

CMU 21-484: Graph Theory, Spring 2022
Notes and Exam Review

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May 4, 2022

1: Basic Concepts

Definition 1.1 (Paths). Two paths are:

- Vertex-disjoint if they have no vertices in common
- Edge-disjoint if they have no edges in common
- Independent if they have no vertices in common, except at the endpoints

If paths are just called disjoint without specification, that would usually mean vertex-disjoint.

Theorem 1.2 (Tree). *The following are equivalent:*

- G is a tree
- G is an edge-minimally connected graph
- G is an edge-maximally acyclic graph
- For all x, y in G , there is a unique path from x to y

Theorem 1.3 (Matrix-tree, no proof). *For any multigraph G , and any vertex $v \in G$,*

$$\tau(G) = \det(L_G[v]),$$

where τ_G is the number of spanning trees of G , and $A[i]$ is the matrix A with row and column i deleted.

Useful properties of determinants:

- Adding one row to another does not change determinant
- Determinant of upper(or lower) triangular matrix is the product of the diagonal elements
- Swapping two rows causes determinant's sign to change
- Multiplying a row by scalar k causes determinant to also be multiplied by k

Definition 1.4 (Bipartite graph). G is bipartite if $V = A \dot{\cup} B$ s.t. $\forall e \in E, |e \cap A| = |e \cap B| = 1$.

Theorem 1.5 (Bipartite characterization in odd cycles). G is bipartite $\iff G$ has no odd cycles.

2: Matchings

Definition 2.1 (Alternating paths). An alternating path with respect to M begins at unmatched vertex a_0 , run alternate edges in and out of M . If it ends at unmatched vertex of B , it is augmenting.

Definition 2.2 (Augmenting Paths). Let $G = (V, E)$ be a graph and $M \subseteq E$ be a matching. A path $P = x_0, c, \dots, x_k$ is called augmenting if:

- $x_{i-1}x_i \in E \setminus M$ for every odd $i \in [k]$;
- $x_{i-1}x_i \in M$ for every even $i \in [k]$;
- Neither x_0 nor x_k is incident to an edge of M .

Theorem 2.3 (Hall's). *Bipartite G on $A \dot{\cup} B$ has a complete matching of A iff Hall's condition holds, i.e*

$$\forall S \subseteq A, |\Gamma(S)| \geq |S|$$

Theorem 2.4 (Konig's, not in scope). *If G is bipartite, then $\mu(G) = h(G)$, where $\mu(G)$ is the size of the maximum matching and $h(G)$ is the size of the minimum hitting set.*

A hitting set is a set $x \subset EV$ s.t. $e \cap x \neq \emptyset \forall e \in E$.

Theorem 2.5 (Tutte's, no proof). *G has a perfect matching iff $q(G \setminus S) \leq |S| \forall S \subseteq V(G)$.*

$q(X)$ is the number of odd components in $G[X]$, the subgraph induced by X .

3: Connectivity

Definition 3.1 (k -vertex connected). *G is k -vertex connected if $G \setminus X$ is connected for all $|X| < k$ and $|V(G)| \geq k + 1$.*

So a k -clique is not k -connected, for instance. Denote

$$\kappa(G) = \max\{k \mid G \text{ is } k\text{-vertex connected}\}.$$

Definition 3.2 (l -edge connected). *G is l -edge connected if $G \setminus F$ is connected for all $F \subseteq E(G)$, $|F| < l$.*

Denote

$$\lambda(G) = \max\{l \mid G \text{ is } l\text{-edge connected}\}.$$

Theorem 3.3 ($\kappa(G) \leq \lambda(G)$). $\kappa(G) \leq \lambda(G) \forall$ graphs G .

Proof. Consider a smallest $F \subseteq E(G)$ such that $G \setminus F$ is disconnected.

Observe, since F is smallest, no edge $e \in F$ lies inside a component C of $G \setminus F$; otherwise, e could be removed from F .

Case 1: G contains a vertex v not incident with any $e \in F$. Let C be the component of $G \setminus F$ containing v . Then the vertices of C incident with an edge in F separate v from $G \setminus C$. Since no edge in F has both ends in C by minimality of F , there are at most $|F|$ such vertices, giving $\kappa(G) \leq |F|$.

Case 2: Every vertex is incident with some $e \in F$. Consider any vertex $v \in V(G)$, and let C be the component of $G \setminus F$ containing v . Then the neighbours w of v with $vw \notin F$ lie in C and are incident to distinct edges in F (again by the minimality of F), giving $\deg(v) \leq |F|$. If v can be separated in this manner we are done; else since v was any vertex, it means that G was a clique because we have $x \cup \Gamma(v) = V(G)$. \square

Theorem 3.4 (Ear Theorem). *A graph is 2-connected iff it can be constructed from a cycle by successively adding H -paths to graphs H already constructed.*

Theorem 3.5 (3-connected). *If G is 3-connected, $\exists e \in E(G)$ s.t. G/e is 3-connected.*

Theorem 3.6 (Menger's, local edge version). *Given $u, v \in G$, the minimum number of edges separating u from v is the maximum number of edge-disjoint paths.*

Theorem 3.7 (Menger's, global edge version). *If G is l -edge connected, every pair of vertices can be joined by l independent paths.*

Theorem 3.8 (Menger's, local vertex version). *If u and v are distinct non-adjacent vertices of G , then the minimum number of vertices $\neq u, v$ separating u from v is equal to the maximum number of independent paths from u to v .*

Theorem 3.9 (Menger's, global vertex version). *If G is k -vertex-connected, then every pair of vertices is joined by k independent paths.*

Theorem 3.10 (Menger's, local vertex set version). *If A and B are (not necessarily disjoint) subsets of $V(G)$, then the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B .*

Theorem 3.11 (Max-flow min-cut). *In every network, the maximum total value of a flow equals the minimum capacity of a cut.*

Theorem 3.12 (Integer flows). *If all capacities are integers, there exists a max flow of integer value.*

Theorem 3.13 (0/1 Flows). *Every 0/1 valued flow can be decomposed into (edge or vertex) disjoint paths and cycles.*

4: Planarity

Theorem 4.1 (Euler's formula). *If G is a connected plane graph with n vertices, m edges, and l faces, then*

$$n - m + l = 2.$$

Proof. By induction on m (n is fixed).

Base case. $m = n - 1$, G is a tree. G has one face, $l = 1$.

Then $n - (n - 1) + 1 = 2$.

Induction Step. $m \geq n$.

G has a cycle. Let e be an edge of this cycle. By JCT, C divides \mathbb{R}^2 into two regions, and e is incident with precisely 2 faces.

Thus $G \setminus e$ has one fewer face. $G \setminus e$ has:

- $n' = n$
- $m' = m - 1$
- $l' = l - 1$

By induction,

$$\begin{aligned} n' - m' + l' &= 2 \\ \implies n - (m - 1) + (l - 1) &= 2 \\ \implies n - m + l &= 2. \end{aligned}$$

□

Remark 4.2. It's easy to derive this formula if you forget it.

Corollary 4.3 (Faces and Edges). *Consider a bipartite graph where each vertex on one side represents a face in G , and each vertex on the other side represents an edge in G . By considering their degrees, we obtain*

$$3l \leq 2m, \text{ so } l \leq \frac{2}{3}m.$$

Corollary 4.4 (Connected plane graph). *In a connected plane graph (if not connected, could just have a bunch of edges), then $2 = n - m + l \leq n - m + \frac{2}{3}m = n - \frac{m}{3}$, so*

$$m \leq 3n - 6.$$

Theorem 4.5 (For 2-connected plane graphs, face boundaries are cycles). *If G is a 2-connected plane graph, then every face boundary is a cycle.*

Proof. We use the ear-theorem.

BC: Trivial.

IS: $G = H \cup P$ where H is a 2-connected plane graph, and P is a H -path.

P lies in some face f' of H . By induction, the boundary of f' is a cycle, as it is the boundary of all other faces. Adding P divides f' into cycles, so all faces are still bounded by cycles. \square

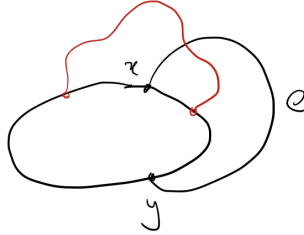
Definition 4.6 (Induced cycle). An induced cycle in G , a cycle in G forming an induced subgraph, is one that has no chords.

Theorem 4.7 (Non-separating induced cycles are faces). *If C is a non-separating induced cycle in a connected planar graph, C is the boundary of a face.*

Proof. By JCT, C divides the plane into two regions. Since C is non-separating, one must contain no vertices. Since C is cordless, it also contains no edges. This region is a face, whose boundary is C . \square

Theorem 4.8 (Faces in 3-connected graphs are non-separating induced cycles). *If G is 3-connected and C is the boundary of a face, C is a non-separating induced cycle.*

Proof. Let G be a 3-connected plane graph, and let $C \subseteq G$. Suppose that C bounds a face f ; then because G is also 2-connected, face boundaries are cycles, so C is a cycle. If C has a chord $e = xy$, then the components of $C - \{x, y\}$ are linked by a C -path in G , because G is 3-connected. This path and e both run through the outer face of C (not f) but do not intersect, a contradiction.



So C is cordless. It remains to show that C does not separate any two vertices $x, y \in G \setminus C$.

By Menger's theorem, x, y are linked in G by three independent paths. But C can only intersect 2 of them, so since f is 3-connected, x, y are connected. \square

Definition 4.9 (Subdivision). A subdivision of X is, informally, any graph obtained from X by “subdividing” some or all of its edges by drawing new vertices on those edges. In other words, we replace some edges of X with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$ or on another new path.

Definition 4.10 (Topological Minor). H is a topological minor of G if there exists J such that:

- J is a subgraph of G ,
- J is a subdivision of H .

Definition 4.11 (Minor). H is a minor of G if H can be obtained from G by deletions (both edges and vertices) and contractions.

Theorem 4.12 (Kuratowski’s). G is planar iff G has no K_5 or $K_{3,3}$ topological minor.

Theorem 4.13 (Wagner’s). G is planar iff G has no K_5 or $K_{3,3}$ minor.

5: Vertex Coloring

Definition 5.1 (Chromatic Number). The chromatic number $\chi(G)$ of G is defined

$$\chi(G) = \min\{k \mid G \text{ is } k\text{-colorable}\}.$$

Notation:

- $\omega(G)$ is the size of the largest clique in G
- $\alpha(G)$ is the size of the largest independent set in G (a set with no edges among its vertices)
- $\Delta(G)$ is the largest degree in G

Basic bounds:

- $\chi(G) \leq 2 \iff G$ has no odd cycles (i.e bipartite)
- $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

Lower bound: just to color the clique, require at least $\omega(G)$ colors.

Upper bound: always can color if more colors available than max. degree

- $\chi(G) \geq \frac{n}{\alpha(G)}$, where $\alpha(G)$ is the size of the largest independent set of G .
- $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.

Definition 5.2 (Coloring Number).

$$\text{col}(G) = \left(\min_{\substack{\text{orderings} \\ v_1, \dots, v_n}} \max_i \deg^>(v_i) \right) + 1,$$

where

$$\deg^<(v_i) = |\Gamma(v_i) \cap \{v_1, \dots, v_{i-1}\}|.$$

Theorem 5.3.

$$\chi(G) \leq \text{col}(G)$$

Definition 5.4 (Greedy Algorithm). For some ordering v_1, v_2, \dots, v_n of $V(G)$, at step i , color v_i with the smallest integer not already used at one of its neighbors in $\{v_1, \dots, v_{i-1}\}$.

Theorem 5.5 (5-color Theorem for Planar Graphs). *If G is planar, $\chi(G) \leq 5$.*

Proof. By induction. Fix a drawing of G .

By Euler's formula, G , has a vertex v of $\deg \leq 5$. (since $m \leq 3n - 6$, AFSOC if all vertices has degree ≥ 6 , then we have $3n \leq 3n - 6$, a contradiction).

By induction, $G \setminus v$ is 5-colorable. Fix some 5-coloring of $G \setminus v$ C . We aim to extend C to v .

If $\deg(v) \leq 4$, v sees ≤ 4 colors in $c(\Gamma(v))$, and we can extend.

So we assume that $\deg(v) = 5$, and moreover, the 5 neighbors of v has distinct colors.

Label the neighbors v_1, \dots, v_5 in cyclic order. WLOG, assume $c(v_i) = i$.

Rest is by case analysis - main idea is we have paths with alternating colors from 1 to 3 and 2 to 4. The paths must intersect by JCT. So can recolor, done. \square

Theorem 5.6 (Brook's, statement only). *If G is connected and not complete or an odd cycle, $\chi(G) \leq \Delta(G)$.*

Definition 5.7 (k -constructible). G is k -constructible if G can be constructed, starting from K_k 's (i.e clique of size k), via the following operations:

- Adding edges
- Identifying two non-adjacent vertices x, y , adding an edge between them, and contracting both vertices
- Hajos sum: Given 2 k -constructible graphs H_1, H_2 with edges x_1y_1 and x_2y_2 respectively, remove edges x_1y_1 and x_2y_2 , identify x_1 and x_2 (i.e adding an edge between them and contracting them), and add edge from y_1 to y_2 .

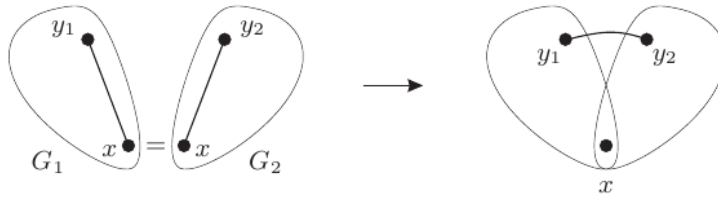


Fig. 5.2.2. The Hajos construction (iii)

Theorem 5.8 (Hajos', statement only). *Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geq k$ if and only if G has a k -constructible subgraph.*

6: Edge Coloring

Definition 6.1 (Edge Coloring). A proper k edge-coloring of G is a function $c : E(G) \rightarrow [k]$ such that $e_1 \cap e_2 \neq \emptyset \implies c(e_1) \neq c(e_2)$.

Then the edge coloring is defined

$$\chi_e(G) = \min\{k \mid G \text{ has a proper coloring.}\}$$

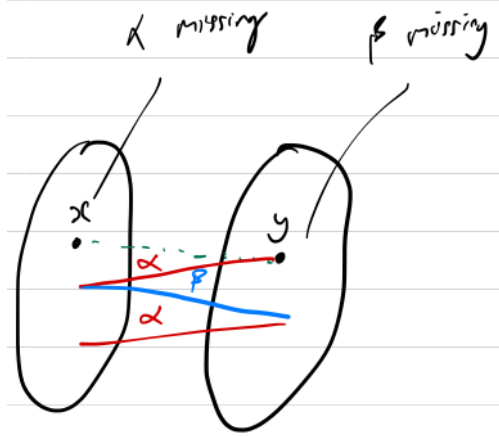
Theorem 6.2 (Konig's). *Every bipartite graph G satisfies $\chi_e(G) = \Delta(G)$.*

N.B: sometimes $\chi_e(G)$ is written $\chi'(G)$.

Proof. By induction on edges. Given graph G , assume theorem holds for graphs with fewer edges.

Choose $xy \in G$. By induction, $G \setminus xy$ can be Δ -colored.

If we can't extend the coloring to xy , some color α is missing at x (but not at y), and some color β is missing at y (but not at x) (because if $\alpha = \beta$, we can extend the coloring and are done).



Consider the longest α/β walk from y ; this is a path. If it does not end at x , we can flip colors on the path (since each vertex along the path after y must have both α, β colors, it does not matter if we flip the order). This means that we can extend the coloring to e .

Otherwise, the path ends at x . But then this is impossible, since we will enter A with an α -edge, and α is missing at x . \square

Theorem 6.3 (Vizing's, statement only).

$$\chi_e(G) \leq \Delta(G) + 1$$

.

Proof. By induction on the number of edges. \square

7: List Coloring

Definition 7.1 (List Coloring). Given G and lists L_v , a coloring from the lists is an assignment $c : V \rightarrow \bigcup L_v$ such that for $c(v) \in L_v$,

$$u \sim v \implies c(u) \neq c(v).$$

G is k -list colorable if for any list L_v satisfying $|L_v| = k \forall v \in G$ can be colored from the lists. The list chromatic number $\chi_l(G)$ is

$$\chi_l(G) = \min\{k \mid G \text{ is } k\text{-list colorable}\}.$$

Basic Properties

- k -list colorable $\implies k$ -colorable, i.e $\chi(G) \leq \chi_l(G)$.

- Other direction DOES NOT hold, can make $\chi_l(G)$ much larger than $\chi(G)$.
I.e from HW, for any k , there are graphs G with $\chi(G) = 2$ and $\chi_l(G) = k$.

Theorem 7.2 (Planar List Coloring). *If G is planar, G is 5-list-colorable.*

Proof. Strengthen the IH. We show that if G is a plane graph, all of whose inner faces are triangles, whose outer boundary is a cycle comprising vertices v_1, v_2, \dots, v_k , where

- $L_{v_1} = \{1\}$,
- $L_{v_2} = \{2\}$,
- $|L_{v_i}| \geq 3 \forall i = 3, \dots, k$,
- $|L_v| \geq 5$ for $v \neq v_i \forall i$,

then G is 5-list-colorable.

(N.B: to meet the triangle requirement, just add edges until it holds.)

We prove this by induction.

1. C has a chord.

It divides G into G_1, G_2 , intersecting just at chord. WLOG, $v_1, v_2 \in G_1$.

Color G_1 by induction, then G_2 , with endpoints of chord as new v_1, v_2 . This produces a valid coloring for G .

2. C has no chord.

Consider neighbors u_1, u_2, \dots, u_l of v_k (other than v_1, \dots, v_{k-1}).

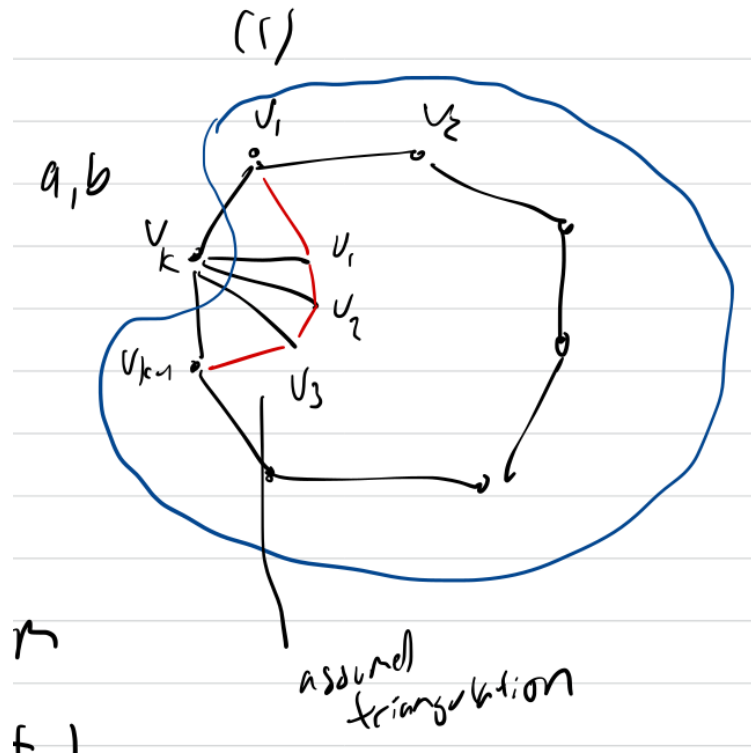
Choose j, l from L_{v_k} s.t. $a, b \neq 1$ (recall 1 already used to color v_1), this is possible since $|L_{v_k}| \geq 3$.

For each u_i , define $L'_{u_i} = L_{u_i} \setminus \{a, b\}$.

So $|L'_{u_i}| \geq 3 \forall i$.

Apply induction on $G \setminus v_k$, which now has all the u_i 's on the outer face by triangulation.

Since a, b not used at v_1 or any u_i , at most one is used at v_{k-1} , so we can extend to v_k .



□

8: Perfect Graphs

Definition 8.1 (Perfect Graphs). A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G .

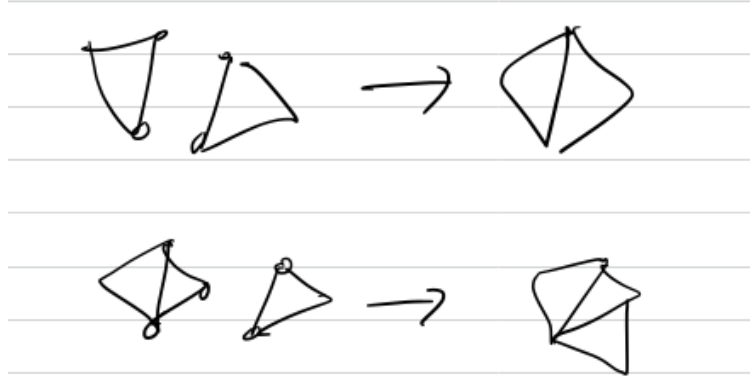
Note that we always have $\omega(G) \leq \chi(G)$.

Definition 8.2 (Chordal Graphs). A graph is chordal if every cycle of length ≥ 4 has a chord. Equivalently, there are no induced cycles of length ≥ 4 vertices.

Theorem 8.3 (Chordal graphs are constructed from cliques by pasting along cliques). G is chordal \iff either

1. G is a clique, or
2. $G = G_1 \cup G_2, G_1 \cap G_2$ is a clique, G_1, G_2 are chordal.

In other words, chordal graphs can be constructed recursively from cliques, by pasting along cliques.



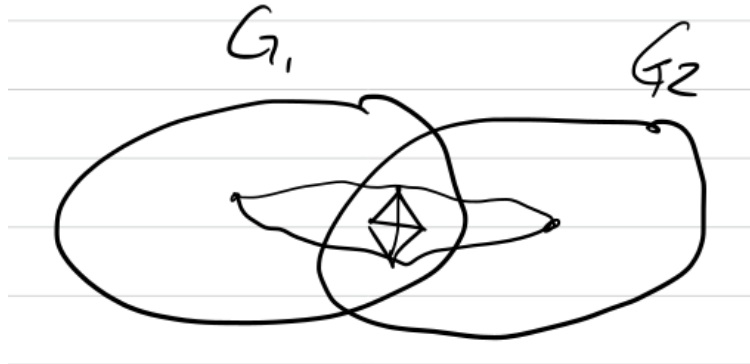
Proof. (\implies)

If G is a clique, then it is chordal, since there are no induced cycles of length 4.

Suppose $G = G_1 \cup G_2$, $G_1 \cap G_2 = K$ is a clique, G_1, G_2 chordal.

Consider any cycle C . If $C \subseteq G_i$ for some i , C has a chord since G_i chordal.

Otherwise, C contains vertices in both $G_1 \setminus G_2$, and $G_2 \setminus G_1$. So the cycle must have non-consecutive vertices in K . Since K is a clique, it means that C must have a chord when it goes through K in this way.



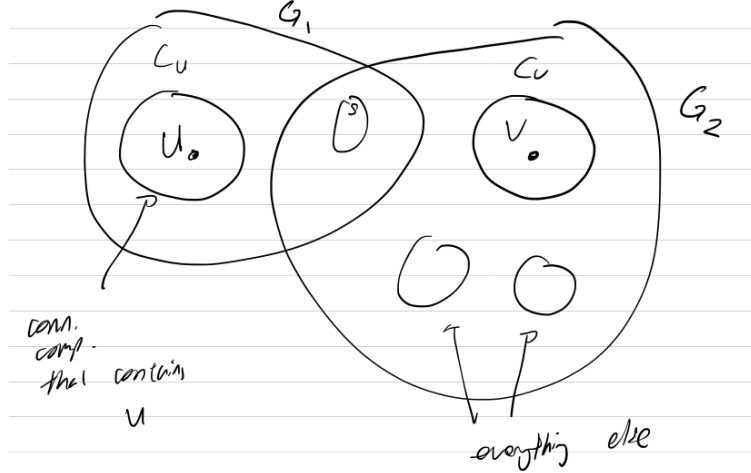
(\impliedby)

Suppose that G is chordal and not a clique.

Since G not a clique, $\exists u, v$ where $u \not\sim v$.

Consider a minimum separating set S for u, v . We claim that S is a clique.

Since S is a minimum, every vertex in S has a neighbor in C_u and C_v (because if not, that vertex can be removed and we still have a separating set, since that vertex disconnects one of the components anyway).



Suppose $x, y \in S$, $x \not\sim y$.

Since x, y both have neighbors in both C_u and C_v , there's a shortest path P from x to y through C_u , and Q from y to x through C_v . If $x \not\sim y$, PQ is an induced cycle of length ≥ 4 (which violates the fact that G is chordal).

Note that G_i must still be chordal since induced subgraphs of chordal graphs are chordal. So we have shown the desired conditions. \square

Definition 8.4 (Perfect Elimination Ordering (p.e.o)). A perfect elimination ordering (p.e.o) of G is an ordering v_1, v_2, \dots, v_n s.t. $\forall i$,

$$\Gamma(v_i) \cap \{v_1, \dots, v_{i-1}\}$$

is a clique.

Remark 8.5. Given a p.e.o, $\omega(G) = \max \text{ back degree} + 1$. Coloring greedily in this order uses $\leq \max \text{ back deg} + 1$ colors.

Definition 8.6 (PEO Graphs). A PEO graph is a graph for which a perfect elimination ordering exists.

Theorem 8.7 (PEO Graphs are Perfect). If G is a PEO graph, $\chi(G) = \omega(G)$.

Proof. Because induced subgraphs of PEO graphs are also PEO graphs (has the same p.e.o minus deleted vertices), and using greedy coloring on p.e.o requires $\omega(G)$ colors. \square

Definition 8.8 (Simplicial vertex). A simplicial vertex is one whose neighbors form a clique.

Theorem 8.9 (PEO graphs \iff chordal). G is a PEO graph $\iff G$ is chordal.

Proof. (\implies)

Suppose we have a PEO graph. Any induced subgraph is also a PEO graph by using the same ordering and dropping the vertices which no longer exists. However, a cycle of length ≥ 4 is not PEO, since there are no simplicial vertices. So G must be chordal.

(\impliedby)

Suppose that G is chordal. Then it contains a simplicial vertex (since if G is chordal and not a clique, G has two non-adjacent simplicial vertices). Then if we delete this vertex, we have a smaller chordal graph. So by induction, the smaller graph has a PEO, so we are done. \square

Remark 8.10 (Constructing a peo from a PEO graph). We can use a greedy algorithm to start from the last vertex in the ordering, and keep choosing a simplicial vertex, working backwards in the ordering.

Definition 8.11 (G_x). Given a graph G and $x \in V(G)$, define G_x to be the graph formed by adding a new vertex x' to G , adding edges $x'y$ whenever $xy \in E(G)$, as well as the edge xx' .

Theorem 8.12 (Expanding vertex preserves property of being a perfect graph). *If G is perfect, then G_x is perfect.*

Proof. We need to show that for all induced $H \subseteq G_x$, $\chi(H) \leq \omega(H)$.

Either H is an induced subgraph of G , or it is obtained by expanding an induced subgraph of G .

So it suffices to show that $\chi(G_x) \leq \omega(G_x)$. Let $w = \omega(G)$.

Case 1: $\omega(G_x) = \omega + 1$. Then we can ω -color G , and use the new color at x' .

Case 2: $\omega(G_x) = \omega$. Define \mathcal{K} to be the set of ω -cliques in G (i.e max-sized cliques in G). We know that $x \notin k_\omega$ for any $k_\omega \in \mathcal{K}$, since otherwise this would have increased the maximum clique size in G_x .

Consider a ω -coloring C of G . Consider the color class C_x of x with respect to C . C_x intersects every $k_\omega \in \mathcal{K}$, since $\chi(G) = \omega(G)$. So deleting C_x decreases the clique number, and so does deleting $C_x \setminus x$ (since x not involved in any max cliques). We know that $C_x \setminus x$ intersects every $k_\omega \in \mathcal{K}$. So delete $C_x \setminus x$ from G to produce G' , where we know $\omega(G') \leq \omega - 1$. Since G' perfect, $\chi(G') \leq \omega - 1$. So, consider an $(\omega - 1)$ coloring C' of G' . We need to extend the coloring to $\{x'\} \cup (C_x \setminus x)$. But this is an independent set, so one color suffices. So we have colored G_x with ω colors.

Thus G_x is perfect. □

Theorem 8.13 (Weak perfect graph theorem, statement only). *G is perfect if and only if \overline{G} is perfect.*

9: Extremal Graph Theory

Definition 9.1 ($ex(n, H)$). Denote $ex(n, H)$ to be the maximum number of edges in any graph on n vertices which does not contain H as a subgraph.

Definition 9.2 (Complete r -partite graphs). The complete r -partite graph is a graph on vertex set $V = V_1 \sqcup V_2 \cdots \sqcup V_r$, where $x \sim y \iff x \in V_i, y \in V_j$ for $i \neq j$.

Definition 9.3 (Turán Graphs). The complete r -partite graph with classes differing by ≤ 1 in size are called Turán graphs, and denoted $T^r(n)$.

Write $t_r(n)$ to denote the number of edges in $T^r(n)$.

We have

$$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1}.$$

Theorem 9.4 (Turán's Theorem).

$$ex(n, K_r) = t_{r-1}(n).$$

Moreover, if G has $ex(n, K_r)$ edges, in fact $G \cong T^{r-1}(n)$.

Proof. By induction. G is edge-maximal without K_r , so G has a clique K on $r - 1$ vertices. \square

Theorem 9.5 ($\text{ex}(n, C_4) \leq Cn^{3/2}$).

$$\text{ex}(n, C_4) \leq Cn^{3/2}.$$

Proof. Key idea: count the number of 2-paths centered at all vertices v , know that this is upper bounded by $\binom{n}{2}$. Use Jensen's inequality, then assume $n < m$ to get result. \square

Definition 9.6 ($K_r(t)$). $K_r(t)$ is the complete r -partite graph in which every partition class contains exactly t vertices.

Theorem 9.7 (Erdős-Stone theorem, statement only). $\forall \varepsilon > 0, \forall t$, there's an N such that for all $n > N$, if G has $\geq \frac{r-1}{r} \frac{n^2}{2} + \varepsilon n^2$ edges and n vertices, then G contains a $K_{r+1}(t)$.

Remark 9.8. The textbook formulates it slightly differently, that if it has at least

$$t_r(n) + \varepsilon n^2$$

edges, then it contains $K_{r+1}(t)$ as a subgraph.

Corollary 9.9 (Important Corollary of Erdős-Stone Theorem).

$$\text{ex}(n, H) \sim \frac{\chi(H) - 2}{\chi(H) - 1} \binom{n}{2}$$

for $\chi(H) \geq 3$.

Proof. For any graph H , there exists an integer r (the independence number $\alpha(H)$ will work) such that H is a subgraph of $K_{\chi(H)}(r)$. Color H , add vertices to the color classes so that they all have the same size r , then join any two different colored vertices. Then apply Erdős-Stone on $K_{\chi(H)}(r)$: we get that if we have at least $\frac{\chi(H)-2}{\chi(H)-1} \frac{n^2}{2} + \varepsilon n^2$ edges, then there will be a $K_{\chi(H)}(r)$, which contains H as a subgraph. This gives an upper bound on the number of edges.

For the lower bound, note that the Turan graph $T_{\chi(H)-1}(n)$ is $\chi(H) - 1$ colorable, and therefore it cannot contain H . \square

10: Hamilton Cycles

Definition 10.1 (Hamilton Cycle). A cycle in G on n vertices where $n = |V(G)|$.

Theorem 10.2 (Dirac). If $\deg(v) \geq \frac{n}{2} \forall v \in G$, G is Hamiltonian, i.e has a Hamilton cycle..

Proof. Proof ideas: pigeonhole principle, considering $I = \{i | v_i \sim x\}, J = \{i | v_{i-1} \sim y\}$. Start by assuming longest path, find a cycle, then realize we could find a longer path, thus contradiction. \square

Theorem 10.3 (Necessary condition for Hamilton cycle). If G has a set S , $|S| = k$,

- $G \setminus S$ has $\geq k + 1$ components $\implies G$ has no Hamilton cycle
- $G \setminus S$ has $\geq k + 2$ components $\implies G$ has no Hamilton path

Proof. Pigeonhole on visiting vertices in S between connected components. \square

11: Forbidden Minors

Theorem 11.1 (Forbidden minor characterization for K_4 minor, statement only). *A graph with at least three vertices is edge-maximal without a K_4 minor if and only if it can be constructed recursively from triangles by pasting along K_2 's.*

Theorem 11.2 (Forbidden minor characterization of K_5 minor, statement only). *Let G be an edge-maximal graph without a K_5 minor. If $|G| \geq 4$ then G can be constructed recursively, by pasting along triangles and K_2 's, from plane triangulations and copies of the graph W .*

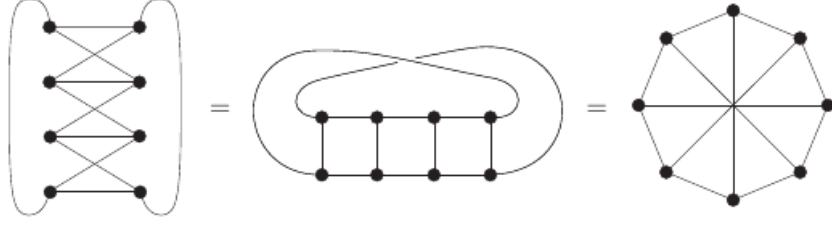


Fig. 7.3.1. Three representations of the Wagner graph W

Conjecture 11.3 (Hadwiger). *If G has no K_r minor, then $\chi(G) < r$.*

12: Random graphs

Definition 12.1 (Erdős-Renyi random graph model). $G_{n,p}$ is the graph-valued random variable drawn from the probability space of graphs on n vertices, where adjacencies are chosen independently, each with probability p .

Theorem 12.2 (Expected size of cliques in $G_{n, \frac{1}{2}}$).

$$\begin{aligned} \Pr[\omega(G_{n, \frac{1}{2}}) \geq k] &\leq \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}} && \text{(Union bound)} \\ &\leq n^k \left(\frac{1}{2}\right)^{k(k-1)/2} \\ &= \left(n \left(\frac{1}{2}\right)^{(k-1)/2}\right)^k \end{aligned}$$

Then only when $n < 2^{(k-1)/2}$, the probability is less than 1.

Theorem 12.3 (Expected size of independent sets in $G_{n,p}$).

$$\begin{aligned}
\Pr[\alpha(G_{n,p}) > r] &\leq \binom{n}{r} (1-p)^{\binom{r}{2}} \\
&= \binom{n}{r} (1-p)^{r(r-1)/2} \\
&\leq n^r (1-p)^{r(r-1)/2} \\
&= \left(n(1-p)^{(r-1)/2} \right)^r \\
&\leq \left(n(e^{-p})^{(r-1)/2} \right)^r \quad (1-p \leq e^{-p}) \\
&\rightarrow 0 \text{ if } n < e^{p(r-1)/2}
\end{aligned}$$

When $n < e^{p(r-1)/2}$, $p > \frac{2 \ln n}{r-1}$. Using $r = \frac{n}{k}$, i.e. $\Pr[\alpha(G_{n,p}) > \frac{n}{k}]$, we get $p > \frac{3 \ln nk}{n}$. So the takeaway is that if you want $\alpha(G) \leq \frac{n}{k}$, set $p > \frac{3 \ln nk}{n}$.

Theorem 12.4 (Chromatic number/girth theorem, statement only). $\forall k, \exists$ graphs G with $\chi(G) = k$ and $\text{girth}(G) \geq k$.

Definition 12.5 (Property $P_{i,j}$ of a graph). A graph has property $P_{i,j}$ if $\forall U, w, U \cap w = \emptyset, |U| = i, |w| = j, \exists v \notin U \cup w, v \sim u \forall u \in U, v \not\sim w \forall w \in W$.

Theorem 12.6 (Rado Graph). Let R be a graph whose vertex set is in \mathbb{N} (the Rado graph), and adjacencies chosen uniformly randomly with probability p . Then

$$\Pr[R \text{ has } P_{i,j} \forall i, j] = 1.$$

Theorem 12.7 (Erdős and Renyi). There exists a unique countable graph R (up to isomorphism) with the property that it satisfies all $P_{i,j}$'s.

13: Expander Graphs

Definition 13.1 (Bipartite Expanders). A (d, β) expander graph is a bipartite graph on $L \sqcup R$ where all vertices in L have degree d and $\forall S \subseteq L$ such that $|S| \leq \frac{n}{d}, |\Gamma(S)| \geq \beta d |S|$.

Remark 13.2. Sometimes instead of $|\Gamma(S)| \geq \beta d |S|$, $|\Gamma(S)| \geq \beta |S|$ is also used.

Definition 13.3 (Expander Graphs). Similar definition to a bipartite expander, except the condition is imposed on all subsets, not just subsets on one side of a bipartite graph.

Remark 13.4 (Existence of Bipartite Expanders). We show that (d, β) expanders exist for $\beta = \frac{1}{4}$.

Proof idea: for each vertex in L , choose a random neighbor in R independently. WThen we show that there are likely to be no bad pairs S, T where $\Gamma(S) \subseteq T, |S| \leq \frac{n}{d}, |T| \leq \frac{d}{4} |S|$ (i.e does not have the expansion property). This can be formulated as

$$\sum_{s=1}^{n/d} \binom{n}{s} \binom{n}{\frac{sd}{4}} \left(\frac{sd}{4n} \right)^{sd},$$

where the summation is performing a union over all sizes of set S , $\binom{n}{s}$ is choosing our S , $\binom{n}{\frac{sd}{4}}$ is choosing T , and the last term is the probability that all neighbors of S are in T .

Perform the computation, you will need to use the inequality $\binom{n}{k} \leq \left(\frac{en}{k} \right)^k$ somewhere. Then eventually you will arrive that the probability is < 1 , which implies that there is such a graph where no bad pair exists, i.e our bipartite expander exists.

Theorem 13.5. *Largest eigenvalue of a d -regular graph is d . The largest eigenvalue of a d -regular graph is d .*

Proposition 13.6 (Multiplicity of largest eigenvalue). *The multiplicity of the largest eigenvalue d of a d -regular graph is the number of connected components in G .*

Theorem 13.7 (Lower bound of expansion parameter, statement only). *Given any $|S| \leq \frac{n}{2}$, we have that*

$$\frac{e(S, \bar{S})}{|S|} \geq \lambda_1 - \lambda_2.$$

Remark 13.8 (Spirit of expanders). A big spectral gap, i.e. $\lambda_1 - \lambda_2$ implies good expansion.

Remark 13.9 (Eigenvalues of the complete bipartite graph $K_{s,t}$). It has eigenvalue 0 with multiplicity $n - 2$, and $\pm\sqrt{st}$.

14: Graph minors and well-quasi-orders

Definition 14.1 (Quasi-order). A quasi order is:

- Reflexive, $x \leq x$
- Transitive, $x \leq y, y \leq z \implies x \leq z$
- Antisymmetric, $x \leq y \wedge y \leq x \implies x = y$

Definition 14.2 (Well-quasi-order). A quasi-order such that in any infinite sequence x_0, x_1, \dots , there exists a “good pair” : an $i < j$ such that $x_i \leq x_j$.

Theorem 14.3 (Sufficient and necessary condition for w.q.o). *A $q.o \leq$ is a w.q.o on $X \iff$ there are no infinite decreasing sequence or infinite antichains.*

Definition 14.4 (Minor-closed properties). A property \mathcal{P} of graphs is minor-closed if every minor of a graph with \mathcal{P} has \mathcal{P} .

So G has \mathcal{P} if and only if it has no minor without \mathcal{P} .

Definition 14.5 (Forbidden Minors). Define $\text{Forb}(\mathcal{P}) = \{H \mid H \text{ does not have } \mathcal{P}\}$.

Define $\overline{\text{Forb}(\mathcal{P})}$ = minimal elements of $\text{Forb}(\mathcal{P})$ with respect to minors.

Lemma 14.6 ($\overline{\text{Forb}(\mathcal{P})}$ forms an antichain). *$\overline{\text{Forb}(\mathcal{P})}$ forms an antichain, since each element is the minimum.*

Theorem 14.7 (Finite subsets of a w.q.o set is also w.q.o under injection ϕ). *Suppose A is w.q.o by \leq . Then $A^{<\omega}$ (finite subsets of A) are w.q.o by the extension of \leq as defined as follows:*

Given $S, T \subseteq A$, there exists an injection ϕ from S to T such that $s \leq \phi(s)$ for all $s \in S$.

Theorem 14.8 (Graph Minor Theorem). *Graphs are well-quasi-ordered by the minor relation.*

Corollary 14.9 (Minor-closed properties are characterized by finite sets of forbidden minors). *Since $\overline{\text{Forb}(\mathcal{P})}$ is an antichain, it must be finite as the minor relationship is a w.q.o. Therefore G has $\mathcal{P} \iff G$ has no $H \in \overline{\text{Forb}(\mathcal{P})}$ as a minor.*