

Game-theoretic Foundations of Multi-agent Systems

Lecture 5: Games in Extensive Form

Seyed Majid Zahedi



Outline

1. Perfect-info Extensive-form Games
2. Pure Strategies in Perfect-info Games
3. Subgame-perfect Equilibrium
4. Imperfect-info Extensive-form Games
5. Randomized Strategies in Extensive-form Games



Extensive-form Games

- So far, we have studied **strategic-form** games
 - Agents take actions once and simultaneously
- Next, we study **extensive-form** games (a.k.a. **sequential** or **multi-stage** games)
 - Extensive-form games can be conveniently represented by **game trees**



(Finite) Perfect-info Extensive-form Game: Definition

- The game consists of a set of agents, $N = \{1, 2, \dots, n\}$
- A is set of actions
- H is set of **choice nodes** (internal nodes of game tree)
- Z is set of **terminal** nodes (leaves of game tree)



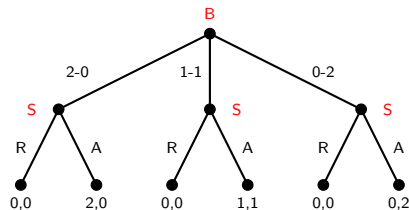
(Finite) Perfect-info Extensive-form Game: Definition (cont.)

- $\alpha : H \rightarrow N$ is **agent function**
 - Maps each choice node to an agent who chooses an action at that node
- $\beta : H \rightarrow 2^A$ is **action function**
 - Maps each choice node to set of actions available at that node
- $\rho : H \times A \rightarrow H \cup Z$ is **successor function**
 - Maps each choice node and action pair to new choice node or terminal node
 - If $\rho(h_1, a_1) = \rho(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
- $u = (u_1, \dots, u_n)$, where $u_i : Z \rightarrow \mathbb{R}$ is agent i 's **utility function**
 - Maps each terminal node to a real value



Example: Sharing Game

- Brother and sister share two gifts
- Brother suggests a split first
- Sister then chooses to accept or reject
- If she accepts, they get suggested gifts
- Otherwise, neither gets any gift



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History in Extensive-form Games

- If height of game tree (i.e, number of stages) is finite, then game is **finite-horizon** game
- Otherwise, the game is called **infinite-horizon** game
- For perfect-information games, each node maps to unique history (and vice versa)
- Since choice nodes form a tree, we can unambiguously identify a node with its history
 - I.e., sequence of choices leading from the root node to it



Pure Strategies

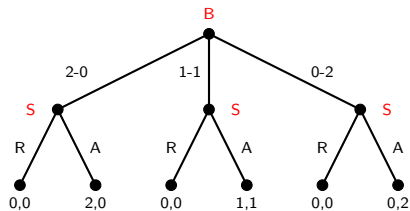
- Agent i 's pure strategy defines contingency plan for all choice nodes mapped to i

$$a_i \in A_i = \prod_{h \in H, \alpha(h)=i} \beta(h)$$

- Strategy must specify a decision at each choice node
 - Regardless of whether it is possible to reach that node



Pure Strategies: Example

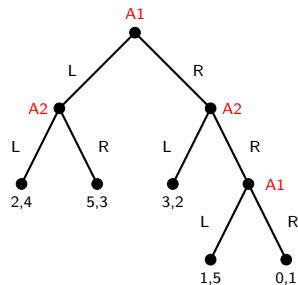


- $A_B = \{ "2-0", "1-1", "0-2" \}$
- $A_S = \{ (R, R, R), (R, R, A), (R, A, R), (A, R, R), (R, A, A), (A, R, A), (A, A, R), (A, A, A) \}$



Pure Strategies: (Another) Example

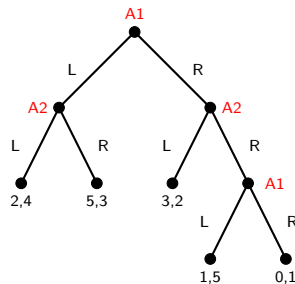
- What are pure strategies for A2?
 - $A_{A2} = \{(L, L), (L, R), (R, L), (R, R)\}$
- What about A1?
 - $A_{A1} = \{(L, L), (L, R), (R, L), (R, R)\}$



Normal-form Representation of Extensive-form Games

- For every perfect-info game, there is corresponding normal-form game

		A2			
		(L, L)	(L, R)	(R, L)	(R, R)
A1	(L, L)	2, 4	2, 4	5, 3	5, 3
	(L, R)	2, 4	2, 4	5, 3	5, 3
	(R, L)	3, 2	1, 5	3, 2	1, 5
	(R, R)	3, 2	0, 1	3, 2	0, 1



Transformation from Extensive form to Normal Form

- It can **always** be performed for perfect-information games
- It can cause redundancy
 - E.g., $(2, 4)$ occurs once in extensive form but 4 times in normal form
- It can result in **exponential blowup** of game representation
- Reverse transformation does not always exist
 - E.g., there is **no** extensive-form representation for Prisoner's Dilemma
 - Perfect-information extensive-form games cannot model **simultaneity**



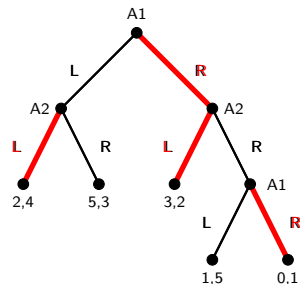
Nash Equilibrium of Perfect-info Games in Extensive Form

- [Theorem] Every (finite) perfect-info extensive-form game has pure-strategy NE
- Agents see everything before each action \Rightarrow randomness is not required
- This is not the case for every finite game in normal form



Nash Equilibrium: An Empty Threat?

		A2			
		(L, L)	(L, R)	(R, L)	(R, R)
A1	(L, L)	2, 4	(2, 4)	5, 3	5, 3
	(L, R)	2, 4	(2, 4)	5, 3	5, 3
	(R, L)	3, 2	1, 5	3, 2	1, 5
	(R, R)	(3, 2) (3, 2)	0, 1	3, 2	0, 1



- Strategy of A1 is called a **threat**
 - Committing to choose R forces A2 to avoid that part of the tree
- A2 may not consider A1's threat to be **credible**
 - Would A1 really follow through on this threat if final decision node is reached?



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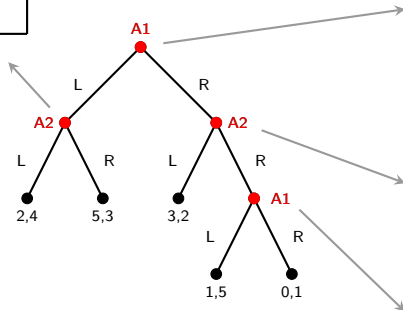
Subgames: Definition

- Let G be a perfect-information extensive-form game
- **Subgame** of G rooted at node h is restriction of G to descendants of h
- Set of subgames of G consists of all of subgames of G rooted at some node in G



Subgames: Example

	A2	
	(*, L)	(*, R)
A1 (*, *)	2,4	5,3



		A2			
		(L, L)	(L, R)	(R, L)	(R, R)
A1	(L, L)	2,4	2,4	5,3	5,3
	(L, R)	2,4	2,4	5,3	5,3
	(R, L)	3,2	1,5	3,2	1,5
	(R, R)	3,2	0,1	3,2	0,1

		A2	
		(*, L)	(*, R)
A1	(*, L)	3,2	1,5
	(*, R)	3,2	0,1

		A2	
		(*, L)	(*, R)
A1	(*, L)	1,5	
	(*, R)	0,1	



Subgame Perfect Equilibrium (SPE)

- Let $s_{G'}$ be restriction of strategy profile s to subgame G'
- Profile s^* is SPE of game G if for every subgame G' of G , $s_{G'}^*$ is NE
- Loosely speaking, subgame perfection removes non-credible threats
 - Non-credible threats are not NE in their subgames
- How to find SPE?
 - One could find all of NE, then eliminate those that are not subgame perfect
 - But there are more economical ways of doing it



Computing Equilibrium: Backward Induction for Finite Games

- (1) Start from “last” subgames (choice nodes with all terminal children)
- (2) Find Nash equilibria of those subgames
- (3) Turn those choice nodes to terminal nodes using NE utilities
- (4) Go to (1) until no choice node remains



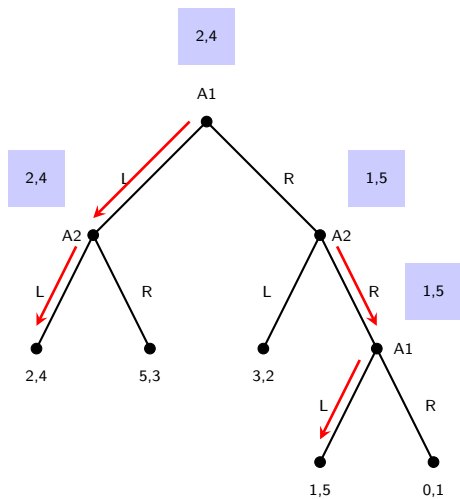
Backward Induction Procedure

Algorithm 1: Finding value of sample SPE of perfect-info extensive-form game

```
procedure Backward_Induction(node  $h$ )  
  if  $h \in Z$  then  
    return  $u(h)$ ;  
   $best\_utility \leftarrow -\infty$ ;  
  forall  $a \in \beta(h)$  do  
     $u = \text{Backward\_Induction}(\rho(h, a))$ ;  
    if  $u_{\alpha(h)} > best\_utility$  then  
       $best\_utility = u_{\alpha(h)}$ ;  
  return  $best\_utility$ 
```

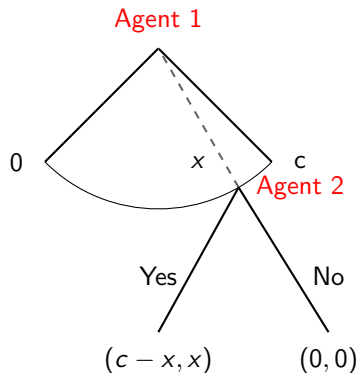


SPE: Example



Example: Ultimatum Game

- Two agents want to **split c dollars**
 - A1 offers A2 some amount $x \leq c$
 - If A2 accepts, outcome is $(c - x, x)$
 - If A2 rejects, outcome is $(0, 0)$
- What is A2's best response if $x > 0$?
 - Yes
- What is A2's best response if $x = 0$?
 - Indifferent between Yes or No
- What are A2's optimal strategies?
 - **Option 1:** Yes for all $x \geq 0$
 - **Option 2:** Yes if $x > 0$, No if $x = 0$



SPE of Ultimatum Game

- What is A1's optimal strategy for each of A2's optimal strategies?
 - For option 1, A1's optimal strategy is to offer $x = 0$
 - For option 2, if A1 offers $x = 0$, then A1's utility is 0
 - If A1 wants to offer any $x > 0$, then A1 must offer

$$\operatorname{argmax}_{x>0}(c - x)$$

- This optimization does not have any optimal solution
 - No offer of agent 1 is optimal
- Unique SPE of ultimatum game is A1 offers 0, and A2 accepts all offers



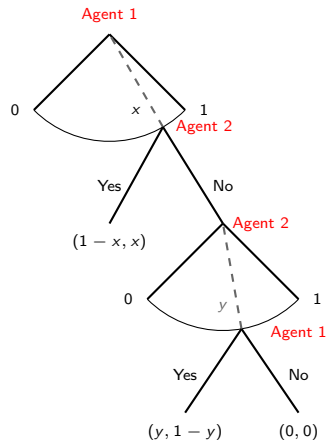
Example: Discrete Ultimatum Game

- What are A2's optimal strategies if c is in multiple of cent?
 - **Option 1:** Yes for all $x \geq 0$
 - **Option 2:** Yes if $x > 0$, No if $x = 0$
- What are A1's optimal strategies for each of A2's?
 - For option 1, offer $x = 0$
 - For option 2, offer $x = 1$ cent
- What are SPE of this modified ultimatum game?
 - A1 offers 0, and A2 accepts all offers
 - A1 offers 1 cent, and A2 accepts all offers except 0
- Show that every $\bar{x} \in [0, c]$, there exists NE in which A1 offers \bar{x}
 - What is agent A2's optimal strategy?



Example: Bargaining Game

- Two agents want to split $c = 1$ dollar
- First, A1 makes her offer
- Then, A2 decides to accept or reject
- If A2 rejects, then A2 makes new offer
- Then, A1 decides to accept or reject
- Let $x = (x_1, x_2)$ denote A1's offer
- Let $y = (y_1, y_2)$ denote A2's offer



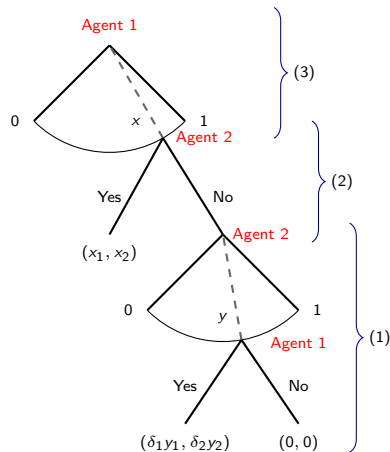
Backward Induction for Bargaining Game

- Second round is ultimatum game with **unique SPE**
 - A2 offers 0, and A1 accepts all offers
- What is A2's optimal strategy in round 1's subgame?
 - **Option 1**: If $x_2 \leq 1$, reject
 - **Option 2**: If $x_2 = 1$, accept, and reject otherwise
- What are A1's optimal strategies in round 1 for each of A2's?
 - For both options, A1 is indifferent between all strategies
- How many SPE does this game have?
 - Infinitely many! In all SPE, A2 gets everything (**Last mover's advantage**)
 - In every SPE, agent who makes offer in last round gets everything



Example: Discounted Bargaining Game

- Utilities are discounted by $0 < \delta_i < 1$
- What is unique SPE of (1)?
 - A2 offers $y_1 = 0$ and A1 accepts all offers
- What are optimal strategies in (2)?
 - **Option 1:** Yes if $x_2 \geq \delta_2$, No otherwise
 - **Option 2:** Yes if $x_2 > \delta_2$, No otherwise
- What are optimal strategies in (3)?
 - For option 1, offer $x_2 = \delta_2$
 - For option 2, there is no optimal strategy



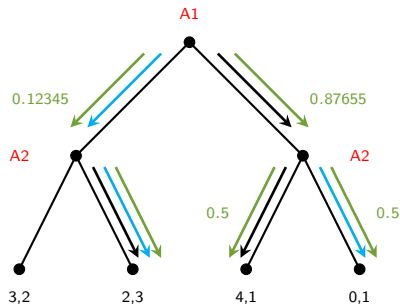
Unique SPE of Discounted Bargaining Game

- What are SPE **strategies**?
 - Agent 1's proposes $(1 - \delta_2, \delta_2)$
 - Agent 2 only accepts proposals with $x_2 \geq \delta_2$
 - Agent 2 proposes $(0, 1)$ after any history in which 1's proposal is rejected
 - Agent 1 accepts all proposals of Agent 2
- What is SPE **outcome** of game?
 - Agent 1 proposes $(1 - \delta_2, \delta_2)$
 - Agent 2 accepts
 - Resulting utilities are $(1 - \delta_2, \delta_2)$
- Desirability of earlier agreement yields positive utility for agent 1



Limitation of Backward Induction

- If there are ties, how they are broken affects what happens up in tree



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Imperfect-info Games: Motivation

- So far, we have allowed agents to specify action they take at every choice node
- This implies that agents know the node they are in and all prior choices
- This is why we call these games **perfect-information** games
- However, this might not be the case in all environments



Imperfect-info Games: Motivation (cont.)

- We may want to model agents with **partial or no knowledge** of others' actions
- We may even want to model agents with **limited memory** of their **own** past actions
- **Imperfect-info** games in extensive form address this limitation
- In such games, each agent's choice nodes are partitioned into **information sets**
- If two nodes are in same info set, then agent cannot distinguish between them



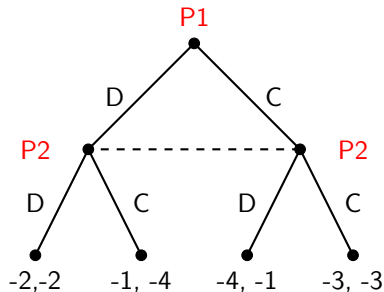
Imperfect-info Extensive-form Games: Definition

- $N, A, H, Z, \alpha, \beta, \rho, u$ are the same as before
- $I = (I_1, \dots, I_n)$, where $I_i = (I_{i,1}, \dots, I_{i,k_i})$ is a partition of $\{h \in H : \alpha(h) = i\}$
- If h, h' are in the same **equivalence class** $I_{i,j}$, then $\beta(h) = \beta(h')$
- Perfect-info games are imperfect-info games with singleton equivalence classes



Example: Prisoners' Dilemma in Extensive Form

- P1 decides on D or C
- P2 then decides on D or C
(without observing P1's decision)



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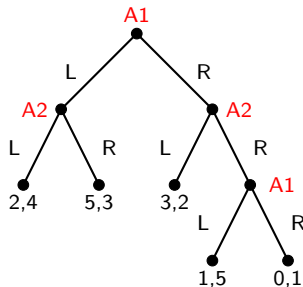
Pure, Mixed, and Behavioral Strategies

- **Pure strategies** of agent i consists of $\prod_{l_{i,j} \in I_i} \beta(l_{i,j})$
- **Mixed strategies** define randomization over pure strategies
- **Behavioral strategy** define independent randomization at each info set
- Mixed strategy is **distribution over vectors** (each vector describing a pure strategy)
- Behavioral strategy is a **vector of distributions**
- In general, expressive power of behavioral and mixed strategies are noncomparable
 - In some games, there are outcomes that are achieved via mixed but not any behavioral strategies
 - And in some games it is the other way around



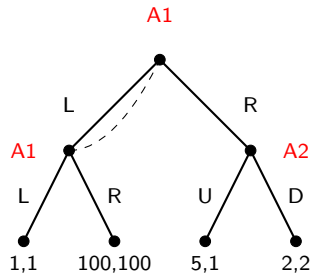
Mixed vs Behavioral Strategies: Example I

- Give behavioral strategy for A1
 - L w.p. 0.2 and R w.p. 0.5
- Give mixed strategy for A1 that is not behavioral strategy
 - (L, L) w.p. 0.4 and (R, R) w.p. 0.6
 - Why this is not behavioral strategy?
- In this game, every behavioral strategy **corresponds to** a mixed strategy and vice versa (more on this soon)



Mixed vs Behavioral Strategies: Example II

- What is mixed-strategy NE of this game?
 - (R, D) with outcome utilities (2,2)
- What is A1's expected utility for $(p, 1 - p)$?
 - $p^2 + 100p(1 - p) + 2(1 - p)$
- What is A1's best response?
 - $p = 98/198$
- What is behavioral NE of this game?
 - $((98/198, 100/198), (0, 1))$

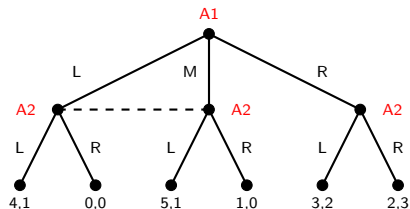


Perfect Recall

- Strategies that induce same distribution on outcomes, for fixed strategy profile of others, are called **equivalent** strategies
- If all agents remember all their own actions, game is a game of **perfect recall**
- In such games, any mixed strategy of given agent can be replaced by an **equivalent** behavioral strategy
- And any behavioral strategy can be replaced by an **equivalent** mixed strategy



Subgame Perfection and Imperfect Information



- There are two subgames: game itself and subgame after agent 1 plays R
 - $(R, (R,R))$ is NE and SPE
- But, why should 2 play R after 1 plays L or M?
 - This is **non-credible threat**
- There are more sophisticated equilibrium refinements that rule this out
 - They explicitly model agents' beliefs on where they are for every info set
 - E.g., sequential equilibrium, perfect Bayesian equilibrium



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