MXB106 Take Home Assignment 4

Question 1

Consider a spherical object whose steady state temperature distribution is described by the equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}u(r)\right) = 0$$

on the domain $1 < r < 2$. Solve for the temperature distribution with boundary conditions

$$u(1) = 0.5 \quad \text{and} \quad u'(2) = 0.5$$

and plot the temperature distribution over the relevant domain.

Question 2

Consider the following non-homogeneous ODE

$$y''(t) - 8y'(t) + 16y(t) = \kappa\delta(t)$$

where κ is a constant and $\delta(t)$ is the Dirac delta function. Solve this ODE using Laplace transforms with the boundary conditions $y(0) = \alpha$ and $y'(0) = \beta$.

Question 3

Consider the second order differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Given that $y(x) = y_1(x)$ is a known solution to this ODE, find a second solution, $y_2(x)$ in terms of $y_1(x)$, $p(x)$, and $q(x)$.

Question 4

Solve

$$y''(x) + 9y(x) = 0$$

where $y(0) + y'(0) = 0$ and $y(\pi) - y'(\pi) = 10$. Plot the solution on the domain $0 < x < \pi$.

Question 5

Solve the following differential equation

$$y''(x) + y'(x) = \frac{1}{1+e^x}$$

via variation of parameters.

Question 6

Given that $y_1(x)$ and $y_2(x)$ satisfy the differential equation

$$f_1(x)y''(x) + f_2(x)y'(x) + f_3(x)y(x) = 0$$

show that

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

also satisfies this differential equation.

Question 7

Find the values of α such that the system

$$\vec{x}'(t) = \begin{pmatrix} 3 & -9 \\ \alpha & -3 \end{pmatrix} \vec{x}(t)$$

will have periodic solutions.

Question 8

Determine the asymptotic stability of the solution $\vec{x} = (0,0)^T$ for the system

$$\vec{x}'(t) = \begin{pmatrix} c & 1 \\ -4 & 3 \end{pmatrix} \vec{x}(t)$$

for $c = 5, 7,$ and 10.

Question 9

Find the Laplace transform of

$$t^n f'(t)$$

where $n \geq 1$.

Question 10

The fundamental set solutions of a third order ODE whose characteristic polynomial has one real root and two complex roots is

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = e^{\alpha t}\cos(\beta t), \quad \text{and} \quad y_3(t) = e^{\alpha t}\sin(\beta t)$$

where $\lambda_{1,2} = \alpha \pm \beta i$. Prove that $y_1(t)$, $y_2(t)$, and $y_3(t)$ indeed form the fundamental set of solutions, i.e. prove they are linearly independent.