Algorithm Design and Analysis
Assignment 6

1. You are given a set S of football teams, where each team $x \in S$ has already accumulated w_x wins so far. For every pair of teams $x, y \in S$ we know that there are g_{xy} games remaining between x and y (that is, if x wins a remaining games against y, then y must win $g_{xy} - a$ remaining games against x). Given a specific team $z \in S$, we would like to decide if z still has a chance to have the maximum number of wins. Alternatively, suppose we have the power to determine who wins each of the remaining games, is there a way such that z would get as many wins as anybody else by the end of the tournament? Give a polynomial-time algorithm for this problem. Hint: We can assume without loss of generality that z wins all remaining games. Is there a way to split the wins of the remaining games among the rest of the teams so that none of them get more wins than z? Consider a reduction to the network flow problem.

Solution. Wlog, we can assume team z wins all the games left and only consider the other games. At first, let us calculate the number of wins z can make finally (use w_z^* to denote the number and $w_z^* = w_z + \sum_{(x \in S)} g_{xz}$). Then, we reduce the problem to a max flow problem by constructing the following network G. (Refer to Figure 1)

1. Construct a source and a sink vertex s and t.
2. For each team x except z, make a vertex v_x and connect it to t via an arc (v_x, t) with capacity $w_z^* - w_x$.
3. For each team pair (x, y) $(x, y \neq z)$ we make a vertex M_{xy} and connect it to v_x and v_y via arcs (M_{xy}, v_x) and (M_{xy}, v_y) (infinity capacity). We also make an arc (s, M_{xy}) with capacity g_{xy} from the source.

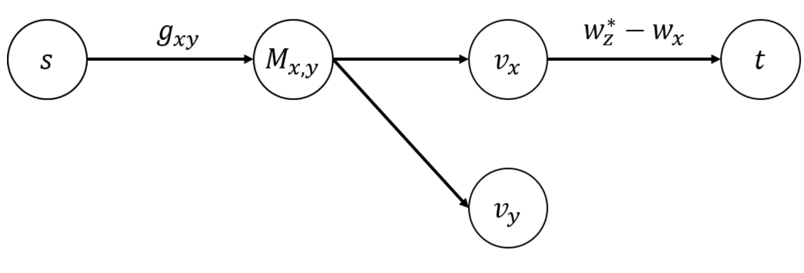

Figure 1: The example of the network.

Roughly speaking, a set of game results can be viewed as a balanced flow in the aforementioned network. One unit of flow goes from s to t via a path (s, M_{xy}, x, t) can be viewed as one game result that x wins one game between x and y. Therefore, consider a flow and a set of game results it represents, the flow on arcs (s, M_{xy}) can be viewed as the number of game results between x and y and the flow on arcs (v_x, t) can be viewed as the number of wins of x. The capacity of (v_x, t) can control the number of wins of x, so that it can not exceed w_z^*, hence it controls the game results are feasible (i.e. i.e. no one wins more than z). Moreover, if the flow is full on each arc (s, M_{xy}) (i.e. $f(e = (s, M_{xy})) = g_{xy}$), it means that the flow can represent a set of game results that contains all the remaining games.

Therefore, we can run the max flow algorithm on G, if and only if all arcs (s, M_{xy}) are full in the max flow, we can find a feasible set of game results that contains all the remaining games. □

2. Let G be a flow network in which each edge e has a capacity $c(e)$ as well as a lower bound $d(e)$ on the flow it must carry. Note that $d(e) \le c(e)$. A feasible flow assigns a value within the range of $[d(e), c(e)]$ to each edge e. Note that assigning a zero flow to every edge may not give a feasible flow of G (due to the lower bound requirement of the edges). Show how to make use a maximum flow algorithm to find a feasible flow of G.

Solution. Let us construct a initialized flow f_0 such that each edge's flow equals to it's lower bound $d(e)$. ($f_0(e) = d(e)$) The problem is that it may be not a feasible flow because the flow of some vertices may be unbalanced. Consider an unbalanced vertex v, let $f_0^{in}(v)$ be the flow go into v and $f_0^{ou}(v)$ be the flow go out of v. Intuitively, to make the flow balanced, if $f_0^{ou}(v) > f_0^{in}(v)$, we should have additional $f_0^{ou}(v) - f_0^{in}(v)$ unit of flow go into vertex v in G. Symmetrically, if $f_0^{ou}(v) < f_0^{in}(v)$, we should have additional $f_0^{in}(v) - f_0^{ou}(v)$ unit of flow go out of vertex v in G.

Follow this intuition, we create a network G' as follows (Refer to Figure 2). Roughly speaking, we want to use it to find a compensation that can make f_0 feasible.

1. Make a source s and a sink t.
2. Copy all vertices and edges from G to G', and set the capacity of each edge e to be $c(e) - d(e)$. (No longer lower bound)
3. For each vertex v with $f_0^{in}(v) > f_0^{ou}(v)$, make a compensation arc (s, v) with capacity $f_0^{in}(v) - f_0^{ou}(v)$.
4. For each vertex v with $f_0^{ou}(v) > f_0^{in}(v)$, make a compensation arc (v, t) with capacity $f_0^{ou}(v) - f_0^{in}(v)$.

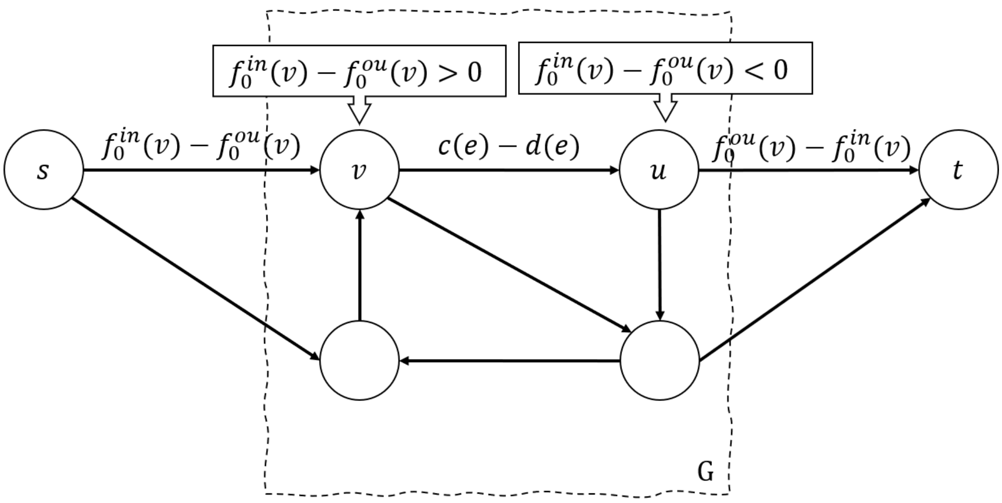

Figure 2: The example of the network.

We claim that G has a feasible flow if and only if all compensation arcs in G' are fully filled in the max flow of G'.

proof of the claim $(\rightarrow)$: If all compensation arcs are full in the max flow f_1 of G', then for each edge e in G, we set $f_2(e) = f_1(e) + f_0(e)$. We have $d(e) \leq f_2(e) \leq c(e)$ simply because the $f_1(e)$ is non-negative and does not exceed $c(e) - d(e)$. Moreover, each vertex is balanced because

$$f_2^{in}(v) - f_2^{ou}(v) = f_1^{in}(v) - f_1^{ou}(v) + f_0^{in}(v) - f_0^{ou}(v) + \begin{cases} -f_1(e = (s, v)) & f_0^{in}(v) - f_0^{ou}(v) > 0 \\ +f_1(e = (v, t)) & f_0^{in}(v) - f_0^{ou}(v) < 0 \\ 0 & f_0^{in}(v) - f_0^{ou}(v) = 0. \end{cases}$$

Notice that it equals zero in all the cases, and hence f_2 is balanced on all vertices.

$(\leftarrow)$: On the other hand, if we have a feasible flow f_2 for G, we will construct the following flow f_1 in G'. (we will prove it's a feasible flow in G' where all compensation arcs are full, and hence it's a max flow.)

For each e in G, let $f_1(e) = f_2(e) - d(e) = f_2(e) - f_0(e)$. For each compensation arc (s, v) (or (v, t)), we set them be full (i.e. $f_1(e) = f_0^{in}(v) - f_0^{ou}(v)$ (or $f_0^{ou}(v) - f_0^{in}(v)$)). For each v other than s and t in G', if $f_0^{in}(v) - f_0^{ou}(v) > 0$, $f_1^{in}(v) - f_1^{ou}(v) = f_2^{in}(v) - f_2^{ou}(v) - f_0^{in}(v) + f_0^{ou}(v) + f_1(e = (s, v)) = 0$. We have it equals to 0 because $f_1(e = (s, v)) = f_0^{in}(v) - f_0^{ou}(v)$. For those v such that $f_0^{ou}(v) - f_0^{in}(v) < 0$, we can also prove $f_1^{in}(v) - f_1^{ou}(v) = 0$ symmetrically. Therefore, we prove that if there is a feasible flow in G, there is at least a feasible flow where all compensation arcs are full, such as f_1.

$\square$

3. How long does it take you to finish the assignment (include thinking and discussing)? Give a score (1,2,3,4,5) to the difficulty.