Game-theoretic Foundations of Multi-agent Systems

Lecture 8: Bayesian Games

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Outline

1. Introduction and Definitions
2. Strategies and Equilibria
3. Auctions
4. Extensive-form Games of Incomplete-Info

Bayesian Games: Games of Incomplete Information

- So far, we assumed all agents know what game they are playing
  - Number of agents
  - Actions available to each agent
  - Utilities associated with each outcome
- In extensive-form games, taken actions could be unknown, but game itself is
- Bayesian games allow us to represent uncertainties about game
  - Commonly known probability distribution over possible games

Assumptions

- All games have same number of agents and same action sets for each agents
- Possible games only differ in agents' utilities for each outcome
- Beliefs are posteriors, obtained by conditioning common prior on private signals

Bayesian Games: Formal Definition

- N is finite set of agents
- A_i is set of actions available to agent i
- Θ_i is type space of agent i
- $p : \Theta \mapsto [0, 1]$ is common prior over types
- $u_i : A \times \Theta \mapsto \mathbb{R}$ is utility function for agent i

Example I: Bayesian Entry-deterrence Game

- Firm 1 decides whether to fight, Firm 2 decides whether to enter
- Firm 1 knows its cost
- Firm 2 is uncertain if 1's cost is 4 w.p. p or 1 w.p. $1-p$
- Game takes one of following two forms

	Enter	Stay out
Fight	$0, -1$	$2, 0$
Don't fight	$2, 1$	$3, 0$

θ_{11}: High Cost

	Enter	Stay out
Fight	$3, -1$	$5, 0$
Don't fight	$2, 1$	$3, 0$

θ_{12}: Low Cost

- $\Theta_1 = \{\theta_{11}, \theta_{12}\}$ and $\Theta_2 = \{\theta_{21}\}$

Example II

	θ_{21}	θ_{22}
θ_{11}	MP $\begin{array}{\|c\|c\|}\hline 2,0 & 0,2 \\ \hline 0,2 & 2,0 \\ \hline\end{array}$ $p = 0.3$	PD $\begin{array}{\|c\|c\|}\hline 2,2 & 0,3 \\ \hline 3,0 & 1,1 \\ \hline\end{array}$ $p = 0.1$
θ_{12}	Coord $\begin{array}{\|c\|c\|}\hline 2,2 & 0,0 \\ \hline 0,0 & 1,1 \\ \hline\end{array}$ $p = 0.2$	BoS $\begin{array}{\|c\|c\|}\hline 2,1 & 0,0 \\ \hline 0,0 & 1,2 \\ \hline\end{array}$ $p = 0.4$

Types: Discussion

- Types encapsulate information possessed by agents that is not common knowledge
  - E.g., agents' knowledge of their private utility function
- Type could also include
  - Agent's beliefs about other agents' utilities
  - Other agents' beliefs about the agent's own utility
  - And any other higher-order beliefs

Outline

Strategies

- Before the game starts, agents only know the common prior
- Agent i's strategy is $s_i : \Theta_i \mapsto \Delta(A_i)$ is contingency plan for all $\theta_i \in \Theta_i$
- $s_i(\theta_i)$ specifies agent i's (mixed) strategy when i's type is θ_i
- $s_i(a_i \mid \theta_i)$ specifies probability of agent i taking action s_i when i's type is θ_i
- Type of agents is revealed to them once the game starts
- Once agents know their type, they follow their strategy for that particular type

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- Ex ante: Agents only know the common prior on types (before the game starts)
- Interim: Agents only knows about their own type (after types are reveals)
- Ex post: Agents know everyone's type (hypothetical – before they take actions)

Expected Utilities (cont.)

- Ex-post expected utility (a):

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)$$

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- Interim expected utility:

$$EU_i(s,\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i) EU_i(s,(\theta_i,\theta_{-i}))$$

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- Ex-ante expected utility:

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)$$

Dominated Strategies

- Ex-ante dominated strategy: Alternative strategy provides greater ex ante utility regardless of all other agents' strategies
- Interim dominated strategy: For a given type, alternative strategy provides greater interim utility regardless of all other agents' strategies

Best Response in Bayesian Games

- Agent i's best response to strategy s_{-i} is

$$BR_i(s_{-i}) = \underset{s_i}{\operatorname{argmax}}\, EU_i(s_i, s_{-i})$$

- To play best response, i must know strategy of all agents for each of their types
- Without this information, it is not possible to evaluate $EU_i(s_i, s_{-i})$

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- So, maximizing $EU_i(s_i, s_{-i})$ is equal to maximizing $EU_i(s_i, s_{-i}, \theta_i)$ for all $\theta_i \in \Theta_i$
- Intuitively, if certain action is best after a signal is revealed, it is also the best conditional plan devised ahead of time for what to do should that signal be received

Bayes-Nash Equilibrium

- Bayes-Nash equilibrium (BNE) is strategy profile s^*, such that

$$s_i^* \in BR_i(s_{-i}^*) \quad \forall i$$

- [Theorem] Any finite Bayesian game has BNE

Example

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$$EU_2(UD, LR) = \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta)$$

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\begin{array}{ccc}
 & \theta_{21} & \theta_{22} \\
\theta_{11} &
\begin{array}{c}\text{MP}\\ \begin{array}{|c|c|}\hline 2,0 & 0,2 \\ \hline 0,2 & 2,0 \\ \hline\end{array}\\ p = 0.3\end{array} &
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\end{array}
$$

$$
\begin{aligned}
EU_2(UD, LR) &= \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta) \\
&= p(\theta_{11}, \theta_{2,1}) u_2(U, L, \theta_{11}, \theta_{2,1}) + p(\theta_{11}, \theta_{2,2}) u_2(U, R, \theta_{11}, \theta_{2,2}) + \\
&\quad p(\theta_{12}, \theta_{2,1}) u_2(D, L, \theta_{12}, \theta_{2,1}) + p(\theta_{12}, \theta_{2,2}) u_2(D, R, \theta_{12}, \theta_{2,2}) \\
&= 0.3 \times 0 + 0.1 \times 3 + 0.2 \times 0 + 0.4 \times 2 = 1.1
\end{aligned}
$$

Example (cont.)

- Continuing in this manner, complete payoff matrix can be constructed as

	LL	LR	RL	RR
UU	$2, 1$	$1, 0.7$	$1, 1.2$	$0, 0.9$
UD	$0.8, 0.2$	$1, 1.1$	$0.4, 1$	$0.6, 1.9$
DU	$1.5, 1.4$	$0.5, 1.1$	$1.7, 0.4$	$0.7, 0.1$
DD	$0.3, 0.6$	$0.5, 1.5$	$1.1, 0.2$	$1.3, 1.1$

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- Note that row agent's best response to RL is DU

Example (cont.)

- Once row agent receives the signal θ_{11}, we can calculate interim utilities

	LL	LR	RL	RR
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- Note that DU is still best response to RL
- What has changed is how much better it is compared to other strategies

Ex-post Equilibrium

- Strategy profile s^* is ex-post equilibrium if

$$s_i^* \in \operatorname*{argmax}_{s_i} EU_i(s_i, s_{-i}^*, \theta) \quad \forall i, \theta \in \Theta$$

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  - Even if they knew θ, agents would never want to deviate
  - Ex-post equilibrium is not guaranteed to exist

Example: Incomplete Information Cournot

- Two firms decide on their production level $q_i \in [0, \infty)$
- Price is given by $P(q)$ where $q = q_1 + q_2$
- Firm 1 has marginal cost equal to c which is common knowledge
- Firm 2's marginal cost is private information
    - c_L with probability x and c_H with probability $(1 - x)$, where $c_L < c_H$
- Utility of agents are ($t \in \{L, H\}$ type of firm 2)
    - $u_1((q_1, q_2), t) = q_1 P(q_1, q_2) - c$
    - $u_2((q_1, q_2), t) = q_2 P(q_1, q_2) - c_t$

Example: Incomplete Information Cournot (cont.)

- What are firms best responses?

$$B_1(q_L, q_H) = \arg\max_{q \geq 0} \Big(\big(xP(q + q_L) + (1 - x)P(q + q_H) - c \big) q \Big)$$

$$B_2^L(q_1) = \arg\max_{q \geq 0} \Big(\big(P(q_1 + q) - c_L \big) q \Big)$$

$$B_2^H(q_1) = \arg\max_{q \geq 0} \Big(\big(P(q_1 + q) - c_H \big) q \Big)$$

- BNE of this game is vector (q_1^*, q_L^*, q_H^*) such that

$$q_1^* \in B_1(q_L^*, q_H^*), q_L^* \in B_2^L(q_1^*), q_H^* \in B_2^H(q_1^*)$$

Example: Incomplete Information Cournot (cont.)

- For example, if $P(q) = \max(\alpha - q, 0)$, then we have:

$$q_1^* = \frac{1}{3}(\alpha - 2c + xc_L + (1-x)c_H)$$

$$q_L^* = \frac{1}{3}(\alpha - 2c_L + c) - \frac{1}{6}(1-x)(c_H - c_L)$$

$$q_H^* = \frac{1}{3}(\alpha - 2c_H + c) + \frac{1}{6}x(c_H - c_L)$$

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- Extracting private valuations could be challenging
- E.g., giving painting for free to bidder with highest valuation would create incentive for all bidders to overstate their valuations

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- Second-price action: similar to first price, except that winner pays second highest bid

Valuations

- Private valuations: valuation of each bidder is independent of others' valuations
- Common valuations: bidders' valuations are correlated to common value

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- Pure strategy for bidder i is map $b_i : [0, \overline{v}] \to \mathbb{R}_+$

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- [Proof] We need to answer following questions
  - If other bidders bid truthfully, does winner want to change their bid?
  - If other bidders bid truthfully, does looser want to change their bid?

Truthful Bidding in Second-price Auction

- Truthful equilibrium is (weak) ex-post equilibrium
- I.e., truthful bidding weakly dominates other strategies even if all values are known
- [Proof sketch] Define maximum bid excluding i's bid as $B^*_{-i} = \max_{j \neq i} b_j$

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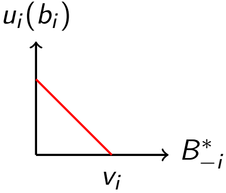

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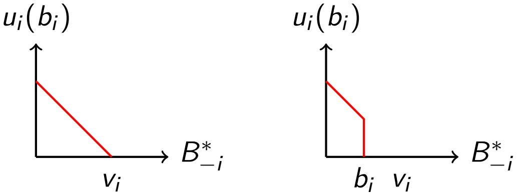

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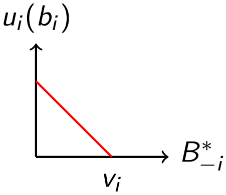

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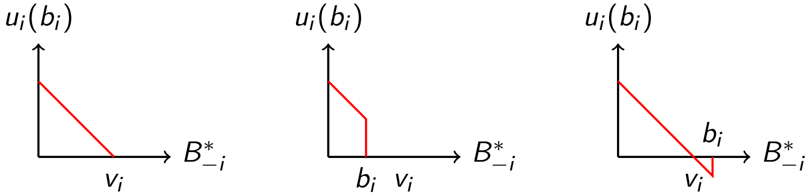

- Truthful equilibrium is also the unique BNE

Expected Payment in Second-price Auctions

- Define random variable $y_i = \max_{j \neq i} v_j$
  - CDF of y_i is $G_{y_i}(v) = F(v)^{N-1}$
  - PDF of y_i is $g_{y_i}(v) = (N-1)f(v)F(v)^{N-2}$
- Expected payment of bidder i with value v_i is given by

$$\begin{aligned} p(v_i) &= P(v_i \text{ wins}) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= P(y_i \leq v_i) \times \mathbb{E}[y_i \mid y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy = \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$

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- This implies that bidder i wins whenever $y_i < \beta^{-1}(b_i)$
- Optimal bid of bidder i is $b_i = \operatorname{argmax}_{b \geq 0} G_{y_i}(\beta^{-1}(b))(v_i - b)$

First-price Auctions (cont.)

- First-order (necessary) optimality conditions imply[1]:

$$\frac{g_{y_1}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}(v_i - b_i) - G_{y_i}(\beta^{-1}(b_i)) = 0$$

- In symmetric equilibrium, $b_i = \beta(v_i)$, therefore we have:

$$v_i g_{y_i}(v_i) = \beta'(v_i) G_{y_i}(v_i) + \beta(v_i) g_{y_i}(v_i) = \frac{d}{dv}\left(\beta(v_i) G_{y_i}(v_i)\right)$$

- With boundary condition $\beta(0) = 0$, we have:

$$\beta(v_i) = G_{y_i}^{-1}(v_i) \int_0^{v_i} y g_{y_i}(y) dy = \mathbb{E}[y_i \mid y_i \leq v_i]$$

[1] Derivative of $\beta^{-1}(b)$ is $1/\beta'(\beta^{-1}(b))$.

Expected Payment in First-price Auctions

- Expected payment of bidder i with value v_i is:

$$
\begin{aligned}
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- This establishes somewhat surprising results that both first and second price auction formats yield same expected revenue to auctioneer

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- Suppose that values are *i.i.d* and all bidders are risk neutral
- [Theorem] Any symmetric and increasing equilibria of any standard auction (such that expected payment of bidder with value zero is zero) yields same expected revenue to auctioneer

Oil-field Example: Common Values with Correlated Recommendations

- Suppose that there are two bidders bidding to lease oil field
- Oil field could be worth \$0, \$25M, or \$50M w.p. 0.25, 0.5, and 0.25, respectively
- Bidders hire their own consultant to evaluate value of oil field
- Bidders get private recommendations, r_1 and r_2
- If field is worth \$0, then $r_1 = r_2 = L$
- If field is worth \$25M, then $r_1 = H$, $r_2 = L$ or $r_1 = L$, $r_2 = H$ (both equally likely)
- If field is worth \$50M, then $r_1 = r_2 = H$
- Given their private recommendation, how should bidders bid?

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- Truthful bidding is not BNE in second-price auction with common values and dependent recommendations

Winner's Curse

- Winning means bidder received highest or most optimistic recommendation
- Condition on winning, value of item is lower than what recommendation says
- Ignoring this leads to paying, on average, more than true value of item
- To avoid this curse, bidders should assume their recommendation is optimistic
- In oil-field example, we can show that the following bidding strategy is BNE
  - Bid 0 upon receiving L
  - Bid \$50M upon receiving H

Oil-field Example II: Common Values and Independent Recommendations

- Consider two bidders interested in buying oil field that has part A and B
- Each bidder values A and B but is more interested in one of them
- Bidders hire their own consultant to evaluate value of their part
- Bidder 1 gets private recommendation r_1 about value of part A
- Bidder 2 gets private recommendation r_2 about value of part B
- Suppose that both recommendations are uniformly distributed over $[0, 1]$
- Suppose value of oil field to each bidder is as follows
  - $v_i = a.r_i + b.r_{-i}$ with $a \geq b \geq 0$
  - Private values are special case where $a = 1$ and $b = 0$

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- Bidder i's payment if i wins is $\beta(r_{-i}) = (a+b)r_{-i}$

Oil Field Example II: Second-price Auction (cont.)

- Expected payment of i condition on i winning is:

$$\mathbb{E}[(a+b)r_{-i} \mid r_{-i} < b_i/(a+b)] = b_i/2$$

- Expected value of $-i$'s signal condition on i winning is:

$$\mathbb{E}[r_{-i} \mid r_{-i} < b_i/(a+b)] = b_i/2(a+b)$$

- Expected utility of bidding b_i for recommendation r_i is

$$\begin{aligned} EU(b_i, r_i) &= P(b_i \text{ wins}) \times (a.r_i + b.\mathbb{E}[r_{-i} \mid b_i \text{ wins}] - \mathbb{E}[(a+b)r_{-i} \mid b_i \text{ wins}]) \\ &= b_i/(a+b) \times (a.r_i + b.b_i/2(a+b) - b_i/2) \end{aligned}$$

- Maximizing this with respect to b_i (for given r_i) leads to $b_i^* = (a+b)r_i$

Oil Field Example II: First-price Auction

- Analysis is similar to that of first-price auctions with private values
- It can be shown that unique symmetric BNE is for each bidder to bid $\beta(r_i) = (a + b)r_i/2$
- It can be shown that expected revenue is equal to first price auction
- Revenue equivalence principle continues to hold for common values

Outline

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- We can use information sets to represent what each agent knows
- We need to modify BNE to include notion of perfection (as in subgame perfection)

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Information \ Timing	Simultaneous	Sequential
Complete	Nash	SPE
Incomplete	Bayesian Nash	

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Incomplete	Bayesian Nash	Perfect Bayesian

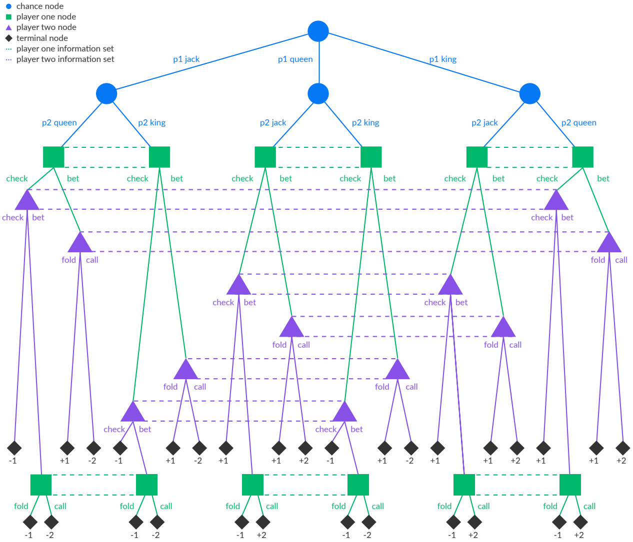
Extensive-form Games of Incomplete Information: Definition

- N, A, H, Z, α, β, ρ, u, and I are the same as extensive-form games
- Θ_i is type space of agent i
- $p : \Theta \mapsto [0, 1]$ is common prior over types
- $u_i : Z \times \Theta \mapsto \mathbb{R}$ is utility function for agent i

The "Nature" with Chance Moves

- To capture common prior, we can add special agent called Nature
- Nature makes probabilistic choices
- Nature does not have utility function (can be viewed as having constant utility)
- Nature has unique strategy of randomizing in commonly known way
- Agents receive individual signals about Nature's choice

Example: Kune Poker



Beliefs and Strategies

- Agents have beliefs about which node they are for each information set (infoset)
- For each infoset, μ defines prob. distribution over all nodes in that infoset
- behavioral strategy, s, maps each infoset prob. distribution over actions

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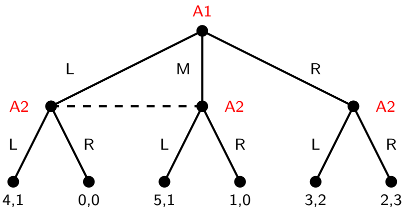
- IV. Off-the-path consistency: For any off-the-equilibrium-path infoset, μ must be derived from s according to Bayes' rule whenever possible

Weak and Strong PBE

- I-III define weak PBE, and I-IV define strong PBE
- PBE is defined for all extensive-form games with imperfect information

Example I (from Lecture 5)



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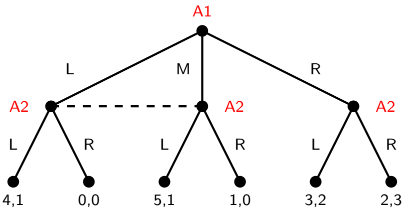

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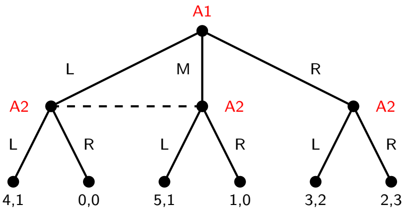

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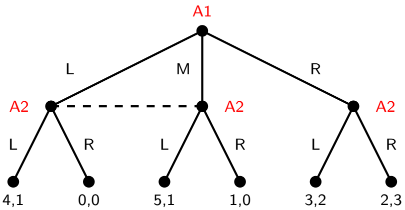

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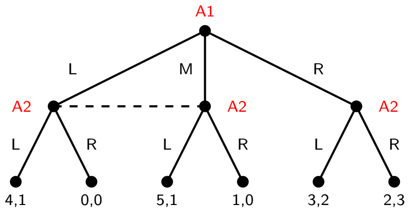

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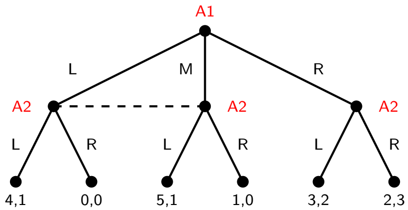

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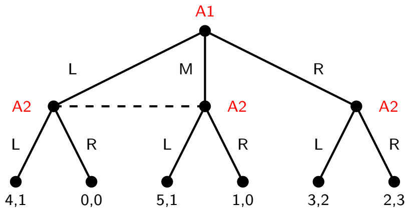
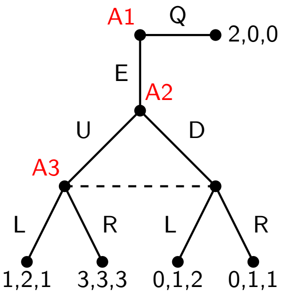


- (R, (R, R)) is NE and SPE, but it is not PBE, why?
  - R in A2's left-side infoset is not optimal for any belief of A2
- (M, (L, R)) + believing that A2 takes M with probability 1 is weak PBE
  - M is best response to (L, R) and (L, R) is best response to M
  - Off-the-path beliefs are consistent with the equilibrium strategy
- (M, (L, R)) + believing that A2 takes M with probability 1 is also strong PBE
  - Off-the-path beliefs are also consistent (right-side infoset has single node)

Example II: Strong vs. Weak PBE

- U is A2's dominant strategy



Example II: Strong vs. Weak PBE

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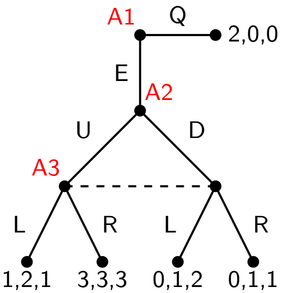
- NE of A2's subgame is (U, R)



Example II: Strong vs. Weak PBE

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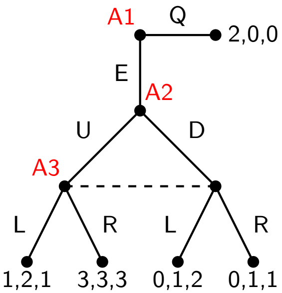
- (E, U, R) is SPE



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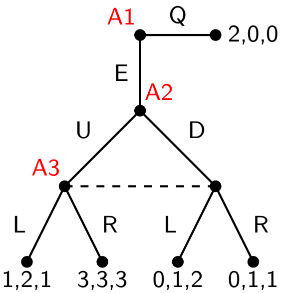
- (E, U, R) + A3 believing that A2 takes U w.p 1 is PBE (S&W)



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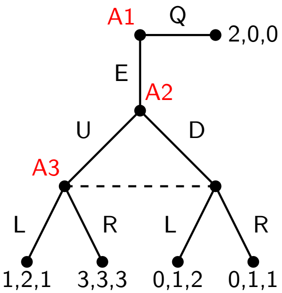

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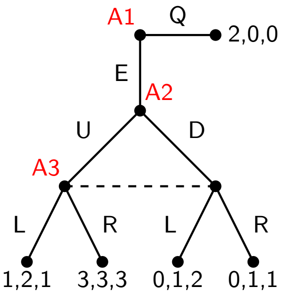
- D is best respond to (U, L) and U is dominant strategy



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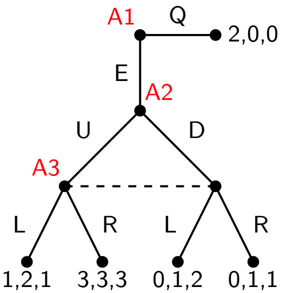

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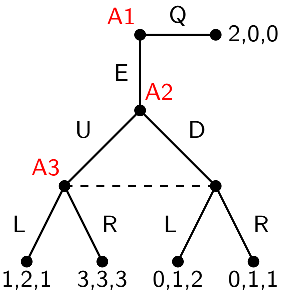

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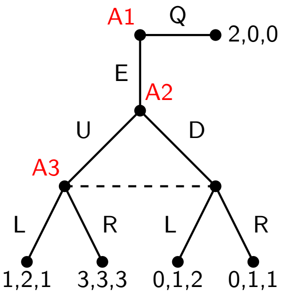
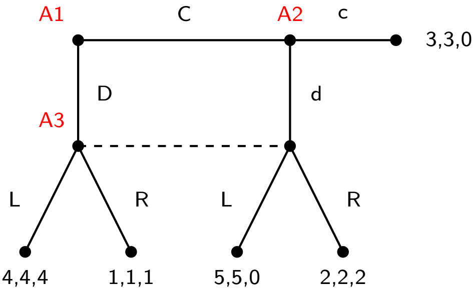
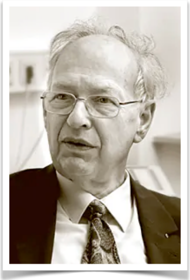
- D is best respond to (U, L) and U is dominant strategy
- L is best respond to believing that A2 takes R w.p. 1
- So, it is weak PBE, but is it also strong PBE?
- No! IV does not hold; A3's belief is inconsistent with A2's strategy



Example II: Selten's Horse

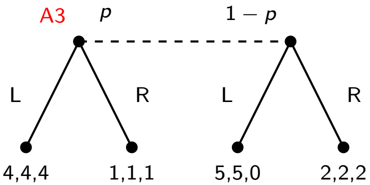




Reinhard Selten[2]
(1930-2016)

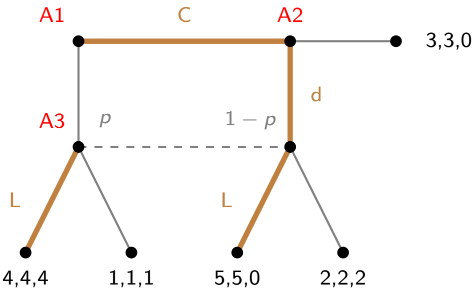
[1]Photograph by Stefan Schickler

Example II: Selten's Horse (cont.)



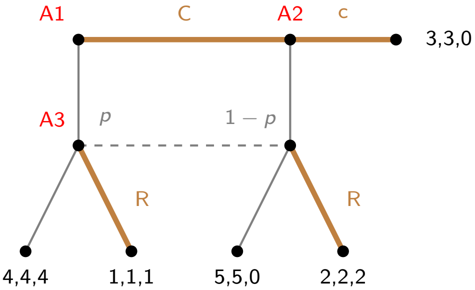
- A3 believes that left and right nodes are reached w.p. p and $1 - p$, respectively
- Utility for playing L is $2p$ and $1 - p$ for playing R
- A3 must play R if $p < 1/3$, R or L if $p = 1/3$, and L if $p > 1/3$

Example II: Selten's Horse (cont.)



- Is there any p with which (C, d, L) is weak PBE?
- Given (C, d), on-the-path belief for A3 must set $p = 0$
- For $p = 0$, A3 must take R, so the answer is NO

Example II: Selten's Horse (cont.)



- Is there any p with which (C, c, R) is weak PBE?
- Given (C, c), A3's infoset is off the equilibrium path
- Consistency does not put any constraint on p; optimality of R requires $p \leq 2/5$
- Is (C, c, R) + $p \leq 2/5$ strong PBE? Why?

Example III: Signaling Games

- Informed agent moves first to signal some information to uninformed agent
- Sending signal is more costly if it conveys false information
- E.g., producer provides warranty to signal that its products are unlikely to break
- E.g., employees acquire college degree to signal their ability to employers
- This is different from sending costless messages in cheap talk games
- Cheap talk is communication between agents that does not directly affect payoffs
- E.g., agents message each other on where they want to go in Battle of the Sexes

PBE Types in Signaling Games

- Separating: Informed agent sends distinct signal for each type

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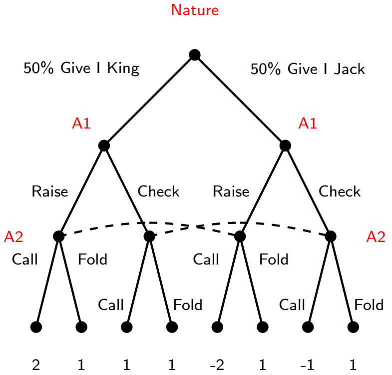
PBE Types in Signaling Games

- Separating: Informed agent sends distinct signal for each type
    - Signal always reveals sender's type
    - Receiver's beliefs become deterministic after seeing the signal
- Pooling: Informed agent sends the same signal for all types
    - Signal does not give any information to receiver
    - Receiver's beliefs are not updated after seeing the signal
- Semi-separating (a.k.a. partially pooling): Informed agent sends same signal for some types distinct signal for some other types

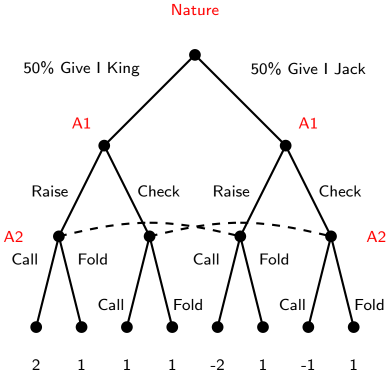
Simple Poker-like Game: Separating PBE

- Consider Raising for King and Checking for Jack



Simple Poker-like Game: Separating PBE

- Consider Raising for King and Checking for Jack
- What is A2's posterior belief?



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- Consider Raising for King and Checking for Jack
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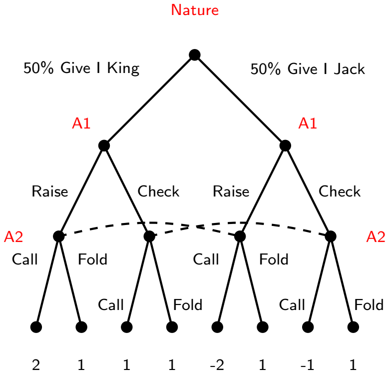
  - If A1 Raises, then A1 has King w.p. 1



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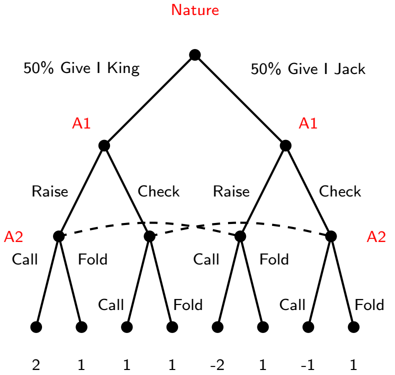

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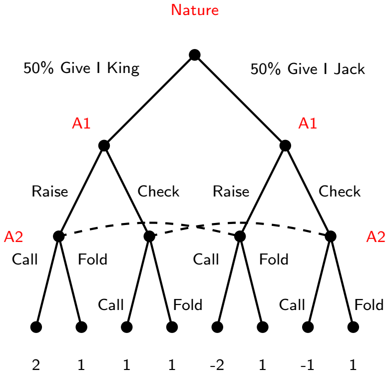
- What is A2's optimal strategy?



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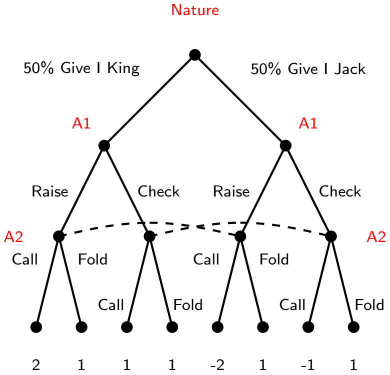
  - Fold if A1 Raises, Call if A1 Checks



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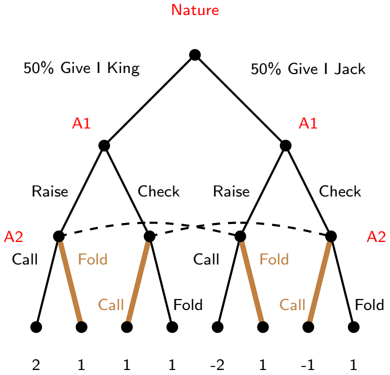

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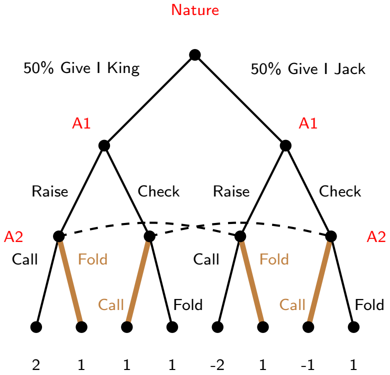
    - Indifferent between Raise and Check if King $(1 = 1)$



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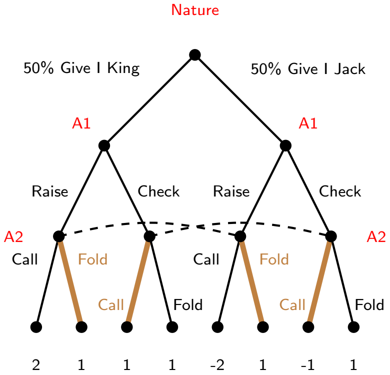

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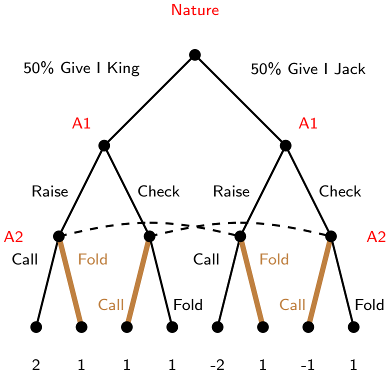

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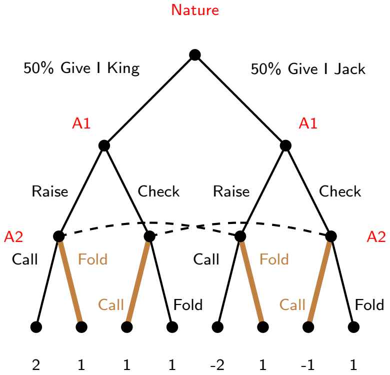
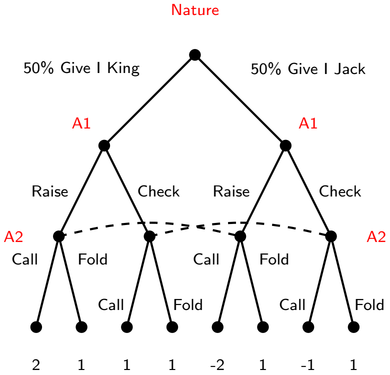
- How about Checking for King and Raising for Jack?



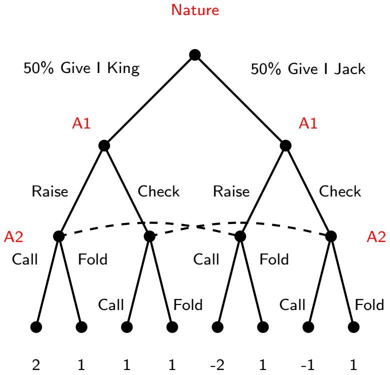
Simple Poker-like Game: Pooling PBE

- Consider Raising for both King and Jack



Simple Poker-like Game: Pooling PBE

- Consider Raising for both King and Jack
- A2's posterior beliefs are the same as prior beliefs



Simple Poker-like Game: Pooling PBE

- Consider Raising for both King and Jack
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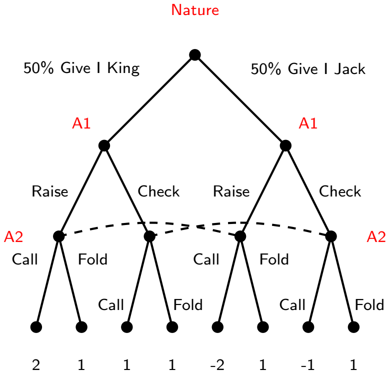
  - King w.p. 0.5 and Jack w.p. 0.5



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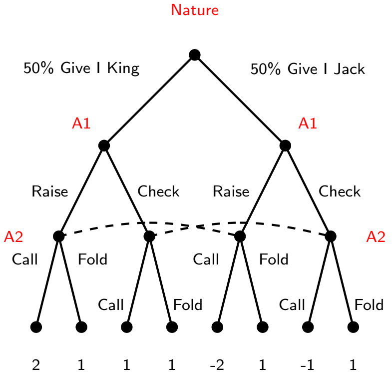

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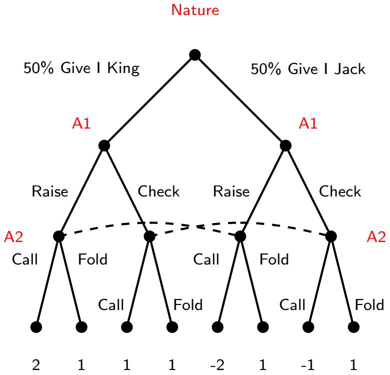
  - Call give 0 (-2 $\times$ 0.5 + 2 $\times$ 0.5), Fold gives -1



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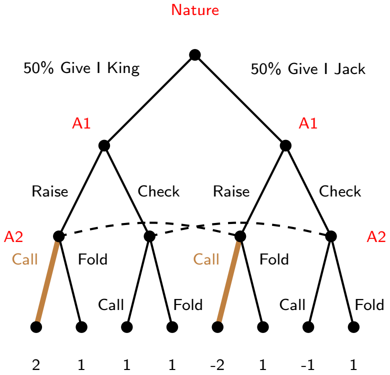
  - A2 prefers Call on the equilibrium path



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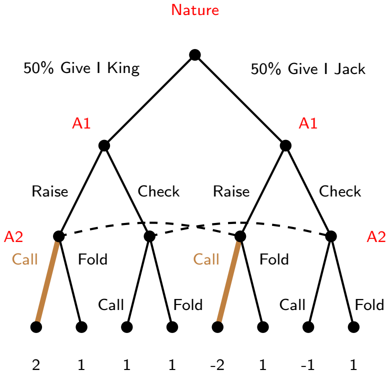
  - Call give 0 (-2 $\times$ 0.5 + 2 $\times$ 0.5), Fold gives -1
  - A2 prefers Call on the equilibrium path
- What is A2's optimal strategy off equilibrium path (Check)?



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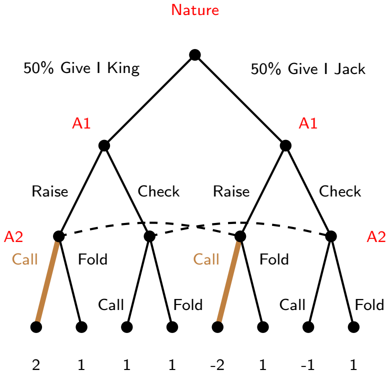
  - A2 prefers Call on the equilibrium path
- What is A2's optimal strategy off equilibrium path (Check)?
  - Consistency does not put any restriction on beliefs



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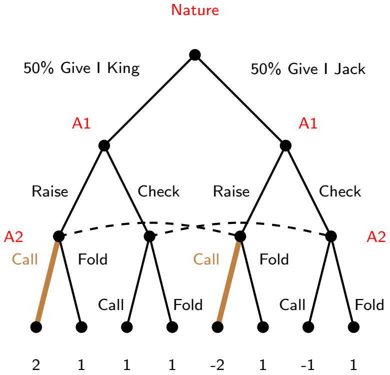

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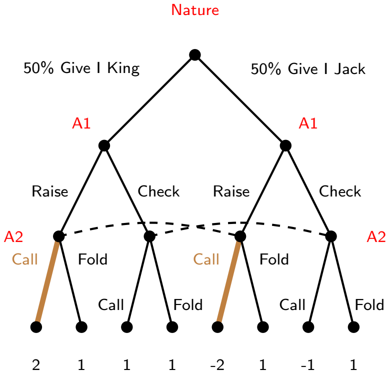
  - Call give $-p + 1 - p$, Fold gives -1 and



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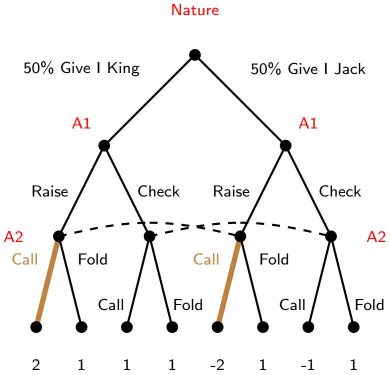
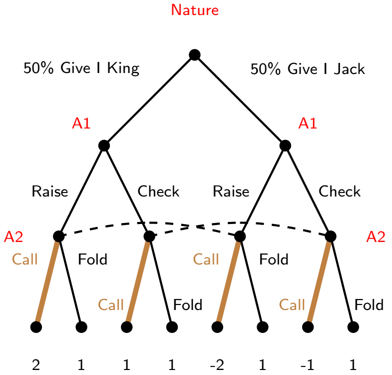
  - Consistency does not put any restriction on beliefs
  - Consider p for King and $1 - p$ for Jack
  - Call give $-p + 1 - p$, Fold gives -1 and
  - For $p < 1$, A2 prefers Call, for $p = 1$, A2 is indifferent



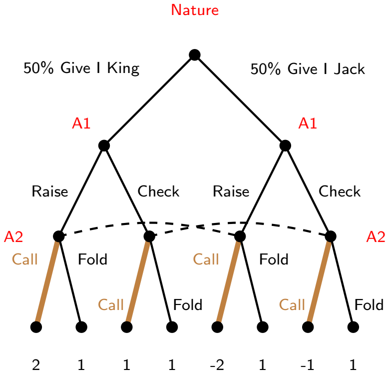
Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?



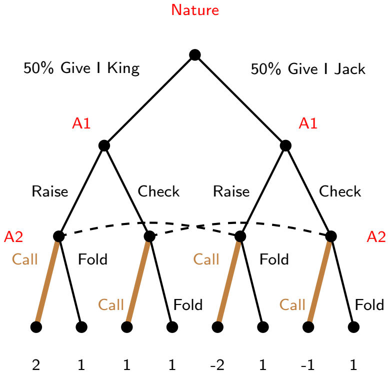
Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?
  - If King, A1 prefers Raise



Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?
  - If King, A1 prefers Raise
  - If Jack, A1 prefers Check



Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?
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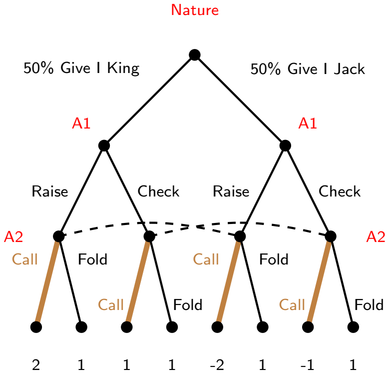
  - A1 wants to deviate from pooling strategy



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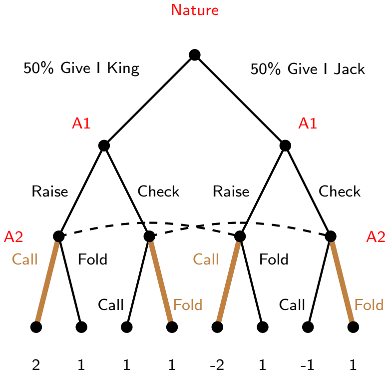
- What if A2 Calls on and Folds off the path (for $p = 1$)?



Simple Poker-like Game: Pooling PBE (cont.)

- If A2 Calls ($p \leq 1$), what is A1's best response?
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- What if A2 Calls on and Folds off the path (for $p = 1$)?
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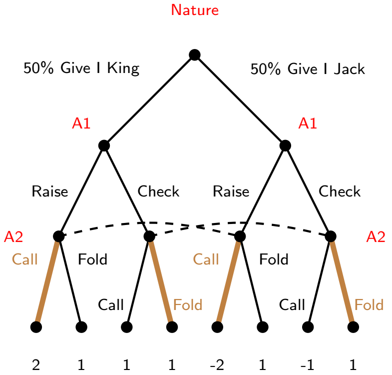

Simple Poker-like Game: Pooling PBE (cont.)

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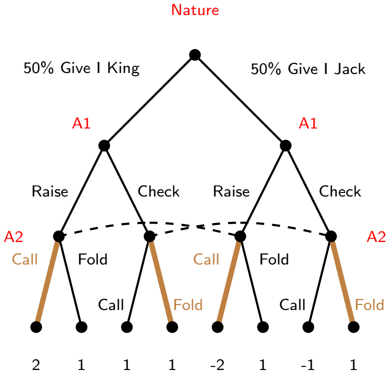

Simple Poker-like Game: Pooling PBE (cont.)

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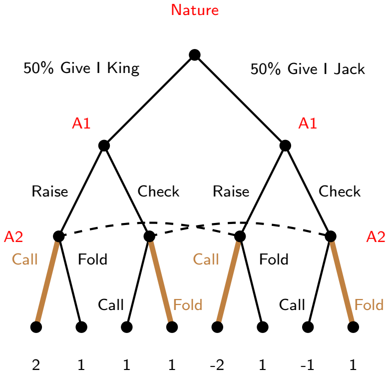

Simple Poker-like Game: Pooling PBE (cont.)

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- What if A2 Calls on and Folds off the path (for $p = 1$)?
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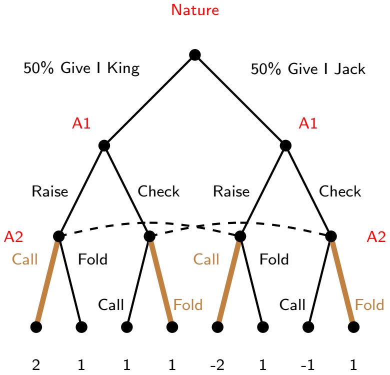
  - If Jack, A1 prefers Check
  - A1 wants to deviate from pooling strategy
- There is no p for which A1 wants to follow pooling



Simple Poker-like Game: Pooling PBE (cont.)

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  - If Jack, A1 prefers Check
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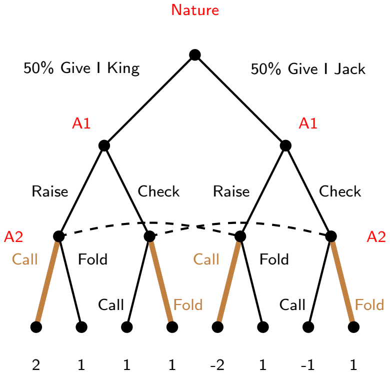
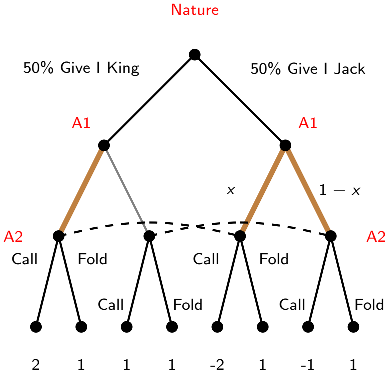
- There is no p for which A1 wants to follow pooling
- What about Checking for both King and Jack?



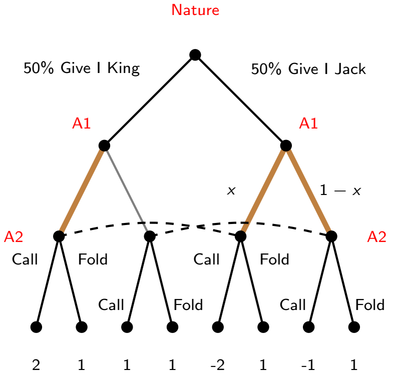
Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x



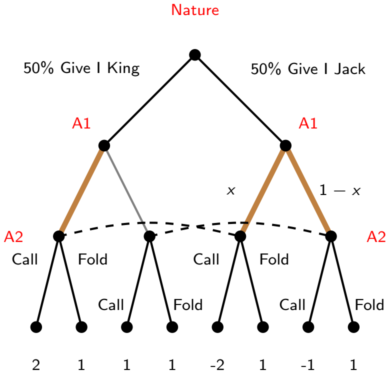
Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?



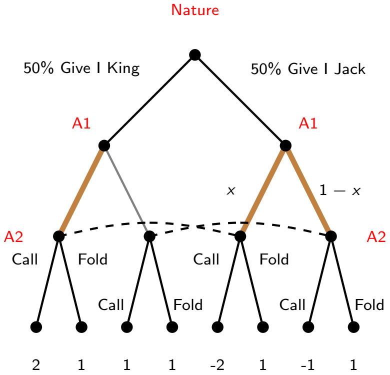
Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1



Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1
  - If Raise, King w.p. $1/(1+x)$ and Jack w.p. $1/(1+x)$



Simple Poker-like Game: Semi-separating PBE

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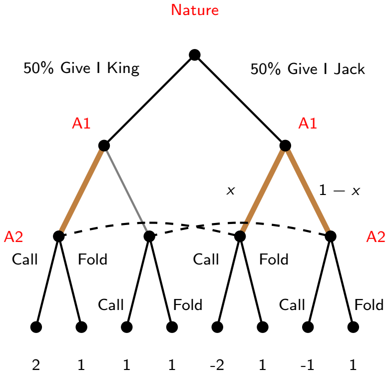
- What is A2's best response if A1 Checks?



Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
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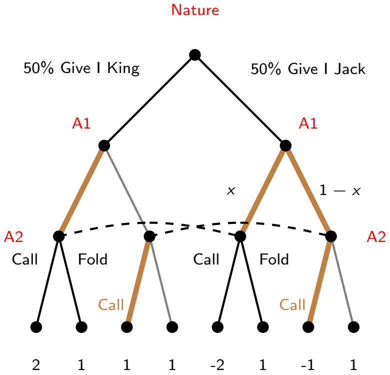
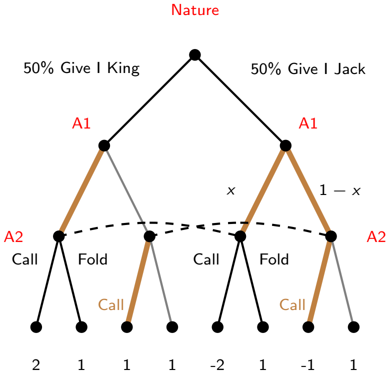


Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1
  - If Raise, King w.p. $1/(1+x)$ and Jack w.p. $1/(1+x)$
- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack



Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1
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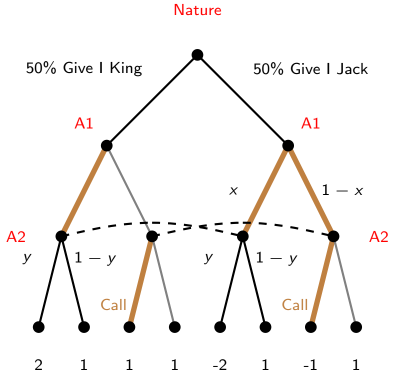
- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
  - Suppose A2 Calls w.p. y if A1 Raises



Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1
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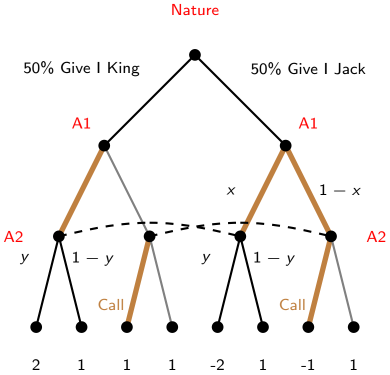
- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
  - Suppose A2 Calls w.p. y if A1 Raises
  - A1's utility for Raise is $-2y+1-y$



Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
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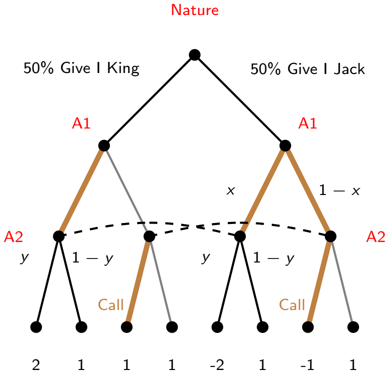
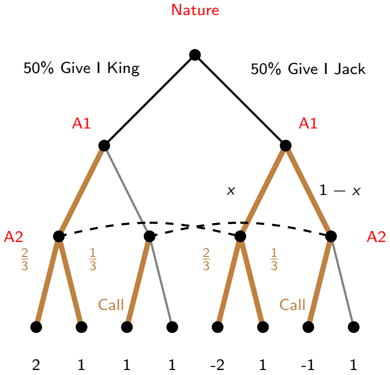
  - If Raise, King w.p. $1/(1+x)$ and Jack w.p. $1/(1+x)$
- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
  - Suppose A2 Calls w.p. y if A1 Raises
  - A1's utility for Raise is $-2y + 1 - y$
  - A1's utility for Check is -1

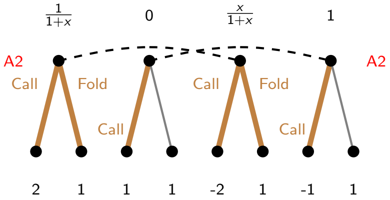


Simple Poker-like Game: Semi-separating PBE

- If King, A1 Raises - If Jack, A1 Raises w.p x
- What is A2's posterior belief?
  - If Check, Jack w.p. 1
  - If Raise, King w.p. $1/(1+x)$ and Jack w.p. $1/(1+x)$
- What is A2's best response if A1 Checks?
  - A2 must Call (A2 believes Jack w.p. 1)
- A2's strategy should make A1 indifferent if Jack
  - Suppose A2 Calls w.p. y if A1 Raises
  - A1's utility for Raise is $-2y+1-y$
  - A1's utility for Check is -1
  - $y = 2/3$ makes A1 indifferent



Simple Poker-like Game: Semi-separating PBE (cont.)



- x should be set s.t. A2 is indifferent between Call and Fold
- If A1 Raises, A2's utility for Call is $(2x-2)/(1+x)$
- If A1 Raises, A2's utility for Fold is -1
- $x = 1/3$ makes A2 indifferent between Call and Fold

Simple Poker-like Game: Final Semi-separating PBE

- A1 Raises w.p. 1 if King and w.p. 1/3 if Jack
- A1 Checks w.p. 0 if King and w.p. 2/3 if Jack
- A2 Calls w.p. 1 if A1 Checks and w.p. 2/3 if A1 Raises
- A2 Folds w.p. 0 if A1 Checks and w.p. 1/3 if A1 Raises
- If A1 Raises, A2 believes King w.p. 3/4 and Jack w.p. 1/4
- If A1 Checks, A2 believes Jack w.p. 1

Acknowledgment

- This lecture is a slightly modified version of ones prepared by
    - Asu Ozdaglar [MIT 6.254]
    - William Spaniel Game Theory 101
- Yiqin Huang helped with importing slides from PowerPoint to LaTeX