CMU 21-355: Real Analysis I, Spring 2022
Notes and Exam Review

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1: Chapter 1 - The Real and Complex Number System

2: Chapter 2 - Basic Topology

Definition 2.1 (Limit Point)**.** x is a **limit point** of E if $\forall \varepsilon > 0\ \exists y \in E \cap N'_\varepsilon(x)$.

Definition 2.2 (Isolated Point)**.** x is an **isolated point** of E if $x \in E$ and $\exists \varepsilon > 0$ such that $E \cap N'_\varepsilon(x) = \emptyset$.

Definition 2.3 (Interior Point)**.** x is an **interior point** of E if $\exists \varepsilon > 0$ such that $N_\varepsilon(x) \subset E$.

Definition 2.4 (Basic Definitions)**.** A set $E \subset X$ is:

- **open** if $\forall x \in E\ \exists \varepsilon > 0$ such that $N_\varepsilon(x) \subset E$.
  In other words, every $x \in E$ is an interior point of E.
- **closed** if E contains all its limit points.
- **dense** in X if every $x \in X$ is in E, a limit point of E, or both.
  In other words, $\overline{E} = X$.
- **bounded** if $\exists M \in \mathbb{R}, x \in E$ such that $d(x,y) < M\ \forall y \in E$.

Definition 2.5 (Basic Definitions, Continued)**.** Given a set E,

- The **closure** $\overline{E}$ of E is $\overline{E} = E \cup E'$, where E' is the set of all limit points of E.
- The **interior**E° of E is the set of all interior points of E.
- An **open cover** of E is a collection $\{G_\alpha\}$ of open sets such that $E \subset \bigcup_\alpha G_\alpha$.
- E is **compact** if every open cover $\{G_\alpha\}$ of E has a finite subcover
- A set $E \subset Y \subset X$ is **relatively open** in Y if $\exists G \subset X$ open such thata $E = G \cap Y$.

Definition 2.6 (Separated Sets)**.** Two sets A and B are **separated** if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

Definition 2.7 (Connected Sets)**.** A set E is connected if it is not the union of two nonempty separated sets.

Theorem 2.8 (Properties of closure of sets)**.** *Let X be a metric space, $E \subset X$. Then*

1. *$\overline{E}$ is closed.*
2. *$E = \overline{E} \iff E$ is closed.*
3. *If $E \subset F$, F closed, then $\overline{E} \subset F$.*

Theorem 2.9 (Intersection of collection of sets where each finite subcollections has nonempty intersection is nonempty)**.** *Let $\{K_\alpha\}$ be a collection of compact sets. Suppose each finite subcollection has nonempty intersection. Then*

$$\bigcap_\alpha K_\alpha \neq \emptyset.$$

Corollary 2.10 (Nested sequence of compact sets must have non-empty intersection). *If $\{K_n\}_{n\in\mathbb{N}}$ is a nested sequence of compact sets, i.e $K_1 \subset K_2 \subset \cdots$. Then*

$$\bigcap_{n=1}^{\infty} K_n$$

is non-empty.

Lemma 2.11 (Nested sequence of rectangles is non-empty). *If $R_1 \subset R_2 \subset \cdots$ is a nested sequence of rectangles in $\mathbb{R}^k$, then $\bigcap_{n\in\mathbb{N}} R_n$ is non-empty.*

Lemma 2.12. *Any rectangle $R \subset \mathbb{R}^k$ is compact.*

Proof. AFSOC not compact, so there exists an open cover with no finite subcover.

□

Theorem 2.13 (Equivalence of compact sets). *Let $E \subset \mathbb{R}^k$. Then the following are equivalent:*

1. *E is closed and bounded.*
2. *E is compact*
3. *Every infinite subset of E has a limit point in E ("sequentially compact")*

Remark: (a) $\iff$ (b) is known as the Heine-Borel theorem.

Corollary 2.14 (Weierstrauss). *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

3: Chapter 3 - Numerical Sequences

Theorem 3.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then the following holds.*

1. *$\{x_n\}$ converges to $x \iff \forall \varepsilon > 0$, all but finitely many terms of $\{x_n\}$ are contained.*
2. *If $x_n \to x$ and $x_n \to x'$, then $x = x'$.*
3. *If $\{x_n\}$ converges, then it is bounded.*
4. *If $E \subset \mathbb{R}$ and x is a limit point of E, then $\exists \{x_n\}$ sequence in E such that $x_n \to x$.*

Theorem 3.2 (There exists a subsequential sequence in a compact subset)**.** *The following holds:*

1. *If $\{x_n\}$ is a sequence in a compact subset $K \subset \mathbb{R}$, then there exists a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\} \to x \in K$.*
2. *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Definition 3.3 (Cauchy Sequences)**.** A sequence $\{x_n\}$ in $\mathbb{R}$ is Cauchy if for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m \geq N$, then $|x_n - x_m| < \varepsilon$.

Theorem 3.4 (Cauchy $\iff$ sequence converges)**.** *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_n\}$ is Cauchy $\iff$ $\{x_n\}$ converges.*

Definition 3.5 ($\limsup, \liminf$)**.** Given a sequence $\{x_n\} \subset \mathbb{R}$,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{m \geq n} x_m \right)$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{m \geq n} x_m \right)$$

Theorem 3.6 (Equivalent characterization of $\limsup, \liminf$)**.** *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then*

$$\limsup_{n\to\infty} x_n = x \in \mathbb{R} \cup \{\pm\infty\}$$

if and only if the following true properties hold:

1. *$\exists$ subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ (either converges to x if x finite, or diverges to $\pm\infty$)*
2. *If $y > x$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n < y$.*

   Equivalently, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n < x + \varepsilon$.

The result holds similarly for $\liminf$, where in (2) we have that if $y < x$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n > y$.

Theorem 3.7 (Cauchy Criteria for Convergence)**.** *Given $\{a_n\}$ sequence in $\mathbb{R}$,*

$$\sum a_n \text{ converges } \iff \forall\, \varepsilon > 0, \exists N \in \mathbb{N}$$

such that if $n, m \geq N$ (WLOG $n \geq m$),

$$\Big| \sum_{k=m}^{n} a_k \Big| < \varepsilon.$$

Corollary 3.8 (If series converges, elements converge to 0). *If $\sum a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

Theorem 3.9 (Comparison Test). *TODO*

Theorem 3.10 (Root Test). *TODO*

Theorem 3.11 (Ratio Test). *TODO*

4: Chapter 4 - Continuity

Theorem 4.1 (Sequential Continuity). *$E \subset \mathbb{R}$, $f : E \to \mathbb{R}$, $x_0 \in E$ a limit point of E. Then f is continuous at $x_0 \iff f(a_n) \to f(x_0)$.*

Theorem 4.2 (Continuity Closed under Composition). *Let $f : E_1 \to E_2$, $g : E_2 \to \mathbb{R}$, $E_1, E_2 \subset \mathbb{R}$. If f is continuous at $x_0 \in E$, and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Theorem 4.3 (Useful Characterization of Continuity). *$f : \mathbb{R} \to \mathbb{R}$ continuous $\iff f^{-1}[G]$ open for any G open.*

Remark 4.4. Pre-image of open is open $\iff$ pre-image of closed is closed. So the same result above holds for any G closed.

Theorem 4.5 (Continuous Image of Compact Sets is Compact). *If $f : E \to \mathbb{R}$, $K \subset E$ compact, f continuous, then $f(K)$ is compact.*

Theorem 4.6 (Inf and Sup Achieved on Continuous Functions on a Compact Set). *Let $E \subset \mathbb{R}$ be compact, $f : E \to \mathbb{R}$ continuous. Then f is bounded. Moreover, the infimum and supremum are achieved.*

Theorem 4.7 (Continuous Functions on Compact Sets are Uniformly Continuous). *$E \subset \mathbb{R}$ compact, $f : E \to \mathbb{R}$ continuous. Then f is uniformly continuous.*

Theorem 4.8 (Monotonic Functions has Left and Right Limits on all points). *$f : [a, b] \to \mathbb{R}$ monotonically increasing (decreasing). Then $f(x^+)$ and $f(x^-)$ exists for every $x \in (a, b)$ and*

$$\sup_{a<t<x} f(t) = f(x^-) \leq f(x) \leq f(x^+) = \inf_{x<t<b} f(t)$$

Corollary 4.9 (Monotonic Functions do not have Discontinuities of the Second Kind). *$f : [a, b] \to \mathbb{R}$ monotonic. Then f has no discontinuities of the second kind.*

Corollary 4.10 (Set of Discontinuities in Monotonic Functions Are At Most Countable). *$f : [a, b] \to \mathbb{R}$ monotonic. Then the set of discontinuities of f is at most countable.*

5: Chapter 5 - Differentiation

Proposition 5.1 (Differentiatable at x implies Continuous at x). *$f : [a, b] \to \mathbb{R}$. If f differentiable at $x \in [a, b]$ then f is continuous at x.*

Proposition 5.2 (Chain Rule). *Suppose $f : [a, b] \to [c, d], g : [c, d] \to \mathbb{R}$, f continuous. Assume f is differentiable at $x \in [a, b]$, g differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.*

Lemma 5.3 (Local Max/Min Has Derivative 0)**.** $f : [a, b] \to \mathbb{R}$. *If f has a local max/min at $x \in (a, b)$, and f is differentiable at x, then $f'(x) = 0$.*

Theorem 5.4 (Mean Value Theorem)**.** *f continuous on $[a, b]$, differentiable on (a, b), then there exists $x \in (a, b)$ such that*

$$f(b) - f(a) = f'(x)(b - a).$$

Corollary 5.5 (Constant Sign of Gradient Implies Monotonicity)**.** *If f differentiable on (a, b) and $f'(x) \geq 0$ $\forall x \in (a, b)$, then f monotone increasing.*

The equivalent formulation holds for decreasing and constant.

Theorem 5.6 (Generalized Mean Value Theorem)**.** *f, g continuous on $[a, b]$, differentiable on (a, b). There exists $x \in (a, b)$ such that*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Remark 5.7. In the non-generalized case, we simply had $g(t) = t$.

Theorem 5.8 (Like MVT but for gradients)**.** *$f : [a, b] \to \mathbb{R}$ differentiable. Suppose $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \lambda$.*

Corollary 5.9 (Differentiable Functions have no Discontinuities of the First Kind)**.** *If f differentiable on $[a, b]$, then f has no discontinuities of the first kind.*

Theorem 5.10 (L'Hospital's Rule)**.** *Suppose g, f differentiable on (a, b), $-\infty \leq a < b \leq +\infty$.*

Suppose

1. *$f(x) \to 0, g(x) \to 0$ as $x \to a$ (Case A), OR*
2. *$g(x) \to +\infty$ as $x \to a$ (Case B).*

If

$$\frac{f'(x)}{g'(x)} \to A \in \mathbb{R} \cup \{\pm\infty\},$$

then

$$\frac{f(x)}{g(x)} \to A.$$

The analogous statement is also true if $x \to b$, or if $g(x) \to -\infty$.

Theorem 5.11 (Taylor's Theorem)**.** *$f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$. Suppose $f^{(k)}(t)$ is continuous on $[a, b]$ for $k = n - 1$, and $f^{(n)}$ exists on (a, b).*

For $x_0 \in [a, b]$ define the Taylor polynomial

$$P_{x_0}(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k$$

for all $t \in [a, b]$.

For any $y \in [a, b]$, $y \neq x$, there exists t_0 in (x_0, y) or (y, x_0) (depending on direction) such that

$$f(y) - P_{x_0}(y) = \frac{f^{(n)}(t_0)}{n!}(y - x_0)^n.$$

6: Chapter 6 - The Riemann-Stieltjes Integral

Lemma 6.1. *If P^* is a common refinement of P, then*

$$L(f,P,\alpha) \leq L(f,P^*,\alpha) \qquad \text{(Lower sum can only go up)}$$
$$U(f,P,\alpha) \geq U(f,P^*,\alpha) \qquad \text{(Upper sum can only go down)}$$

Lemma 6.2 (Lower Riemann Integral Bounded By Upper Riemann Integral)**.**

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$$

Theorem 6.3 (Useful Characterization of Riemann-Stieltjes Integrability)**.**

$$f \in \mathscr{R}(\alpha)$$

on $[a,b] \iff \forall \varepsilon > 0 \; \exists$ *partition* P *of* $[a,b]$ *such that*

$$0 \leq U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon.$$

Theorem 6.4 (Continuous Functions are Integrable)**.** *If f is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ on $[a,b]$.*

Theorem 6.5 (Integrability Closed under Composition with Continuous Functions)**.** *Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$. Suppose g continuous on $[m,M]$. Then*

$$h = g \circ f \in \mathscr{R}(\alpha).$$

Theorem 6.6 (Functions with only Finitely Many Discontinuities are Integrable)**.** *Suppose f has only finitely many points of discontinuities, suppose α is continuous at each discontinuity point of f. Then $f \in \mathscr{R}(\alpha)$.*

Proposition 6.7 (Linearity Properties of Integrals)**.** *Pretty much what you would expect, omitted for brevity.*

Theorem 6.8 (Other Properties of Integrals)**.** *If $f, g \in \mathscr{R}(\alpha)$, then*

1. *$fg \in \mathscr{R}(\alpha)$,*
2. *$|f| \in \mathscr{R}(\alpha)$, and*
$$\Big| \int_a^b f d\alpha \Big| \leq \int_a^b |f| d\alpha.$$

Theorem 6.9 (Change of Variables)**.** *Suppose φ continuous, stricly increasing on $[A,B]$ with $\varphi([A,B]) = [a,b]$. Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and set $g = f \circ \varphi$, $\beta = \alpha \circ \varphi$.*

Then $g \in \mathscr{R}(\beta)$ on $[A,B]$, and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Theorem 6.10. *Suppose $f \in \mathscr{R}$ on $[a, b]$. For $x \in [a, b]$, let*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $x_0 \in [a, b]$, then F differentiable at x_0, with $F'(x_0) = f(x_0)$.

Theorem 6.11 (Fundamental Theorem of Calculus)**.** *If $f \in \mathscr{R}$ on $[a, b]$ and if F is differentiable on $[a, b]$ with $F' = f$, then*

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Theorem 6.12 (Integration by Parts)**.** *Suppose F, G differentiable on $[a, b]$, $F' = f \in \mathscr{R}$, $G' = g \in \mathscr{R}$ on $[a, b]$. Then*

$$\int_a^b f(x)G(x)\, dx = -\int_a^b F(x)g(x)\, dx + F(b)G(b) - F(a)G(a).$$

7: Chapter 7 - Sequences and Series of Functions

Definition 7.1 (Pointwise Convergence)**.** Suppose $\{f_n\}$ sequence of functions on $E \subseteq \mathbb{R}$ (for each $n \in \mathbb{N}, f_n : E \to \mathbb{R}$), and suppose that for each fixed $x \in E$, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges.

Then we define

$$f = \lim_{n\to\infty} f_n : E \to \mathbb{R}$$

by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We say that "f_n converges pointwise to f", or "f is the pointwise limit of $\{f_n\}$".

Similarly, if $\sum f_n(x)$ converges for each fixed $x \in E$, then we can define

$$g = \sum f_n : E \to \mathbb{R}$$

by

$$g(x) = \sum f_n(x) \qquad \forall x \in E.$$

Example 7.2. Continuity, integral, and derivatives are not necessarily preserved when limits are taken.

Continuity: consider

$$f_n(x) = \begin{cases} 1 - nx & \text{on } [0,\, 1/n], \\ 0 & \text{on } [1/n,\, 1]. \end{cases}$$

Derivatives: consider

$$f_n(x) = \frac{1}{\sqrt{n}\sin(nx)}.$$

Integrals: consider

$$f_n(x) = \begin{cases} 0 & \text{on } [0,\, 0], \\ n - n^2 x & \text{on } (0,\, 1/n), \\ 0 & \text{on } [1/n,\, 1]. \end{cases}$$

7.1: Uniform Convergence

Definition 7.3 (Uniform Convergence)**.** A sequence of functions $\{f_n\}$ **converges uniformly** on E to a function $f : E \to \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N, \forall x \in E$,

$$|f_n(x) - f(x)| < \varepsilon.$$

(So for uniform convergence, N is uniform for all $x \in E$.)

Likewise, the series $\sum\limits_{n=1}^{\infty} f_n(x)$ **converges uniformly** on E if the partial sum $g_n(x) = \sum\limits_{i=1}^{\infty} f_i(x)$ converge uniformly.

Theorem 7.4 (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions $\{f_n\}$ defined on $E \subseteq \mathbb{R}$ **converges uniformly** on E if and only if $\forall \varepsilon > 0, \exists N\ N(\varepsilon) \in \mathbb{N}$ such that $\forall n, m \geq N, \forall x \in E$,*

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Proposition 7.5. *Suppose $f_n \to f$ pointwise. Let*

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly $\iff M_n \to 0$.

Theorem 7.6 (Weierstrauss M-Test)**.** *Suppose $\{f_n\}$ sequence of functions on E, define*

$$M_n \coloneqq \sup_{x \in E} |f_n(x)|.$$

Then

$$\sum f_n \text{ converges uniformly} \iff \sum M_n \text{ converges.}$$

7.2: Uniform Convergence and Continuity

Theorem 7.7 (Limit Exchange Theorem)**.** *Suppose $f_n \to f$ **uniformly** on $E \subseteq \mathbb{R}$. Let x be a limit point of E and suppose that*

$$\lim_{t \to x} f_n(t) = A_n \qquad \forall n \in \mathbb{N}.$$

Then $A_n \to A$ and $\lim_{t \to x} f(t) = A$.

In other words,

$$\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t).$$

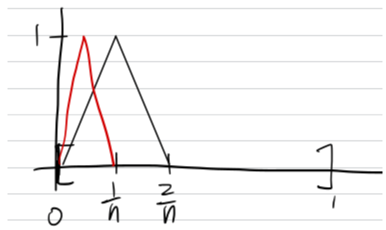
Corollary 7.8 (Important Corollary)**.** *Suppose $\{f_n\}$ is a sequence of functions on E and $\{f_n\} \to f$ **uniformly**. Then f is continuous.*

Remark 7.9. The uniformity condition is required, else we can use the counter-examples from before.

Remark 7.10. The converse statement is not true: suppose $f_n \to f$ pointwise, f_n, f both continuous, it can be the case that f_n does not converge uniformly to f.

Consider the spike function defined by

$$f_n(x) \begin{cases} nx & x \in [0, \frac{1}{n}], \\ 2 - nx & x \in [\frac{1}{n}, \frac{2}{n}], \\ 0 & x \in [\frac{2}{n}, 1]. \end{cases}$$



Then it converges to $f \equiv 0$ pointwise, but it is not uniform by the Weierstrauss M-Test, since

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = M_n = 1.$$

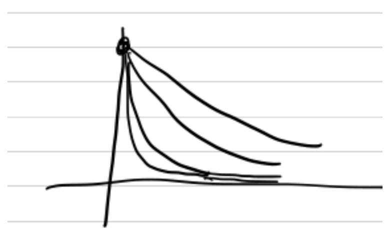
However, we do have a partial converse, which requires that the domain is compact.

Theorem 7.11 (Partial Converse of Important Corollary)**.** *Suppose $K \subset \mathbb{R}$ compact. Let $\{f_n\}$ be a sequence of continuous functions on K, f continuous on K, $f_n \to f$ pointwise. Assume $f_n(x) \geq f_{n+1}(x)$. $\forall x \in K, n \in \mathbb{N}$. Then $f_n \to f$ uniformly.*

Remark 7.12. Compactness is necessary for the previous theorem, i.e consider

$$f_n(x) = \frac{1}{1+nx}$$

on $(0,1)$. Then $f_n \to 0$ on $(0,1)$ but $f_n \not\to 0$ uniformly.



7.3: Uniform Convergence and Integration

Theorem 7.13. *Suppose α monotonically increasing on $[a,b]$, $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, $f_n \to f$* ***uniformly*** *on $[a,b]$.*

Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha.$$

Corollary 7.14. *If $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\sum\limits_{n=1}^{\infty} f_n(x)$* ***converges uniformly*** *on $[a,b]$. Then*

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right) d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha.$$

7.4: Uniform Convergence and Differentiation

Theorem 7.15. *$\{f_n\}$ differentiable on $[a,b]$, and suppose $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then $\{f_n\}$ converges uniformly to a function f and*

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

Note that the anchor point x_0 is required, else it may not hold.

Theorem 7.16 (Weierstrauss Monster Function)**.** *There exists a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and nowhere differentiable.*

7.5: Equicontinuous Families of Functions

Theorem 7.17 (Uniform convergence on compact domain implies equicontinuous)**.** *$K \subset \mathbb{R}$ compact, $\{f_n\}$ sequence of continuous functions on K. If $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous.*

Lemma 7.18 (Key Idea of Arzela-Ascoli)**.** *If $\{f_n\}$ is a pointwise bounded sequence of functions on a countable set E, then $\{f_n\}$ has a subsequence that converges pointwise on E.*

Theorem 7.19 (Arzela-Ascoli)**.** *$K \subset \mathbb{R}$ compact, $\{f_n\}$ sequence of functions on K that are **pointwise bounded** and **equicontinuous** on K. Then:*

1. *$\{f_n\}$ is uniformly bounded,*
2. *(Arzela-Ascoli) $\{f_n\}$ has a uniformly convergent subsequence.*

Theorem 7.20 (Stone-Weierstrass Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly.*