A Multiscale Finite Element Method for Elliptic Problems in Composite Materials and Porous Media

BigSmall

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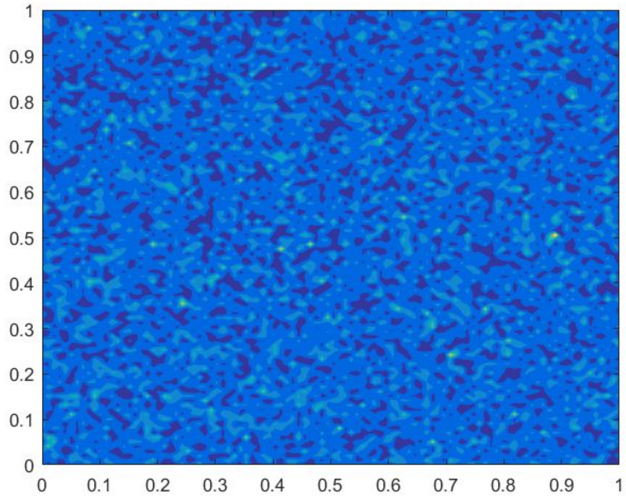
The Problem

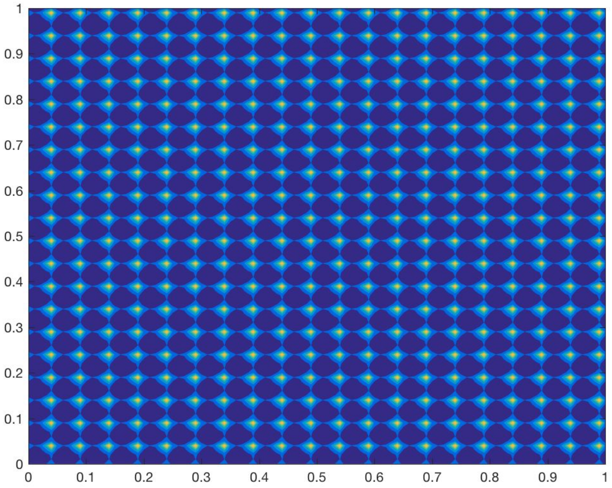
$$\begin{cases} -\nabla \cdot a \nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$a(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$$

Heterogeneous Media

a(**x**) can be random or periodic





Periodical Media

$$\begin{cases} -\nabla \cdot a(\frac{x}{\epsilon})\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

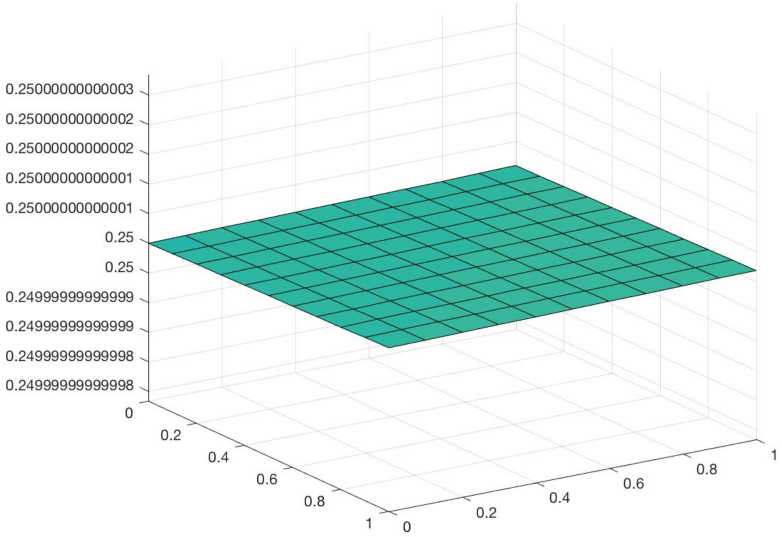
a(**y**) periodic in **y** with period Y and smooth

Hard to Capture (Aliasing)

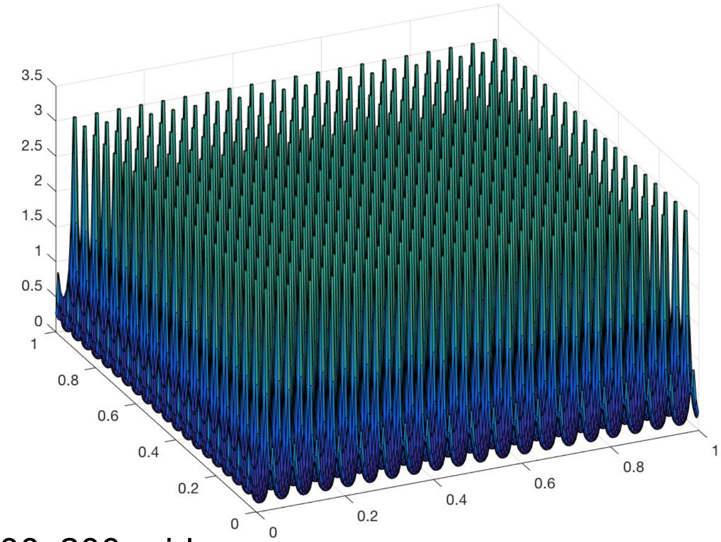
$$P = 1.5$$

$$\epsilon = 0.05$$

$$a(\mathbf{x}) = \frac{1}{(1 + Psin(2\pi x/\epsilon))(1 + Psin(2\pi y/\epsilon))}$$



10x10 grid

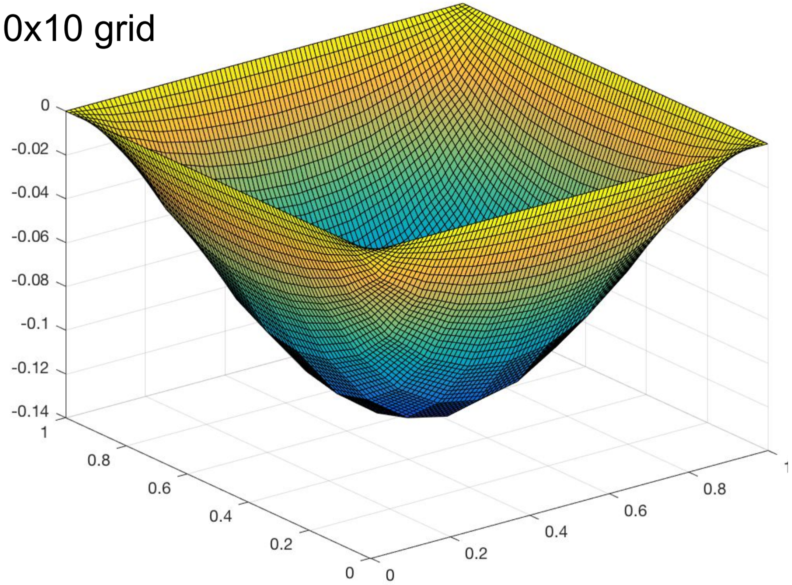


200x200 grid

For the solvers to not see the fictitious aliasing behavior, we need sampling rate higher twice the frequency of a(**x**)

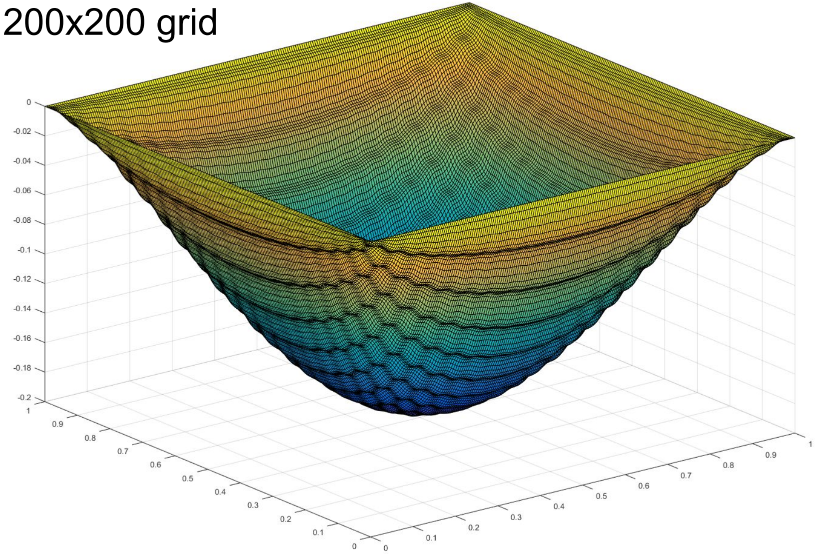
Difficulty with Traditional FEM

10x10 grid



Time: ≈ 2.79 s
Mem: ≈ 78.13 KB

200x200 grid



Time: ≈ 223.71 s
Mem: ≈ 11.92 GB

Intuition

- Difficulties
  - Finer grid => Size of problem is tremendous.
  - Coarser grid => Bad accuracy
    - Discretization doesn’t capture the small scale information imbedded in the operator $-\nabla \cdot a\nabla$.

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  - Finer grid => Size of problem is tremendous.
  - Coarser grid => Bad accuracy
    - Discretization doesn’t capture the small scale information imbedded in the operator $-\nabla \cdot a\nabla$.
- Solution
  - Solve the problem on courser grid, but
  - Capture the fine features in a(**x**) into to the basis function and thus include these info into the discrete operator.

Basis Construction

Construct basis functions ϕ for a cell based on PDE local to the cell (adaptive to structure in a(**x**)):

$$-\nabla \cdot a \nabla \phi_K^i = 0$$

In element K for node i

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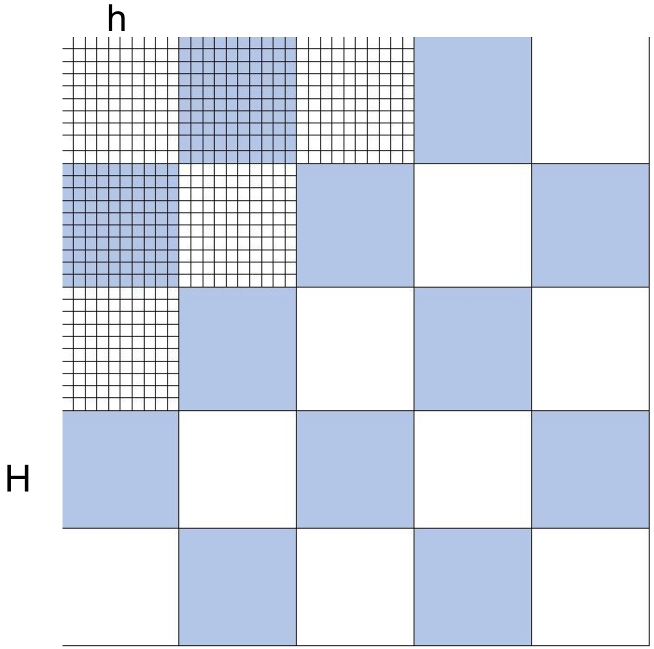


Basis Boundaries

"The reduced problems are obtained by deleting terms with partial derivatives in the direction normal to ∂K and having the coordinate normal to ∂K as a parameter."

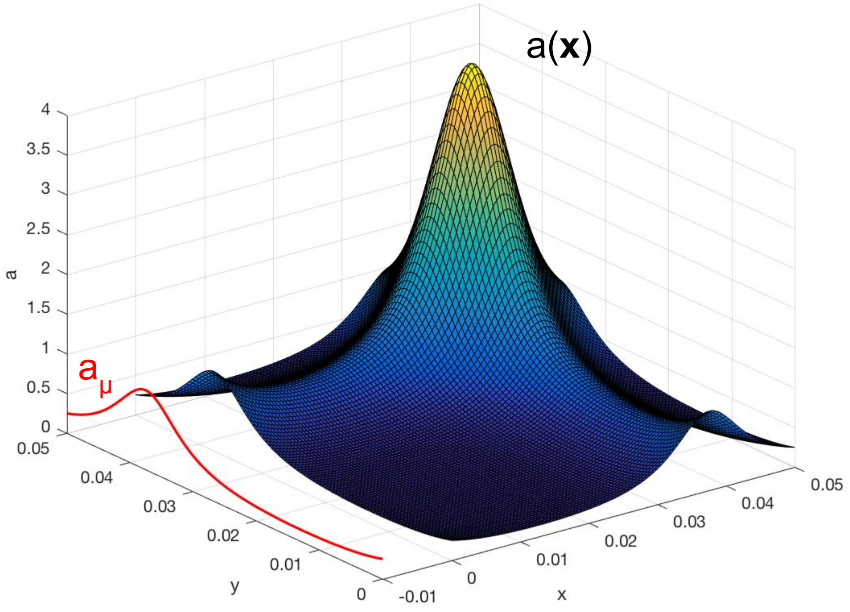
Basis Boundaries

On a rectangular cell, for example

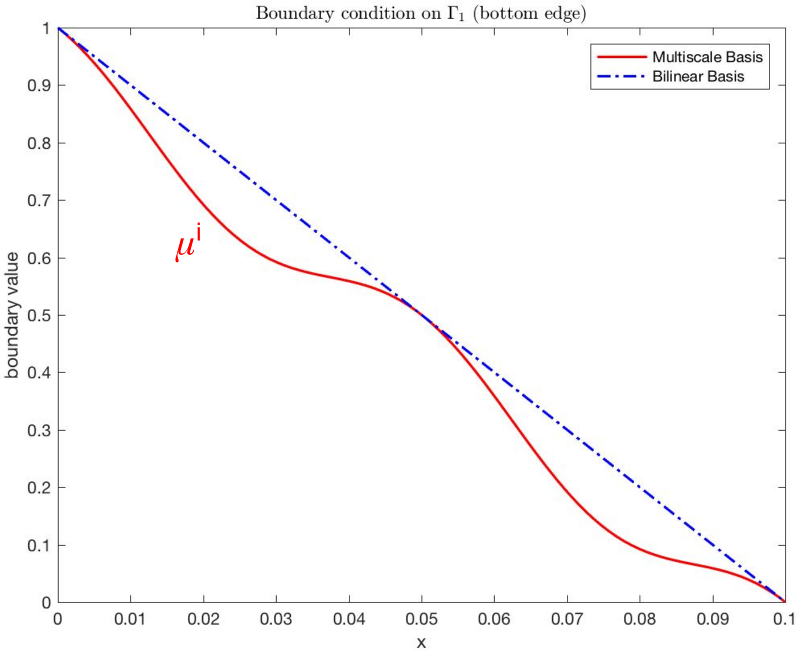
Obtaining a boundary condition μ^i along y direction

$$\frac{\partial}{\partial y} a_\mu(y) \frac{\partial \mu^i(y)}{\partial y} = 0$$

$$\mu^i(y_i) = 1, \mu^i(y_j) = 0$$



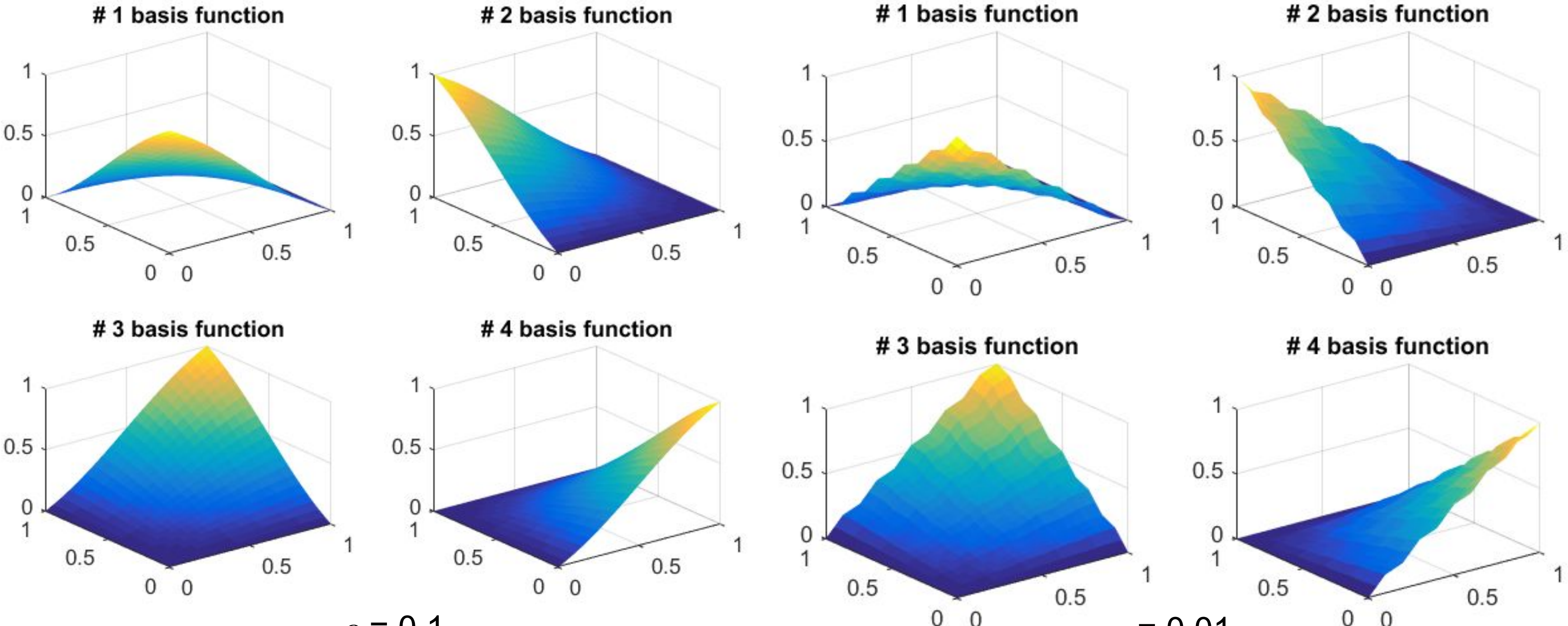
Basis Boundaries



Basis Construction

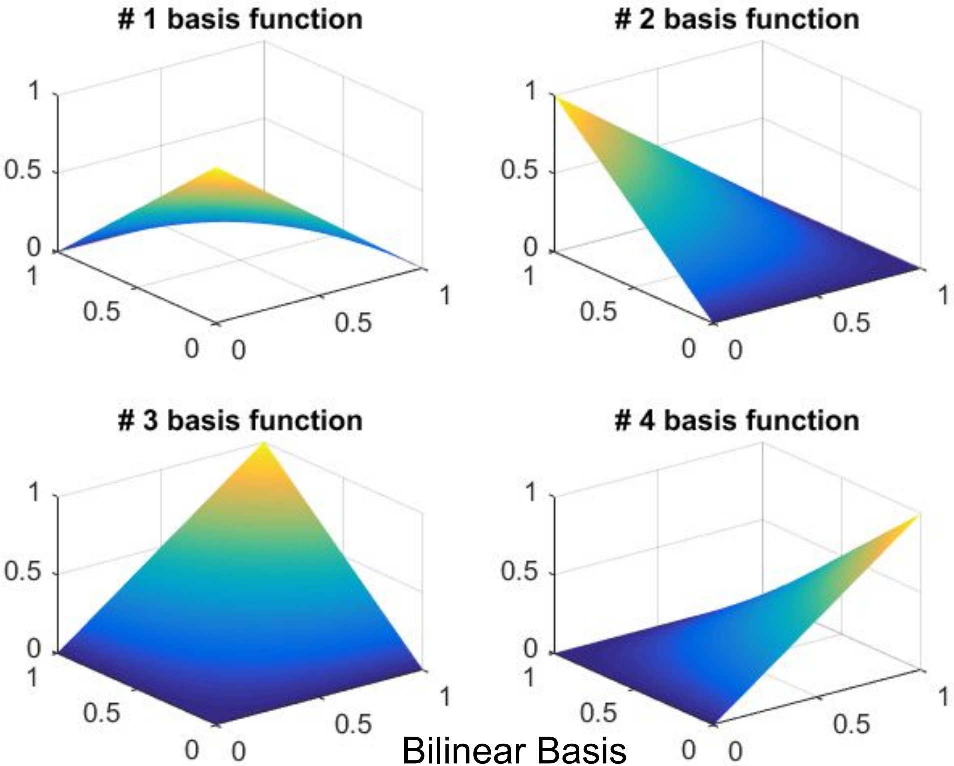
$$a(\mathbf{x}) = \frac{1}{(1 + Psin(2\pi x/\epsilon))(1 + Psin(2\pi y/\epsilon))}$$

$$P = 1.5$$



ε = 0.1 ε = 0.01

Basis Construction



Bilinear Basis

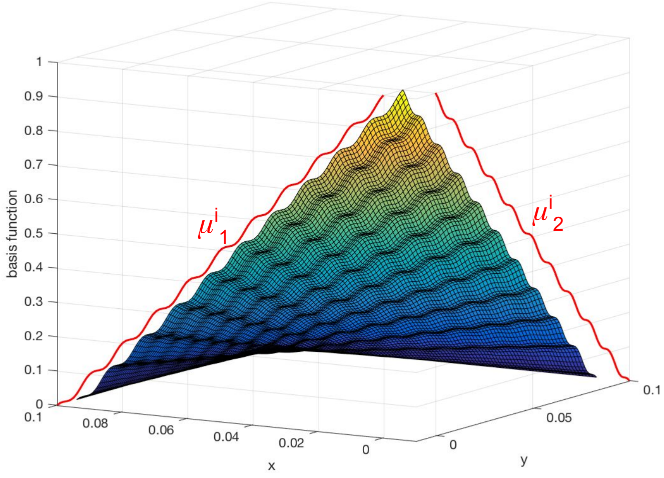
Basis Function Properties

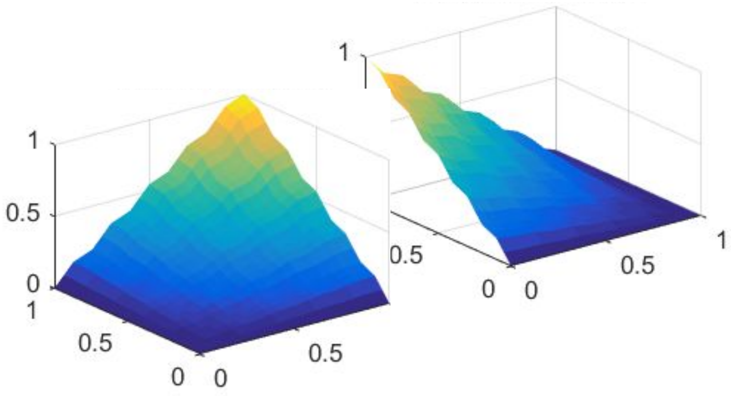
For rectangular cells

- If a(**x**) is separable (i.e. a(x, y) = b(x)c(y)).

$$\phi^i(x,y) = \mu_1^i(x)\mu_2^i(y)$$

- $\sum_i \phi_i = 1$
- ϕ_i are compatible across the boundaries.





Variational form

Seek for $u \in H_0^1(\Omega)$

Such that $b(u, v) = l(v) \forall v \in H_0^1(\Omega)$

Where
$$b(u, v) = \int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i},$$
$$l(v) = \int_\Omega f v dx$$

Seek for $u \in V_h$

Such that $b(u, v) = l(v) \forall v \in V_h$

Where
$$b(u, v) = \int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i},$$
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Assemble A and f

$$A_{ij} = \sum_{K \in \mathscr{K}_{ij}} \int_K (\nabla \phi_K^i)^T a \nabla \phi_K^j dx$$

$$f_i = \sum_{K \in \mathscr{K}_i} \int_K f \phi_K^i dx$$

We assemble our A in a node based fashion
Where curly Kij and Ki are set of elements containing node i,j

Numerical Experiment

Media

$$a(\mathbf{x}) = \frac{1}{(1 + 1.5sin(2\pi x/0.05))(1 + 1.5sin(2\pi y/0.05))}$$

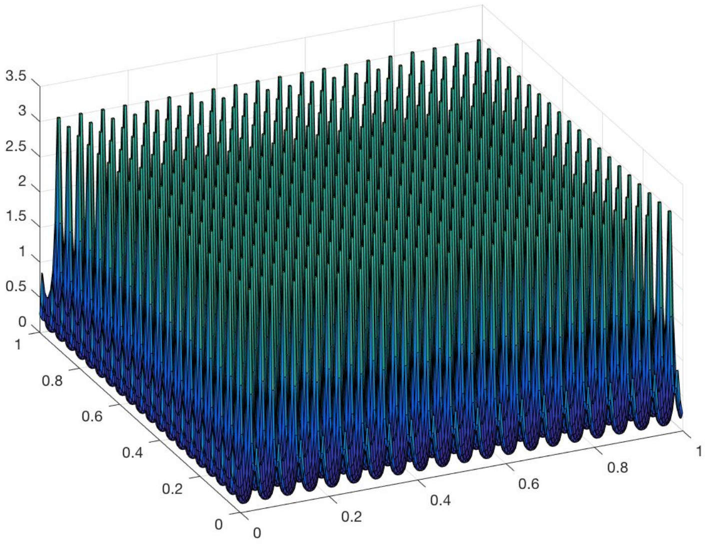
Grid

[0,1]x[0,1] Rectangular domain.

Machine

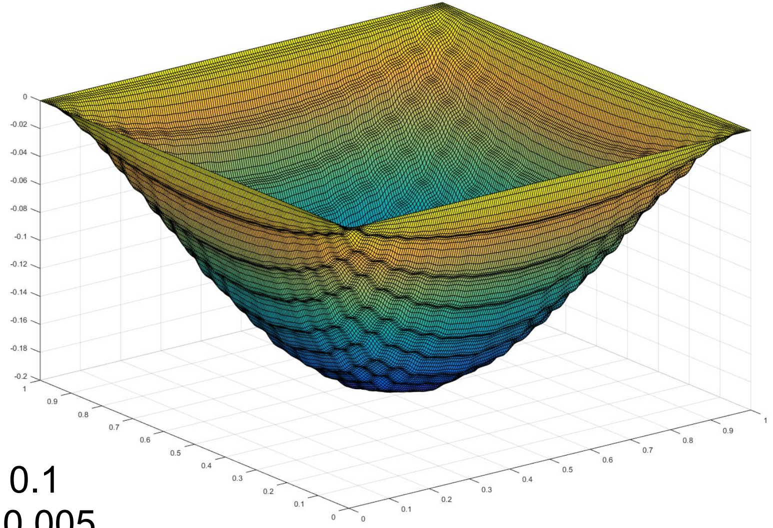
intel i7-6700k

16 Gb Memory



Results

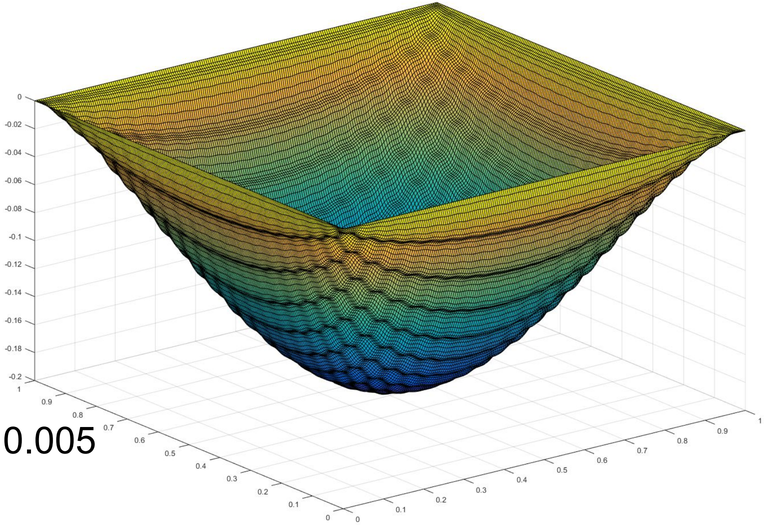
Multi-Scale



H = 0.1
h = 0.005

Time: 27.01 s / 6.93 s
Mem: 1.22 Mb

Bilinear base FEM

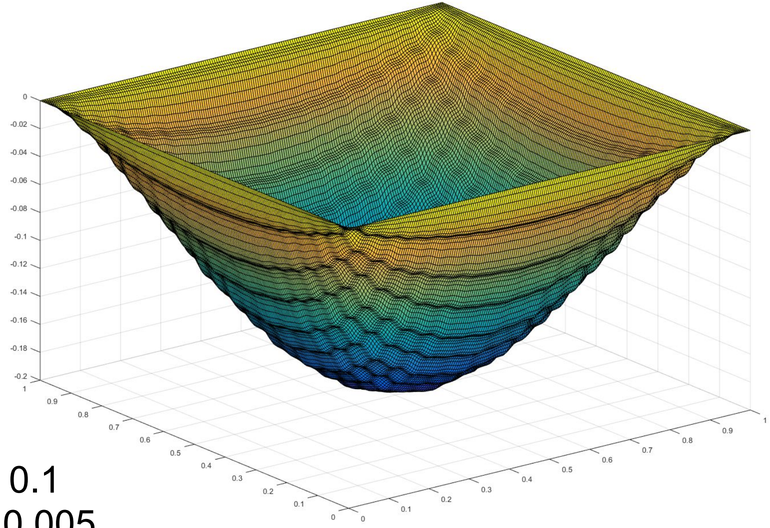


h = 0.005

Time: 223.75 s
Mem: 11.92 Gb

Results

Multi-Scale



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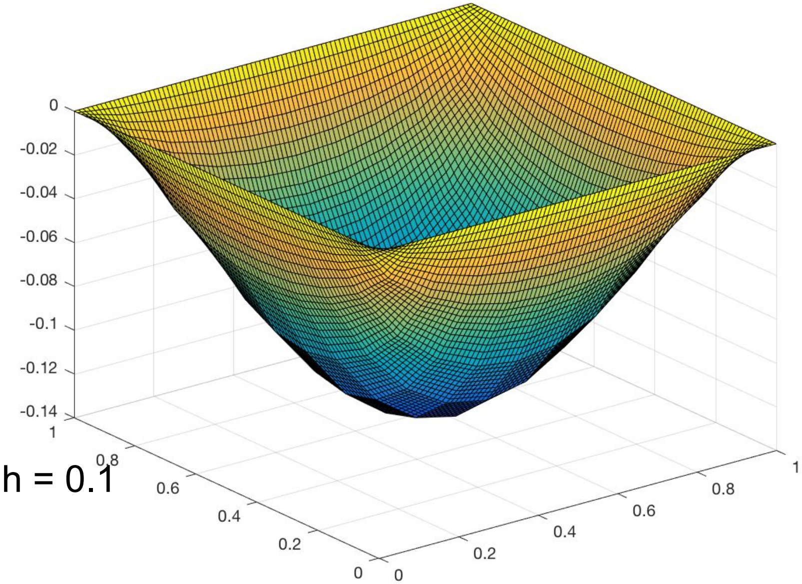

h = 0.1

Time: 2.71 s
Mem: 78.13 Kb

Discussion

In general

- Basis construction is highly parallelizable, since only the local info of the cell is required to compute the basis functions.
- The nodal basis function only overlaps among neighboring nodes (A sparse)
- Spatial complexity is $O(M^d + N^d)$.
    - If A is assembled in a cell-based fashion, the basis in previous cells can be dumped in a serial implementation.

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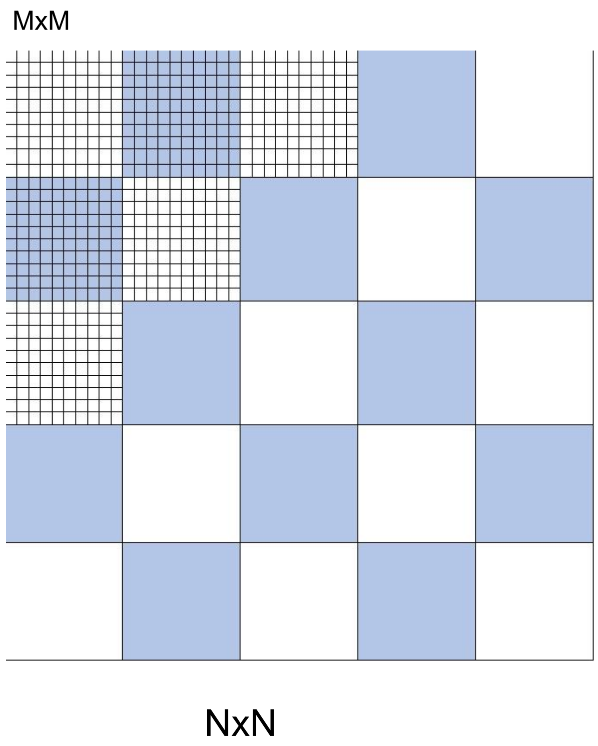

Discussion

Periodicity

- If discretized at integer multiples of period (i.e. $H = k\varepsilon$). The set of basis functions are the same in each cell.
- Space complexity $O(M^d + N^d)$

$a(x,y)$ Separable ($a(x, y) = b(x)c(y)$) with rectangular cells

- Don’t need to solve the PDE on the cell.

Q&A

Thanks!