CMU 21-484: Graph Theory, Spring 2022
Notes and Exam Review

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1: Basic Concepts

Definition 1.1 (Paths)**.** Two paths are:

- Vertex-disjoint if they have no vertices in common
- Edge-disjoint if they have no edges in common
- Independent if they have no vertices in common, except at the endpoints

If paths are just called disjoint without specification, that would usually mean vertex-disjoint.

Theorem 1.2 (Tree)**.** *The following are equivalent:*

- *G is a tree*
- *G is an edge-minimally connected graph*
- *G is an edge-maximally acyclic graph*
- *For all x, y in G, there is a unique path from x to y*

Theorem 1.3 (Matrix-tree, no proof)**.** *For any multigraph G, and any vertex $v \in G$,*

$$\tau(G) = \det(L_G[v]),$$

where τ_G is the number of spanning trees of G, and $A[i]$ is the matrix A with row and column i deleted.

Useful properties of determinants:

- Adding one row to another does not change determinant
- Determinant of upper(or lower) triangular matrix is the product of the diagonal elements
- Swapping two rows causes determinant's sign to change
- Multiplying a row by scalar k causes determinant to also be multiplied by k

Definition 1.4 (Bipartite graph)**.** G is bipartite if $V = A \dot\cup B$ s.t. $\forall e \in E$, $|e \cap A| = |e \cap B| = 1$.

Theorem 1.5 (Bipartite characterization in odd cycles)**.** *G is bipartite $\iff$ G has no odd cycles.*

2: Matchings

Definition 2.1 (Alternating paths)**.** An alternating path with respect to M begins at unmatched vertex a_0, run alternate edges in and out of M. If it ends at unmatched vertex of B, it is augmenting.

Definition 2.2 (Augmenting Paths)**.** Let $G = (V, E)$ be a graph and $M \subseteq E$ be a matching. A path $P = x_0, c \ldots, x_k$ is called augmenting if:

- $x_{i-1}x_i \in E \setminus M$ for every odd $i \in [k]$;
- $x_{i-1}x_i \in M$ for every even $i \in [k]$;
- Neither x_0 nor x_k is incident to an edge of M.

Theorem 2.3 (Hall's)**.** *Bipartite G on $A\dot{\cup}B$ has a complete matching of A iff Hall's condition holds, i.e*

$$\forall S \subseteq A, |\Gamma(S)| \geq |S|$$

Theorem 2.4 (Konig's, not in scope)**.** *If G is bipartite, then $\mu(G) = h(G)$, where $\mu(G)$ is the size of the maximum matching and $h(G)$ is the size of the minimum hitting set.*

A hitting set is a set $x \subset EV$ s.t. $e \cap X \neq \emptyset\ \forall e \in E$.

Theorem 2.5 (Tutte's, no proof)**.** *G has a perfect matching iff $q(G \setminus S) \leq |S|\ \forall S \subseteq V(G)$.*

$q(X)$ is the number of odd components in $G[X]$, the subgraph induced by X.

3: Connectivity

Definition 3.1 (k-vertex connected)**.** G is k-vertex connected if $G \setminus X$ is connected for all $|X| < k$ and $|V(G)| \geq k + 1$.

So a k-clique is not k-connected, for instance. Denote

$$\kappa(G) = \max\{k| \text{ is } k\text{-vertex connected}\}.$$

Definition 3.2 (l-edge connected)**.** G is l-edge connected if $G \setminus F$ is connected for all $F \subseteq E(G)$, $|F| < l$.

Denote

$$\lambda(G) = \max\{l| \text{ is } l\text{-edge connected}\}.$$

Theorem 3.3 ($\kappa(G) \leq \lambda(G)$)**.** *$\kappa(G) \leq \lambda(G)\ \forall$ graphs G.*

Proof. Consider a smallest $F \subseteq E(G)$ such that $G \setminus F$ is disconnected.

Observe, since F is smallest, no edge $e \in F$ lies inside a component C of $G \setminus F$; otherwise, e could be removed from F.

Case 1: G contains a vertex v not incident with any $e \in F$. Let C be the component of $G \setminus F$ containing v. Then the vertices of C incident with an edge in F separate v from $G \setminus C$. Since no edge in F has both ends in C by minimality of F, there are at most F such vertices, giving $\kappa(G) \leq |F|$.

Case 2: Every vertex is incident with some $e \in F$. Consider any vertex $v \in V(G)$, and let C be the component of $G \setminus F$ containing v. Then the neighbours w of v with $vw \notin F$ lie in C and are incident to distinct edges in F (again by the minimality of F), giving $\deg(G) \leq |F|$. If v can be separated in this manner we are done; else since v was any vertex, it means that G was a clique because we have $x \cup \Gamma(v) = V(G)$. □

Theorem 3.4 (Ear Theorem)**.** *A graph is 2-connected iff it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.*

Theorem 3.5 (3-connected)**.** *If G is 3-connected, $\exists e \in E(G)$ s.t. G/e is 3-connected.*

Theorem 3.6 (Menger's, local edge version)**.** *Given $u, v \in G$, the minimum number of edges separating u from v is the maximum number of edge-disjoint paths.*

Theorem 3.7 (Menger's, global edge version)**.** *If G is l-edge connected, every pair of vertices can be joined by l independent paths.*

Theorem 3.8 (Menger's, local vertex version)**.** *If u and v are distinct non-adjacent vertices of G, then the minimum number of vertices $\neq u, v$ separating u from v is equal to the maximum number of independent paths from u to v.*

Theorem 3.9 (Menger's, global vertex version)**.** *If G is k-vertex-connected, then every pair of vertices is joined by k independent paths.*

Theorem 3.10 (Menger's, local vertex set version)**.** *If A and B are (not necessarily disjoint) subsets of $V(G)$, then the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B.*

Theorem 3.11 (Max-flow min-cut)**.** *In every network, the maximum total value of a flow equals the minimum capacity of a cut.*

Theorem 3.12 (Integer flows)**.** *If all capacities are integers, there exists a max flow of integer value.*

Theorem 3.13 (0/1 Flows)**.** *Every 0/1 valued flow can be decomposed into (edge or vertex) disjoint paths and cycles.*

4: Planarity

Theorem 4.1 (Euler's formula)**.** *If G is a connected plane graph with n vertices, m edges, and l faces, then*

$$n - m + l = 2.$$

Proof. By induction on m (n is fixed).

Base case. $m = n - 1$, G is a tree. G has one face, $l = 1$.

Then $n - (n - 1) + 1 = 2$.

Induction Step. $m \geq n$.

G has a cycle. Let e be an edge of this cycle. By JCT, C divides $\mathbb{R}^2$ into two regions, and e is incident with precisely 2 faces.

Thus $G \setminus e$ has one fewer face. $G \setminus e$ has:

- $n' = n$
- $m' = m - 1$
- $l' = l - 1$

By induction,

$$\begin{aligned} & n' - m' + l' = 2 \\ \implies & n - (m - 1) + (l - 1) = 2 \\ \implies & n - m + l = 2. \end{aligned}$$

□

Remark 4.2. It's easy to derive this formula if you forget it.

Corollary 4.3 (Faces and Edges)**.** *Consider a bipartite graph where each vertex on one side represents a face in G, and each vertex on the other side represents an edge in G. By considering their degrees, we obtain*

$3l \leq 2m$, so $l \leq \frac{2}{3}m$.

Corollary 4.4 (Connected plane graph)**.** *In a connected plane graph (if not connected, could just have a bunch of edges), then $2 = n - m + l \leq n - m + \frac{2}{3}m = n - \frac{m}{3}$, so*

$$m \leq 3n - 6.$$

Theorem 4.5 (For 2-connected plane graphs, face boundaries are cycles)**.** *If G is a 2-connected plane graph, then every face boundary is a cycle.*

Proof. We use the ear-theorem.

BC: Trivial.

IS: $G = H \cup P$ where H is a 2-connected plane graph, and P is a H-path.

P lies in some face f' of H. By induction, the boundary of f' is a cycle, as it is the boundary of all other faces. Adding P divides f' into cycles, so all faces are still bounded by cycles. □

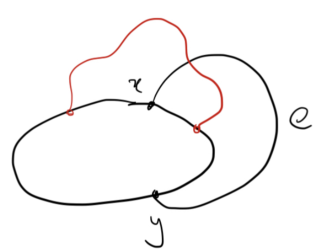
Definition 4.6 (Induced cycle)**.** An induced cycle in G, a cycle in G forming an induced subgraph, is one that has no chords.

Theorem 4.7 (Non-separating induced cycles are faces)**.** *If C is a non-separating induced cycle in a connected planar graph, C is the boundary of a face.*

Proof. By JCT, C divides the plane into two regions. Since C is non-separating, one must contain no vertices. Since C is cordless, it also contains no edges. This region is a face, whose boundary is C. □

Theorem 4.8 (Faces in 3-connected graphs are non-separating induced cycles)**.** *If G is 3-connected and C is the boundary of a face, C is a non-separating induced cycle.*

Proof. Let G be a 3-connected plane graph, and let $C \subseteq G$. Suppose that C bounds a face f; then because G is also 2-connected, face boundaries are cycles, so C is a cycle. If C has a chord $e = xy$, then the components of $C - \{x, y\}$ are linked by a C-path in G, because G is 3-connected. This path and e both run through the outer face of C (not f) but do not intersect, a contradiction.



So C is cordless. It remains to show that C does not separate any two vertices $x, y \in G \setminus C$.

By Menger's theorem, x, y are linked in G by three independent paths. But C can only intersect 2 of them, so since f is 3-connected, x, y are connected. □

Definition 4.9 (Subdivision)**.** A subdivision of X is, informally, any graph obtained from X by "subdividing" some or all of its edges by drawing new vertices on those edges. In other words, we replace some edges of X with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$ or on another new path.

Definition 4.10 (Topological Minor)**.** H is a topological minor of G if there exists J such that:

- J is a subgraph of G,
- J is a subdivision of H.

Definition 4.11 (Minor)**.** H is a minor of G if H can be obtained from G by deletions (both edges and vertices) and contractions.

Theorem 4.12 (Kuratowski's)**.** *G is planar iff G has no K_5 or $K_{3,3}$ topological minor.*

Theorem 4.13 (Wagner's)**.** *G is planar iff G has no K_5 or $K_{3,3}$ minor.*

5: Vertex Coloring

Definition 5.1 (Chromatic Number)**.** The chromatic number $\chi(G)$ of G is defined

$$\chi(G) = \min\{k \mid G \text{ is } k\text{-colorable}\}.$$

Notation:

- $\omega(G)$ is the size of the largest clique in G
- $\alpha(G)$ is the size of the largest independent set in G (a set with no edges among its vertices)
- $\Delta(G)$ is the largest degree in G

Basic bounds:

- $\chi(G) \leq 2 \iff G$ has no odd cycles (i.e bipartite)
- $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

  Lower bound: just to color the clique, require at least $\omega(G)$ colors.

  Upper bound: always can color if more colors available than max. degree
- $\chi(G) \geq \frac{n}{\alpha(G)}$, where $\alpha(G)$ is the size of the largest independent set of G.
- $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.

Definition 5.2 (Coloring Number)**.**

$$\mathrm{col}(G) = \left(\min_{\substack{\text{orderings} \\ v_1,\ldots,v_n}} \max_i \deg^{>}(v_i) \right) + 1,$$

where

$$\deg^{<}(v_i) = |\Gamma(v_i) \cap \{v_1, \ldots, v_{i-1}\}|.$$

Theorem 5.3.

$$\chi(G) \le \operatorname{col}(G)$$

Definition 5.4 (Greedy Algorithm)**.** For some ordering $v_1, v_2, \dots, v_n$ of $V(G)$, at step i, color v_i with the smallest integer not already used at one of its neighbors in $\{v_1, \dots, v_{i-1}\}$.

Theorem 5.5 (5-color Theorem for Planar Graphs)**.** *If G is planar, $\chi(G) \le 5$.*

Proof. By induction. Fix a drawing of G.

By Euler's formula, G, has a vertex v of deg ≤ 5. (since $m \le 3n-6$, AFSOC if all vertices has degree ≥ 6, then we have $3n \le 3n-6$, a contradiction).

By induction, $G \setminus v$ is 5-colorable. Fix some 5-coloring of $G \setminus v$ C. We aim to extend C to v.

If $\deg(v) \le 4$, v sees ≤ 4 colors in $c(\Gamma(v))$, and we can extend.

So we assume that $\deg(v) = 5$, and moreover, the 5 neighbors of v has distinct colors.

Label the neighbors $v_1, \dots, v_5$ in cyclic order. WLOG, assume $c(v_i) = i$.

Rest is by case analysis - main idea is we have paths with alternating colors from 1 to 3 and 2 to 4. The paths must intersect by JCT. So can recolor, done. □

Theorem 5.6 (Brook's, statement only)**.** *If G is connected and not complete or an odd cycle, $\chi(G) \le \Delta(G)$.*

Definition 5.7 (k-constructible)**.** G is k-constructible if G can be constructed, starting from K_k's (i.e clique of size k), via the following operations:

- Adding edges
- Identifying two non-adjacent vertices x, y, adding an edge between them, and contracting both vertices
- Hajos sum: Given 2 k-constructible graphs H_1, H_2 with edges x_1y_1 and x_2y_2 respectively, remove edges x_1y_1 and x_2y_2, identify x_1 and x_2 (i.e adding an edge between them and contracting them), and add edge from y_1 to y_2.

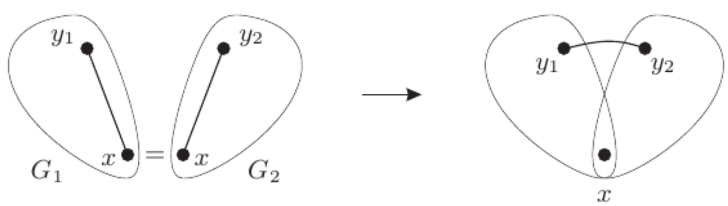


Fig. 5.2.2. The *Hajós construction* (iii)

Theorem 5.8 (Hajos', statement only)**.** *Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \ge k$ if and only if G has a k-constructible subgraph.*

6: Edge Coloring

Definition 6.1 (Edge Coloring)**.** A proper k edge-coloring of G is a function $c : E(G) \to [k]$ such that $e_1 \cap e_2 \ne \emptyset \implies c(e_1) \ne c(e_2)$.

Then the edge coloring is defined

$$\chi_e(G) = \min\{k \mid G \text{ has a proper coloring.}\}$$

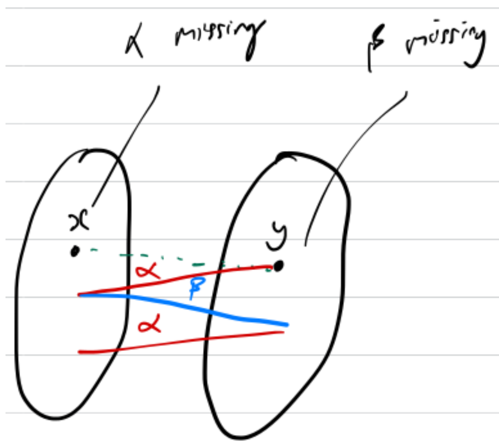
Theorem 6.2 (Konig's)**.** *Every bipartite graph G satisfies $\chi_e(G) = \Delta(G)$.*

N.B: sometimes $\chi_e(G)$ is written $\chi'(G)$.

Proof. By induction on edges. Given graph G, assume theorem holds for graphs with fewer edges.

Choose $\sim y \in G$. By induction, $G \setminus xy$ can be Δ-colored.

If we can't extend the coloring to xy, some color α is missing at x (but not at y), and some color β is missing at y (but not at x) (because if $\alpha = \beta$, we can extend the coloring and are done).



Consider the longest α/β walk from y; this is a path. If it does not end at x, we can flip colors on the path (since each vertex along the path after y must have both α, β colors, it does not matter if we flip the order). This means that we can extend the coloring to e.

Otherwise, the path ends at x. But then this is impossible, since we will enter A with an α-edge, and α is missing at x. □

Theorem 6.3 (Vizing's, statement only)**.**

$$\chi_e(G) \leq \Delta(G) + 1$$

.

Proof. By induction on the number of edges. □

7: List Coloring

Definition 7.1 (List Coloring)**.** Given G and lists L_v, a coloring from the lists is an assignment $c : V \to \bigcup L_v$ such that for $c(v) \in L_v$,

$$u \sim v \implies c(u) \neq c(v).$$

G is k-list colorable if for any list L_v satisfying $|L_v| = k \,\forall V \in G$ can be colored from the lists. The list chromatic number $\chi_l(G)$ is

$$\chi_l(G) = \min\{k \,|\, G \text{ is } k\text{-list colorable}.$$

Basic Properties

- k-list colorable $\implies$ k-colorable, i.e $\chi(G) \leq \chi_l(G)$.

- Other direction DOES NOT hold, can make $\chi_l(G)$ much larger than $\chi(G)$.

  I.e from HW, for any k, there are graphs G with $\chi(G) = 2$ and $\chi_l(G) = k$.

Theorem 7.2 (Planar List Coloring)**.** *If G is planar, G is 5-list-colorable.*

Proof. Strengthen the IH. We show that if G is a plane graph, all of whose inner faces are triangles, whose outer boundary is a cycle comprising vertices $v_1, v_2, \ldots, v_k$, where

- $L_{v_1} = \{1\}$,
- $L_{v_2} = \{2\}$,
- $|L_{v_i}| \geq 3 \, \forall i = 3, \ldots, k$,
- $|L_v| \geq 5$ for $v \neq v_i \, \forall i$,

then G is 5-list-colorable.
(N.B: to meet the triangle requirement, just add edges until it holds.)
We prove this by induction.

1. *C has a chord.*

   It divides G into G_1, G_2, intersecting just at chord. WLOG, $v_1, v_2 \in G_1$.

   Color G_1 by induction, then G_2, with endpoints of chord as new v_1, v_2. This produces a valid coloring for G.

2. *C has no chord.*

   Consider neighbors $u_1, u_2, \ldots, u_l$ of v_k (other than $v_1, \ldots, v_{k-1}$).

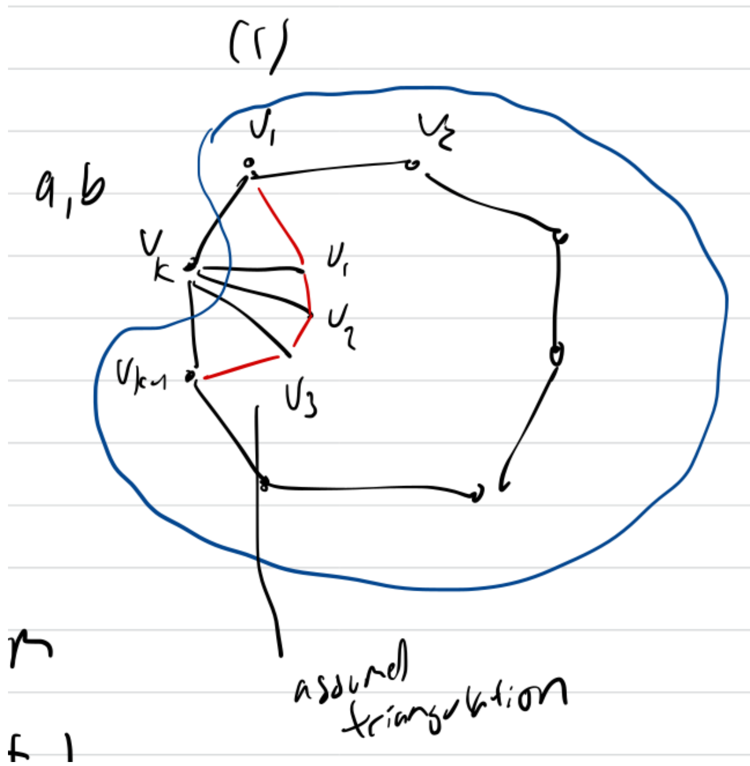
   Choose j, l from L_{v_k} s.t. $a, b \neq 1$ (recall 1 already used to color v_1), this is possible since $|L_{v_k}| \geq 3$.

   For each u_i, define $L'_{u_i} = L_{u_i} \setminus \{a, b\}$.

   So $|L'_{u_i}| \geq 3 \, \forall i$.

   Apply induction on $G \setminus v_k$, which now has all the u_i's on the outer face by triangulation.

   Since a, b not used at v_1 or any u_i, at most one is used at v_{k-1}, so we can extend to v_k.



□

8: Perfect Graphs

Definition 8.1 (Perfect Graphs)**.** A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G.

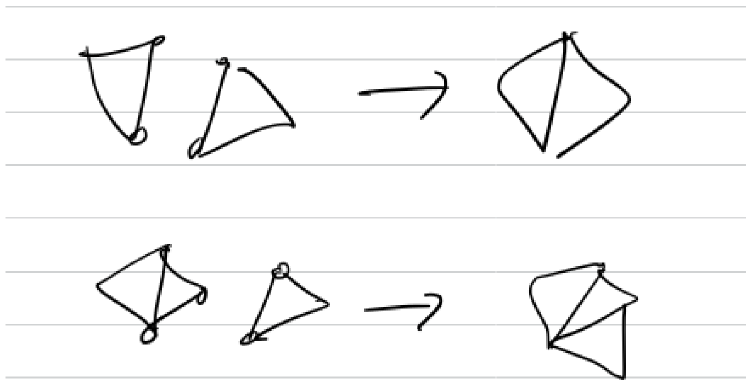
Note that we always have $\omega(G) \leq \chi(G)$.

Definition 8.2 (Chordal Graphs)**.** A graph is chordal if every cycle of length ≥ 4 has a chord. Equivalently, there are no induced cycles of length ≥ 4 vertices.

Theorem 8.3 (Chordal graphs are constructed from cliques by pasting along cliques)**.** *G is chordal $\iff$ either*

1. *G is a clique, or*
2. *$G = G_1 \cup G_2, G_1 \cap G_2$ is a clique, G_1, G_2 are chordal.*

In other words, chordal graphs can be constructed recursively from cliques, by pasting along cliques.



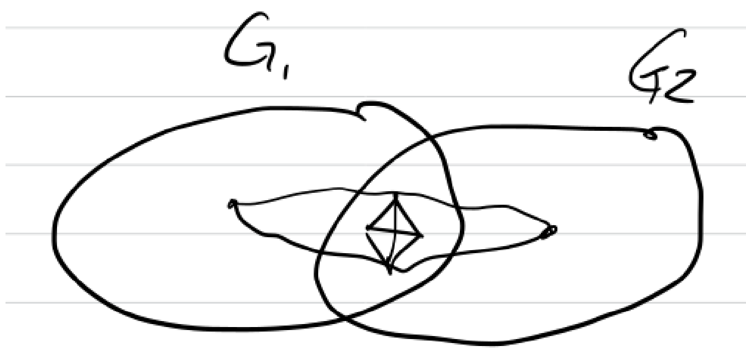
Proof. ($\implies$)

If G is a clique, then it is chordal, since there are no induced cycles of length 4.

Suppose $G = G_1 \cup G_2, G_1 \cap G_2 = K$ is a clique, G_1, G_2 chordal.

Consider any cycle C. If $C \subseteq G_i$ for some i, C has a chord since G_i chordal.

Otherwise, C contains vertices in both $G_1 \setminus G_2$, and $G_2 \setminus G_1$. So the cycle must have non-consecutive vertices in K. Since K is a clique, it means that C must have a chord when it goes through K in this way.



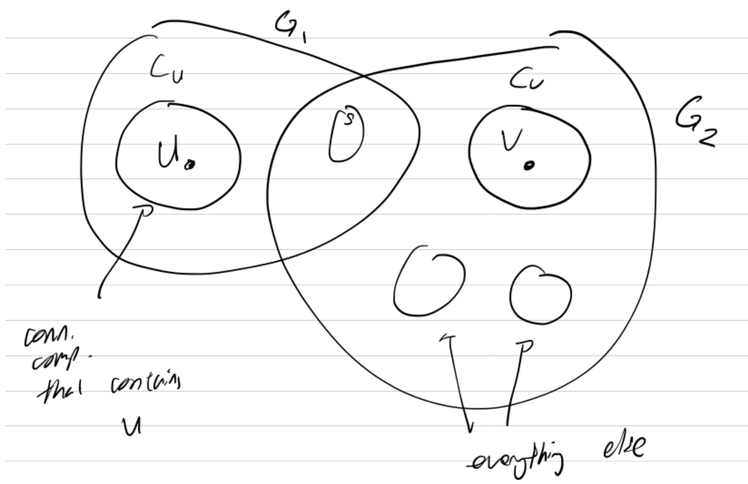
($\impliedby$)

Suppose that G is chordal and not a clique.

Since G not a clique, $\exists u, v$ where $u \not\sim v$.

Consider a minimum separating set S for u, v. We claim that S is a clique.

Since S is a minimum, every vertex in S has a neighbor in C_u and C_v (because if not, that vertex can be removed and we still have a separating set, since that vertex disconnects one of the components anyway).



Suppose $x, y \in S$, $x \not\sim y$.

Since x, y both have neighbors in both C_u and C_v, there's a shortest path P from x to y through C_u, and Q from y to x through C_v. If $x \not\sim y$, PQ is an induced cycle of length ≥ 4 (which violates the fact that G is chordal).

Note that G_i must still be chordal since induced subgraphs of chordal graphs are chordal. So we have shown the desired conditions. □

Definition 8.4 (Perfect Elimination Ordering (p.e.o)). A perfect elimination ordering (p.e.o) of G is an ordering $v_1, v_2, \ldots, v_n$ s.t. $\forall i$,

$$\Gamma(v_i) \cap \{v_1, \ldots, v_{i-1}\}$$

is a clique.

Remark 8.5. Given a p.e.o, $\omega(G)$=max back degree +1. Coloring greedily in this order uses $\leq$ max back deg + 1 colors.

Definition 8.6 (PEO Graphs). A PEO graph is a graph for which a perfect elimination ordering exists.

Theorem 8.7 (PEO Graphs are Perfect). *If G is a PEO graph, $\chi(G) = \omega(G)$.*

Proof. Because induced subgraphs of PEO graphs are also PEO graphs (has the same p.e.o minus deleted vertices), and using greedy coloring on p.e.o requires $\omega(G)$ colors. □

Definition 8.8 (Simplicial vertex). A simplicial vertex is one whose neighbors form a clique.

Theorem 8.9 (PEO graphs $\iff$ chordal). *G is a PEO graph $\iff$ G is chordal.*

Proof. ($\implies$)

Suppose we have a PEO graph. Any induced subgraph is also a PEO graph by using the same ordering and dropping the vertices which no longer exists. However, a cycle of length ≥ 4 is not PEO, since there are no simplicial vertices. So G must be chordal.

($\impliedby$)

Suppose that G is chordal. Then it contains a simplicial vertex (since if G is chordal and not a clique, G has two non-adjacent simplicial vertices). Then if we delete this vertex, we have a smaller chordal graph. So by induction, the smaller graph has a PEO, so we are done. □

Remark 8.10 (Constructing a peo from a PEO graph)**.** We can use a greedy algorithm to start from the last vertex in the ordering, and keep choosing a simplicial vertex, working backwards in the ordering.

Definition 8.11 (G_x)**.** Given a graph G and $x \in V(G)$, define G_x to be the graph formed by adding a new vertex x' to G, adding edges $x'y$ whenever $xy \in E(G)$, as well as the edge xx'.

Theorem 8.12 (Expanding vertex preserves property of being a perfect graph)**.** *If G is perfect, then G_x is perfect.*

Proof. We need to show that for all induced $H \subseteq G_X$, $\chi(H) \leq \omega(H)$.

Either H is an induced subgraph of G, or it is obtained by expanding an induced subgraph of G.

So it suffices to show that $\chi(G_x) \leq \omega(G_x)$. Let $w = \omega(G)$.

Case 1: $\omega(G_x) = \omega + 1$. Then we can ω-color G, and use the new color at x'.

Case 2: $\omega(G_x) = \omega$. Define $\mathcal{K}$ to be the set of ω-cliques in G (i.e max-sized cliques in G). We know that $x \notin k_\omega$ for any $k_\omega \in \mathcal{K}$, since otherwise this would have increased the maximum clique size in G_x.

Consider a ω-coloring C of G. Consider the color class C_x of x with respect to C. C_x intersects every $k_\omega \in \mathcal{K}$, since $\chi(G) = \omega(G)$. So deleting C_x decreases the clique number, and so does deleting $C_x \setminus x$ (since x not involved in any max cliques). We know that $C_x \setminus x$ intersects every $k_\omega \in \mathcal{K}$. So delete $C_x \setminus x$ from G to produce G', where we know $\omega(G') \leq \omega - 1$. Since G' perfect, $\chi(G') \leq \omega - 1$. So, consider an $(\omega - 1)$ coloring C' of G'. We need to extend the coloring to $\{x'\} \cup (C_x \setminus x)$. But this is an independent set, so one color suffices. So we have colored G_x with ω colors.

Thus G_x is perfect. □

Theorem 8.13 (Weak perfect graph theorem, statement only)**.** *G is perfect if and only if $\overline{G}$ is perfect.*

9: Extremal Graph Theory

Definition 9.1 ($ex(n, H)$)**.** Denote $\mathrm{ex}(n, H)$ to be the maximum number of edges in any graph on n vertices which does not contain H as a subgraph.

Definition 9.2 (Complete r-partite graphs)**.** The complete r-partite graph is a graph on vertex set $V = V_1 \sqcup V_2 \cdots \sqcup V_r$, where $x \sim y \iff x \in V_i, y \in V_j$ for $i \neq j$.

Definition 9.3 (Turán Graphs)**.** The complete r-partite graph with classes differing by ≤ 1 in size are called Turán graphs, and denoted $T^r(n)$.

Write $t_r(n)$ to denote the number of edges in $T^r(n)$.

We have

$$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1}.$$

Theorem 9.4 (Turán's Theorem)**.**

$$ex(n, K_r) = t_{r-1}(n).$$

Moreover, if G has $ex(n, K_r)$ edges, in fact $G \cong T^{r-1}(n)$.

Proof. By induction. G is edge-maximal without K_r, so G has a clique K on $r-1$ vertices. $\square$

Theorem 9.5 ($\text{ex}(n, C_4) \leq Cn^{3/2}$)**.**

$$ex(n, C_4) \leq Cn^{3/2}.$$

Proof. Key idea: count the number of 2-paths centered at all vertices v, know that this is upper bounded by $\binom{n}{2}$. Use Jensen's inequality, then assume $n < m$ to get result. $\square$

Definition 9.6 ($K_r(t)$)**.** $K_r(t)$ is the complete r-partite graph in which every partition class contains exactly t vertices.

Theorem 9.7 (Erdős-Stone theorem, statement only)**.** $\forall \varepsilon > 0, \forall t$, *there's an* N *such that for all* $n > N$, *if* G *has* $\geq \frac{r-1}{r}\frac{n^2}{2} + \varepsilon n^2$ *edges and* n *vertices, then* G *contains a* $K_{r+1}(t)$.

Remark 9.8. The textbook formulates it slightly differently, that if it has at least

$$t_r(n) + \varepsilon n^2$$

edges, then it contains $K_{r+1}(t)$ as a subgraph.

Corollary 9.9 (Important Corollary of Erdős-Stone Theorem)**.**

$$ex(n, H) \sim \frac{\chi(H) - 2}{\chi(H) - 1}\binom{n}{2}$$

for $\chi(H) \geq 3$.

Proof. For any graph H, there exists an integer r (the independence number $\alpha(H)$ will work) such that H is a subgraph of $K_{\chi(H)}(r)$. Color H, add vertices to the color classes so that they all have the same size r, then join any two different colored vertices. Then apply Erdos-Stone on $K_{\chi(H)}(r)$: we get that if we have at least

$$\frac{\chi(H) - 2}{\chi(H) - 1}\frac{n^2}{2} + \varepsilon n^2$$

edges, then there will be a

$$K_{\chi(H)}(r)$$

, which contains

$$H$$

as a subgraph. This gives an upper bound on the number of edges.

For the lower bound, note that the Turan graph $T_{r-1}(n)$ is $r-1$ colorable, and therefore it cannot contain H.

$\square$

10: Hamilton Cycles

Definition 10.1 (Hamilton Cycle)**.** A cycle in G on n vertices where $n = |V(G)|$.

Theorem 10.2 (Dirac)**.** *If* $\deg(v) \geq \frac{n}{2}$ $\forall v \in G$, *G is Hamiltonian, i.e has a Hamilton cycle..*

Proof. Proof ideas: pigeonhole principle, considering $I = \{i|v_i \sim x\}\}, J = \{i|v_{i-1} \sim y\}\}$. Start by assuming longest path, find a cycle, then realize we could find a longer path, thus contradiction. $\square$

Theorem 10.3 (Necessary condition for Hamilton cycle)**.** *If G has a set S, $|S| = k$,*

- *$G \setminus S$ has $\geq k+1$ components $\implies$ G has no Hamilton cycle*
- *$G \setminus S$ has $\geq k+2$ components $\implies$ G has no Hamilton path*

Proof. Pigeonhole on visiting vertices in S between connected components. □

11: Forbidden Minors

Theorem 11.1 (Forbidden minor characterization for K_4 minor, statement only)**.** *A graph with at least three vertices is edge-maximal without a K_4 minor if and only if it can be constructed recursively from triangles by pasting along K_2's.*

Theorem 11.2 (Forbidden minor characterization of K_5 minor, statement only)**.** *Let G be an edge-maximal graph without a K_5 minor. If $|G| \geq 4$ then G can be constructed recursively, by pasting along triangles and K_2's, from plane triangulations and copies of the graph W.*

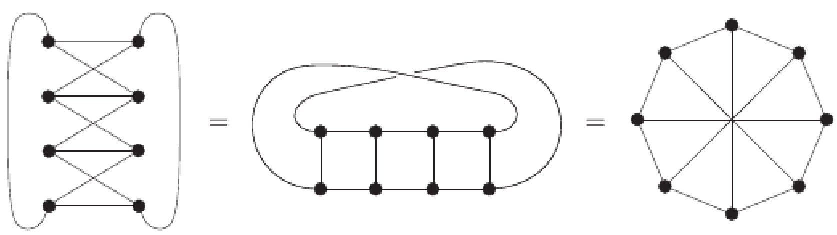


Fig. 7.3.1. Three representations of the *Wagner graph W*

Conjecture 11.3 (Hadwiger)**.** *If G has no K_r minor, then $\chi(G) < r$.*

12: Random graphs

Definition 12.1 (Erdős-Renyi random graph model)**.** $G_{n,p}$ is the graph-valued random variable drawn from the probability space of graphs on n vertices, where adjacencies are chosen independently, each with probability p.

Theorem 12.2 (Expected size of cliques in $G_{n,\frac{1}{2}}$)**.**

$$\begin{aligned}
\mathbf{Pr}[\omega(G_{n,\frac{1}{2}}) \geq k] &\leq \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} && \text{(Union bound)} \\
&\leq n^k \left(\frac{1}{2}\right)^{k(k-1)/2} \\
&= \left(n\left(\frac{1}{2}\right)^{(k-1)/2}\right)^k
\end{aligned}$$

Then only when $n < 2^{(k-1)/2}$, the probability is less than 1.

Theorem 12.3 (Expected size of independent sets in $G_{n,p}$).

$$\begin{aligned}
\mathbf{Pr}[\alpha(G_{n,p}) > r] &\leq \binom{n}{r}(1-p)^{\binom{r}{2}} \\
&= \binom{n}{r}(1-p)^{r(r-1)/2} \\
&\leq n^r(1-p)^{r(r-1)/2} \\
&= \left(n(1-p)^{(r-1)/2}\right)^r \\
&\leq \left(n(e^{-p})^{(r-1)/2}\right)^r && (1-p \leq e^{-p}) \\
&\to 0 \textit{ if } n < e^{p(r-1)/2}
\end{aligned}$$

When $n < e^{p(r-1)/2}$, $p > \frac{2\ln n}{r-1}$. *Using* $r = \frac{n}{k}$, *i.e* $\mathbf{Pr}[\alpha(G_{n,p}) > \frac{n}{k}]$, *we get* $p > \frac{3\ln nk}{n}$. *So the takeaway is that if you want* $\alpha(G) \leq \frac{n}{k}$, *set* $p > \frac{3\ln nk}{n}$.

Theorem 12.4 (Chromatic number/girth theorem, statement only). $\forall k$, $\exists$ *graphs* G *with* $\chi(G) = k$ *and girth(G)*$\geq k$.

Definition 12.5 (Property $P_{i,j}$ of a graph). A graph has property $P_{i,j}$ if $\forall U, w$, $U \cap w = \emptyset$, $|U| = i, |w| = j$, $\exists v \notin U \cup w$, $v \sim u \, \forall u \in U$, $v \not\sim w \, \forall w \in W$.

Theorem 12.6 (Rado Graph). *Let* R *be a graph whose vertex set is in* $\mathbb{N}$ *(the Rado graph), and adjacencies chosen uniformly randomly with probability* p. *Then*

$$\mathbf{Pr}[R \textit{ has } P_{i,j} \, \forall i,j] = 1.$$

Theorem 12.7 (Erdős and Renyi). *There exists a unique countable graph* R *(up to isomorphism) with the property that it satisfies all* $P_{i,j}$*'s.*

13: Expander Graphs

Definition 13.1 (Bipartite Expanders). A (d,β) expander graph is a bipartite graph on $L \sqcup R$ where all vertices in L have degree d and $\forall S \subseteq L$ such that $|S| \leq \frac{n}{d}$, $|\Gamma(S)| \geq \beta d|S|$.

Remark 13.2. Sometimes instead of $|\Gamma(S)| \geq \beta d|S|$, $|\Gamma(S)| \geq \beta|S|$ is also used.

Definition 13.3 (Expander Graphs). Similar definition to a bipartite expander, except the condition is imposed on all subsets, not just subsets on one side of a bipartite graph.

Remark 13.4 (Existence of Bipartite Expanders). We show that (d,β) expanders exist for $\beta = \frac{1}{4}$.

Proof idea: for each vertex in L, choose a random neighbor in R independently. WThen we show that there are likely to be no bad pairs S, T where $\Gamma(S) \subseteq T$, $|S| \leq \frac{n}{d}$, $|T| \leq \frac{d}{4}|S|$ (i.e does not have the expansion property). This can be formulated as

$$\sum_{s=1}^{n/d} \binom{n}{s}\binom{n}{\frac{sd}{4}}\left(\frac{sd}{4n}\right)^{sd},$$

where the summation is performing a union over all sizes of set S, $\binom{n}{s}$ is choosing our S, $\binom{n}{\frac{sd}{4}}$ is choosing T, and the last term is the probability that all neighbors of S are in T.

Perform the computation, you will need to use the inequality $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ somewhere. Then eventually you will arrive that the probability is < 1, which implies that there is such a graph where no bad pair exists, i.e our bipartite expander exists.

Theorem 13.5. *Largest eigenvalue of a d-regular graph is d The largest eigenvalue of a d-regular graph is d*

Proposition 13.6 (Multiplicity of largest eigenvalue)**.** *The multiplicity of the largest eigenvalue d of a d-regular graph is the number of connected components in G.*

Theorem 13.7 (Lower bound of expansion parameter, statement only)**.** *Given any $|S| \leq \frac{n}{2}$, we have that*

$$\frac{e(S, \overline{S})}{|S|} \geq \lambda_1 - \lambda_2.$$

Remark 13.8 (Spirit of expanders)**.** A big spectral gap, i.e $\lambda_1 - \lambda_2$ implies good expansion.

Remark 13.9 (Eigenvalues of the complete bipartite graph $K_{s,t}$)**.** It has eigenvalue 0 with multiplicity $n-2$, and $\pm\sqrt{st}$.

14: Graph minors and well-quasi-orders

Definition 14.1 (Quasi-order)**.** A quasi order is:

- Reflextive, $x \leq x$
- Transitive, $x \leq y, y \leq z \implies x \leq z$
- Antisymmetric, $x \leq y \wedge y \leq x \implies x \leq y$

Definition 14.2 (Well-quasi-order)**.** A quasi-order such that in any infinite sequence $x_0, x_1, \ldots$, there exists a "good pair" : an $i < j$ such that $x_i \leq x_j$.

Theorem 14.3 (Sufficient and necessary condition for w.q.o)**.** *A q.o $\leq$ is a w.q.o on $X \iff$ there are no infinite decreasing seqeunce or infinite antichains.*

Definition 14.4 (Minor-closed properties)**.** A property $\mathcal{P}$ of graphs is minor-closed if every minor of a graph with $\mathcal{P}$ has $\mathcal{P}$.

So G has $\mathcal{P}$ if and only if it has no minor without $\mathcal{P}$.

Definition 14.5 (Forbidden Minors)**.** Define $\mathsf{Forb}(\mathcal{P}) = \{H \mid H \text{ does not have } \mathcal{P}\}$.

Define $\overline{\mathsf{Forb}(\mathcal{P})}$ = minimal elements of $\mathsf{Forb}(\mathcal{P})$ with respect to minors.

Lemma 14.6 ($\overline{\mathsf{Forb}(\mathcal{P})}$ forms an antichain)**.** $\overline{\mathsf{Forb}(\mathcal{P})}$ *forms an antichain, since each element is the minimum.*

Theorem 14.7 (Finite subsets of a w.q.o set is also w.q.o under injection ϕ)**.** *Suppose A is w.q.o by $\leq$. Then $A^{<\omega}$ (finite subsets of A) are w.q.o by the extension of $\leq$ as defined as follows:*

Given $S, T \subseteq A$, there exists an injection ϕ from S to T such that $s \leq \phi(s)$ for all $s \in S$.

Theorem 14.8 (Graph Minor Theorem)**.** *Graphs are well-quasi-ordered by the minor relation.*

Corollary 14.9 (Minor-closed properties are characterized by finite sets of forbidden minors)**.** *Since* $\overline{\mathsf{Forb}(\mathcal{P})}$ *is an antichain, it must be finite as the minor relationship is a w.q.o. Therefore G has $\mathcal{P} \iff G$ has no $H \in \overline{\mathsf{Forb}(\mathcal{P})}$ as a minor.*