Survey on available model problems for the elliptic HMM and MsFEM

December 6, 2012

In the following $\Omega \subset \mathbb{R}^2$ is a given domain that is related to a fixed model problem. The parameter ϵ is a (variable) given constant. The larger ϵ, the coarser the structure, the smaller ϵ the finer the structure. It can take any value in a given range or with given restrictions.

Model Problem 1 (ONE - *linear* - *periodic*)**.**
Let $\Omega :=]0,2[^2$ and $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x) \nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the scalar diffusion coefficient A^ϵ is given by

$$A^\epsilon(x_1, x_2) := 2 + \sin(2\pi \frac{x_1}{\epsilon}).$$

The exact solution is unknown.

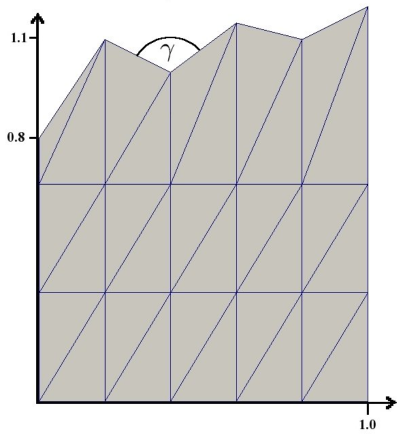
Model Problem 2 (TWO - *linear* - *heterogeneous*)**.**
Let Ω be as depicted in the figure below and let $\epsilon > 0$. Find u^ϵ with

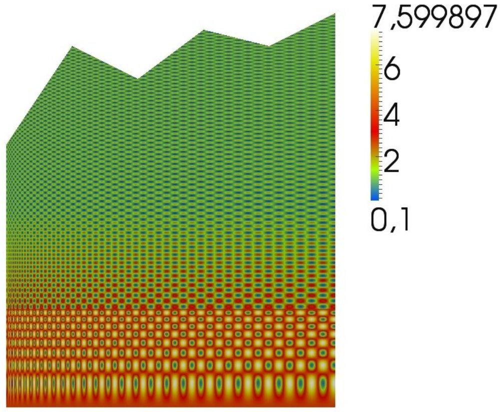
$$\begin{aligned} -\nabla \cdot (A^\epsilon(x) \nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the scalar diffusion coefficient A^ϵ is given by

$$A^\epsilon(x_1, x_2) :=:= \begin{cases} 4 + \frac{18}{5}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \leq 0.3 \\ 1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \geq 0.6 \\ (3 - \frac{10x_2}{3}) \cdot \left(1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2)\right) & \text{else.} \end{cases}$$

The exact solution is unknown.





In the left figure we see the computational domain Ω, as well as a corresponding initial triangulation (the coarsest macro grid). In the right figure we display the diffusion coefficient A^ϵ for $\epsilon = 0.05$. It rapidly takes values between 0.1 and 7.6.

Model Problem 3 (THREE - *nonlinear* - *heterogeneous*)**.**
This is a nonlinear version of TWO. Let Ω be as depicted in the figure below and let $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x \nabla u^\epsilon(x))) &= f(x) \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } x_2 \leq \frac{1}{10} \\ 1 & \text{else.} \end{cases}$$

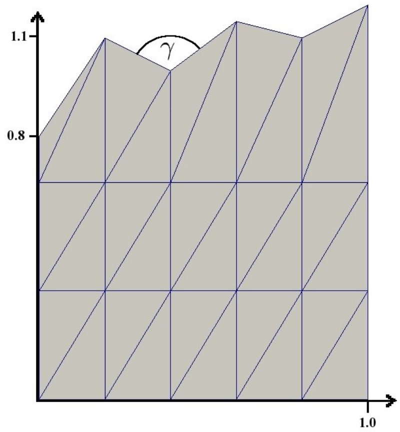
where the nonlinear diffusion operator A^ϵ is given by

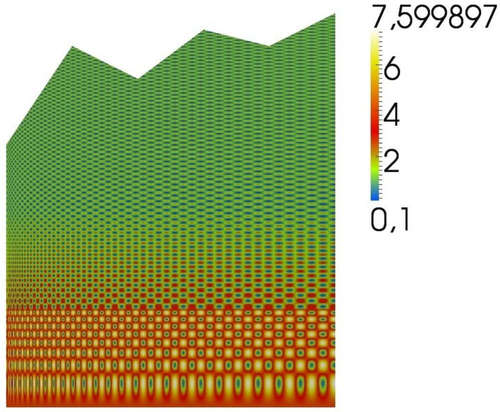
$$A^\epsilon(x, \xi) := c^\epsilon(x_1, x_2) \begin{pmatrix} \xi_1 + \frac{1}{3}\xi_1^3 \\ \xi_2 + \frac{1}{3}\xi_2^3 \end{pmatrix},$$

with

$$c^\epsilon(x_1, x_2) := \begin{cases} 4 + \frac{18}{5}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \leq 0.3 \\ 1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \geq 0.6 \\ (3 - \frac{10x_2}{3}) \cdot \left(1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2)\right) & \text{else.} \end{cases}$$

The exact solution is unknown.





In the left figure we see the computational domain Ω, as well as a corresponding initial triangulation (the coarsest macro grid). In the right figure we display the diffusion coefficient A^ϵ for $\epsilon = 0.05$. It rapidly takes values between 0.1 and 7.6.

Model Problem 4 (Four)**.**

Model Problem 5 (Five)**.**

Model Problem 6 (Six)**.**

Model Problem 7 (Seven)**.**

Model Problem 8 (EIGHT - *nonlinear* - *periodic* - *exact solution*)**.** Let $\Omega :=]0,1[^2$ and ϵ such that $\epsilon^{-1} \in \mathbb{N}$. The problem reads: find u^ϵ with

$$\begin{aligned} -\nabla \cdot A^\epsilon(x, \nabla u^\epsilon(x)) &= f(x) \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with

$$f(x) := -\sum_{i,j=1, i\neq j}^{2} 2(x_i - x_j^2) - 12(2x_i - 1)^2(x_j^2 - x_j)^3$$

and where the nonlinear diffusion operator A^ϵ is given by

$$A^\epsilon(x,\xi) := \begin{pmatrix} \xi_1 + (2 + \sin(2\pi \frac{x_1+x_2}{\epsilon}))\xi_1^3 - d_{12}^\epsilon(x) \\ \xi_2 + (2 + \sin(2\pi \frac{x_1+x_2}{\epsilon}))\xi_2^3 - d_{21}^\epsilon(x) \end{pmatrix},$$

with

$$\begin{aligned} h_{ij}^\epsilon(x) &:= \Big(3(2x_i - 1)(x_j^2 - x_j) + 3(x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big) \\ &\quad \cdot (2x_i - 1)(x_j^2 - x_j)(x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon}); \\ g_{ij}^\epsilon(x) &:= (2 + \sin(2\pi\frac{x_i + x_j}{\epsilon}))\left(h_{ij}^\epsilon(x) + \Big((x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big)^3\right); \\ d_{ij}^\epsilon(x) &:= (x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon}) + \sin(2\pi\frac{x_i + x_j}{\epsilon})(2x_i - 1)^3(x_j^2 - x_j)^3 + g_{ij}^\epsilon(x). \end{aligned}$$

The solution u^ϵ of this problem has an asymptotic expansion $u^\epsilon(x) = u_0(x) + \epsilon u_1(x, \frac{x}{\epsilon})$ where

$$u_0(x) = -(x_1^2 - x_1)(x_2^2 - x_2) \quad \text{and} \quad u_1(x, y) = -(x_1 + x_2)\sin(2\pi y_1)\sin(2\pi y_2).$$

Model Problem 9 (NINE - *linear - periodic - exact solution*)**.**
Let $\Omega :=]0, 1[^2$ and ϵ such that $\epsilon^{-1} \in \mathbb{N}$. The problem reads: find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x)\nabla u^\epsilon(x)) &= f^\epsilon(x) \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}$$

where A^ϵ is given by

$$A^\epsilon(x_1, x_2) := \frac{1}{8\pi^2} \begin{pmatrix} 2(2 + \cos(2\pi\frac{x_1}{\epsilon}))^{-1} & 0 \\ 0 & 1 + \frac{1}{2}\cos(2\pi\frac{x_1}{\epsilon}) \end{pmatrix}$$

and f^ϵ by

$$f^\epsilon(x) := -\nabla \cdot (A^\epsilon(x)\nabla v^\epsilon(x)) \approx \sin(2\pi x_1)\sin(2\pi x_2)$$

with

$$v^\epsilon(x_1, x_2) := \sin(2\pi x_1)\sin(2\pi x_2) + \frac{\epsilon}{2}\cos(2\pi x_1)\sin(2\pi x_2)\sin(2\pi\frac{x_1}{\epsilon}).$$

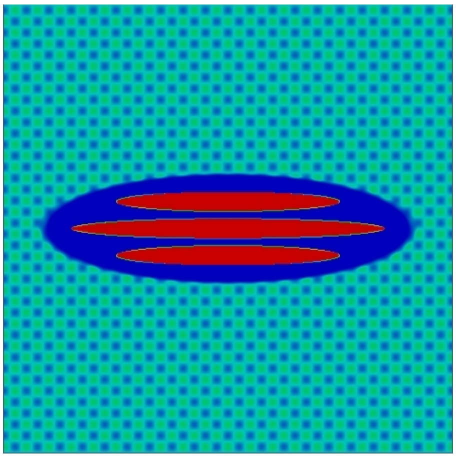
The exact solution is known and we have

$$u^\epsilon(x) = v^\epsilon(x).$$

Model Problem 10 (TEN - *linear - heterogeneous*)**.**
Let $\Omega :=]0, 1[^2$ and $\epsilon > 0$. The problem reads: find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x)\nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

Here, the synthetic scalar coefficient A^ϵ is depicted in the figure below for the special choice $\epsilon = 5 \cdot 10^{-2}$. For small ϵ, A^ϵ is rapidly oscillating in an outer region. In an inner region, the the conductivity is very low ($5 \cdot 10^{-4}$) but still contains layers of constant high conductivity ($5 \cdot 10^{-2}$).



In the figure we can see a plot of the diffusion coefficient A^ϵ. The colorshading is from red (0.05) to blue (0.0005). The micro structure outside the inner patch is periodic and given by $(8\pi^2)^{-2}\left(1 + 2^{-1}\cos(2\pi\frac{x_0}{\epsilon})\sin(2\pi\frac{x_1}{\epsilon})\right)$ with $\epsilon = 5 \cdot 10^{-2}$. The transition is smooth. The exact solution is unknown.
