# Game-theoretic Foundations of Multi-agent Systems

## Lecture 3: Games in Normal Form

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# Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium

# Normal-form Games

- Let's start with games in which all agents act simultaneously
- Agents choose their actions without knowledge of other agents' actions
- Such games are referred to as strategic-form games or normal-form games

# Normal-form Games: Definition

- The game consists of a set of agents, $N = \{1, 2, \ldots, n\}$
- Set of available actions to agent $i$ is denoted by $A_i$
- Action taken by agent $i$ is denoted by $a_i \in A_i$
- Outcome of game is an action profile of all agents, $a = (a_1, \ldots, a_n)$
- Set of all action profiles is denoted by $A = \prod A_i$
- Agent $i$ has a utility function $u_i$ that maps outcomes to real numbers

# Some Notations

- $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ is an action profile of all agents except $i$
- $A_{-i} = \prod_{j \neq i} A_j$ is set of action profiles of all agents except $i$
- $a = (a_i, a_{-i}) \in A$ is another way of denoting an action profile (or an outcome)

# Matrix-form Representation

- When $A_i$ is finite for all $i$, we call the game finite game
- For 2 agents and small action sets, game can be represented in matrix form

Agent 2

| Agent 1 | | x | y |
|---|---|---|---|
| | m | $a, b$ | $e, f$ |
| | n | $c, d$ | $g, h$ |

- Each cell indexed by row $r$ and column $c$ contains a pair, $(p, q)$, where $p = u_1(r, c)$ and $q = u_2(r, c)$.

# Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 1 pockets both, otherwise agent 2 pockets them

| | Heads | Tails |
|---|---|---|
| Heads | $-1, 1$ | $1, -1$ |
| Tails | $1, -1$ | $-1, 1$ |

- Zero-sum game: Utility of one agent is negative of utility of other agent

# Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | $0, 0$ | $-1, 1$ | $1, -1$ |
| Paper | $1, -1$ | $0, 0$ | $-1, 1$ |
| Scissors | $-1, 1$ | $1, -1$ | $0, 0$ |

# Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

| | Left | Right |
|---|---|---|
| Left | $1, 1$ | $-1, -1$ |
| Right | $-1, -1$ | $1, 1$ |

- Team game: For all outcomes $s$, and any pair of agents $i$ and $j$, it is the case that $u_i(a) = u_j(a)$ (also known as common-payoff game or pure-coordination game)

# Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ($a_i \in [0, \infty]$)
- Utility of each firm is its total revenue minus its total cost

$$u_i(a_1, a_2) = a_i p(a_1 + a_2) - c a_i$$

  - $p(\cdot)$ is the price function that maps total production to a price
  - $c$ is a unit cost
  - E.g., $p(x) = \max(0, 2 - x)$ and $c = 1$

# Outline

# Mixed and Pure Strategies

- Let $\Delta(X)$ be set of all probability distributions over $X$
- Set of (mixed) strategies for agent $i$ is denoted by $S_i = \Delta(A_i)$
- For mixed strategy $s_i \in S_i$, $s_i(a)$ is probability that action $a$ is played under $s_i$
- Pure strategy is a mixed strategy that puts probability 1 on a single action
- Support of mixed strategy $s_i$ is set of pure strategies, $a_i$, such that $s_i(a_i) > 0$
- Expected utility of agent $i$ for a (mixed) strategy profile $s = (s_1, \ldots, s_n)$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

# Example

| | | Agent 2 | | |
|---|---|---|---|---|
| | | R ($\frac{2}{3}$) | P (0) | S ($\frac{1}{3}$) |
| | R ($\frac{1}{3}$) | $0, 0$ | $-1, 1$ | $1, -1$ |
| Agent 1 | P ($\frac{2}{3}$) | $1, -1$ | $0, 0$ | $-1, 1$ |
| | S (0) | $-1, 1$ | $1, -1$ | $0, 0$ |

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$

# Dominant and Dominated Strategies

- Let $s_i$ and $s_i'$ be two strategies of agent $i$
- $s_i$ strictly dominates $s_i'$ if
  - $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$
- $s_i$ weakly dominates $s_i'$ if
  - $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$, and
  - $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for at least one $s_{-i} \in S_{-i}$
- $s_i$ is strictly/weakly dominant if it strictly/weakly dominates all other strategy
- $s_i$ is strictly/weakly dominated if another strategy strictly/weakly dominates it
- $s = (s_1, \ldots, s_n)$ is dominant strategy equilibrium if $s_i$ is dominant strategy for all $i$

# Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

| | D | C |
|---|---|---|
| D | $-2, -2$ | $-4, -1$ |
| C | $-1, -4$ | $-3, -3$ |

- Absolute value of utilities are the length of jail term each prisoner gets
- Confess is strictly dominant strategy for both prisoners
- (C,C) is a strict dominant strategy equilibrium
- The dilemma: (D,D) is better for both prisoners, but they won't play it!

# Iterated Elimination of Strictly Dominated Strategies

- All strictly dominated pure strategies can be ignored

$$
\begin{array}{c|c|c|c|}
 & L & C & R \\ \hline
U & 3,1 & 0,2 & 0,0 \\ \hline
M & 1,2 & 1,1 & 5,0 \\ \hline
D & 0,1 & 4,2 & 0,0 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|}
 & L & C \\ \hline
U & 3,1 & 0,2 \\ \hline
M & 1,2 & 1,1 \\ \hline
D & 0,1 & 4,2 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|}
 & L & C \\ \hline
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\end{array}
\Rightarrow
\begin{array}{c|c|}
 & C \\ \hline
U & 0,2 \\ \hline
D & 4,2 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|}
 & C \\ \hline
D & 4,2 \\ \hline
\end{array}
$$

- Column R can be eliminated, since it is dominated by, for example, column L
- M is not dominated by U or D but is dominated by 0.5U + 0.5D mixed strategy
- Note, however, that it was not dominated before the elimination of the R column

# Iterated Elimination of Strictly Dominated Strategies (cont.)

- Once one pure strategy is eliminated, another strategy that was not dominated can become dominated
- In finite games, iterated elimination of strictly dominated strategies ends after finite number of iterations
- Order of elimination does not matter when removing strictly dominated strategies (Church–Rosser property)
- Elimination order can make a difference in final outcome when removing weakly dominated strategies
- If the procedure ends with a single strategy for each agent, then the game is said to be dominance solvable

# Existence of Dominant Strategy Equilibrium

- Dominant strategy equilibrium does not always exist
- Example: Matching pennies

| | Heads | Tails |
|---|---|---|
| Heads | $-1, 1$ | $1, -1$ |
| Tails | $1, -1$ | $-1, 1$ |

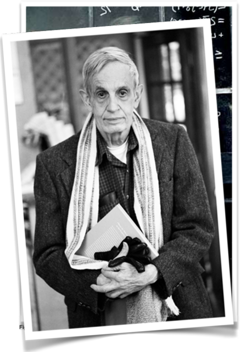
# Outline

# Best Response

- Picking a strategy would be simple if an agent knew how others were going to act
- Best response: $s_i^* \in BR_i(s_{-i})$ iff $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$
- Best response is not necessarily unique
  - If there is more than one best response, any mixed strategy over those must be a best response as well
- Best response is not a solution concept
  - I.e., it does not identify an interesting set of outcomes
  - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the Nash equilibrium

# Nash Equilibrium - Intersection of Best Responses

- $s^* = (s_1^*, ..., s_n^*)$ is a Nash equilibrium iff $\forall i,\ s_i^* \in Br_i(s_{-i}^*)$
- No agent can profitably deviate given strategies of others
- Nash equilibrium is a stable strategy profile
- Nash theorem: Every finite game has a Nash equilibrium



**John Forbes Nash Jr.**
**1928-2015**

# Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

| | | Wife: Football | Wife: Opera |
|---|---|---|---|
| Husband | Football | 2,1 $\boxed{2, 1}$ | $0, 0$ |
| Husband | Opera | $0, 0$ | 1,2 $\boxed{1, 2}$ |

- Are these the only Nash equilibria?

# Example: Battle of Sexes (cont.)

| | F ($p$) | O ($1-p$) |
|---|---|---|
| F | $2, 1$ | $0, 0$ |
| O | $0, 0$ | $1, 2$ |

- In general, it is tricky to compute mixed-strategy equilibria (will discuss this later)
- It becomes easy when we know (or can guess) support of equilibrium strategies
- Let us now guess that both agents randomize over both F and O
- Wife's strategy is to play F w.p. $p$ and O w.p. $1-p$
- Husband must be indifferent between F and O (why?):

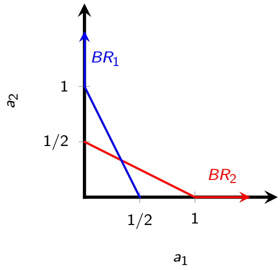
$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1-p) \Rightarrow p = 1/3$$

- You can show that the unique mixed-strategy NE is $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$

# Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$BR_i(a_{-i}) = \underset{a_i \geq 0}{\operatorname{argmax}}\, a_i(2 - a_i - a_{-i}) - a_i$$
$$= \begin{cases} (1 - a_{-i})/2 & \text{if } a_{-i} < 1, \\ 0 & \text{Otherwise.} \end{cases}$$

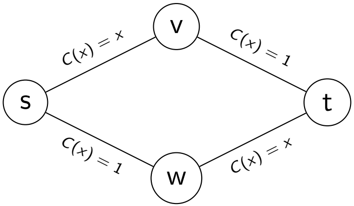


# The "Equilibrium Selection Problem"

- You are about to play a game that you have never played before with a person that you have never met
- According to which equilibrium should you play?
  - Equilibrium that maximizes the sum of utilities (social welfare)
  - Or, at least not a Pareto-dominated equilibrium
  - So-called focal equilibria (e.g., "Meet in Paris" game - you and a friend were supposed to meet in Paris at noon on Sunday, but you forgot to discuss where and you cannot communicate. Where will you go?)
  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - ...
- Equilibrium selection is a difficult problem

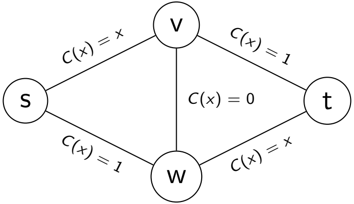
# Outline

# Braess's Paradox




- Suppose there are $2k$ drivers commuting from $s$ to $t$
- $C(x)$ indicates travel time in hours for $x$ fraction of drivers
- $k$ drivers going through $v$, and $k$ going through $w$ is NE (why?)

# Braess's Paradox (cont.)




- Suppose we install a teleportation device allowing instant travel from $v$ to $w$
- What is new NE?
- What is optimal commute time?
- Price of anarchy: ratio between (worst) NE performance and optimal performance
  - Ratio between 2 and 3/2 in Braess's Paradox

# Outline

# Maxmin Strategy

- Maxmin strategy for agent $i$ is

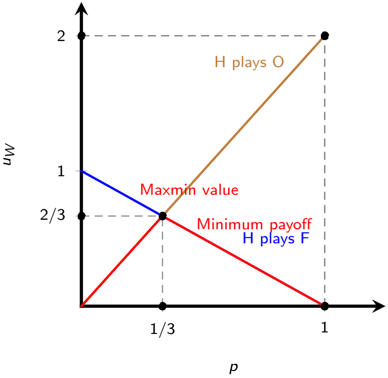
$$\underset{s_i}{\operatorname{argmax}} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Maxmin value for agent $i$ is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- If $i$ plays maxmin strategy and others play arbitrarily, $i$ still receives expected payoff of at least their maxmin value

# Example: Battle of Sexes




W

| | | F $(1-p)$ | O $(p)$ |
|---|---|---|---|
| H | F | $2, 1$ | $0, 0$ |
| | O | $0, 0$ | $1, 2$ |

# Minmax Strategy

- Minmax strategy against against $i$ is

$$\underset{s_{-i}}{\operatorname{argmin}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax value for agent $i$ is

$$\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax strategy against $i$ keeps maximum payoff of agent $i$ at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)

# Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium[1], each agent receives a payoff that is equal to both their maxmin value and their minmax value

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minimax theorem does not hold with pure strategies only (example?)

[1] You might wonder how a theorem from 1928 can use the term "Nash equilibrium," when Nash's work was published in 1950. John von Neumann used different terminology and proved the theorem in a different way; however, the given presentation is probably clearer in the context of modern game theory

# Example

| | | Agent 2 | |
|---|---|---|---|
| | | Left | Right |
| Agent 1 | Up | $20, -20$ | $0, 0$ |
| | Down | $0, 0$ | $10, -10$ |

- What is maximin value of agent 1 with and without mixed strategies?
- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?

# Outline

# Rationalizability

- Rationalizable strategy: Perfectly rational agent could justifiably play it
  - Best response to some beliefs about strategies of others
- Agents cannot have arbitrary beliefs about other agents
- Agent $i$'s beliefs must take into account:
  - Other agents' rationality
  - Other agents' knowledge of agent $i$'s rationality
  - Other agents' knowledge of agent $i$'s knowledge of their rationality
  - ... (infinite regress)

# Example: Matching Pennies

| | H | T |
|---|---|---|
| H | $-1, 1$ | $1, -1$ |
| T | $1, -1$ | $-1, 1$ |

- Col playing H is rationalizable
  - Col could believe Row plays H
- Col believing that Row plays H is rationalizable
  - Col could believe Row believes Col plays T
- Col believing that Row believes that Col plays T is rationalizable
  - Col could believe Row believes Col believes Row plays T
- ...
- In this game, all pure strategies are rationalizable

# Rationalizability: Properties

- Nash equilibrium strategies are always rationalizable
- Some rationalizable strategies are not Nash
  - Set of rationalizable strategies in finite games is nonempty
- To find rationalizable strategies:
  - In 2-player games, use iterated elimination of strictly dominated strategies
  - In $n$-player games, iterated elimination of never-best response strategies
    - Eliminate strategies that are not optimal against any belief about others' strategies

# Example: 2/3-Beauty Contest Game

- No agent plays more than 100
- 2/3 of average is strictly less than 67 ($100 \times 2/3$)
- Any integer $> 67$ is never-best response to any belief about others' strategy
- No agent plays more than 67
- 2/3 of average is less than 45 ($67 \times 2/3$)
- Any integer $> 45$ is never-best response to any belief about others' strategy
- ...
- Only rationalizable action is playing 1

# Outline

# Example: Battle of Sexes

W

| | | Football | Opera |
|---|---|---|---|
| H | F | $2, 1$ | $0, 0$ |
| | O | $0, 0$ | $1, 2$ |

- Unique mixed strategy NE yields each agent expected payoff of $2/3$
- In NE, agents randomize over strategies independently
- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome

# Example: Battle of Sexes (cont.)

- Suppose there is publicly observable fair coin
- If it is heads/tails, they both get recommendation to go to football/opera
- If they see heads, they believe that the other one goes to football
- Going to football is best response, agents have no incentive to deviate
- Similar argument can be made when they see tails
- Expected utilities for this play of game increases to $(1.5, 1.5)$

# Correlated Recommendations

- Let $R = (R_1, \ldots, R_n)$ be random variable taking values in $A = \prod_i A_i$
- Let $R$ be distributed according to $\pi \in \Delta(A)$
- $r = (r_1, \ldots, r_n)$ is an instantiatation of $R$ and a pure strategy profile
- $r_i \in A_i$ is called recommendation to agent $i$
- $\pi(r_i)$ represents marginal probability for $R_i = r_i$
- Given $r_i$, agent $i$ can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$

# Correlated Equilibrium: Formal Definition

- Correlated equilibrium of finite game is joint probability distribution $\pi \in \Delta(A)$ such that if $R$ is random variable distributed according to $\pi$, then for all $i, r_i \in A_i$ with $\pi(r_i) > 0$, and $r_i' \in A_i$

$$\sum_{r_{-i} \in A_{-i}} \pi(r_{-i} \mid r_i) \left[ u_i(r_i, r_{-i}) - u_i(r_i', r_{-i}) \right] \geq 0$$

- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations

# Example: Game of Chicken

| | | Driver 2 | |
|---|---|---|---|
| | | Dare | Yield |
| Driver 1 | D | $-5, -5$ | $1, -1$ |
| | Y | $-1, 1$ | $0, 0$ |

- (D,Y) and (Y,D) are strict pure-strategy NE
- Assume Driver 1 yields w.p. $p$ and Driver 2 yields w.p. $q$
- Using mixed equilibrium characterization, we have

$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

$$q - 5 \times (1 - q) = -(1 - q) \implies q = 4/5$$

- Mixed-strategy NE utilities are $(-0.2, -0.2)$, people die with probability 0.04

# Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is $1/3$
- Expected utility of Y is $1 \times$ -$2/3$
- Expected utility of D is $1 \times 1/3$ - $5 \times 2/3$
- Following the recommendation is better
- If D1 gets D recommendation, D2 must yield
- Following recommendation is again better
- Similar analysis works for D2
- Expected utilizes are $(0, 0)$, so nobody dies!

| | | D2: D | D2: Y |
|---|---|---|---|
| D1 | D | $-5, -5$<br>0% | $1, -1$<br>40% |
| D1 | Y | $-1, 1$<br>40% | $0, 0$<br>20% |

# Characterization of Correlated Equilibrium

- Join distribution $\pi \in \Delta(S)$ is correlated equilibrium of finite game iff

$$\sum_{r_{-i} \in A_{-i}} \pi(r) \left[ u_i(r) - u_i(r_i', r_{-i}) \right] \geq 0, \quad \forall i, r_i, r_i' \in A_i \tag{1}$$

- Proof (only for one side):
  - Correlated equilibrium can be written for all $i, r_i, r_i' \in A_i$ with $\pi(r_i) > 0$ as:

$$\sum_{r_{-i} \in A_{-i}} \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})} [u_i(r_i, r_{-i}) - u_i(r_i', r_{-i})] \geq 0$$

  - Denominator does not depend on variable of sum
  - So it can be factored and canceled
  - If $\pi(r_i) = 0$, LHS of (1) is zero regardless of $i$ and $r_i'$, so equation always holds

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