1 One-Factor Model

The one-factor model is an asset pricing model that acknowledges the market return as the only factor that influences the performance of a given stock. It draws from *simple linear regression*, a regression model with least-squares estimator that analyses how a predictor *(independent variable)* affects the criterion *(dependent variable)*. The resulting linearly regressed line would minimize the set of distances between it and all points on the graph.

We will apply this statistical technique to the behaviour of securities in the financial market. We are interested in the analysis of variability of risky assets, in this case a stock, in response to changes in prices of the market index. We assume the market index to be reflective of the performance of the overall stock market.

1.1 Mathematical Modelling

Throughout this report, the time scale will be restricted from the first day for which we have the historical price for a given stock, where time $n = 0$, to any arbitrary point in future T.

The price of an equity on day n is given by $P(n)$, and the rate of return of an equity on day n is given by:

$$\frac{P(n) - P(n-1)}{P(n-1)}.$$

The rate of return of one stock at time n will be denoted by $S(n)$, and the rate of return of the market index at time n will be denoted by $M(n)$.

The formula for simple linear regression is hence given by:

$$S(n) = \beta_0 + \beta_1 M(n) + e,$$

where β_0 is interpreted as the mean of $S(n)$ when $M(n) = 0$ (intercept), and β_1 is interpreted as the mean difference in S between those who differ by one unit in M. e is the random error that denotes the deviation of the observed value from the expected value. Rearranging the equation, we have:

$$e = S(n) - (\beta_0 + \beta_1 M(n)).$$

To minimise the risk of the investment, we need to solve for β_0 and β_1 (refer to Appendix) such that e^2 is a minimum:

$$\text{where } e^2 = \sum_{i=1}^{n} [S(i) - (\beta_0 + \beta_1 M(i))]^2.$$

Applying partial differentiation, we obtain

$$\beta_0 = \overline{S(i)} - \beta_1 \overline{M(i)}$$

$$\text{and } \beta_1 = \frac{\sum\limits_{i=1}^{n} [M(i) - \overline{M(i)}][S(i) - \overline{S(i)}]}{\sum\limits_{i=1}^{n} [M(i) - \overline{M(i)}]^2},$$

where $\overline{x}$ denotes the average value of a set of value x.

1.2 Choice of Market Index and Stock

Our choice of market index M will be the Standard & Poor's 500 (SPY), which tracks the stocks of the 500 largest publicly traded companies based on market capitalization[1]. It is considered by experts to be among the best benchmarks available to judge the overall U.S equity market performance[2].

Our choice of stock S will be that of American multinational oil and gas corporation ExxonMobil (XOM), the second largest publicly traded company by market capitalization[3]. As the prices of oil influences the cost of production of most firms, we can deduce that there should be strong correlation between prices of S and M. Our calculations of the Pearson product-moment correlation coefficient between S and M over a five-year period (2008-2013) shows this to be true with a very strong positive coefficient of 0.8115 being found. Completeness of its historical price data was another reason XOM was chosen, as it was not the target of any trading halts or suspensions. Complete data allows for easier calculation and analysis.

1.3 Ideal Time Frame

In this section, we investigate the reliability of the predictions of the one-factor model from input data for various time periods, so as to find a suitable input data time range for our future predictions in the later sections.

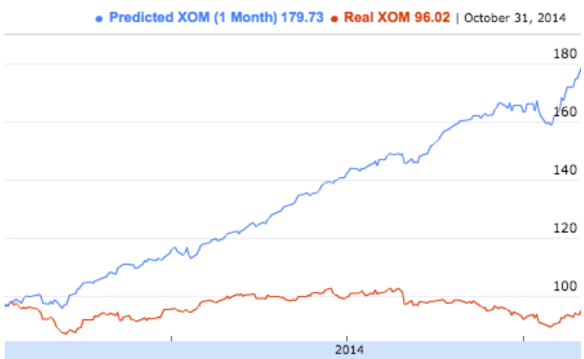

Figure 1: Predicted vs Actual XOM prices with 1 month of training data.

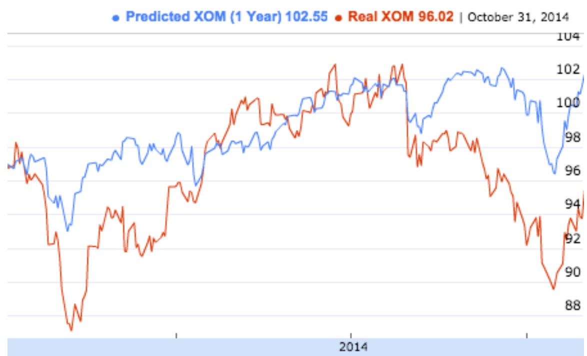

Figure 2: Predicted vs Actual XOM prices with 1 year of training data.

The training data will be $S(n)$ and $M(n)$ over the given period of time ending on 31st October 2014, allowing us to determine the values of β_0 and β_1 (see section 1.1), hence completing our linear regression. As we know the rates of return for SPY, we can then use the calculated values of β_0 and β_1 to predict the rate of return of XOM everyday, for the same time period.

Given that the base price of XOM on Jan 1 is \$97.00, we are able to compound the predicted rate of return for XOM every day up until Oct 31st. The plotted graph of the predicted versus expected prices is shown in Figures 1-3.

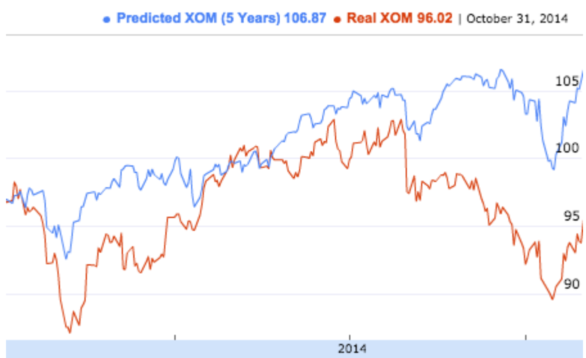

Figure 3: Predicted vs Actual XOM prices with 5 years of training data.

It is immediately clear that there is a huge deviation from the predicted results when only one month of training data was used. For a quantitative measure of comparing the deviations of the predicted prices of XOM from that of the real value, the standard deviation between the values was found (Table 1). We can see that any input data between 1 to 5 years allows for predictions with moderately acceptable deviations.

Time Period	Standard Deviation
1 Month	24.97466808
3 Months	15.69990719
6 Months	4.926836351
1 Year	3.41288401
1.5 Years	3.369686931
2 Years	3.14824598
2.5 Years	3.935556201
3 Years	4.373239687
5 Years	3.183845281

Table 1: Standard deviation of the predicted and actual XOM prices with different input data time ranges

1.4 Applications of the One-Factor Model

We will apply the one-factor model in use of predicting the future price of XOM on SSEF Judging Day, on 11th March 2015. As shown in section 1.3, the standard deviation of the actual and predicted stock prices is the lowest when input data of the range 1-5 years are used. Hence, 2 years of training data (1st Nov 2012 – 31st Oct 2014) is used in our prediction to increase the accuracy of our prediction.

As SPY's performance is reflective of the overall market performance, its price does not tend to fluctuate widely due to the counteracting movements of various industries. By plotting its rate of return against time over the past two years ending 31st Oct 2014 (Figure 4), from the general trendline of its rate of return we can assume it to be constant for the short period of time in the future that we are interested in. We will take this constant as the average rate of return for the past two years. Using this value, as well as the values of β_0 and β_1 obtained from the 2 years of training data, the rate of return and the future stock price of XOM can be thus be predicted by the formulae:

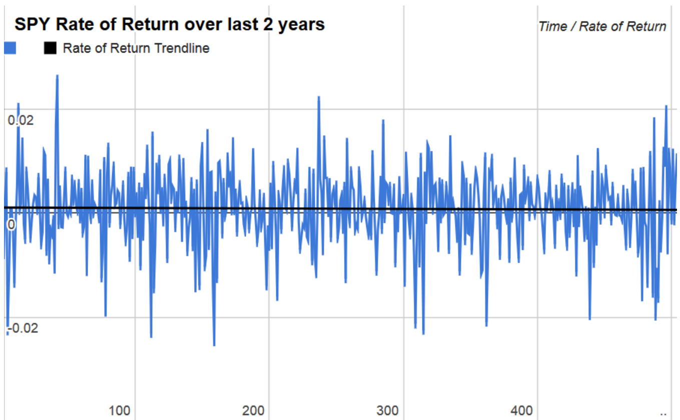

Figure 4: Trendline for Rate of Return for SPY (black line near the x-axis).

$$S(n) = \beta_0 + \beta_1 \overline{M(n)}$$
$$\text{and } P(n) = P(n-1)[1 + S(n)].$$

Applying these formulae, we obtain 0.000254580 as the rate of return for XOM. Using the starting price of XOM on 31 Oct 2014 of \$96.02 and compounding it with the expected daily rate of return of the 88 trading days[4] in NYSE(New York Stock Exchange) ending on 11 March 2015 (Figure 5), we obtain \$98.20 as the future price of XOM on that day.

Should we have invested in XOM stocks on 1 Nov 2014, we would expect to have received a return of \$2.175 and a rate of return of 2.265% on our investment!

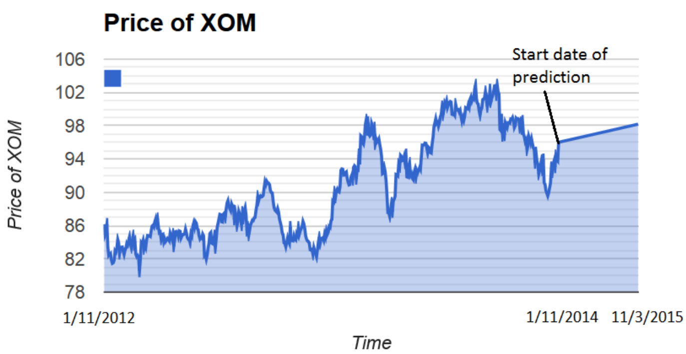

Figure 5: Historical price of XOM ending on 31/10/14, followed by our predicted price until 11/3/15.

Furthermore, when the future prices are plotted in a graph along with the training data from the past two years, it is evident that the predicted future prices are consistent with the trend of previous prices, making our prediction rather reasonable. To make it more exciting, the accuracy of our prediction can be verified on the day of judging itself.

1.5 Limitations of the One-Factor Model

An underlying limitation of this model is that the relationship between M and S is assumed to follow strictly to a linear model. By conducting several event studies, it is apparent that the magnitude of price changes of M and S varies differently, hence making this relationship not completely realistic as compared to what happens in real financial markets. Multi-factor models exist to factor in more variables affecting prices of stocks[5]. Next, we will look at a model that helps to optimise portfolio risk and returns.

2 The Efficient Frontier

First introduced by Nobel Laureate Harry Markowitz, and later developed by economist William Sharpe[11], the efficient frontier is a statistical model that revolutionised the finance sector for its ability to show investors the exact fraction of money to invest in each stocks for a specified return and risk level. By using standard deviation as a proxy for risk in this model[6], we allow the determination for the lowest possible risk of an investment which one has to take for a particular expected return in simple numerical terms[7]. The efficient frontier is also the upper bound of the minimum variance set, which contains the feasible set (the set of all possible portfolios with different expected rate of return and risk) of the investment.

2.1 Mathematical Modelling

The variance of the portfolio return is given as

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij}$$

$$\text{subject to } \sum_{i=1}^{n} \omega_i \overline{r_i} = \overline{r}, \ \ \sum_{i=1}^{n} \omega_i = 1.$$

As such, we obtain the Lagrangian function, L of the Markowitz problem which is given as

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij} - \lambda(\sum_{i=1}^{n} \omega_i \overline{r_i} - \overline{r}) - \mu(\sum_{i=1}^{n} \omega_i - 1),$$

where ω_i and r_i denote the weight of money invested on and the rate of return the i^{th} stock respectively, and λ and μ are the Lagrange multipliers.

Applying partial differentiation on L with respect to ω_i for $i = 1, 2, 3..., n$ and equating the first derivative to 0 so as to find the minimum variance (*risk*), we obtain

$$\sum_{j=1}^{n} \omega_j \sigma_{ij} - \lambda r_i - \mu = 0 \qquad (1)$$

By matrix multiplication (see Appendix), we obtain the equation

$$W = \alpha \frac{\mathbf{C^{-1}r}}{b} + (1-\alpha)\frac{\mathbf{C^{-1}1}}{c},$$

where $\mathbf{W}$ is a column vector of the weights of the stocks,
$\mathbf{C^{-1}}$ is the inverse covariance matrix of the n stocks,
$\mathbf{r}$ is a column vector of the expected returns of the n stocks,
$\mathbf{1}$ is a column vector with n entries of 1,
$b = \mathbf{r}^T\mathbf{C}^{-1}\mathbf{1} = \mathbf{1}^T\mathbf{C}^{-1}\mathbf{r}$,
$c = \mathbf{1^T C^{-1} 1}$,
$\alpha = \mathrm{b}\lambda$ (weight of fund 1) and
$1-\alpha = \mathrm{c}\mu$ (weight of fund 2).

This is in accordance with the Two Funds Separation Theorem[8] , which states that the efficient frontier of any risky assets can be obtained with any two risky portfolios on the frontier. It can be noted that by setting $\lambda = 0$, we will be removing the constraint of expected returns as $\alpha \cdot \frac{\mathbf{C^{-1}r}}{b} = 0$, allowing the point of minimum variance on the efficient frontier to be determined. Furthermore, it is observed that $\lambda = \frac{c\overline{r}-b}{ac-b^2}$, showing that the minimum variance point occurs when $\overline{r} = \frac{b}{c}$.

2.2 Choice of Stock

To investigate the effectiveness of using the efficient frontier to maximize returns and minimize risk for investors, we will construct it using three stocks, which are Wal-Mart Stores Inc. (WMT), China Biologic Products, Inc. (CBPO) and Exxon Mobil Corporation (XOM).

2.3 Application of the Efficient Frontier

With input data over two years, the mean of the daily rate of return for each company is calculated and assumed to be the expected rate of return as well in the future.

A covariance matrix between the daily rate of return of all companies is calculated, providing the values required to construct a system of linear equations derived from the first derivative of the Langrangian L (see 1.1). We then find the inverse of the covariance matrix, allowing us to find the weights of the individual stocks of the two funds (refer to Appendix).

In order to construct the efficient frontier, we need to perform an exhaustive search over all possible portfolio combinations with weight divisions of 0.01, where $\alpha = [-0.99, 1.99]$, for $\alpha + (1-\alpha) = 1$. We are therefore able to find the expected rate of return for every set of weights of the two funds that satisfies this constraint, via the relation

$$E_{\text{rate of return}} = \alpha E_{\text{fund 1}} + (1-\alpha)E_{\text{fund 2}},$$

as well as the standard deviation of returns of the two funds, which is given by

$$\sigma = \sqrt{\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_{12}},$$

where σ is the standard deviation of the portfolio returns,
α_1 is the standard deviation of fund 1,
α_2 is the standard deviation of fund 2,
and σ_{12} is the covariance between fund 1 and fund 2.

W(WMT)	W(CBPO)	W(XOM)	Rate of Ret	Std Dev
0.531367	0.301609	0.167023	0.0012158	0.010584528
0.481919	0.199043	0.319039	0.0009877	0.008284204
0.43247	0.096476	0.471054	0.0007596	0.006860206
0.408346	0.046474	0.545162	0.0006484	0.006668387
0.384258	-0.00353	0.619269	0.0005372	0.006860206
0.334809	-0.10609	0.771285	0.0003091	0.008284204
0.28536	-0.20866	0.923301	0.000081	0.010584528

Table 2: Points on the Minimum Variance Set

We plot a graph of $E_{\text{rate of return}}$ against σ to find the minimum variance set. We then truncate the curve at the point where $\alpha = 0$ or $1 - \alpha = 1$ to construct the efficient frontier (Figure 6). A non-satiated and risk-averse investor can easily identify a point (of suitable risk and returns) and construct their desired portfolio with the corresponding equity weights.

Several points are selected on the Minimum Variance Set and shown in Table 2, which depicts how certain combination of weights of stocks can result in a higher rate of return despite having the same risk associated with the investment. In this paper, we allow short selling [9] (where the investor borrows the stock and sells it to make use of the proceeds to invest in some other investments) which is represented by the negative weights of the stocks.

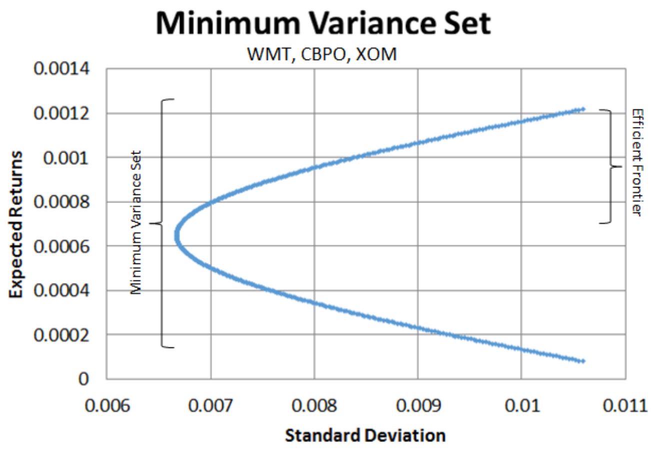

Figure 6: Minimum variance set containing efficient frontier for stocks WMT, CBPO and XOM.

2.4 Impact of Stock Correlation on Portfolio

2.4.1 Low Correlation Stocks

We now investigate how the correlation of rate of return between various stocks affect the risk of the portfolio. We are not going to compare the rate of returns of the two portfolios as it is heavily influenced by the performance of the industries of the stocks.

The stocks are chosen from different industries, and as such price movements for the stocks are lowly correlated. The stocks picked are NOK(Nokia), WMT (Wal-Mart), NAT(Nordic American Tanker), LRCDF(Laurentian Bank of Canada) and CBPO(China Biologic Products Inc).

Stocks	NOK	WMT	NAT	LRCDF	CBPO
NOK	1	0.1811	0.2020	0.01836	0.0439
WMT	0.1811	1	-0.0782	0.1792	-0.0926
NAT	0.2020	-0.0782	1	-0.0525	0.0397
LRCDF	0.0184	0.1792	-0.0525	1	0.0069
CBPO	0.0439	-0.0926	0.0397	0.0069	1

Table 3: Correlation matrix of low correlation stocks.

The efficient frontier for these stocks is shown in Figure 7.

2.4.2 High Correlation Stocks

For comparison, we also construct the efficient frontier for stocks of high correlation, *i.e.* from the same industry. The stocks chosen are WFC(Wells Fargo & Co), PNC(PNC Financial Services), USB(U.S. Bancorp), JPM(JPMorgan Chase & Co.) and STI(SunTrust Banks, Inc.).

The correlation matrix for the stocks, all large American banks, is shown in Table 4.

Stocks	WFC	PNC	USB	JPM	STI
WFC	1	0.7493	0.7998	0.7067	0.6946
PNC	0.7493	1	0.7801	0.6926	0.7781
USB	0.7998	0.7801	1	0.7282	0.7590
JPM	0.7067	0.6926	0.7282	1	0.6707
STI	0.6946	0.7781	0.7590	0.6707	1

Table 4: Correlation matrix of high correlation stocks.

2.4.3 Analysis of Results

The efficient frontier of the lowly (Figure 7) and highly (Figure 8) correlated stocks were found to have a slope of 0.157 and 0.138 along the tangent of their curves respectively.

From here, we can deduce that for every additional unit of risk that we are willing to take, we would gain a greater additional amount of return for the lowly correlated portfolio as compared to the highly correlated portfolio. In other words, a change in the risk we are willing to take brings about a greater change in return for the first portfolio.

This is likely to be because in a diversified portfolio, when a particular industry is performing poorly or struck by unfavourable events, the impact of the loss on the overall portfolio is mitigated and cushioned by other stocks; whereas equity price movements of that of a highly correlated portfolio generally respond similarly to such events, as such introducing greater risk for the investor[10].

The concept of diversification to reduce risk goes back at least to Bernoulli(1738)[11], but even Shakespeare knew it (see The Merchant of Venice).

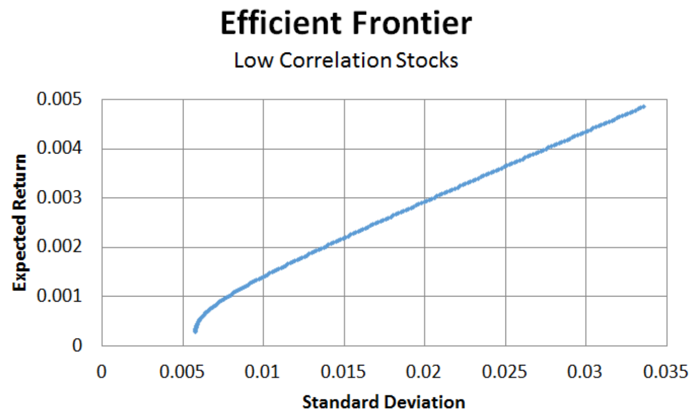

Figure 7: Efficient Frontier for low correlation stocks.

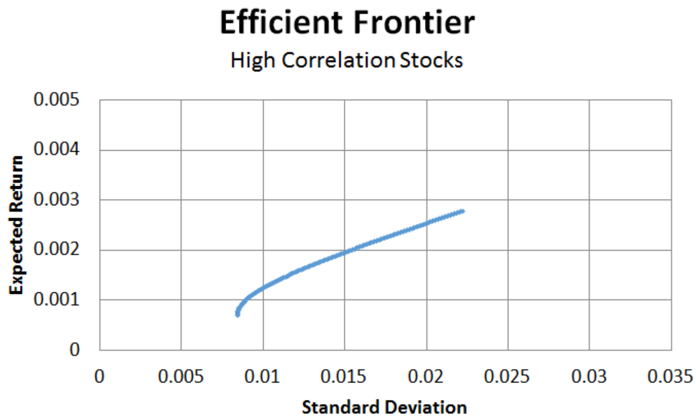

Figure 8: Efficient Frontier for high correlation stocks.

2.5 Limitations of the Efficient Frontier

In this model, the deviation of actual returns from the expected returns is taken as risk for investment. However, most often, investors only consider downward deviation as risk, which is against the fundamental assumption of the efficient frontier and makes the model not entirely realistic. Furthermore, the transaction costs which include taxes further complicate the model as they reduce the expected returns of investors[12].

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Appendix

A Derivation of the One Factor Model

To minimise the risk of the investment, we need to minimise the sum of the squared error (let the function be f), that is,

$$f = \sum_{i=1}^{n} [S(i) - (\beta_0 + \beta_1 M(i))]^2 \tag{2}$$

Applying partial differentiation to (1) with respect to $\beta_0$andβ_1 , we obtain

$$\frac{\partial f}{\partial \beta_0} = -2 \sum_{i=1}^{n} [S(i) - (\beta_0 + \beta_1 M(i))] \text{ and} \tag{3}$$

$$\frac{\partial f}{\partial \beta_1} = -2 \sum_{i=1}^{n} [S(i) - (\beta_0 + \beta_1 M(i))] M(i) \tag{4}$$

Equating (2) to zero, we obtain

$$-2 \sum_{i=1}^{n} [S(i) - (\beta_0 + \beta_1 M(i))] = 0$$

$$\sum_{i=1}^{n} S(i) - n\beta_0 - \beta_1 \sum_{i=1}^{n} M(i) = 0$$

$$n\bar{S} - n\beta_0 - n\beta_1 \bar{M} = 0$$

By simple algebraic manipulation, we express $beta_0$ as the subject of other variables, obtaining

$$\beta_0 = \bar{S} - \beta_1 \bar{M} \tag{5}$$

Equating (3) to zero, we obtain

$$-2 \sum_{i=1}^{n} [S(i) - [\beta_0 + \beta_1 M(i)]] M(i) = 0$$

$$\sum_{i=1}^{n} S(i)M(i) - \beta_0 \sum_{i=1}^{n} M(i) - \beta_1 \sum_{i=1}^{n} [M(i)]^2 = 0$$

Substituting (4) into the above equation, we obtain

$$\sum_{i=1}^{n} S(i)M(i) - (\bar{S} - \beta_1 \bar{M}) \sum_{i=1}^{n} M(i) - \beta_1 \sum_{i=1}^{n} [M(i)]^2 = 0$$

$$\sum_{i=1}^{n} S(i)M(i) - \bar{S} \sum_{i=1}^{n} M(i) = \beta_1 [\sum_{i=1}^{n} [M(i)]^2 - \bar{M} \sum_{i=1}^{n} M(i)]$$

$$\beta_1 = \frac{\sum_{i=1}^{n} S(i)M(i) - \bar{S} \sum_{i=1}^{n} M(i)}{\sum_{i=1}^{n} [M(i)]^2 - \bar{M} \sum_{i=1}^{n} M(i)}$$

$$\beta_1 = \frac{\sum\limits_{i=1}^{n} M(i)[S(i) - \bar{S}]}{\sum\limits_{i=1}^{n} M(i)[M(i) - \bar{M}]} \tag{6}$$

Since

$$\begin{aligned} \sum_{i=1}^{n} \bar{M}[S(i) - \bar{S}] &= \bar{M} \sum_{i=1}^{n} S(i) - \sum_{i=1}^{n} \bar{x}\bar{y} \\ &= n\bar{M}\bar{S} - n\bar{M}\bar{S} \\ &= 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \bar{M}[M(i) - \bar{M}] &= \bar{M} \sum_{i=1}^{n} S(i) - n(\bar{M})^2 \\ &= n\bar{M}\bar{M} - n(\bar{M})^2 \\ &= 0, \end{aligned} \tag{8}$$

by substituting (6) and (7) into (5), we obtain

$$\beta_1 = \frac{\sum\limits_{i=1}^{n} M(i)[S(i) - \bar{S}] - \sum\limits_{i=1}^{n} \bar{M}[S(i) - \bar{S}]}{\sum\limits_{i=1}^{n} M(i)[M(i) - \bar{M}] - \sum\limits_{i=1}^{n} \bar{M}[M(i) - \bar{M}]}$$

$$\beta_1 = \frac{\sum_{i=1}^{n} [M(i) - \bar{M}][S(i) - \bar{S}]}{\sum_{i=1}^{n} [M(i) - \bar{M}]^2}$$

B Solving the Markowitz Problem

The Markowitz model finds the minimum variance portfolio with a given mean. To find the weights for a portfolio of 3 stocks with minimum variance that has a fixed expected return $\overline{r}$:

$$\text{minimize } \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\omega_i\omega_j\sigma_{ij}$$

$$\text{subject to } \sum_{i=1}^{3}\omega_i\overline{r_i} = \overline{r}, \tag{9}$$

$$\sum_{i=1}^{3}\omega_i = 1. \tag{10}$$

Therefore, we obtain the Lagrangian function, L

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j\sigma_{ij} - \sum_{i=1}^{n}\omega_i\overline{r_i} - \sum_{i=1}^{n}\omega_i \tag{11}$$

Applying partial differentiation on (10) with respect to ω_i for $i = 1, 2, 3..., n$ and equating the first derivative to 0, we obtain

$$\sum_{j=1}^{3} w_j\sigma_{ij} - \lambda r_i - \mu = 0 \tag{12}$$

$$\begin{aligned}
i = 1: \quad & \omega_1\sigma_{11} + \omega_2\sigma_{12} + \omega_3\sigma_{13} - \lambda r_1 - \mu = 0\\
i = 2: \quad & \omega_1\sigma_{21} + \omega_2\sigma_{22} + \omega_3\sigma_{23} - \lambda r_2 - \mu = 0\\
i = 3: \quad & \omega_1\sigma_{31} + \omega_2\sigma_{32} + \omega_3\sigma_{33} - \lambda r_3 - \mu = 0
\end{aligned}$$

The covariance matrix **C** is given by $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$.

The matrix of weights, ω is given by $\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$.

r , the rates of return of the respective stocks is given by $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$.

1 , is a column matrix $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

We thus have

$$\mathbf{C}\omega = \lambda\mathbf{r} + \mu\mathbf{1}$$

$$\omega = \mathbf{C}^{-1}(\lambda\mathbf{r} + \mu\mathbf{1}) \tag{13}$$

From (8):

$$\omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3 = \overline{r}$$

$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \mathbf{r} = \overline{r}$$
$$\omega^T \mathbf{r} = \overline{r} \text{ or } \mathbf{r}^T \omega = \overline{r} \tag{14}$$

Substituting (12) into (13):

$$\mathbf{r}^T \mathbf{C}^{-1}(\lambda \mathbf{r} + \mu \mathbf{1}) = \overline{r}$$
$$(\mathbf{r}^T \mathbf{C}^{-1} \mathbf{r})\lambda + (\mathbf{r}^T \mathbf{C}^{-1} \mathbf{1})\mu = \overline{r} \tag{15}$$

From (9):

$$\omega_1 + \omega_2 + \omega_3 = 1$$
$$\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \mathbf{1} = 1$$
$$\omega^T \mathbf{1} = 1 \text{ or } \mathbf{1}^T \omega = 1 \tag{16}$$

Substituting (12) into (15)

$$\mathbf{1}^T \mathbf{C}^{-1}(\lambda \mathbf{r} + \mu \mathbf{1}) = 1$$
$$(\mathbf{1}^T \mathbf{C}^{-1} \mathbf{r})\lambda + (\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1})\mu = 1 \tag{17}$$

Let $(\mathbf{r}^T \mathbf{C}^{-1} \mathbf{r}) = a$,
$((\mathbf{r}^T \mathbf{C}^{-1} \mathbf{1}) = (\mathbf{1}^T \mathbf{C}^{-1} \mathbf{r}) = b$,
and $(\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}) = c$.

From (14):

$$a\lambda + b\mu = \overline{r}$$

From (16):

$$b\lambda + c\mu = 1$$

Through algebraic manipulation,

$$\lambda = \frac{c\overline{r} - b}{ac - b^2} \tag{18}$$

and

$$\mu = \frac{a - b\overline{r}}{ac - b^2} \tag{19}$$

Substituting (17) and (18) into (11)

$$\omega = \mathbf{C}^{-1} \left(\frac{c\overline{r} - b}{ac - b^2} \mathbf{r} + \frac{a - b\overline{r}}{ac - b^2} \mathbf{1} \right)$$

$$\omega = \frac{b\,(c\overline{r} - b)}{ac - b^2} \cdot \frac{\mathbf{C}^{-1}\mathbf{r}}{b} + \frac{c\,(a - b\overline{r})}{ac - b^2} \cdot \frac{\mathbf{C}^{-1}\mathbf{1}}{c}$$

Since $\frac{b(c\overline{r}-b)}{ac-b^2} + \frac{c(a-b\overline{r})}{ac-b^2} = 1$, we let

$$\omega = \alpha \cdot \frac{\mathbf{C}^{-1}\mathbf{r}}{b} + (1 - \alpha) \cdot \frac{\mathbf{C}^{-1}\mathbf{1}}{c}$$

Since

$$\frac{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{r}}{b} = \frac{b}{b} = 1 = \mathbf{1}^T \omega^1,$$

where ω^1 is the weight of fund 1, we let

$$\frac{\mathbf{C}^{-1}\mathbf{r}}{b}$$

be the weight of fund 1.

Since

$$\frac{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}{c} = \frac{c}{c} = 1 = \mathbf{1}^T\omega^2,$$

where ω^2 is the weight of fund 2, we let

$$\frac{\mathbf{C}^{-1}\mathbf{1}}{c}$$

be the weight of fund 2.