

# 1 Effective parameters

Consider a photonic crystal composed of layers of two media  $a$  and  $b$ . Layer  $a$  has electric permittivity  $\epsilon_a$ , magnetic permeability  $\mu_a$  and width  $\ell_a$ . Similarly, layer "b" has permittivity  $\epsilon_b$ , permeability  $\mu_b$  and width  $\ell_b$ .

Now let us look at fields present in the photonic crystal. Specifically, the component of  $\mathbf{E}$  parallel to the layer boundaries  $\mathbf{E}_{\parallel}$  and the component of  $\mathbf{D}$  perpendicular to layer boundaries  $\mathbf{D}_{\perp}$ . From continuity conditions of these fields we can equate the fields at both sides and their average (which is equal to them, since electric intensity is conserved across the boundary):

$$\mathbf{E}_{\parallel a} = \mathbf{E}_{\parallel b} = \mathbf{E}_{\parallel \text{ave}} \quad (1)$$

and utilising the material relation  $\mathbf{D} = \epsilon \mathbf{E}$  we can average out the vector of electric induction through the two layers:

$$\mathbf{D}_{\parallel \text{ave}} = \frac{\ell_a \epsilon_a \mathbf{E}_{\parallel a} + \ell_b \epsilon_b \mathbf{E}_{\parallel b}}{\ell_a + \ell_b} \quad (2)$$

$$\mathbf{D}_{\parallel \text{ave}} = \frac{\ell_a \epsilon_a + \ell_b \epsilon_b}{\ell_a + \ell_b} \mathbf{E}_{\parallel \text{ave}} \quad (3)$$

thus we can conclude the derivation of electric permittivity in direction parallel to the interface:

$$\epsilon_{\parallel \text{eff}} = \frac{\ell_a \epsilon_a + \ell_b \epsilon_b}{\ell_a + \ell_b} \quad (4)$$

Now we can repeat this procedure for  $\mathbf{D}_{\perp}$ . Similarly, perpendicular component of electric induction is conserved across the boundary, so

$$\mathbf{D}_{\perp \text{ave}} = \mathbf{D}_{\perp a} = \mathbf{D}_{\perp b} \quad (5)$$

and using the same material relation as before we can average out the perpendicular component of the  $\mathbf{E}$  field:

$$\mathbf{E}_{\perp \text{ave}} = \frac{\ell_a \mathbf{E}_{\perp a} + \ell_b \mathbf{E}_{\perp b}}{\ell_a + \ell_b} \quad (6)$$

$$\mathbf{E}_{\perp \text{ave}} = \frac{\ell_a \frac{\mathbf{D}_{\perp a}}{\epsilon_a} + \ell_b \frac{\mathbf{D}_{\perp b}}{\epsilon_b}}{\ell_a + \ell_b} \quad (7)$$

$$\mathbf{E}_{\perp \text{ave}} = \frac{\frac{\ell_a}{\epsilon_a} + \frac{\ell_b}{\epsilon_b}}{\ell_a + \ell_b} \mathbf{D}_{\perp \text{ave}} \quad (8)$$

and since  $\mathbf{E} = \frac{1}{\epsilon} \mathbf{D}$

$$\epsilon_{\perp \text{eff}} = \frac{\ell_a + \ell_b}{\frac{\ell_a}{\epsilon_a} + \frac{\ell_b}{\epsilon_b}} = \epsilon_a \epsilon_b \frac{\ell_a + \ell_b}{\ell_a \epsilon_b + \ell_b \epsilon_a} \quad (9)$$

Thus, we can conclude that effective permittivity is a tensor:

$$\epsilon_{\text{eff}} = \begin{pmatrix} \epsilon_{\parallel} & 0 & 0 \\ 0 & \epsilon_{\parallel} & 0 \\ 0 & 0 & \epsilon_{\perp} \end{pmatrix} \quad (10)$$

Similarly we can derive the effective description of magnetic permeability. Again, we will only consider the parallel component of magnetic intensity  $\mathbf{H}_{\parallel}$  and the perpendicular component of magnetic induction  $\mathbf{B}_{\perp}$ . The material relation connecting them is  $\mathbf{B} = \mu\mathbf{H}$ . Start with  $\mathbf{B}_{\perp}$ . Since this component is continuous across the interface, the average will simply be

$$\mathbf{B}_{\perp\text{ave}} = \mathbf{B}_{\perp a} = \mathbf{B}_{\perp b} \quad (11)$$

and for the average of  $\mathbf{H}_{\perp}$  using the material relation we get

$$\mathbf{H}_{\perp\text{ave}} = \frac{\ell_a \mathbf{H}_{\perp a} + \ell_b \mathbf{H}_{\perp b}}{\ell_a + \ell_b} = \frac{\ell_a \frac{\mathbf{B}_{\perp a}}{\mu_a} + \ell_b \frac{\mathbf{B}_{\perp b}}{\mu_b}}{\ell_a + \ell_b} = \frac{\frac{\ell_a}{\mu_a} + \frac{\ell_b}{\mu_b}}{\ell_a + \ell_b} \mathbf{B}_{\perp\text{ave}} \quad (12)$$

hence

$$\mu_{\perp\text{eff}} = \frac{\ell_a + \ell_b}{\frac{\ell_a}{\mu_a} + \frac{\ell_b}{\mu_b}} = \mu_a \mu_b \frac{\ell_a + \ell_b}{\ell_a \mu_b + \ell_b \mu_a} \quad (13)$$

And now similarly for  $\mathbf{H}_{\parallel}$ :

$$\mathbf{H}_{\parallel\text{ave}} = \mathbf{H}_{\parallel a} = \mathbf{H}_{\parallel b} \quad (14)$$

$$\mathbf{B}_{\parallel\text{ave}} = \frac{\ell_a \mathbf{B}_{\parallel a} + \ell_b \mathbf{B}_{\parallel b}}{\ell_a + \ell_b} \quad (15)$$

again, using the material relation:

$$\mathbf{B}_{\parallel\text{ave}} = \frac{\ell_a \mu_a \mathbf{H}_{\parallel a} + \ell_b \mu_b \mathbf{H}_{\parallel b}}{\ell_a + \ell_b} = \frac{\ell_a \mu_a + \ell_b \mu_b}{\ell_a + \ell_b} \mathbf{H}_{\parallel\text{ave}} \quad (16)$$

and comparing with the material relation

$$\mu_{\parallel\text{eff}} = \frac{\ell_a \mu_a + \ell_b \mu_b}{\ell_a + \ell_b} \quad (17)$$

finally, putting relationships (13) and (17) together we get the permeability tensor:

$$\mu_{\text{eff}} = \begin{pmatrix} \mu_{\parallel} & 0 & 0 \\ 0 & \mu_{\parallel} & 0 \\ 0 & 0 & \mu_{\perp} \end{pmatrix} \quad (18)$$