# 1 Effective parameters

Consider a photonic crystal composed of layers of two media $a$ and $b$. Layer $a$ has electric permittivity $\epsilon_a$, magnetic permeability $\mu_a$ and width $\ell_a$. Similarly, layer "b" has permittivity $\epsilon_b$, permeability $\mu_b$ and width $\ell_b$.
Now let us look at fields present in the photonic crystal. Specifically, the component of $\mathbf{E}$ parallel to the layer boundaries $\mathbf{E}_{\|}$ and the component of $\mathbf{D}$ perpendicular to layer boundaries $\mathbf{D}_{\perp}$. From continuity conditions of these fields we can equate the fields at both sides and their average (which is equal to them, since electric intensity is conserved across the boundary):

$$\mathbf{E}_{\|a} = \mathbf{E}_{\|b} = \mathbf{E}_{\|\mathrm{ave}} \tag{1}$$

and utilising the material relation $\mathbf{D} = \epsilon \mathbf{E}$ we can average out the vector of electric induction through the two layers:

$$\mathbf{D}_{\|\mathrm{ave}} = \frac{\ell_a \epsilon_a \mathbf{E}_{\|a} + \ell_b \epsilon_b \mathbf{E}_{\|b}}{\ell_a + \ell_b} \tag{2}$$

$$\mathbf{D}_{\|\mathrm{ave}} = \frac{\ell_a \epsilon_a + \ell_b \epsilon_b}{\ell_a + \ell_b} \mathbf{E}_{\|\mathrm{ave}} \tag{3}$$

thus we can conclude the derivation of electric permittivity in direction parallel to the interface:

$$\epsilon_{\|\mathrm{eff}} = \frac{\ell_a \epsilon_a + \ell_b \epsilon_b}{\ell_a + \ell_b} \tag{4}$$

Now we can repeat this procedure for $\mathbf{D}_{\perp}$. Similarly, perpendicular component of electric induction is conserved across the boundary, so

$$\mathbf{D}_{\perp\mathrm{ave}} = \mathbf{D}_{\perp a} = \mathbf{D}_{\perp b} \tag{5}$$

and using the same material relation as before we can average out the perpendicular component of the $\mathbf{E}$ field:

$$\mathbf{E}_{\perp\mathrm{ave}} = \frac{\ell_a \mathbf{E}_{\perp a} + \ell_b \mathbf{E}_{\perp b}}{\ell_a + \ell_b} \tag{6}$$

$$\mathbf{E}_{\perp\mathrm{ave}} = \frac{\ell_a \frac{\mathbf{D}_{\perp a}}{\epsilon_a} + \frac{\mathbf{D}_{\perp b}}{\epsilon_b}}{\ell_a + \ell_b} \tag{7}$$

$$\mathbf{E}_{\perp\mathrm{ave}} = \frac{\frac{\ell_a}{\epsilon_a} + \frac{\ell_b}{\epsilon_b}}{\ell_a + \ell_b} \mathbf{D}_{\perp\mathrm{ave}} \tag{8}$$

and since $\mathbf{E} = \frac{1}{\epsilon}\mathbf{D}$

$$\epsilon_{\perp\mathrm{eff}} = \frac{\ell_a + \ell_b}{\frac{\ell_a}{\epsilon_a} + \frac{\ell_b}{\epsilon_b}} = \epsilon_a \epsilon_b \frac{\ell_a + \ell_b}{\ell_a \epsilon_b + \ell_b \epsilon_a} \tag{9}$$

Thus, we can conclude that effective permittivity is a tensor:

$$\epsilon_{\mathrm{eff}} = \begin{pmatrix} \epsilon_{\|} & 0 & 0 \\ 0 & \epsilon_{\|} & 0 \\ 0 & 0 & \epsilon_{\perp} \end{pmatrix} \tag{10}$$

Similarly we can derive the effective description of magnetic permeability. Again, we will only consider the parallel component of magnetic intensity $\mathbf{H}_{||}$ and the perpendicular component of magnetic induction $\mathbf{B}_{\perp}$. The material relation connecting them is $\mathbf{B} = \mu\mathbf{H}$
Start with $\mathbf{B}_{\perp}$. Since this component is continuous across the interface, the average will simply be

$$\mathbf{B}_{\perp\text{ave}} = \mathbf{B}_{\perp a} = \mathbf{B}_{\perp b} \tag{11}$$

and for the average of $\mathbf{H}_{\perp}$ using the material relation we get

$$\mathbf{H}_{\perp\text{ave}} = \frac{\ell_a \mathbf{H}_{\perp a} + \ell_b \mathbf{H}_{\perp b}}{\ell_a + \ell_b} = \frac{\ell_a \dfrac{\mathbf{B}_{\perp a}}{\mu_a} + \ell_b \dfrac{\mathbf{B}_{\perp b}}{\mu_b}}{\ell_a + \ell_b} = \frac{\dfrac{\ell_a}{\mu_a} + \dfrac{\ell_b}{\mu_b}}{\ell_a + \ell_b} \mathbf{B}_{\perp\text{ave}} \tag{12}$$

hence

$$\mu_{\perp\text{eff}} = \frac{\ell_a + \ell_b}{\dfrac{\ell_a}{\mu_a} + \dfrac{\ell_b}{\mu_b}} = \mu_a \mu_b \frac{\ell_a + \ell_b}{\ell_a \mu_b + \ell_b \mu_a} \tag{13}$$

And now similarly for $\mathbf{H}_{||}$:

$$\mathbf{H}_{||\text{ave}} = \mathbf{H}_{||a} = \mathbf{H}_{||b} \tag{14}$$

$$\mathbf{B}_{||\text{ave}} = \frac{\ell_a \mathbf{B}_{||a} + \ell_b \mathbf{B}_{||b}}{\ell_a + \ell_b} \tag{15}$$

again, using the material relation:

$$\mathbf{B}_{||\text{ave}} = \frac{\ell_a \mu_a \mathbf{H}_{||a} + \ell_b \mu_b \mathbf{H}_{||b}}{\ell_a + \ell_b} = \frac{\ell_a \mu_a + \ell_b \mu_b}{\ell_a + \ell_b} \mathbf{H}_{||\text{ave}} \tag{16}$$

and comparing with the material relation

$$\mu_{||\text{eff}} = \frac{\ell_a \mu_a + \ell_b \mu_b}{\ell_a + \ell_b} \tag{17}$$

finally, putting relationships (13) and (17) together we get the permeability tensor:

$$\mu_{\text{eff}} = \begin{pmatrix} \mu_{||} & 0 & 0 \\ 0 & \mu_{||} & 0 \\ 0 & 0 & \mu_{\perp} \end{pmatrix} \tag{18}$$