

# Game-theoretic Foundations of Multi-agent Systems

## Lecture 3: Games in Normal Form

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# Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium



# Normal-form Games

- Let's start with games in which all agents act simultaneously
- Agents choose their actions without knowledge of other agents' actions
- Such games are referred to as **strategic-form games** or **normal-form games**



## Normal-form Games: Definition

- The game consists of a set of agents,  $N = \{1, 2, \dots, n\}$
- Set of available actions to agent  $i$  is denoted by  $A_i$
- Action taken by agent  $i$  is denoted by  $a_i \in A_i$
- Outcome of game is an **action profile** of all agents,  $a = (a_1, \dots, a_n)$
- Set of all action profiles is denoted by  $A = \prod A_i$
- Agent  $i$  has a utility function  $u_i$  that maps outcomes to real numbers



## Some Notations

- $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  is an action profile of all agents except  $i$
- $A_{-i} = \prod_{j \neq i} A_j$  is set of action profiles of all agents except  $i$
- $a = (a_i, a_{-i}) \in A$  is another way of denoting an action profile (or an outcome)



# Matrix-form Representation

- When  $A_i$  is finite for all  $i$ , we call the game **finite game**
- For 2 agents and small action sets, game can be represented in **matrix form**

		Agent 2	
		x	y
Agent 1	m	$a, b$	$e, f$
	n	$c, d$	$g, h$

- Each cell indexed by row  $r$  and column  $c$  contains a pair,  $(p, q)$ , where  $p = u_1(r, c)$  and  $q = u_2(r, c)$ .



## Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 1 pockets both, otherwise agent 2 pockets them

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

- **Zero-sum game:** Utility of one agent is negative of utility of other agent



## Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0





## Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

	Left	Right
Left	1, 1	-1, -1
Right	-1, -1	1, 1

- **Team game**: For all outcomes  $s$ , and any pair of agents  $i$  and  $j$ , it is the case that  $u_i(a) = u_j(a)$  (also known as **common-payoff game** or **pure-coordination game**)



## Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ( $a_i \in [0, \infty]$ )
- Utility of each firm is its total revenue minus its total cost

$$u_i(a_1, a_2) = a_i p(a_1 + a_2) - c a_i$$

- $p(\cdot)$  is the price function that maps total production to a price
- $c$  is a unit cost
- E.g.,  $p(x) = \max(0, 2 - x)$  and  $c = 1$



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## Mixed and Pure Strategies

- Let  $\Delta(X)$  be set of all probability distributions over  $X$
- Set of (mixed) strategies for agent  $i$  is denoted by  $S_i = \Delta(A_i)$
- For mixed strategy  $s_i \in S_i$ ,  $s_i(a)$  is probability that action  $a$  is played under  $s_i$
- Pure strategy is a mixed strategy that puts probability 1 on a single action
- Support of mixed strategy  $s_i$  is set of pure strategies,  $a_i$ , such that  $s_i(a_i) > 0$
- Expected utility of agent  $i$  for a (mixed) strategy profile  $s = (s_1, \dots, s_n)$  is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$



## Example

		Agent 2		
		R ( $\frac{2}{3}$ )	P (0)	S ( $\frac{1}{3}$ )
Agent 1	R ( $\frac{1}{3}$ )	0, 0	-1, 1	1, -1
	P ( $\frac{2}{3}$ )	1, -1	0, 0	-1, 1
	S (0)	-1, 1	1, -1	0, 0

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$



# Dominant and Dominated Strategies

- Let  $s_i$  and  $s'_i$  be two strategies of agent  $i$
- $s_i$  **strictly dominates**  $s'_i$  if
  - $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$
- $s_i$  **weakly dominates**  $s'_i$  if
  - $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$ , and
  - $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$  for at least one  $s_{-i} \in S_{-i}$
- $s_i$  is strictly/weakly **dominant** if it strictly/weakly dominates all other strategy
- $s_i$  is strictly/weakly **dominated** if another strategy strictly/weakly dominates it
- $s = (s_1, \dots, s_n)$  is **dominant strategy equilibrium** if  $s_i$  is dominant strategy for all  $i$



## Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

	D	C
D	-2, -2	-4, -1
C	-1, -4	-3, -3

- Absolute value of utilities are the length of jail term each prisoner gets
- Confess is strictly dominant strategy for both prisoners
- (C,C) is a strict dominant strategy equilibrium
- The dilemma: (D,D) is better for both prisoners, but they won't play it!



# Iterated Elimination of Strictly Dominated Strategies

- All **strictly dominated pure strategies** can be ignored

	L	C	R		L	C		L	C		C	
U	3, 1	0, 2	0, 0		U	3, 1	0, 2		U	3, 1	0, 2	
M	1, 2	1, 1	5, 0	$\Rightarrow$	M	1, 2	1, 1	$\Rightarrow$	D	0, 1	4, 2	
D	0, 1	4, 2	0, 0		D	0, 1	4, 2	$\Rightarrow$	D	0, 1	4, 2	

The diagram illustrates the iterative elimination of strictly dominated strategies. It shows a sequence of four game matrices connected by  $\Rightarrow$  symbols. In the first matrix, column R is dominated by column L. In the second matrix, column R has been removed. In the third matrix, row M is dominated by a mixed strategy of U and D. In the fourth matrix, row M has been removed. In the final matrix, only row D and column C remain, as column L has been eliminated.

- Column R can be eliminated, since it is dominated by, for example, column L
- M is not dominated by U or D but is dominated by  $0.5U + 0.5D$  mixed strategy
- Note, however, that it was not dominated before the elimination of the R column





## Iterated Elimination of Strictly Dominated Strategies (cont.)

- Once one pure strategy is eliminated, another strategy that was not dominated can become dominated
- In finite games, iterated elimination of strictly dominated strategies ends after finite number of iterations
- Order of elimination does not matter when removing strictly dominated strategies (Church–Rosser property)
- Elimination order can make a difference in final outcome when removing weakly dominated strategies
- If the procedure ends with a single strategy for each agent, then the game is said to be dominance solvable



# Existence of Dominant Strategy Equilibrium

- Dominant strategy equilibrium does not always exist
- Example: Matching pennies

	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$



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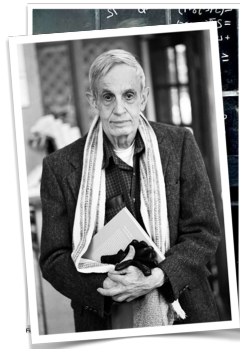
# Best Response

- Picking a strategy would be simple if an agent knew how others were going to act
- **Best response**:  $s_i^* \in BR_i(s_{-i})$  iff  $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$  for all strategies  $s_i \in S_i$
- Best response **is not** necessarily unique
  - If there is more than one best response, any mixed strategy over those must be a best response as well
- Best response is not a **solution concept**
  - I.e., it does not identify an interesting set of outcomes
  - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the **Nash equilibrium**



# Nash Equilibrium - Intersection of Best Responses

- $s^* = (s_1^*, \dots, s_n^*)$  is a **Nash equilibrium** iff  $\forall i, s_i^* \in Br_i(s_{-i}^*)$
- No agent can profitably deviate given strategies of others
- Nash equilibrium is a **stable** strategy profile
- **Nash theorem**: Every finite game has a Nash equilibrium



John Forbes Nash Jr.  
1928-2015



## Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

		Wife	
		Football	Opera
Husband	Football	2,1 <span style="border: 1px solid red; border-radius: 50%; padding: 2px;">2,1</span>	0,0
	Opera	0,0	1,2 <span style="border: 1px solid red; border-radius: 50%; padding: 2px;">1,2</span>

- Are these the only Nash equilibria?



## Example: Battle of Sexes (cont.)

	F ( $p$ )	O ( $1 - p$ )
F	2, 1	0, 0
O	0, 0	1, 2

- In general, it is tricky to compute mixed-strategy equilibria (will discuss this later)
- It becomes easy when we know (or can guess) support of equilibrium strategies
- Let us now guess that both agents randomize over both F and O
- Wife's strategy is to play F w.p.  $p$  and O w.p.  $1 - p$
- Husband must be indifferent between F and O (why?):

$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1 - p) \Rightarrow p = 1/3$$

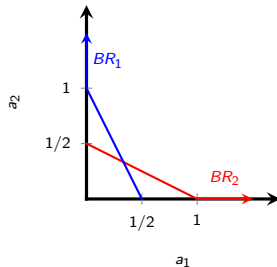
- You can show that the unique mixed-strategy NE is  $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$



## Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$\begin{aligned} BR_i(a_{-i}) &= \operatorname{argmax}_{a_i \geq 0} a_i(2 - a_i - a_{-i}) - a_i \\ &= \begin{cases} (1 - a_{-i})/2 & \text{if } a_{-i} < 1, \\ 0 & \text{Otherwise.} \end{cases} \end{aligned}$$





# The "Equilibrium Selection Problem"

- You are about to play a game that you have never played before with a person that you have never met
- According to which equilibrium should you play?
  - Equilibrium that maximizes the sum of utilities (**social welfare**)
  - Or, at least not a **Pareto-dominated** equilibrium
  - So-called focal equilibria (e.g., "Meet in Paris" game - you and a friend were supposed to meet in Paris at noon on Sunday, but you forgot to discuss where and you cannot communicate. Where will you go?)
  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - ...
- Equilibrium selection is a difficult problem

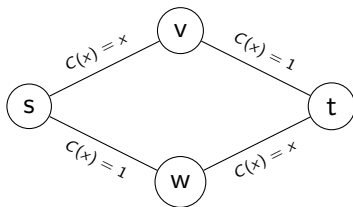


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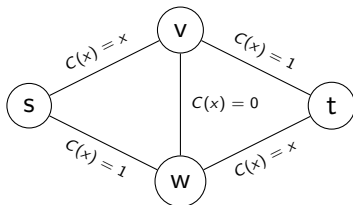


# Braess's Paradox



- Suppose there are  $2k$  drivers commuting from  $s$  to  $t$
- $C(x)$  indicates travel time in hours for  $x$  fraction of drivers
- $k$  drivers going through  $v$ , and  $k$  going through  $w$  is NE (why?)

## Braess's Paradox (cont.)



- Suppose we install a teleportation device allowing instant travel from  $v$  to  $w$
- What is new NE?
- What is optimal commute time?
- **Price of anarchy**: ratio between (worst) NE performance and optimal performance
  - Ratio between 2 and  $3/2$  in Braess's Paradox



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# Maxmin Strategy

- **Maxmin strategy** for agent  $i$  is

$$\operatorname{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

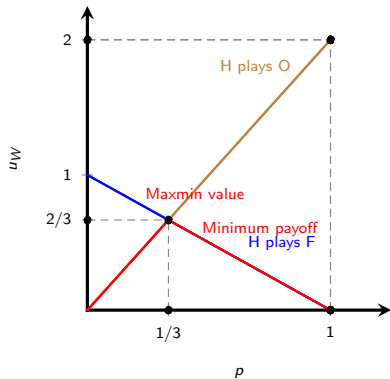
- Maxmin value for agent  $i$  is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- If  $i$  plays maxmin strategy and others play arbitrarily,  $i$  still receives expected payoff of at least their maxmin value



## Example: Battle of Sexes



		W	
		F ( $1 - p$ )	O ( $p$ )
H	F	2, 1	0, 0
	O	0, 0	1, 2

# Minmax Strategy

- **Minmax strategy** against agent  $i$  is

$$\arg\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax value for agent  $i$  is

$$\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minmax strategy against  $i$  keeps maximum payoff of agent  $i$  at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)





# Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium<sup>1</sup>, each agent receives a payoff that is equal to both their maxmin value and their minmax value

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- Minimax theorem does not hold with pure strategies only (example?)



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<sup>1</sup>You might wonder how a theorem from 1928 can use the term "Nash equilibrium," when Nash's work was published in 1950. John von Neumann used different terminology and proved the theorem in a different way; however, the given presentation is probably clearer in the context of modern game theory

## Example

		Agent 2	
		Left	Right
Agent 1	Up	20, -20	0, 0
	Down	0, 0	10, -10

- What is maximin value of agent 1 with and without mixed strategies?
- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?



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# Rationalizability

- **Rationalizable** strategy: Perfectly rational agent could justifiably play it
  - Best response to some beliefs about strategies of others
- Agents cannot have arbitrary beliefs about other agents
- Agent  $i$ 's beliefs must take into account:
  - Other agents' rationality
  - Other agents' knowledge of agent  $i$ 's rationality
  - Other agents' knowledge of agent  $i$ 's knowledge of their rationality
  - ... (infinite regress)



## Example: Matching Pennies

	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

- Col playing H is rationalizable
  - Col could believe Row plays H
- Col believing that Row plays H is rationalizable
  - Col could believe Row believes Col plays T
- Col believing that Row believes that Col plays T is rationalizable
  - Col could believe Row believes Col believes Row plays T
- ...
- In this game, all pure strategies are rationalizable



# Rationalizability: Properties

- Nash equilibrium strategies are always rationalizable
- Some rationalizable strategies are not Nash
  - Set of rationalizable strategies in finite games is nonempty
- To find rationalizable strategies:
  - In **2-player** games, use iterated elimination of strictly dominated strategies
  - In  **$n$ -player** games, iterated elimination of **never-best response** strategies
    - Eliminate strategies that are not optimal against any belief about others' strategies



## Example: 2/3-Beauty Contest Game

- No agent plays more than 100
- $2/3$  of average is strictly less than 67 ( $100 \times 2/3$ )
- Any integer  $> 67$  is never-best response to any belief about others' strategy
- No agent plays more than 67
- $2/3$  of average is less than 45 ( $67 \times 2/3$ )
- Any integer  $> 45$  is never-best response to any belief about others' strategy
- ...
- Only rationalizable action is playing 1



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## Example: Battle of Sexes

		W	
		Football	Opera
H	F	2, 1	0, 0
	O	0, 0	1, 2

- Unique mixed strategy NE yields each agent expected payoff of  $2/3$
- In NE, agents randomize over strategies **independently**
- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome



## Example: Battle of Sexes (cont.)

- Suppose there is **publicly observable** fair coin
- If it is heads/tails, they both get **recommendation** to go to football/opera
- If they see heads, they believe that the other one goes to football
- Going to football is best response, agents have **no incentive to deviate**
- Similar argument can be made when they see tails
- Expected utilities for this play of game **increases** to  $(1.5, 1.5)$



## Correlated Recommendations

- Let  $R = (R_1, \dots, R_n)$  be random variable taking values in  $A = \prod_i A_i$
- Let  $R$  be distributed according to  $\pi \in \Delta(A)$
- $r = (r_1, \dots, r_n)$  is an instantiation of  $R$  and a pure strategy profile
- $r_i \in A_i$  is called **recommendation to agent  $i$**
- $\pi(r_i)$  represents marginal probability for  $R_i = r_i$
- Given  $r_i$ , agent  $i$  can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$



## Correlated Equilibrium: Formal Definition

- **Correlated equilibrium** of finite game is joint probability distribution  $\pi \in \Delta(A)$  such that if  $R$  is random variable distributed according to  $\pi$ , then for all  $i, r_i \in A_i$  with  $\pi(r_i) > 0$ , and  $r'_i \in A_i$

$$\sum_{r_{-i} \in A_{-i}} \pi(r_{-i} \mid r_i) [u_i(r_i, r_{-i}) - u_i(r'_i, r_{-i})] \geq 0$$

- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations



## Example: Game of Chicken

		Driver 2	
		Dare	Yield
Driver 1	D	-5, -5	1, -1
	Y	-1, 1	0, 0

- (D,Y) and (Y,D) are **strict** pure-strategy NE
- Assume Driver 1 yields w.p.  $p$  and Driver 2 yields w.p.  $q$
- Using mixed equilibrium characterization, we have

$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

$$q - 5 \times (1 - q) = -(1 - q) \implies q = 4/5$$

- Mixed-strategy NE utilities are  $(-0.2, -0.2)$ , people **die** with probability 0.04



## Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation
- Conditional probability that D2 yields is  $1/3$
- Expected utility of Y is  $1 \times -2/3$
- Expected utility of D is  $1 \times 1/3 - 5 \times 2/3$
- Following the recommendation is better
- If D1 gets D recommendation, D2 must yield
- Following recommendation is again better
- Similar analysis works for D2
- Expected utilities are  $(0, 0)$ , so nobody dies!

		D2	
		D	Y
D1	D	$-5, -5$ 0%	$1, -1$ 40%
	Y	$-1, 1$ 40%	$0, 0$ 20%



# Characterization of Correlated Equilibrium

- Joint distribution  $\pi \in \Delta(S)$  is correlated equilibrium of finite game iff

$$\sum_{r_{-i} \in A_{-i}} \pi(r) [u_i(r) - u_i(r'_i, r_{-i})] \geq 0, \quad \forall i, r_i, r'_i \in A_i \quad (1)$$

- Proof (only for one side):
  - Correlated equilibrium can be written for all  $i, r_i, r'_i \in A_i$  with  $\pi(r_i) > 0$  as:

$$\sum_{r_{-i} \in A_{-i}} \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})} [u_i(r_i, r_{-i}) - u_i(r'_i, r_{-i})] \geq 0$$

- Denominator does not depend on variable of sum
- So it can be factored and canceled
- If  $\pi(r_i) = 0$ , LHS of (1) is zero regardless of  $i$  and  $r'_i$ , so equation always holds



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