Ex 8.1 $H(X,Y) = H((u,v),(v,w)) = \sum_{x,y} p(u,v,w) \log \frac{1}{p(u,v,w)}$

$= \sum_{u,v,w} p(u)p(v)p(w) \left(\log \frac{1}{p(u)} + \log \frac{1}{p(v)} + \log \frac{1}{p(w)} \right)$

$= H_u + H_v + H_w$

$H(X|Y) = \sum_{x,y} p((u,v)|(v,w)) \log \frac{1}{p(u,v)|(v,w)}$

$= \sum_{u,v,w} p(u|v,w) \log \frac{1}{p(u|v,w)} = H_u$

$I(X;Y) = H(X) - H(X|Y)$

$= H_u + H_v - H_u = H_v$

Ex 8.2 $H(X|y=b_k) = \sum_{x \in X} p(x|y=b_k) \log \frac{1}{p(x|y=b_k)}$

$H(X) = \sum_{x \in X} p(x) \log \frac{1}{p(x)}$

Y \ X	0	1
0		0.5
1	0.25	0.25

$H(x) = 0.811278$

$H(X|Y=1) = 1$

$H(X|Y) = \sum_{x,y} p(x|y) \log \frac{1}{p(x|y)}$

$H(X) - H(X|Y) = H(X) - (H(X,Y) - H(Y))$

$= H(X) + H(Y) - H(X,Y)$

$= \sum_x p(x) \log \frac{1}{p(x)} + \sum_y p(y) \log \frac{1}{p(y)} - \sum_{x,y} p(x,y) \log \frac{1}{p(x,y)}$

$= \sum_{x,y} p(x,y) \left(\log \frac{1}{p(x)} + \log \frac{1}{p(y)} \right) - \sum_{x,y} p(x,y) \log \frac{1}{p(x,y)}$

$$= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = KL\left(p(x,y), p(x)p(y)\right) \geq 0.$$

Ex 8.3

$$H(X,Y) = \sum_{x,y} p(x,y) \log \frac{1}{p(x,y)}$$

$$= \sum_{x,y} p(y|x)p(x) \left(\log \frac{1}{p(y|x)} + \log \frac{1}{p(x)}\right)$$

$$= H(Y|X) + \sum_{x} p(x) \log \frac{1}{p(x)} \sum_{y} p(y|x)$$

$$= H(Y|X) + H(X)$$

Ex 8.4 $H(X,Y) = H(Y) + H(X|Y) = H(X) + H(Y|X)$

$\therefore H(Y) - H(Y|X) = H(X) - H(X|Y)$

$\therefore I(Y;X) = I(X;Y)$

$H(X|Y) \leq H(X)$ Ex 8.2 证明过.

$\therefore I(X;Y) \geq 0$

Ex 8.5.

$$D_H(X,Y) = H(X,Y) - I(X;Y) = H(Y) + H(X|Y) - (H(Y) - H(Y|X))$$

$$= H(X|Y) + H(Y|X)$$

显然 $D_H(X,Y) = D_H(Y,X)$, $D_H(X,X) = 0$

$\because$ 熵不小于0 $\therefore D_H(X,Y) \geq 0$

$$D_H(X,Z) = H(X|Z) + H(Z|X)$$

$$D_H(X,Y) + D_H(Y,Z) = H(X|Y) + H(Y|X) + H(Y|Z) + H(Z|Y)$$

$$D_H(X,Z) = D_H(X,Y) + D_H(Y,Z) + H(Y) - \cancel{H(Z)} + \cancel{H(X)} - H(X,Y) + \cancel{H(Z)} - H(Y,Z) + H(Y) - \cancel{H(X)}$$

$$= D_H(X,Y) + D_H(Y,Z) + 2H(Y) - H(X,Y) - H(Y,Z).$$

$$= D_H(X,Y) + D_H(Y,Z) - H(X|Y) - H(Z|Y)$$

$$\leq D_H(X,Y) + D_H(Y,Z)$$

Ex 8.7.

	0	1
0	0	1
1	1	0

$$P_Z = \{ \tfrac{1}{2} \quad \tfrac{1}{2} \}.$$

$$I(Z;X) = H(Z) - H(Z|X)$$

$$= 1 - 1 = 0$$

(b) $P_Z = [\, qp + (1-q)(1-p) \;,\; q(1-p) + (1-q)p \,].$

$$I(Z;X) = H(Z) - H(Y)$$

Ex 8.9. $I(W;R) = H(W) - H(W|R) = H(R) - H(R|W)$

$$= H(R)$$

$$I(W;D) = H(W) - H(W|D)$$

要证 $H(W|R) \geq H(W|D)$

即 $\sum_{w,r} p(w,r) \log \frac{1}{p(w|r)} - \sum_{w,d} p(w,d) \log \frac{1}{p(w|d)} \geq 0$

$$\sum_{w,d,r} p(w,d,r)\left(\log \frac{1}{p(w|r)} - \log \frac{1}{p(w|d)}\right)$$

$$= \sum_{w,d,r} p(w,d,r)\left(\log \frac{p(w|d)}{p(w|r)}\right)$$

$$= \sum_{w,d,r} p(w,d,r) \log\left(\frac{\cancel{p(d|w)}\,\cancel{p(w)}\; p(r)}{p(d)\, p(r|d,w)\,\cancel{p(d|w)}\,\cancel{p(w)}}\right)$$

$$= \sum_{w,d,r} p(w,d,r) \log \frac{p(r|d)\,p(d)}{p(r|d,w)\,p(d)} = \sum_{w,d,r} p(w,d,r) \log \frac{p(r|d)}{p(r|d,w)}$$

$\because p(r|d) \geq p(r|d,w)$

$\therefore \sum_{w,d,r} p(w,d,r) \log \frac{p(r|d)}{p(r|d,w)} \geq \sum_{w,d,r} p(w,d,r) \log 1 = 0$