrete open question on the boundary of P today” [Papadimitriou STOC'01]
- Sequence of papers showed that computing any NE is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require **exponential time** (in worst case)



Hardness of Computing NE for General-Sum Games (cont.)

- What about computing NE with **specific property**?
 - NE that is not Pareto-dominated
 - NE that maximizes expected social welfare (i.e., sum of all agents' utilities)
 - NE that maximizes expected utility of given agent
 - NE that maximizes expected utility of worst-off player
 - NE in which given pure strategy is played with positive probability
 - NE in which given pure strategy is played with zero probability
 - ...
- All of these are NP-hard (and the optimization questions are inapproximable assuming $P \neq NP$), even in 2-player games

[Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]



Search-based Approaches (for Two-player Games)

- We can use LP, if we know support X_i of each player i 's mixed strategy

$$\begin{aligned} \text{find } & (U_1, U_2) \\ \text{s.t. } & p_{a_i} \geq 0, & \forall i, a_i \in A_i \\ & \sum_{a_i \in A_i} p_{a_i} = 1, & \forall i \\ & p_{a_i} = 0, & \forall i, a_i \in A_i / X_i \\ & \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = U_i, & \forall i, a_i \in X_i \\ & \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) \leq U_i, & \forall i, a_i \in A_i / X_i \end{aligned}$$

- Thus, we can search over possible supports, which is basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]



NE using MILP (for Two-player Games)

[Sandholm, Gilpin, Conitzer AAAI05]

$$\begin{array}{ll}\text{max.} & \text{whatever you like (e.g., social welfare)} \\ \text{s.t.} & p_{a_i} \geq 0, \quad \forall i, a_i \in A_i \\ & \sum_{a_i \in A_i} p_{a_i} = 1, \quad \forall i \\ & \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = u_{a_i}, \quad \forall i, a_i \in A_i \\ & u_{a_i} \leq u_i, \quad \forall i, a_i \in A_i \\ & p_{a_i} \leq b_{a_i}, \quad \forall i, a_i \in A_i \\ & u_i - u_{a_i} \leq M(1 - b_{a_i}), \quad \forall i, a_i \in A_i \\ & b_{a_i} \in \{0, 1\}, \quad \forall i, a_i \in A_i\end{array}$$

- b_{a_i} indicates whether a_i is in support of i 's mixed strategy, and M is large number



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Correlated Equilibrium (N-player Games!)

- Variables are now p_a for all action profiles a (i.e., outcome)

max. whatever you like (e.g., social welfare)

$$\text{s.t.} \quad \sum_{a_{-i} \in A_{-i}} p_a u_i(a) \geq \sum_{a_{-i} \in A_{-i}} p_a u_i(t_i, a_{-i}) \quad \forall i, a_i t_i \in A_i$$

$$\sum_{a \in A} p_a = 1$$

$$p_a \geq 0, \quad \forall a \in A$$



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