Game-theoretic
Foundations of Multi-agent Systems

Lecture 3: Games in Normal Form

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Outline

Normal-form Games

- Let's start with games in which all agents act simultaneously
- Agents choose their actions without knowledge of other agents' actions
- Such games are referred to as strategic-form games or normal-form games

Normal-form Games: Definition

- The game consists of a set of agents, $N = \{1, 2, \ldots, n\}$
- Set of available actions to agent i is denoted by A_i
- Action taken by agent i is denoted by $a_i \in A_i$
- Outcome of game is an action profile of all agents, $a = (a_1, \ldots, a_n)$
- Set of all action profiles is denoted by $A = \prod A_i$
- Agent i has a utility function u_i that maps outcomes to real numbers

Some Notations

- $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is an action profile of all agents except i
- $A_{-i} = \prod_{j \neq i} A_j$ is set of action profiles of all agents except i
- $a = (a_i, a_{-i}) \in A$ is another way of denoting an action profile (or an outcome)

Matrix-form Representation

- When A_i is finite for all i, we call the game finite game
- For 2 agents and small action sets, game can be represented in matrix form

Agent 2

		x	y
Agent 1	m	a, b	e, f
	n	c, d	g, h

- Each cell indexed by row r and column c contains a pair, (p, q), where $p = u_1(r, c)$ and $q = u_2(r, c)$.

Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 1 pockets both, otherwise agent 2 pockets them

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

- Zero-sum game: Utility of one agent is negative of utility of other agent

Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

	Left	Right
Left	$1, 1$	$-1, -1$
Right	$-1, -1$	$1, 1$

- Team game: For all outcomes s, and any pair of agents i and j, it is the case that $u_i(a) = u_j(a)$ (also known as common-payoff game or pure-coordination game)

Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ($a_i \in [0, \infty]$)
- Utility of each firm is its total revenue minus its total cost

$$u_i(a_1, a_2) = a_i p(a_1 + a_2) - c a_i$$

  - $p(\cdot)$ is the price function that maps total production to a price
  - c is a unit cost
  - E.g., $p(x) = \max(0, 2 - x)$ and $c = 1$

Outline

Mixed and Pure Strategies

- Let $\Delta(X)$ be set of all probability distributions over X
- Set of (mixed) strategies for agent i is denoted by $S_i = \Delta(A_i)$
- For mixed strategy $s_i \in S_i$, $s_i(a)$ is probability that action a is played under s_i
- Pure strategy is a mixed strategy that puts probability 1 on a single action
- Support of mixed strategy s_i is set of pure strategies, a_i, such that $s_i(a_i) > 0$
- Expected utility of agent i for a (mixed) strategy profile $s = (s_1, \ldots, s_n)$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

Example

			Agent 2		
			R ($\frac{2}{3}$)	P (0)	S ($\frac{1}{3}$)
	R ($\frac{1}{3}$)		$0, 0$	$-1, 1$	$1, -1$
Agent 1	P ($\frac{2}{3}$)		$1, -1$	$0, 0$	$-1, 1$
	S (0)		$-1, 1$	$1, -1$	$0, 0$

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$

Dominant and Dominated Strategies

- Let s_i and s_i' be two strategies of agent i
- s_i strictly dominates s_i' if
  - $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$
- s_i weakly dominates s_i' if
  - $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$, and
  - $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for at least one $s_{-i} \in S_{-i}$
- s_i is strictly/weakly dominant if it strictly/weakly dominates all other strategy
- s_i is strictly/weakly dominated if another strategy strictly/weakly dominates it
- $s = (s_1, \ldots, s_n)$ is dominant strategy equilibrium if s_i is dominant strategy for all i

Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

- Absolute value of utilities are the length of jail term each prisoner gets
- Confess is strictly dominant strategy for both prisoners
- (C,C) is a strict dominant strategy equilibrium
- The dilemma: (D,D) is better for both prisoners, but they won't play it!

Iterated Elimination of Strictly Dominated Strategies

- All strictly dominated pure strategies can be ignored

$$
\begin{array}{c|c|c|c|}
 & L & C & R \\ \hline
U & 3,1 & 0,2 & 0,0 \\ \hline
M & 1,2 & 1,1 & 5,0 \\ \hline
D & 0,1 & 4,2 & 0,0 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|}
 & L & C \\ \hline
U & 3,1 & 0,2 \\ \hline
M & 1,2 & 1,1 \\ \hline
D & 0,1 & 4,2 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|}
 & L & C \\ \hline
U & 3,1 & 0,2 \\ \hline
D & 0,1 & 4,2 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|}
 & C \\ \hline
U & 0,2 \\ \hline
D & 4,2 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|}
 & C \\ \hline
D & 4,2 \\ \hline
\end{array}
$$

- Column R can be eliminated, since it is dominated by, for example, column L
- M is not dominated by U or D but is dominated by 0.5U + 0.5D mixed strategy
- Note, however, that it was not dominated before the elimination of the R column

Iterated Elimination of Strictly Dominated Strategies (cont.)

- Once one pure strategy is eliminated, another strategy that was not dominated can become dominated
- In finite games, iterated elimination of strictly dominated strategies ends after finite number of iterations
- Order of elimination does not matter when removing strictly dominated strategies (Church–Rosser property)
- Elimination order can make a difference in final outcome when removing weakly dominated strategies
- If the procedure ends with a single strategy for each agent, then the game is said to be dominance solvable

Existence of Dominant Strategy Equilibrium

- Dominant strategy equilibrium does not always exist
- Example: Matching pennies

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

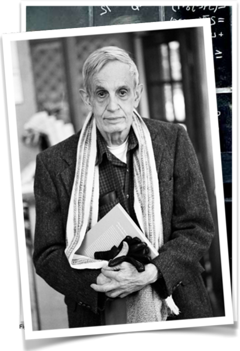
Outline

Best Response

- Picking a strategy would be simple if an agent knew how others were going to act
- Best response: $s_i^* \in BR_i(s_{-i})$ iff $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$
- Best response is not necessarily unique
  - If there is more than one best response, any mixed strategy over those must be a best response as well
- Best response is not a solution concept
  - I.e., it does not identify an interesting set of outcomes
  - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the Nash equilibrium

Nash Equilibrium - Intersection of Best Responses

- $s^* = (s_1^*, ..., s_n^*)$ is a Nash equilibrium iff $\forall i,\ s_i^* \in Br_i(s_{-i}^*)$
- No agent can profitably deviate given strategies of others
- Nash equilibrium is a stable strategy profile
- Nash theorem: Every finite game has a Nash equilibrium



John Forbes Nash Jr.
1928-2015

Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

		Wife: Football	Wife: Opera
Husband	Football	2,1 $\boxed{2, 1}$	$0, 0$
	Opera	$0, 0$	1,2 $\boxed{1, 2}$

- Are these the only Nash equilibria?

Example: Battle of Sexes (cont.)

	F (p)	O ($1-p$)
F	$2, 1$	$0, 0$
O	$0, 0$	$1, 2$

- In general, it is tricky to compute mixed-strategy equilibria (will discuss this later)
- It becomes easy when we know (or can guess) support of equilibrium strategies
- Let us now guess that both agents randomize over both F and O
- Wife's strategy is to play F w.p. p and O w.p. $1-p$
- Husband must be indifferent between F and O (why?):

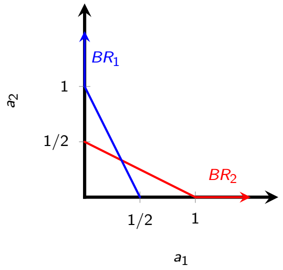
$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1-p) \Rightarrow p = 1/3$$

- You can show that the unique mixed-strategy NE is $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$

Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$BR_i(a_{-i}) = \underset{a_i \geq 0}{\operatorname{argmax}}\, a_i(2 - a_i - a_{-i}) - a_i$$
$$= \begin{cases} (1 - a_{-i})/2 & \text{if } a_{-i} < 1, \\ 0 & \text{Otherwise.} \end{cases}$$



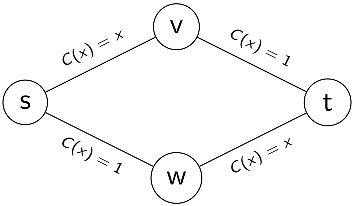
The "Equilibrium Selection Problem"

- You are about to play a game that you have never played before with a person that you have never met
- According to which equilibrium should you play?
  - Equilibrium that maximizes the sum of utilities (social welfare)
  - Or, at least not a Pareto-dominated equilibrium
  - So-called focal equilibria (e.g., "Meet in Paris" game - you and a friend were supposed to meet in Paris at noon on Sunday, but you forgot to discuss where and you cannot communicate. Where will you go?)
  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - ...
- Equilibrium selection is a difficult problem

Outline

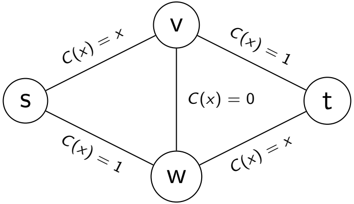
1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium

Braess's Paradox



- Suppose there are $2k$ drivers commuting from s to t
- $C(x)$ indicates travel time in hours for x fraction of drivers
- k drivers going through v, and k going through w is NE (why?)

Braess's Paradox (cont.)



- Suppose we install a teleportation device allowing instant travel from v to w
- What is new NE?
- What is optimal commute time?
- Price of anarchy: ratio between (worst) NE performance and optimal performance
  - Ratio between 2 and 3/2 in Braess's Paradox

Outline

Maxmin Strategy

- Maxmin strategy for agent i is

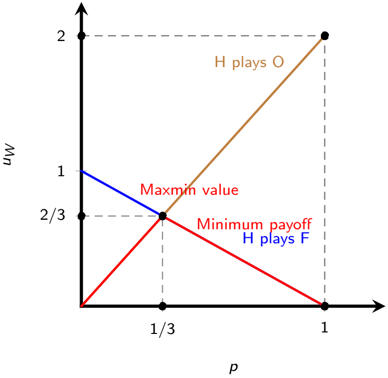
$$\underset{s_i}{\operatorname{argmax}} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Maxmin value for agent i is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- If i plays maxmin strategy and others play arbitrarily, i still receives expected payoff of at least their maxmin value

Example: Battle of Sexes



W

		F $(1-p)$	O (p)
H	F	$2, 1$	$0, 0$
	O	$0, 0$	$1, 2$

Minmax Strategy

- Minmax strategy against against i is

$$\underset{s_{-i}}{\operatorname{argmin}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax value for agent i is

$$\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax strategy against i keeps maximum payoff of agent i at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)

Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium[1], each agent receives a payoff that is equal to both their maxmin value and their minmax value

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minimax theorem does not hold with pure strategies only (example?)

[1] You might wonder how a theorem from 1928 can use the term "Nash equilibrium," when Nash's work was published in 1950. John von Neumann used different terminology and proved the theorem in a different way; however, the given presentation is probably clearer in the context of modern game theory

Example

		Agent 2	
		Left	Right
Agent 1	Up	$20, -20$	$0, 0$
	Down	$0, 0$	$10, -10$

- What is maximin value of agent 1 with and without mixed strategies?
- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?

Outline

Rationalizability

- Rationalizable strategy: Perfectly rational agent could justifiably play it
  - Best response to some beliefs about strategies of others
- Agents cannot have arbitrary beliefs about other agents
- Agent i's beliefs must take into account:
  - Other agents' rationality
  - Other agents' knowledge of agent i's rationality
  - Other agents' knowledge of agent i's knowledge of their rationality
  - ... (infinite regress)

Example: Matching Pennies

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

- Row playing H is rationalizable (Row could believe Col plays H)
- Col playing H is rationalizable (Col could believe Row plays T)
- Row playing T is rationalizable (Col could believe Row believes Col plays T)
- ...
- In this game, all pure strategies are rationalizable

Rationalizability: Properties

- Nash equilibrium strategies are always rationalizable
- Some rationalizable strategies are not Nash
  - Set of rationalizable strategies in finite games is nonempty
- To find rationalizable strategies:
  - In 2-player games, use iterated elimination of strictly dominated strategies
  - In n-player games, iterated elimination of never-best response strategies
    - Eliminate strategies that are not optimal against any belief about others' strategies

Example: 2/3-Beauty Contest Game

- No agent plays more than 100
- 2/3 of average is strictly less than 67 $(100 \times 2/3)$
- Any integer > 67 is never-best response to any belief about others' strategy
- No agent plays more than 67
- 2/3 of average is less than 45 $(67 \times 2/3)$
- Any integer > 45 is never-best response to any belief about others' strategy
- ...
- Only rationalizable action is playing 0

Outline

Example: Battle of Sexes

W

		Football	Opera
H	F	$2, 1$	$0, 0$
	O	$0, 0$	$1, 2$

- Unique mixed strategy NE yields each agent expected payoff of $2/3$
- In NE, agents randomize over strategies independently
- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome

Example: Battle of Sexes (cont.)

- Suppose there is publicly observable fair coin
- If it is heads/tails, they both get recommendation to go to football/opera
- If they see heads, they believe that the other one goes to football
- Going to football is best response, agents have no incentive to deviate
- Similar argument can be made when they see tails
- Expected utilities for this play of game increases to $(1.5, 1.5)$

Correlated Recommendations

- Let $R = (R_1, \ldots, R_n)$ be random variable taking values in $A = \prod_i A_i$
- Let R be distributed according to $\pi \in \Delta(A)$
- $r = (r_1, \ldots, r_n)$ is an instantiatation of R and a pure strategy profile
- $r_i \in A_i$ is called recommendation to agent i
- $\pi(r_i)$ represents marginal probability for $R_i = r_i$
- Given r_i, agent i can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$

Correlated Equilibrium: Formal Definition

- Correlated equilibrium of finite game is joint probability distribution $\pi \in \Delta(A)$ such that if R is random variable distributed according to π, then for all $i, r_i \in A_i$ with $\pi(r_i) > 0$, and $r_i' \in A_i$

$$\sum_{r_{-i} \in A_{-i}} \pi(r_{-i} \mid r_i) \left[u_i(r_i, r_{-i}) - u_i(r_i', r_{-i}) \right] \geq 0$$

- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations

Example: Game of Chicken

		Driver 2	
		Dare	Yield
Driver 1	D	$-5, -5$	$1, -1$
	Y	$-1, 1$	$0, 0$

- (D,Y) and (Y,D) are strict pure-strategy NE
- Assume Driver 1 yields w.p. p and Driver 2 yields w.p. q
- Using mixed equilibrium characterization, we have

$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

$$q - 5 \times (1 - q) = -(1 - q) \implies q = 4/5$$

- Mixed-strategy NE utilities are $(-0.2, -0.2)$, people die with probability 0.04

Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is $1/3$
- Expected utility of Y is $1 \times 2/3$
- Expected utility of D is $1 \times 1/3$ - $5 \times 2/3$
- Following the recommendation is better
- If D1 gets D recommendation, D2 must yield
- Following recommendation is again better
- Similar analysis works for D2
- Expected utilizes is $(0, 0)$, so nobody dies!

		D2: D	D2: Y
D1	D	$-5, -5$ 0%	$1, -1$ 40%
D1	Y	$-1, 1$ 40%	$0, 0$ 20%

Characterization of Correlated Equilibrium

- Join distribution $\pi \in \Delta(S)$ is correlated equilibrium of finite game iff

$$\sum_{r_{-i} \in A_{-i}} \pi(r) \left[u_i(r) - u_i(r_i', r_{-i}) \right] \geq 0, \quad \forall i, r_i, r_i' \in A_i \tag{1}$$

- Proof (only for one side):
  - Correlated equilibrium can be written for all $i, r_i, r_i' \in A_i$ with $\pi(r_i) > 0$ as:

$$\sum_{r_{-i} \in A_{-i}} \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})} [u_i(r_i, r_{-i}) - u_i(r_i', r_{-i})] \geq 0$$

  - Denominator does not depend on variable of sum
  - So it can be factored and canceled
  - If $\pi(r_i) = 0$, LHS of (1) is zero regardless of i and r_i', so equation always holds

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