Game-theoretic Foundations of Multi-agent Systems

Lecture 6: Repeated Games

Seyed Majid Zahedi

UNIVERSITY OF WATERLOO

Outline

1. Finitely Repeated Games
2. Infinitely Repeated Games
3. Folk Theorem
4. Repeated Games with Imperfect Monitoring

Repeated Games

- In a (typical) repeated game:
  - Agents play a given game (aka. stage game)
  - Then, they get their utilities
  - And, they play again ...

Repeated Games

- In a (typical) repeated game:
    - Agents play a given game (aka. stage game)
    - Then, they get their utilities
    - And, they play again ...
- Can be repeated finitely or infinitely many times

Repeated Games

- In a (typical) repeated game:
  - Agents play a given game (aka. stage game)
  - Then, they get their utilities
  - And, they play again . . .
- Can be repeated finitely or infinitely many times
- Really, an extensive form game
  - Would like to find subgame-perfect equilibria

Repeated Games (cont.)

- One subgame-perfect equilibrium:
  - Keep repeating some Nash equilibrium of the stage game
  - Memoryless strategy, called a stationary strategy

Repeated Games (cont.)

- One subgame-perfect equilibrium:
  - Keep repeating some Nash equilibrium of the stage game
  - Memoryless strategy, called a stationary strategy
- But are there other equilibria?
  - Strategy space of repeated game is much richer than that of stage game

Key Questions

- Do agents see what the other agents played earlier?
- Do they remember what they knew?
- Given utility of each stage game, what is the utility of the entire repeated game?

Finitely Repeated Games (with Perfect Monitoring)

- Agents play stage game G for R rounds

Finitely Repeated Games (with Perfect Monitoring)

- Agents play stage game G for R rounds
- At each round, outcomes of all past rounds are observed by all agents

Finitely Repeated Games (with Perfect Monitoring)

- Agents play stage game G for R rounds
- At each round, outcomes of all past rounds are observed by all agents
- Agents' overall utility is sum of discounted utilities at each round
  - Discount factor is $0 \leq \delta \leq 1$
  - Game is denoted by $G^R(\delta)$

Finitely Repeated Games (with Perfect Monitoring)

- Agents play stage game G for R rounds
- At each round, outcomes of all past rounds are observed by all agents
- Agents' overall utility is sum of discounted utilities at each round
  - Discount factor is $0 \leq \delta \leq 1$
  - Game is denoted by $G^R(\delta)$
- Given sequence of utilities $u_i^{(1)}, ..., u_i^{(R)}$, $u_i = \sum_{r=1}^{R} \delta^{r-1} u_i^{(r)}$

Example: Finitely Repeated Prisoner's Dilemma

- Two agents play Prisoner's Dilemma for R rounds ($\delta = 1$)

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

Example: Finitely Repeated Prisoner's Dilemma

- Two agents play Prisoner's Dilemma for R rounds ($\delta = 1$)

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

- Starting from last round, (C, C) is dominant strategy

Example: Finitely Repeated Prisoner's Dilemma

- Two agents play Prisoner's Dilemma for R rounds ($\delta = 1$)

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

- Starting from last round, (C, C) is dominant strategy
- Hence, in second-to-last round, there is no way to influence what will happen

Example: Finitely Repeated Prisoner's Dilemma

- Two agents play Prisoner's Dilemma for R rounds ($\delta = 1$)

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

- Starting from last round, (C, C) is dominant strategy
- Hence, in second-to-last round, there is no way to influence what will happen
- So, (C, C) is dominant strategy at this round as well

Example: Finitely Repeated Prisoner's Dilemma

- Two agents play Prisoner's Dilemma for R rounds ($\delta = 1$)

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

- Starting from last round, (C, C) is dominant strategy
- Hence, in second-to-last round, there is no way to influence what will happen
- So, (C, C) is dominant strategy at this round as well
- The unique SPE is (C, C) at each round

SPE in Finitely Repeated Games

[Theorem]

- If stage game G has unique strategy equilibrium s^*, then $G^R(\delta)$ has unique SPE in which $s^{(r)} = s^*$ for all $r = 1, ..., R$, regardless of history

[Proof]

- By backward induction, at round R, we have $s^{(R)} = s^*$
- Given this, then we have $s^{(R-1)} = s^*$, and continuing inductively, $s^{(r)} = s^*$ for all $r = 1, ..., R$, regardless of history

SPE: Example I

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D1	D2	C
D1	$4, 4$	$1, 1$	$6, 0$
D2	$1, 1$	$2, 2$	$6, 0$
C	$0, 6$	$0, 6$	$5, 5$

SPE: Example I

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D1	D2	C
D1	$4, 4$	$1, 1$	$6, 0$
D2	$1, 1$	$2, 2$	$6, 0$
C	$0, 6$	$0, 6$	$5, 5$

- Consider the following strategy:
  - In round 1, cooperate;
  - In round 2, if someone defected in round 1, play D2; otherwise, play D1

SPE: Example I

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D1	D2	C
D1	$4, 4$	$1, 1$	$6, 0$
D2	$1, 1$	$2, 2$	$6, 0$
C	$0, 6$	$0, 6$	$5, 5$

- Consider the following strategy:
  - In round 1, cooperate;
  - In round 2, if someone defected in round 1, play D2; otherwise, play D1
- If both agents play this, is that SPE?

SPE: Example II

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D	Crazy	C
D	$4, 4$	$1, 0$	$6, 0$
Crazy	$0, 1$	$0, 0$	$0, 1$
C	$0, 6$	$1, 0$	$5, 5$

SPE: Example II

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D	Crazy	C
D	$4, 4$	$1, 0$	$6, 0$
Crazy	$0, 1$	$0, 0$	$0, 1$
C	$0, 6$	$1, 0$	$5, 5$

- What are the subgame perfect equilibria?

SPE: Example II

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D	Crazy	C
D	$4, 4$	$1, 0$	$6, 0$
Crazy	$0, 1$	$0, 0$	$0, 1$
C	$0, 6$	$1, 0$	$5, 5$

- What are the subgame perfect equilibria?
- Consider the following strategy:
  - In round 1, cooperate;
  - In round 2, if someone played D or Crazy in round 1, play Crazy; otherwise, play D

SPE: Example II

- Two agents play the following game for 2 rounds ($\delta = 1$)

	D	Crazy	C
D	$4, 4$	$1, 0$	$6, 0$
Crazy	$0, 1$	$0, 0$	$0, 1$
C	$0, 6$	$1, 0$	$5, 5$

- What are the subgame perfect equilibria?
- Consider the following strategy:
  - In round 1, cooperate;
  - In round 2, if someone played D or Crazy in round 1, play Crazy; otherwise, play D
- If both agents play this, is that NE (not SPE)?

TSPE: Example III

- If G has multiple equilibria, then $G^R(\delta)$ does not have unique SPE
- Consider following example

	x	y	z
x	$3, 3$	$0, 4$	$-2, 0$
y	$4, 0$	$1, 1$	$-2, 0$
z	$0, -2$	$0, -2$	$-1, -1$

- Stage game has two pure NE: (y, y) and (z, z)
- Consider the following policy:
  - Play x in first round
  - Play y in second round if opponent played x; otherwise, play z
- Is both agents playing this SPE?

Outline

1. Finitely Repeated Games
2. Infinitely Repeated Games
3. Folk Theorem
4. Repeated Games with Imperfect Monitoring

Utilities in Infinitely Repeated Games

- Limit-average utility:

$$u_i = \lim_{R\to\infty} \frac{\sum_{r=1}^{R} u_i^{(r)}}{R}$$

- Future-discounted utility:

$$u_i = (1-\delta)\sum_{r=1}^{\infty} \delta^{r-1} u_i^{(r)},$$

for some $0 \geq \delta < 1$

Subgame Perfection in Infinitely Repeated Games

- One-shot deviation from strategy s means deviating from s in single stage and conforming to it thereafter
- Strategy profile s^* is SPE if and only if there are no profitable one-shot deviation for each subgame and every agent
- This follows from principle of optimality of dynamic programming
- This applies to finitely repeated games as well

Trigger Strategies (TS)

- Agents get punished if they deviate from agreed profile
- In non-forgiving TS (or grim TS), punishment continues forever

$$s_i^{(t)} = \begin{cases} s_i^* & \text{if } s^{(r)} = s^* \ \ \forall r < t, \\ \underline{s}_i^j & \text{otherwise} \end{cases}$$

- Here, s^* is agreed profile, and $\underline{s}_i^j$ is punishment strategy of i against agent j
- Single deviation by j triggers agent i to switch to $\underline{s}_i^j$ forever

Example: Infinitely Repeated Prisoner's Dilemma

- Consider trigger strategy:
  - Cooperate as long as everyone cooperates
  - Once a player defects, defect forever

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

Example: Infinitely Repeated Prisoner's Dilemma

- Consider trigger strategy:
  - Cooperate as long as everyone cooperates
  - Once a player defects, defect forever
- Is both agents playing this SPE?

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

Example: Infinitely Repeated Prisoner's Dilemma

- Consider trigger strategy:
  - Cooperate as long as everyone cooperates
  - Once a player defects, defect forever
- Is both agents playing this SPE?
- Does it depend on δ?

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)
  - Utility from no deviation: $(1-\delta)(-2-2\delta-2\delta^2+\dots)=-2$

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)
  - Utility from no deviation: $(1-\delta)(-2-2\delta-2\delta^2+\dots)=-2$
  - Utility from on-shot deviation: $(1-\delta)(-1+(-3\delta-3\delta^2+\dots)=-(1-\delta)-3\delta$

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)
  - Utility from no deviation: $(1-\delta)(-2-2\delta-2\delta^2+\dots)=-2$
  - Utility from on-shot deviation: $(1-\delta)(-1+(-3\delta-3\delta^2+\dots)=-(1-\delta)-3\delta$
  - Deviation is not beneficial if $\delta \geq 1/2$

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)
  - Utility from no deviation: $(1-\delta)(-2-2\delta-2\delta^2+\dots)=-2$
  - Utility from on-shot deviation: $(1-\delta)(-1+(-3\delta-3\delta^2+\dots)=-(1-\delta)-3\delta$
  - Deviation is not beneficial if $\delta \geq 1/2$
- Type-2 subgames: (C is best response to C)

Trigger Strategy for Infinitely Repeated Prisoners' Dilemma

- We can use one-stage deviation principle
- There are two types of subgames:
  - Type 1: Both agents cooperated so far
  - Type 2: At least one agent defeated in the past
- Type-1 subgames: (D is best response to D)
  - Utility from no deviation: $(1-\delta)(-2-2\delta-2\delta^2+\dots)=-2$
  - Utility from on-shot deviation: $(1-\delta)(-1+(-3\delta-3\delta^2+\dots)=-(1-\delta)-3\delta$
  - Deviation is not beneficial if $\delta \geq 1/2$
- Type-2 subgames: (C is best response to C)
  - Other agents will always play C, thus C is best response

Tit-for-tat Strategy

- Consider tit-for-tat strategy:
  - Cooperate in 1st round
  - Then, do whatever other agent did in previous round

Tit-for-tat Strategy

- Consider tit-for-tat strategy:
  - Cooperate in 1st round
  - Then, do whatever other agent did in previous round
- Is both agents playing this NE?

Tit-for-tat Strategy

- Consider tit-for-tat strategy:
  - Cooperate in 1st round
  - Then, do whatever other agent did in previous round
- Is both agents playing this NE?
- Is both agents playing this SPE?

Tit-for-tat Strategy

- Consider tit-for-tat strategy:
    - Cooperate in 1st round
    - Then, do whatever other agent did in previous round
- Is both agents playing this NE?
- Is both agents playing this SPE?
- What about one playing TFT and other trigger?

Remarks

- If s^* is NE of G, then "each agent plays s_i^*" is SPE of $G^R(\delta)$
  - Future play of other agents is independent of how each agent plays
  - Optimal play is to maximize current utility, i.e., play static best response
- Sets of equilibria for finite and infinite horizon versions can be different
  - Multiplicity of equilibria in repeated prisoner's dilemma only occurs at $R = \infty$
  - For any finite R (thus for $R \to \infty$), repeated prisoners' dilemma has unique SPE

Repetition Could Lead to Bad Outcomes

- Consider the following game

	x	y	z
x	$2, 2$	$2, 1$	$0, 0$
y	$1, 2$	$1, 1$	$-1, 0$
z	$0, 0$	$0, -1$	$-1, -1$

- Strategy x strictly dominates y and z for both agents
- Unique NE of stage game is (x, x)
- If $\delta \geq 1/2$, this game has SPE in which (y, y) is played in every round
- It is supported by slightly more complicated strategy than grim trigger
  - I. Play y in every round unless someone deviates, then go to II
  - II. Play z. If no one deviates go to I. If someone deviates stay in II

Outline

1. Finitely Repeated Games

2. Infinitely Repeated Games

3. Folk Theorem

4. Repeated Games with Imperfect Monitoring

Characterizing NE of Infinitely Repeated Games

- Characterizing all equilibrium strategy profiles might be challenging

Characterizing NE of Infinitely Repeated Games

- Characterizing all equilibrium strategy profiles might be challenging
- Instead, we can characterize utilities obtained in them

Characterizing NE of Infinitely Repeated Games

- Characterizing all equilibrium strategy profiles might be challenging
- Instead, we can characterize utilities obtained in them
- Such utilities must be feasible
  - There must be outcomes of game such that agents, on average, get these utilities

Characterizing NE of Infinitely Repeated Games

- Characterizing all equilibrium strategy profiles might be challenging
- Instead, we can characterize utilities obtained in them
- Such utilities must be feasible
  - There must be outcomes of game such that agents, on average, get these utilities
- They must also be enforceable
  - Deviation should lead to punishment that outweighs benefits of deviation

Characterizing NE of Infinitely Repeated Games

- Characterizing all equilibrium strategy profiles might be challenging
- Instead, we can characterize utilities obtained in them
- Such utilities must be feasible
  - There must be outcomes of game such that agents, on average, get these utilities
- They must also be enforceable
  - Deviation should lead to punishment that outweighs benefits of deviation
- Folk theorem states that utility vector can be realized by some NE iff it is both feasible and enforceable

Feasible Utilities: Formal Definition

- Utility profile $u = (u_1, u_2, \ldots, u_n)$ is feasible if there exist rational, non-negative values $\{\alpha_a\}$ such that for all i, $u_i = \sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$

Feasible Utilities: Formal Definition

- Utility profile $u = (u_1, u_2, \ldots, u_n)$ is feasible if there exist rational, non-negative values $\{\alpha_a\}$ such that for all i, $u_i = \sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$
- You could think of feasible utilities as convex hull of possible outcomes:

$$U = \text{Conv}\{u \in \mathbb{R}^{|N|} \mid \text{ there exists } a \in A \text{ such that } u(a) = u\}$$

Feasible Utilities: Formal Definition

- Utility profile $u = (u_1, u_2, \ldots, u_n)$ is feasible if there exist rational, non-negative values $\{\alpha_a\}$ such that for all i, $u_i = \sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$
- You could think of feasible utilities as convex hull of possible outcomes:

$$U = \text{Conv}\{u \in \mathbb{R}^{|N|} \mid \text{ there exists } a \in A \text{ such that } u(a) = u\}$$

- Note that $U \neq \{u \in \mathbb{R}^{|N|} \mid \text{ there exists } s \in S \text{ such that } u(s) = u\}$

Feasibility: Example

	Left	Right
Left	$2, 2$	$0, 3$
Right	$3, 0$	$1, 1$

- Utility vector (2, 2) is feasible as it is one of outcomes of game

Feasibility: Example

	Left	Right
Left	$2, 2$	$0, 3$
Right	$3, 0$	$1, 1$

- Utility vector (2, 2) is feasible as it is one of outcomes of game
- Utility vector (1, 2.5) is feasible as agents can alternate between (2, 2) and (0, 3)

Feasibility: Example

	Left	Right
Left	$2, 2$	$0, 3$
Right	$3, 0$	$1, 1$

- Utility vector (2, 2) is feasible as it is one of outcomes of game
- Utility vector (1, 2.5) is feasible as agents can alternate between (2, 2) and (0, 3)
- What about (0.5, 2.75)?

Feasibility: Example

	Left	Right
Left	$2, 2$	$0, 3$
Right	$3, 0$	$1, 1$

- Utility vector (2, 2) is feasible as it is one of outcomes of game
- Utility vector (1, 2.5) is feasible as agents can alternate between (2, 2) and (0, 3)
- What about (0.5, 2.75)?
- What about (3, 0.1)?

Enforceable and Individually Rational Utilities

- Recall minmax value of agent i:

$$\underline{v}_i = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

Enforceable and Individually Rational Utilities

- Recall minmax value of agent i:

$$\underline{v}_i = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Utility profile $u \in \mathbb{R}^{|N|}$ is individually rational if $u_i \geq \underline{v}_i$ for all i

Enforceable and Individually Rational Utilities

- Recall minmax value of agent i:

$$\underline{v}_i = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Utility profile $u \in \mathbb{R}^{|N|}$ is individually rational if $u_i \geq \underline{v}_i$ for all i
- Utility profile $u = (u_1, u_2, \ldots, u_n)$ is enforceable if it is individually rational

Nash Folk Theorem

- Consider infinitely repeated game G played by agents with average utilities

Nash Folk Theorem

- Consider infinitely repeated game G played by agents with average utilities
- If u is utility profile for any NE of repeated G, then u_i is enforceable for all i

Nash Folk Theorem

- Consider infinitely repeated game G played by agents with average utilities
- If u is utility profile for any NE of repeated G, then u_i is enforceable for all i
- If u is both feasible and enforceable, then u is utility profile for some NE of G

Nash Folk Theorem

- Consider infinitely repeated game G played by agents with average utilities
- If u is utility profile for any NE of repeated G, then u_i is enforceable for all i
- If u is both feasible and enforceable, then u is utility profile for some NE of G
- Folk theorem can be stated for agents with discounted utilities as well

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy
  - Play certain strategy as long as no one deviates

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy
  - Play certain strategy as long as no one deviates
  - If some agent j deviates, then play minmax strategy against that agent thereafter

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy
  - Play certain strategy as long as no one deviates
  - If some agent j deviates, then play minmax strategy against that agent thereafter
- NE involves non-forgiving TS which may be costly for punishers

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy
  - Play certain strategy as long as no one deviates
  - If some agent j deviates, then play minmax strategy against that agent thereafter
- NE involves non-forgiving TS which may be costly for punishers
- NE may include non-credible threats

Problems with Nash Folk Theorem

- Any feasible and enforceable utility can be achieved (for patient enough agents)
- Enforcement is often done by grim trigger strategy
  - Play certain strategy as long as no one deviates
  - If some agent j deviates, then play minmax strategy against that agent thereafter
- NE involves non-forgiving TS which may be costly for punishers
- NE may include non-credible threats
- NE may not be subgame perfect

Example

	L	R
U	$6, 6$	$0, -100$
D	$7, 1$	$0, -100$

- Unique NE in this game is (D, L)
- Minmax values are given by $\underline{v}_1 = 0$ and $\underline{v}_2 = 1$
- Minmax strategy against agent 1 requires agent 2 to play R
- R is strictly dominated by L for agent 2

Outline

1. Finitely Repeated Games
2. Infinitely Repeated Games
3. Folk Theorem
4. Repeated Games with Imperfect Monitoring

Motivation

- So far, we assumed that agents observe actions of others at each round of game
- Next, we consider games where agents' actions may not be directly observable
- We assume that agents observe only an imperfect signal of stage game actions

Example: Cournot Competition with Noisy Demand

[Green and Porter, Non-cooperative Collusion under Imperfect Price Information, 1984]

- Firms set production levels $q_1^{(r)}, \ldots, q_n^{(r)}$ privately at round r
- Firms do not observe each others' output levels
- Market demand is stochastic
- Market price depends on total production and market demand
- Low price could be due to high production or low demand
- Firms utility depends on their own production and market price

Model

- We focus on game with public information
- At each round, all agents observe some public outcome
- Let $y^{(r)} \in Y$ denote publicly observed outcome at round r
- Each action profile a induces probability distribution over y
- Let $\pi(y, a)$ denote probability distribution of y under action profile a
- Public information at round r is $h^{(r)} = (y^{(1)}, \dots, y^{(r-1)})$
- Strategy of agent i is sequence of maps $s_i^{(r)} : h^{(r)} \to S_i$

Model (cont.)

- Agents utility depends only on their own action and public outcome

Model (cont.)

- Agents utility depends only on their own action and public outcome
- Dependence on actions of others is through their effect on distribution of y

Model (cont.)

- Agents utility depends only on their own action and public outcome
- Dependence on actions of others is through their effect on distribution of y
- Agent i's realized utility at round r is $u_i(a_i^{(r)}, y^{(r)})$

Model (cont.)

- Agents utility depends only on their own action and public outcome
- Dependence on actions of others is through their effect on distribution of y
- Agent i's realized utility at round r is $u_i(a_i^{(r)}, y^{(r)})$
- Agent i's expected stage utility is

$$u_i(a) = \sum_{y \in Y} \pi(y, a) u_i(a_i, y)$$

Model (cont.)

- Agents utility depends only on their own action and public outcome
- Dependence on actions of others is through their effect on distribution of y
- Agent i's realized utility at round r is $u_i(a_i^{(r)}, y^{(r)})$
- Agent i's expected stage utility is

$$u_i(a) = \sum_{y \in Y} \pi(y, a) u_i(a_i, y)$$

- Agent i's average discounted utility when sequence $\{a^{(t)}\}$ is played is

$$(1 - \delta) \sum_{r=1}^{\infty} \delta^{r-1} u_i(a^{(r)})$$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y \qquad u_1(C, y) = 4 + y$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y$ $\quad$ $u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y$ $\quad$ $u_2(C, y) = 4 + y$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y \qquad u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y \qquad u_2(C, y) = 4 + y$
- Signal y is defined by cont. random variable X with CDF $F(x)$ and $\mathbb{E}[X] = 0$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y$ $\qquad$ $u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y$ $\qquad$ $u_2(C, y) = 4 + y$
- Signal y is defined by cont. random variable X with CDF $F(x)$ and $\mathbb{E}[X] = 0$
  - If $a = (D, D)$, then $y = X$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y \qquad u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y \qquad u_2(C, y) = 4 + y$
- Signal y is defined by cont. random variable X with CDF $F(x)$ and $\mathbb{E}[X] = 0$
  - If $a = (D, D)$, then $y = X$
  - If $a = (D, C)$ or (C, D), then $y = X - 2$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y$ $\qquad$ $u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y$ $\qquad$ $u_2(C, y) = 4 + y$
- Signal y is defined by cont. random variable X with CDF $F(x)$ and $\mathbb{E}[X] = 0$
  - If $a = (D, D)$, then $y = X$
  - If $a = (D, C)$ or (C, D), then $y = X - 2$
  - If $a = (D, D)$, then $y = X - 4$

Simpler Example: Noisy Prisoner's Dilemma

- Prisoners do not observe each others actions, instead, they observe signal y
  - $u_1(D, y) = 1 + y$ $\quad u_1(C, y) = 4 + y$
  - $u_2(D, y) = 1 + y$ $\quad u_2(C, y) = 4 + y$
- Signal y is defined by cont. random variable X with CDF $F(x)$ and $\mathbb{E}[X] = 0$
  - If $a = (D, D)$, then $y = X$
  - If $a = (D, C)$ or (C, D), then $y = X - 2$
  - If $a = (D, D)$, then $y = X - 4$
- Normal-form stage game is

	D	C
D	$1 + X, 1 + X$	$-1 + X, 2 + X$
C	$2 + X, -1 + X$	X, X

Trigger-price Strategy

- Consider following trigger strategy
  - (I) - Play (D, D) until $y \leq y^*$, then go to (II)
  - (II) - Play(C, C) for R rounds, then go back to (I)
- Notice that strategy is stationary and symmetric
- Also notice that punishment uses NE of stage game
- We can choose y^* and R such that this strategy profile is SPE

Trigger-price Strategy (cont.)

- We use one-shot deviation principle
- Deviation in (II) is obviously not beneficial
- In (I), if agents do not deviate, their expected utility is

$$v = (1-\delta)\left((1+0) + \delta\left(F(y^*)\delta^R v + (1-F(y^*))v\right)\right)$$

- From this, we obtain

$$v = \frac{1-\delta}{1-\delta(1-\delta)\big(1-F(y^*)(1-\delta^R)\big)}$$

Trigger-price Strategy (cont.)

- If some agent deviates in (1), then her expected utility is

$$v_d = (1-\delta)\left((2+0) + \delta\left(F(y^*+2)\delta^R v + (1-F(y^*+2))v\right)\right)$$

- Deviation provides immediate utility, but increases probability of entering (II)
- To have SPE, we mush have $v \geq v_d$ which means

$$v \geq \frac{2(1-\delta)}{1-\delta(1-\delta)\left(1-F(y^*+2)(1-\delta^R)\right)}$$

$$\Rightarrow F(y^*+2) - 2F(y^*) \geq \frac{1-\delta(1-\delta)}{\delta(1-\delta)(1-\delta^R)}$$

- Any R and y^* that satisfy this constraint construct SPE
- Best trigger-price strategy can be found by maximizing v s.t. this constraint

Acknowledgment

- This lecture is a slightly modified version of ones prepared by
  - Asu Ozdaglar [MIT 6.254]
  - Vincent Conitzer [Duke CPS 590.4]
- Elly Khodaie helped with importing slides from PowerPoint to LaTeX