Algorithm Design and Analysis
Assignment 2

1. You are given two sorted lists of size m and n. Give an $O(\log m + \log n)$ time algorithm for computing the k-th smallest element in the union of the two lists.

2. A k-way merge operation. Suppose you have k sorted arrays, each with n elements, and you want to combine them into a single sorted array of kn elements. Design a efficient algorithm using divide-and-conquer.

3. Recall that we have learned how to find the k-th element in a list with a randomized algorithm (randomly choose a pivot), can we do it deterministically? In this exercise, we will develop one called the *Median of Medians* algorithm, invented by Blum, Floyd, Pratt, Rivest, and Tarjan.

   Why we need to pick the *pivot* x randomly in our randomized algorithm? This is to guarantee, at least in expectation, that the numbers less than x and the numbers greater than x are in close proportion. In fact, this task is quite similar to the task of "finding the k-th largest number" itself, and therefore we can bootstrap and solve it recursively!

   Assume we have an array A of n distinct numbers and would like to find its k-th largest number.

   (a) Consider that we line up elements in groups of three and find the median of each group. Let x be the median of these $n/3$ medians. Show that x is close to the median of A, in the sense that a constant fraction of numbers in a is less than x and a constant fraction of numbers is greater than x as well.

   (b) Design a recursive algorithm by the above idea and analyze the running time.

   (c) Can we improve the running time by increasing the number of elements (e.g. 4,5,6?) in each group? What is the best choice? Give the answer and the proof. Note that in this problem, we drop the big-O notation and discuss about the constants.

4. The Hadamard matrices $H_0,\ H_1,\ H_2, \ldots\ldots$ are defined as follows:

   - H_0 is the 1×1 matrix [1].
   - For $k > 0$, H_k is the $2^k \times 2^k$ matrix

$$H_k = \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix}$$

   Show that if $\vec{v}$ is a column vector of length $n = 2^k$, then the matrix-vector product $H_k\vec{v}$ can be calculated using $O(n \log n)$ operations in word RAM.

5. How long does it take you to finish the assignment (include thinking and discussing)? Give a score (1,2,3,4,5) to the difficulty.