

# Game-theoretic Foundations of Multi-agent Systems

## Lecture 8: Bayesian Games

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# Outline

1. Introduction and Definitions
2. Strategies and Equilibria
3. Auctions
4. Extensive-form Games of Incomplete-Info



# Bayesian Games: Games of Incomplete Information

- So far, we assumed **all agents know** what game they are playing
  - Number of agents
  - Actions available to each agent
  - Utilities associated with each outcome
- In extensive-form games, **taken actions** could be unknown, but **game itself** is
- **Bayesian games** allow us to represent uncertainties about game
  - **Commonly known probability distribution** over possible games



# Assumptions

- All games have **same number of agents** and **same action sets** for each agents
- Possible games only differ in agents' utilities for each outcome
- Beliefs are **posteriors**, obtained by conditioning common prior on private signals



# Bayesian Games: Formal Definition

- $N$  is finite set of agents
- $A_i$  is set of actions available to agent  $i$
- $\Theta_i$  is type space of agent  $i$
- $p : \Theta \mapsto [0, 1]$  is common prior over types
- $u_i : A \times \Theta \mapsto \mathbb{R}$  is utility function for agent  $i$



## Example I: Bayesian Entry-deterrence Game

- Firm 1 decides whether to fight, Firm 2 decides whether to enter
- Firm 1 knows its cost
- Firm 2 is uncertain if 1's cost is 4 w.p.  $p$  or 1 w.p.  $1 - p$
- Game takes one of following two forms

	Enter	Stay out
Build	0, -1	2, 0
Don't build	2, 1	3, 0

$\theta_{11}$ : High Cost

	Enter	Stay out
Build	3, -1	5, 0
Don't build	2, 1	3, 0

$\theta_{12}$ : Low Cost

- $\Theta_1 = \{\theta_{11}, \theta_{1,2}\}$  and  $\Theta_2 = \{\theta_{21}\}$



## Example II

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## Types: Discussion

- Types encapsulate information possessed by agents that is **not** common knowledge
  - E.g., agents' knowledge of their private utility function
- Type could also include
  - Agent's beliefs about other agents' utilities
  - Other agents' beliefs about the agent's own utility
  - And any other higher-order beliefs





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# Strategies

- Before the game starts, agents only know the common prior
- Agent  $i$ 's strategy is  $s_i : \Theta_i \mapsto \Delta(A_i)$  is contingency plan for all  $\theta_i \in \Theta_i$
- $s_i(\theta_i)$  specifies agent  $i$ 's (mixed) strategy when  $i$ 's type is  $\theta_i$
- $s_i(a_i | \theta_i)$  specifies probability of agent  $i$  taking action  $s_i$  when  $i$ 's type is  $\theta_i$
- Type of agents is revealed to them once the game starts
- Once agents know their type, they follow their strategy for that particular type



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- We can calculate expected utility depending on what agents know



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- **Ex ante**: Agents only know the common prior on types (before the game starts)
- **Interim**: Agents only knows about their own type (after types are reveals)
- **Ex post**: Agents know everyone's type (hypothetical – before they take actions)

## Expected Utilities (cont.)

- **Ex-post** expected utility (a):

$$EU_i(s, \theta) = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)$$



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- **Interim** expected utility:

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- **Ex-ante** expected utility:

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)$$

# Dominated Strategies

- **Ex-ante dominated strategy:** Alternative strategy provides greater ex ante utility regardless of all other agents' strategies
- **Interim dominated strategy:** For a given type, alternative strategy provides greater interim utility regardless of all other agents' strategies



# Best Response in Bayesian Games

- Agent  $i$ 's **best response** to strategy  $s_{-i}$  is

$$BR_i(s_{-i}) = \operatorname{argmax}_{s_i} EU_i(s_i, s_{-i})$$

- To play best response,  $i$  must know strategy of **all agents** for **each of their types**
- Without this information, it is not possible to evaluate  $EU_i(s_i, s_{-i})$



## Best Response in Bayesian Games (cont.)

- Best response is defined based on agent  $i$ 's **ex ante** expected utility,  $EU_i(s_i, s_{-i})$



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$$BR_i(s_{-i}) = \operatorname{argmax}_{s_i} \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s_i, s_{-i}, \theta_i)$$



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- Observe that  $EU_i(s_i, s_{-i}, \theta_i)$  **does not depend on**  $s_i(\theta'_i)$  for all  $\theta'_i \neq \theta_i$



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- So, maximizing  $EU_i(s_i, s_{-i})$  is equal to maximizing  $EU_i(s_i, s_{-i}, \theta_i)$  for all  $\theta_i \in \Theta_i$



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- So, maximizing  $EU_i(s_i, s_{-i})$  is equal to maximizing  $EU_i(s_i, s_{-i}, \theta_i)$  for all  $\theta_i \in \Theta_i$
- Intuitively, if certain action is best after a signal is revealed, it is also the best **conditional plan** devised **ahead of time** for what to do should that signal be received



# Bayes-Nash Equilibrium

- Bayes-Nash equilibrium (BNE) is strategy profile  $s^*$ , such that

$$s_i^* \in BR_i(s_{-i}^*) \quad \forall i$$

- [Theorem] Any finite Bayesian game has BNE



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$$EU_2(UD, LR) = \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta)$$

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$$\begin{aligned}
 EU_2(UD, LR) &= \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta) \\
 &= p(\theta_{11}, \theta_{2,1}) u_2(U, L, \theta_{11}, \theta_{2,1}) + p(\theta_{11}, \theta_{2,2}) u_2(U, R, \theta_{11}, \theta_{2,2}) + \\
 &\quad p(\theta_{12}, \theta_{2,1}) u_2(D, L, \theta_{12}, \theta_{2,1}) + p(\theta_{12}, \theta_{2,2}) u_2(D, R, \theta_{12}, \theta_{2,2}) \\
 &= 0.3 \times 0 + 0.1 \times 3 + 0.2 \times 0 + 0.4 \times 2 = 1.1
 \end{aligned}$$



## Example (cont.)

- Continuing in this manner, complete payoff matrix can be constructed as

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1



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- Note that row agent's best response to RL is DU

## Example (cont.)

- Once row agent receives the signal  $\theta_{11}$ , we can calculate interim utilities

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	105, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
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- Row agent's payoffs are now **independent** of action taken upon observing  $\theta_{12}$
- Note that DU is **still best response** to RL
- What has changed is how much better it is compared to other strategies

# Ex-post Equilibrium

- Strategy profile  $s^*$  is **ex-post equilibrium** if

$$s_i^* \in \operatorname{argmax}_{s_i} EU_i(s_i, s_{-i}^*, \theta) \quad \forall i, \theta \in \Theta$$



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  - Even if they knew  $\theta$ , agents would never want to deviate
  - Ex-post equilibrium is **not guaranteed** to exist

## Example: Incomplete Information Cournot

- Two firms decide on their production level  $q_i \in [0, \infty)$
- Price is given by  $P(q)$  where  $q = q_1 + q_2$
- Firm 1 has marginal cost equal to  $c$  which is common knowledge
- Firm 2's marginal cost is private information
  - $c_L$  with probability  $x$  and  $c_H$  with probability  $(1 - x)$ , where  $c_L < c_H$
- Utility of agents are ( $t \in \{L, H\}$  type of firm 2)
  - $u_1((q_1, q_2), t) = q_1 P(q_1, q_2) - c$
  - $u_2((q_1, q_2), t) = q_2 P(q_1, q_2) - c_t$



## Example: Incomplete Information Cournot (cont.)

- What are firms best responses?

$$B_1(q_L, q_H) = \arg \max_{q \geq 0} \left( (xP(q + q_L) + (1 - x)P(q + q_H) - c)q \right)$$

$$B_2^L(q_1) = \arg \max_{q \geq 0} \left( (P(q_1 + q) - c_L)q \right)$$

$$B_2^H(q_1) = \arg \max_{q \geq 0} \left( (P(q_1 + q) - c_H)q \right)$$

- BNE of this game is vector  $(q_1^*, q_L^*, q_H^*)$  such that

$$q_1^* \in B_1(q_L^*, q_H^*), q_L^* \in B_2^L(q_1^*), q_H^* \in B_2^H(q_1^*)$$





## Example: Incomplete Information Cournot (cont.)

- For example, if  $P(q) = \max(\alpha - q, 0)$ , then we have:

$$q_1^* = \frac{1}{3}(\alpha - 2c + xc_L + (1-x)c_H)$$

$$q_L^* = \frac{1}{3}(\alpha - 2c_L + c) - \frac{1}{6}(1-x)(c_H - c_L)$$

$$q_H^* = \frac{1}{3}(\alpha - 2c_H + c) + \frac{1}{6}x(c_H - c_L)$$



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- Allocating items to bidders with **highest valuations** is often desirable
- Extracting private valuations could be challenging
- E.g., giving painting for free to bidder with highest valuation would create incentive for all bidders to overstate their valuations

# Different Auctions

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- **Japanese auction:** price rises, bidders drop out, last bidder wins at price of last dropout
- **First-price auction:** bidders bid simultaneously, highest bid wins, winner pays winning bid
- **Second-price action:** similar to first price, except that winner pays second highest bid

# Valuations

- **Private valuations:** valuation of each bidder is independent of others' valuations
- **Common valuations:** bidders' valuations are correlated to common value



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- Utility of bidder  $i$  is  $v_i - p_i$ , where  $p_i$  is bidder  $i$ 's payment



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- Bidders only know their own realized value (type)



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- Suppose  $v$ 's are drawn *i.i.d.* from  $[0, \bar{v}]$  with commonly known CDF  $F$
- Bidders only know their own realized value (type)
- Bidders are risk neutral, maximizing their expected utility



## Sealed-bid Auctions (First- and Second-price Auctions)

- Suppose that there are  $N$  bidders and single object for sale
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- Bidders only know their own realized value (type)
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- Pure strategy for bidder  $i$  is map  $b_i : [0, \bar{v}] \rightarrow \mathbb{R}_+$

## Second-price Auction

- Agent  $i$  submit bid  $b_i$  simultaneously with other agents



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  - If other bidders bids truthfully, does loser want to change their bid?

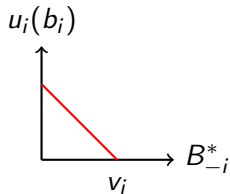
# Truthful Bidding in Second-price Auction

- Truthful equilibrium is (weak) **ex-post equilibrium**
- I.e., truthful bidding weakly dominates other strategies even if all values are known
- **[Proof sketch]** Define maximum bid excluding  $i$ 's bid as  $B_{-i}^* = \max_{j \neq i} b_j$



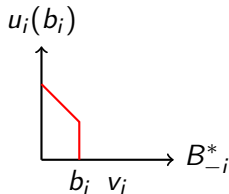
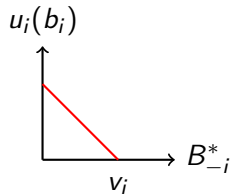
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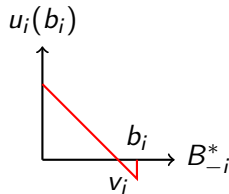
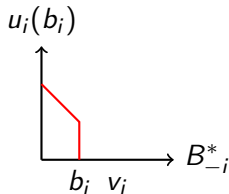
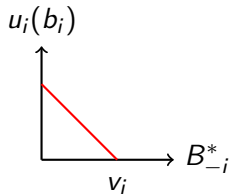
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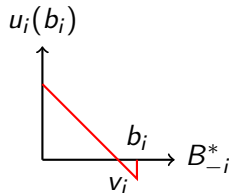
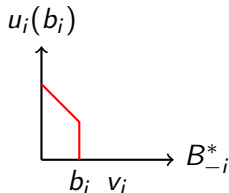
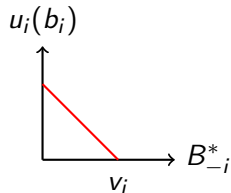
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- Truthful equilibrium is also the unique BNE



## Expected Payment in Second-price Auctions

- Define random variable  $y_i = \max_{j \neq i} v_j$ 
  - CDF of  $y_i$  is  $G_{y_i}(v) = F(v)^{N-1}$
  - PDF of  $y_i$  is  $g_{y_i}(v) = (N-1)f(v)F(v)^{N-2}$
- Expected payment of bidder  $i$  with value  $v_i$  is given by

$$\begin{aligned} p(v_i) &= P(v_i \text{ wins}) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= P(y_i \leq v_i) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy = \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$



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- This implies that bidder  $i$  wins whenever  $y_i < \beta^{-1}(b_i)$
- Optimal bid of bidder  $i$  is  $b_i = \operatorname{argmax}_{b \geq 0} G_{y_i}(\beta^{-1}(b))(v_i - b)$

## First-price Auctions (cont.)

- First-order (necessary) **optimality conditions** imply<sup>1</sup>:

$$\frac{g_{y_i}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}(v_i - b_i) - G_{y_i}(\beta^{-1}(b_i)) = 0$$

- In symmetric equilibrium,  $b_i = \beta(v_i)$ , therefore we have:

$$v_i g_{y_i}(v_i) = \beta'(v_i) G_{y_i}(v_i) + \beta(v_i) g_{y_i}(v_i) = \frac{d}{dv} (\beta(v_i) G_{y_i}(v_i))$$

- With **boundary condition**  $\beta(0) = 0$ , we have:

$$\beta(v_i) = G_{y_i}^{-1}(v_i) \int_0^{v_i} y g_{y_i}(y) dy = \mathbb{E}[y_i \mid y_i \leq v_i]$$

---

<sup>1</sup>Derivative of  $\beta^{-1}(b)$  is  $1/\beta'(\beta^{-1}(b))$ .



## Expected Payment in First-price Auctions

- Expected payment of bidder  $i$  with value  $v_i$  is:

$$\begin{aligned} p(v_i) &= P(v_i \text{ wins}) \times \beta(v_i) \\ &= P(y_i \leq v_i) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy = \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$

- This establishes somewhat surprising results that both first and second price auction formats yield **same expected revenue** to auctioneer



# Revenue Equivalence

- In **standard auctions**, item is sold to bidder with highest submitted bid



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- In **standard auctions**, item is sold to bidder with highest submitted bid
- Suppose that values are *i.i.d* and all bidders are risk neutral
- **[Theorem]** Any symmetric and increasing equilibria of any standard auction (such that expected payment of bidder with value zero is zero) yields same expected revenue to auctioneer

## Oil-field Example: Common Values with Correlated Recommendations

- Suppose that there are two bidders bidding to lease oil field
- Oil field could be worth \$0, \$25M, or \$50M w.p. 0.25, 0.5, and 0.25, respectively
- Bidders hire their own consultant to evaluate value of oil field
- Bidders get private recommendations,  $r_1$  and  $r_2$
- If field is worth \$0, then  $r_1 = r_2 = L$
- If field is worth \$25M, then  $r_1 = H, r_2 = L$  or  $r_1 = L, r_2 = H$  (both equally likely)
- If field is worth \$50M, then  $s_1 = s_2 = H$
- Given their private recommendation, how should bidders bid?



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- Bidding \$0 leads to utility \$0 and is **profitable deviation**
- **Truthful bidding is not BNE in second-price auction with common values and dependent recommendations**

# Winner's Curse

- Winning means bidder received highest or **most optimistic** recommendation
- Condition on winning, value of item is lower than what recommendation says
- Ignoring this leads to paying, on average, **more than** true value of item
- To avoid this curse, bidders should assume their recommendation is optimistic
- In oil-field example, we can show that the following bidding strategy is BNE
  - Bid 0 upon receiving  $L$
  - Bid \$50M upon receiving  $H$



## Oil-field Example II: Common Values and Independent Recommendations

- Consider two bidders interested in buying oil field that has part A and B
- Each bidder values A and B but is more interested in one of them
- Bidders hire their own consultant to evaluate value of their part
- Bidder 1 gets private recommendation  $r_1$  about value of part A
- Bidder 2 gets private recommendation  $r_2$  about value of part B
- Suppose that both recommendations are **uniformly distributed** over  $[0, 1]$
- Suppose value of oil field to each bidder is as follows
  - $v_i = a.r_i + b.r_{-i}$  with  $a \geq b \geq 0$
  - Private values are **special case** where  $a = 1$  and  $b = 0$





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- If  $-i$  follows this, then probability that  $i$  wins by bidding  $b_i$  is:

$$P(\beta(r_{-i}) < b_i) = P((a + b)r_{-i} < b_i) = b_i / (a + b)$$



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- Bidder  $i$ 's payment if  $i$  wins is  $\beta(r_{-i}) = (a + b)r_{-i}$

## Oil Field Example II: Second-price Auction (cont.)

- Expected payment of  $i$  condition on  $i$  winning is:

$$\mathbb{E}[(a + b)r_{-i} \mid r_{-i} < b_i/(a + b)] = b_i/2$$

- Expected value of  $-i$ 's signal condition on  $i$  winning is:

$$\mathbb{E}[r_{-i} \mid r_{-i} < b_i/(a + b)] = b_i/2(a + b)$$

- Expected utility of bidding  $b_i$  for recommendation  $r_i$  is

$$\begin{aligned} EU(b_i, r_i) &= P(b_i \text{ wins}) \times (a.r_i + b.\mathbb{E}[r_{-i} \mid b_i \text{ wins}] - \mathbb{E}[(a + b)r_{-i} \mid b_i \text{ wins}]) \\ &= b_i/(a + b) \times (a.r_i + b.b_i/2(a + b) - b_i/2) \end{aligned}$$

- Maximizing this with respect to  $b_i$  (for given  $r_i$ ) leads to  $b_i^* = (a + b)r_i$



## Oil Field Example II: First-price Auction

- Analysis is similar to that of first-price auctions with private values
- It can be shown that unique symmetric BNE is for each bidder to bid  $\beta(r_i) = (a + b)r_i/2$
- It can be shown that expected revenue is equal to first price auction
- Revenue equivalence principle **continues to hold** for common values



# Outline

1. Introduction and Definitions
2. Strategies and Equilibria
3. Auctions
4. Extensive-form Games of Incomplete-Info



# Incomplete Information in Extensive-form Games

- Incomplete-information games cannot always be represented as static games





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# Incomplete Information in Extensive-form Games

- Incomplete-information games cannot always be represented as static games
- Extensive-form games can capture explicit order of moves or dynamic games
- We can use information sets to represent what each agent knows
- We need to modify BNE to include notion of perfection (as in subgame perfection)

# Equilibrium Concepts

		Timing	
		Simultaneous	Sequential
Information	Complete	Nash	SPE
	Incomplete	Bayesian Nash	



# Equilibrium Concepts

		Timing	
		Simultaneous	Sequential
Information	Complete	Nash	SPE
	Incomplete	Bayesian Nash	Perfect Bayesian



## Extensive-form Games of Incomplete Information: Definition

- $N, A, H, Z, \alpha, \beta, \rho, u$ , and  $I$  are the same as extensive-form games
- $\Theta_i$  is type space of agent  $i$
- $p : \Theta \mapsto [0, 1]$  is common prior over types
- $u_i : Z \times \Theta \mapsto \mathbb{R}$  is utility function for agent  $i$

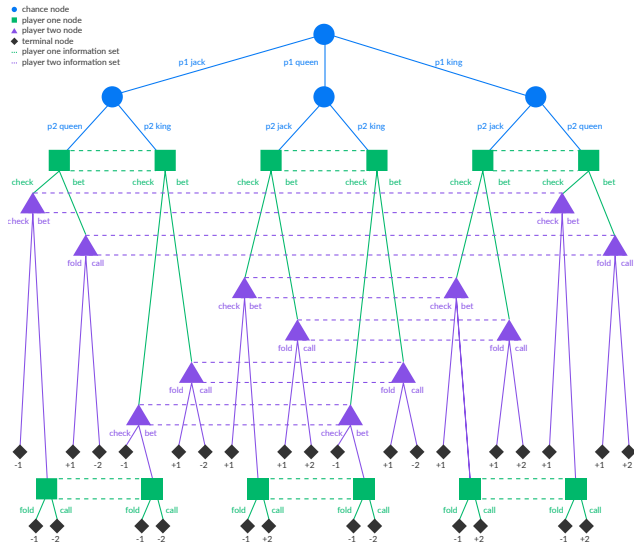


# The “Nature” with Chance Moves

- To capture common prior, we can add special agent called **Nature**
- Nature makes **probabilistic choices**
- Nature **does not** have utility function (can be viewed as having constant utility)
- Nature has unique strategy of randomizing in **commonly known** way
- Agents receive individual signals about Nature's choice



# Example: Kune Poker





# Beliefs and Strategies

- Agents have **beliefs** about which node they are for each information set (**info**set)
- For each info



# Requirements for Perfect Bayesian Equilibrium (PBE)

- I. Beliefs: In addition to strategy profile  $s$ , beliefs  $\mu$  must be specified



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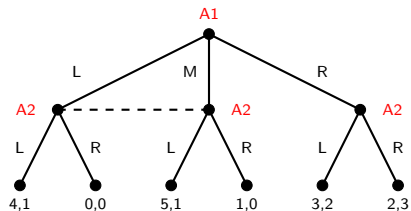
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- IV. **Off-the-path consistency**: For any off-the-equilibrium-path info set,  $\mu$  must be derived from  $s$  according to Bayes' rule **whenever possible**

# Weak and Strong PBE

- I-III define **weak PBE**, and I-IV define strong PBE
- PBE is defined for all extensive-form games with imperfect information

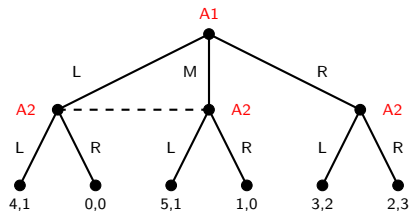


## Example I (from Lecture 5)



- $(R, (R, R))$  is NE and SPE, but it is not PBE, why?

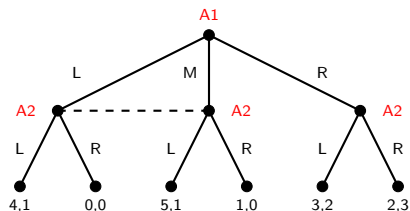
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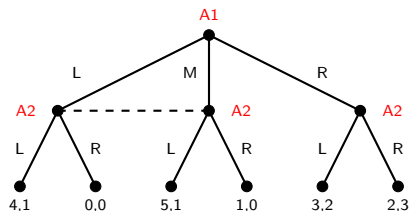


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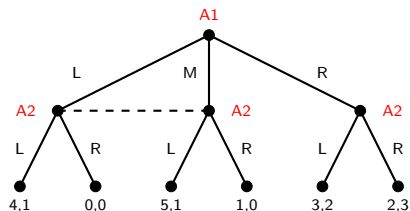
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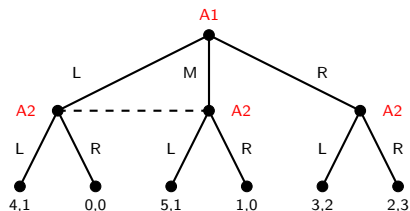
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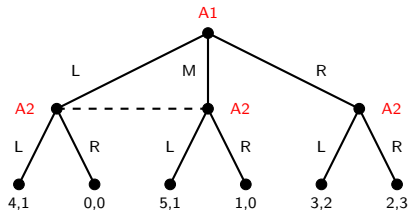
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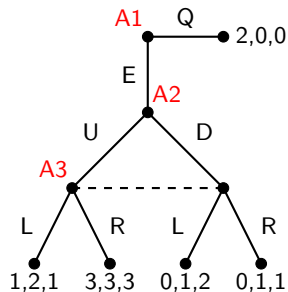
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  - Off-the-path beliefs are **consistent** with the equilibrium strategy
- $(M, (L, R))$  + believing that A2 takes M with probability 1 is also strong PBE
  - Off-the-path beliefs are also consistent (right-side info set has single node)

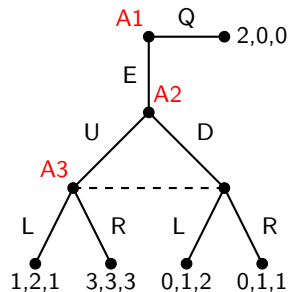
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- U is A2's dominant strategy



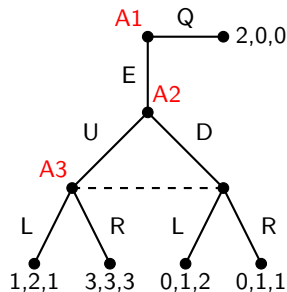
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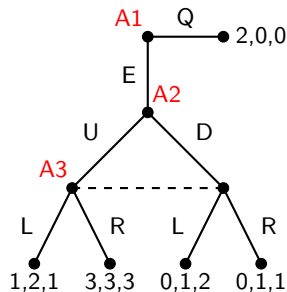
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- (E, U, R) is SPE





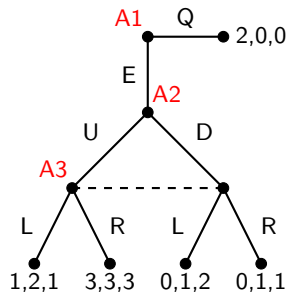
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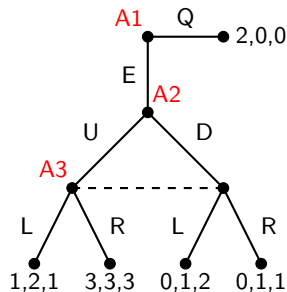
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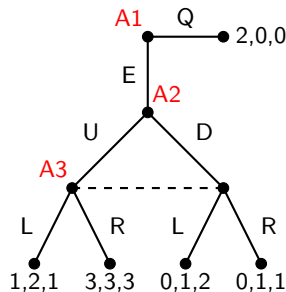
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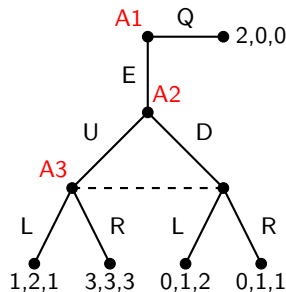
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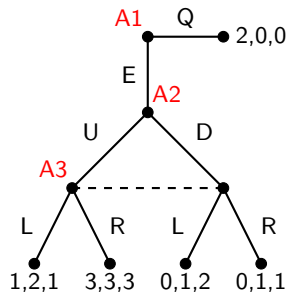
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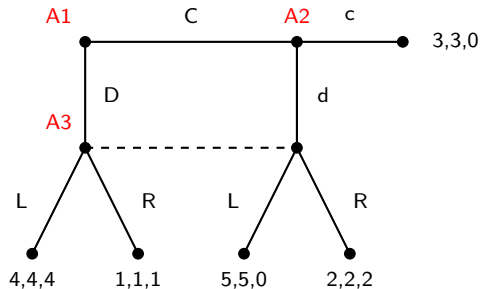


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- D is best respond to (U, L) and U is dominant strategy
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- So, it is weak PBE, but is it also strong PBE?
- No! IV does not hold; A3's belief is **inconsistent** with A2's strategy



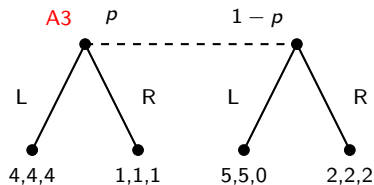
## Example II: Selten's Horse



Reinhard Selten<sup>2</sup>  
(1930-2016)

<sup>1</sup>Photograph by Stefan Schickler

## Example II: Selten's Horse (cont.)

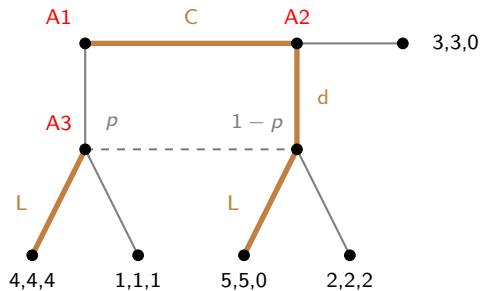


- A3 believes that left and right nodes are reached w.p.  $p$  and  $1 - p$ , respectively
- Utility for playing L is  $2p$  and  $1 - p$  for playing R
- A3 must play R if  $p < 1/3$ , R or L if  $p = 1/3$ , and L if  $p > 1/3$



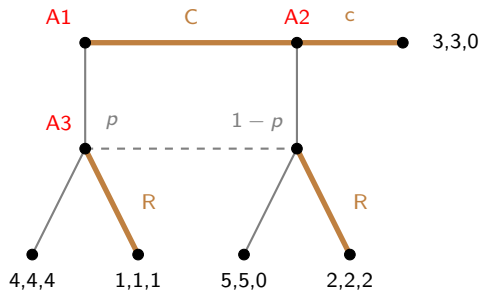


## Example II: Selten's Horse (cont.)



- Is there any  $p$  with which (C, d, L) is weak PBE?
- Given (C, d), on-the-path belief for A3 must set  $p = 0$
- For  $p = 0$ , A3 must take R, so the answer is **NO**

## Example II: Selten's Horse (cont.)



- Is there any  $p$  with which (C, c, R) is weak PBE?
- Given (C, c), A3's info set is off the equilibrium path
- **Consistency** does not put any constraint on  $p$ ; **optimality** of R requires  $p \leq 2/5$
- Is (C, c, R) +  $p \leq 2/5$  strong PBE? Why?



## Example III: Signaling Games

- **Informed** agent moves first to **signal** some information to uninformed agent
- Sending signal is more costly if it conveys false information
- E.g., producer provides warranty to signal that its products are unlikely to break
- E.g., employees acquire college degree to signal their ability to employers
- This is different from sending costless **messages** in **cheap talk** games
- Cheap talk is communication between agents that does not directly affect payoffs
- E.g., agents message each other on where they want to go in Battle of the Sexes



# PBE Types in Signaling Games

- **Separating**: Informed agent sends distinct signal for each type



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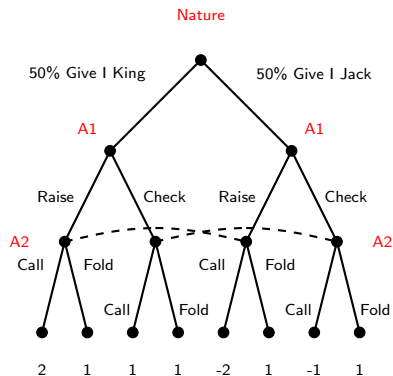


# PBE Types in Signaling Games

- **Separating**: Informed agent sends distinct signal for each type
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- **Pooling**: Informed agent sends the same signal for all types
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- **Semi-separating** (a.k.a. partially pooling): Informed agent sends same signal for some types distinct signal for some other types

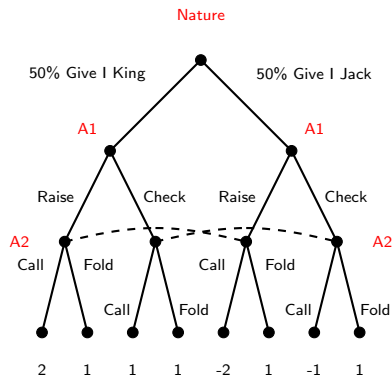
# Simple Poker-like Game: Separating PBE

- Consider Raising for King and Checking for Jack



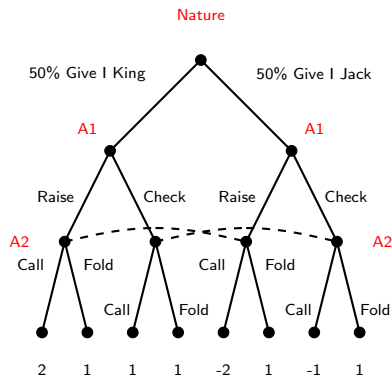
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- What is A2's posterior belief?



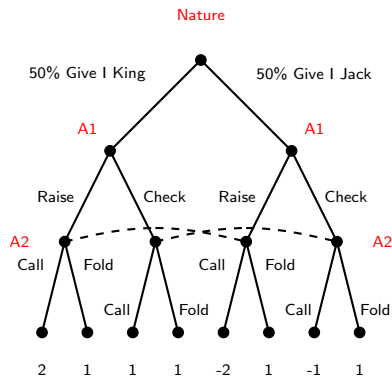
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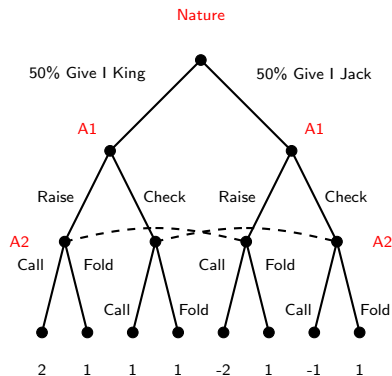
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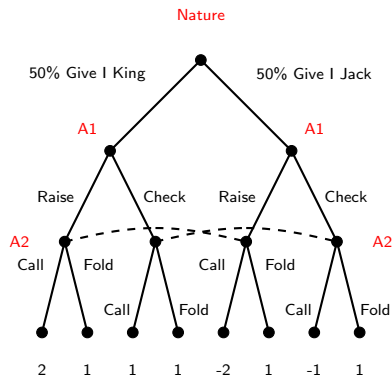
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- What is A2's optimal strategy?



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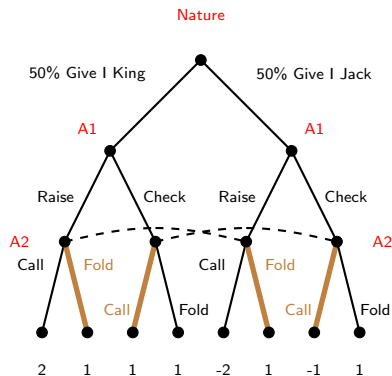
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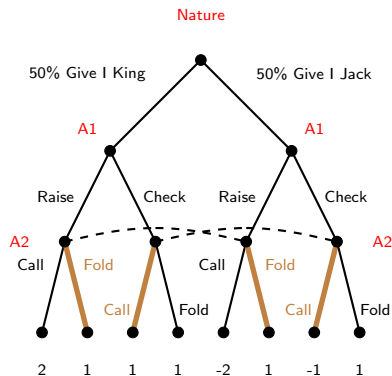
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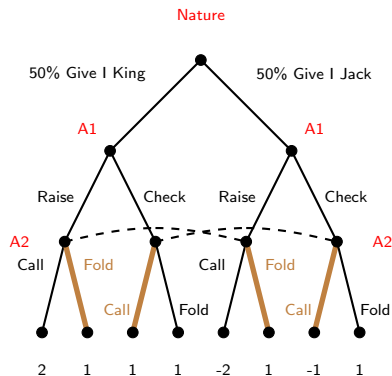
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- Given A2's optimal strategy, what is A1's best response?
  - Indifferent between Raise and Check if King ( $1 = 1$ )



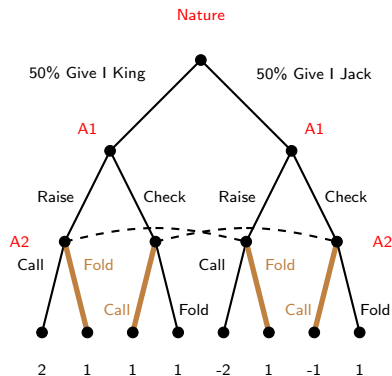
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- Given A2's optimal strategy, what is A1's best response?
  - Indifferent between Raise and Check if King ( $1 = 1$ )
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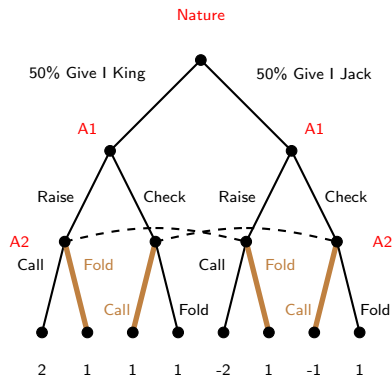
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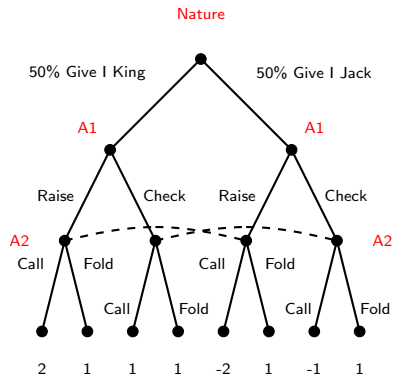
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- How about Checking for King and Raising for Jack?



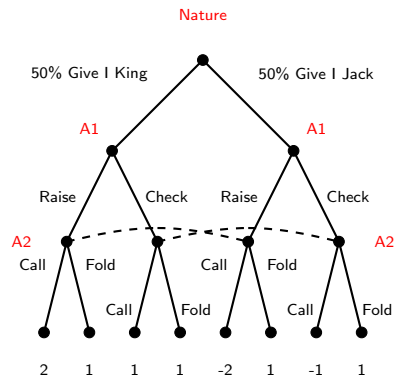
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- Consider Raising for both King and Jack



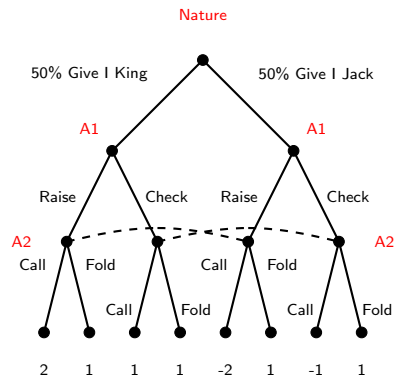
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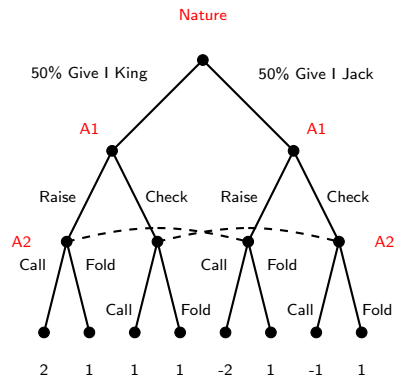
- Consider Raising for both King and Jack
- A2's posterior beliefs are the same as prior beliefs
  - King w.p. 0.5 and Jack w.p. 0.5





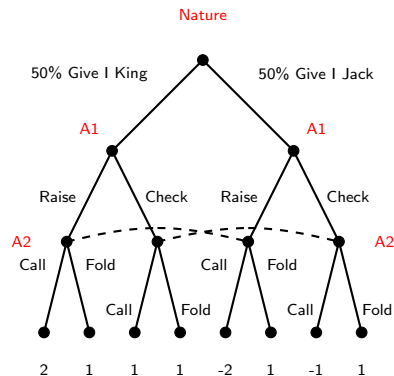
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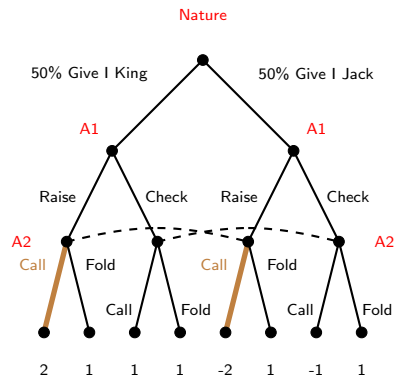
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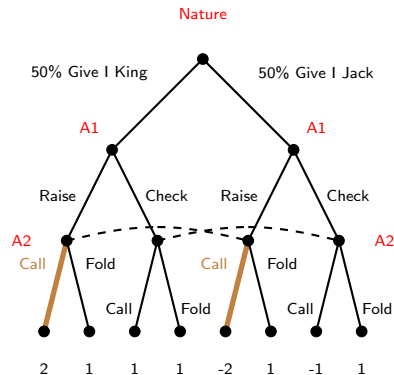
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  - Call give 0 ( $-2 \times 0.5 + 2 \times 0.5$ ), Fold gives -1
  - A2 prefers Call on the equilibrium path



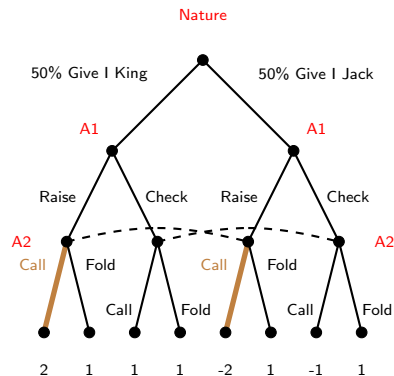
# Simple Poker-like Game: Pooling PBE

- Consider Raising for both King and Jack
- A2's posterior beliefs are the same as prior beliefs
  - King w.p. 0.5 and Jack w.p. 0.5
- What is A2's optimal strategy on equilibrium path (Raise)?
  - Call give 0 ( $-2 \times 0.5 + 2 \times 0.5$ ), Fold gives -1
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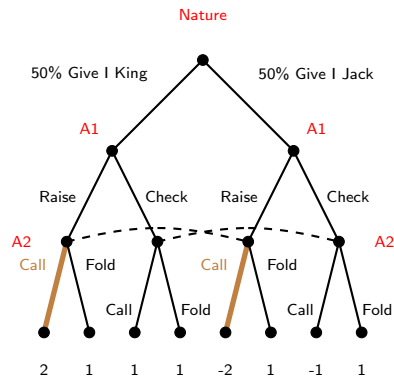
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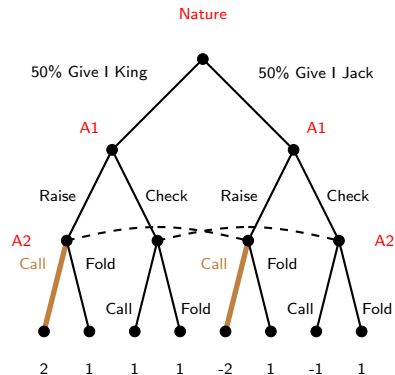
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  - Consider  $p$  for King and  $1 - p$  for Jack



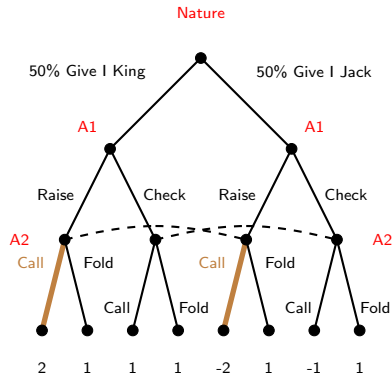
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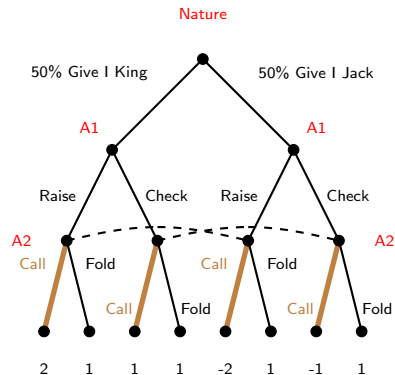
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  - Consider  $p$  for King and  $1 - p$  for Jack
  - Call give  $-p + 1 - p$ , Fold gives -1 and
  - For  $p < 1$ , A2 prefers Call, for  $p = 1$ , A2 is indifferent





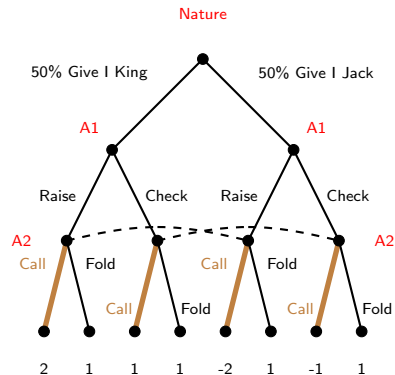
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- If A2 Calls ( $p \leq 1$ ), what is A1's best response?



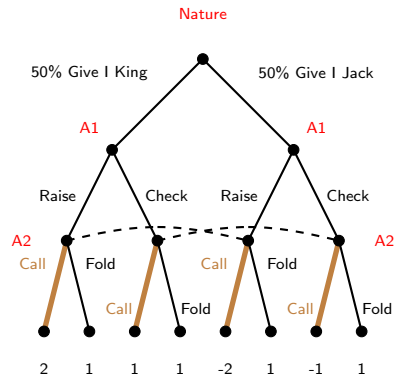
## Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ( $p \leq 1$ ), what is A1's best response?
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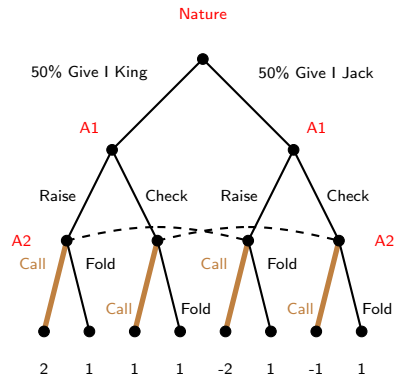
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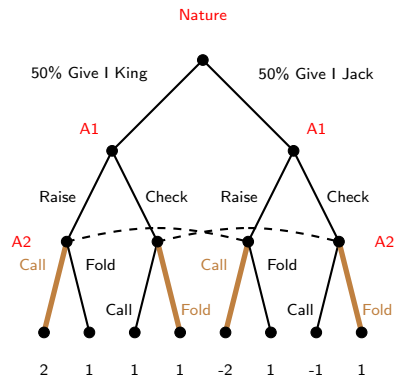
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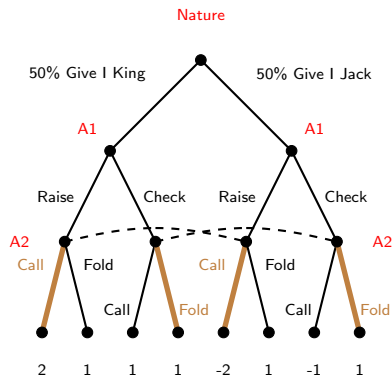
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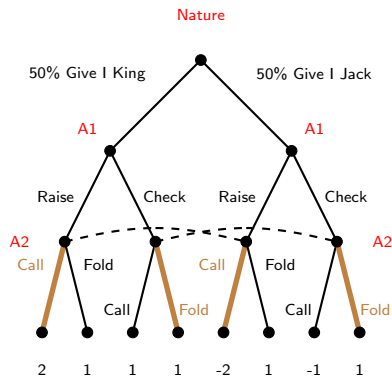
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- What if A2 Calls on and Folds off the path (for  $p = 1$ )?
  - If King, A1 prefers Raise



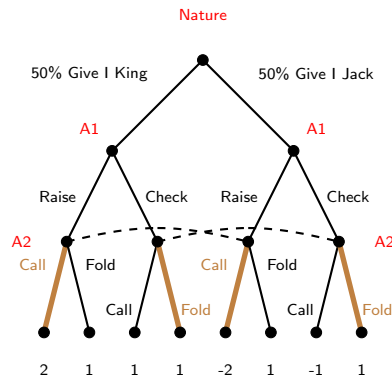
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# Simple Poker-like Game: Pooling PBE (cont.)

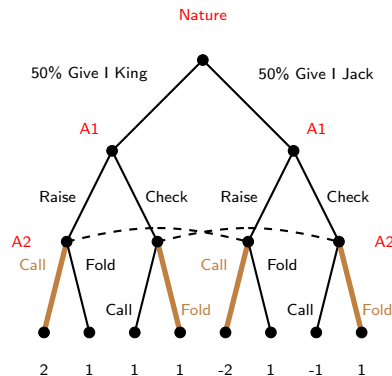
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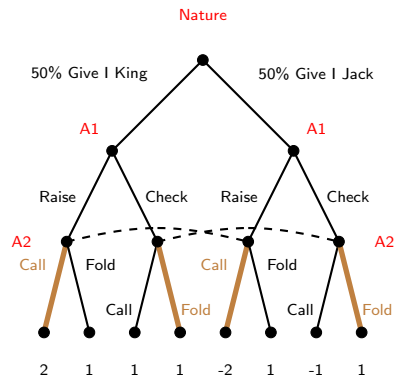
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  - If King, A1 prefers Raise
  - If Jack, A1 prefers Check
  - A1 wants to **deviate** from pooling strategy
- There is no  $p$  for which A1 wants to follow pooling



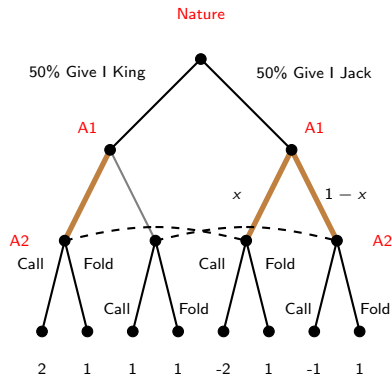
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- There is no  $p$  for which A1 wants to follow pooling
- What about Checking for both King and Jack?



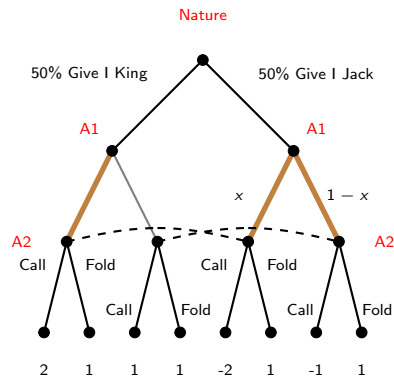
# Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p  $x$



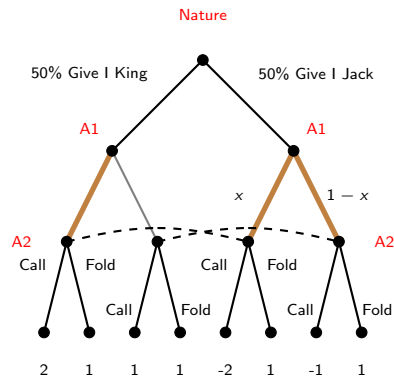
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- If King, A1 Raises - If Jack, A1 Raises w.p  $x$
- What is A2's posterior belief?



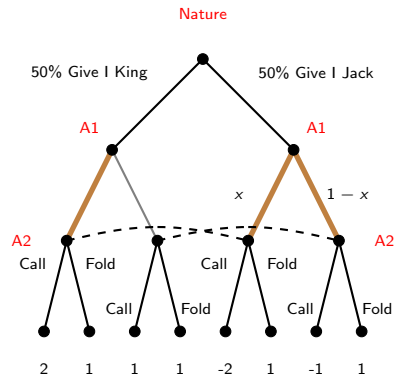
# Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p  $x$
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  - If Check, Jack w.p. 1



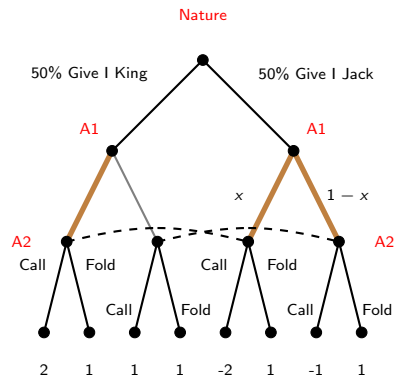
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- If King, A1 Raises - If Jack, A1 Raises w.p  $x$
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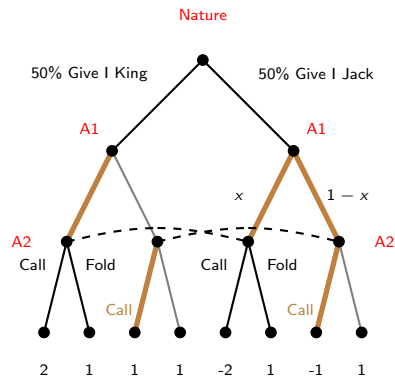
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- What is A2's best response if A1 Checks?



# Simple Poker-like Game: Semi-separating PBE

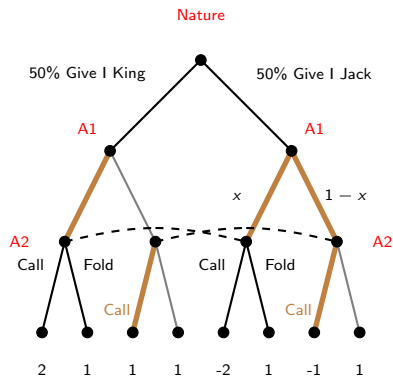
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- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)





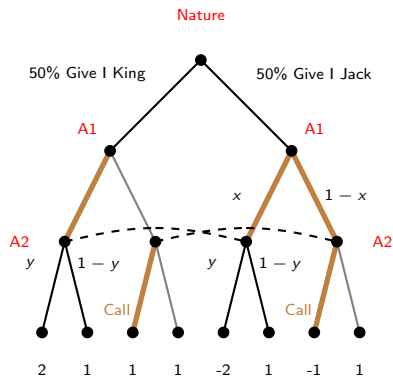
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- A2's strategy should make A1 indifferent if Jack



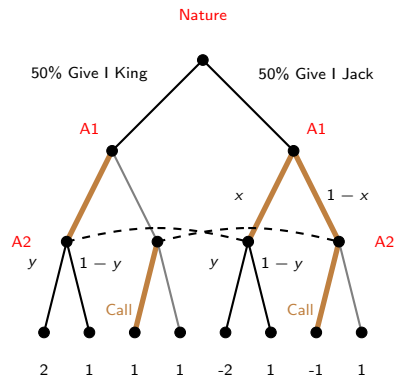
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  - Suppose A2 Calls w.p.  $y$  if A1 Raises



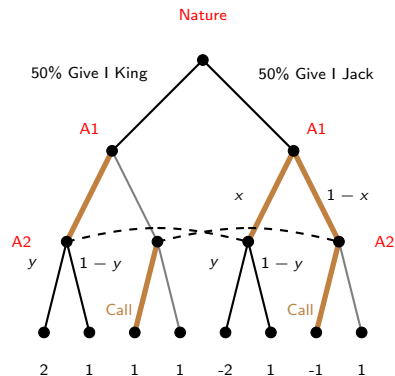
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  - Suppose A2 Calls w.p.  $y$  if A1 Raises
  - A1's utility for Raise is  $-2y + 1 - y$



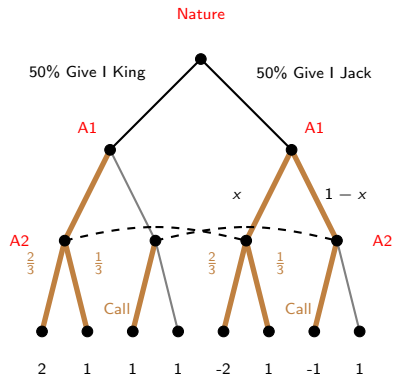
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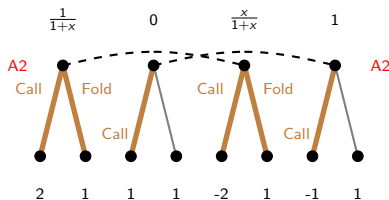


# Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p.  $x$
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- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
  - Suppose A2 Calls w.p.  $y$  if A1 Raises
  - A1's utility for Raise is  $-2y + 1 - y$
  - A1's utility for Check is  $-1$
  - $y = 2/3$  makes A1 indifferent



## Simple Poker-like Game: Semi-separating PBE (cont.)



- $x$  should be set s.t. A2 is indifferent between Call and Fold
- If A1 Raises, A2's utility for Call is  $(2x - 2)/(1 + x)$
- If A1 Raises, A2's utility for Fold is  $-1$
- $x = 1/3$  makes A2 indifferent between Call and Fold



## Simple Poker-like Game: Final Semi-separating PBE

- A1 Raises w.p. 1 if King and w.p.  $1/3$  if Jack
- A1 Checks w.p. 0 if King and w.p.  $2/3$  if Jack
- A2 Calls w.p. 1 if A1 Checks and w.p.  $2/3$  if A1 Raises
- A2 Folds w.p. 0 if A1 Checks and w.p.  $1/3$  if A1 Raises
- If A1 Raises, A2 believes King w.p.  $3/4$  and Jack w.p.  $1/4$
- If A1 Checks, A2 believes Jack w.p. 1



# Acknowledgment

- This lecture is a slightly modified version of ones prepared by
  - Asu Ozdaglar [MIT 6.254]
  - William Spaniel [Game Theory 101](#)
- Yiqin Huang helped with importing slides from PowerPoint to  $\text{\LaTeX}$

