Survey on available model problems for the elliptic HMM and MsFEM

December 8, 2012

In the following $\Omega \subset \mathbb{R}^2$ is a given domain that is related to a fixed model problem. The parameter ϵ is a (variable) given constant. The larger ϵ, the coarser the structure, the smaller ϵ the finer the structure. It can take any value in a given range or with given restrictions.

Model Problem 1 (ONE - *linear* - *periodic*)**.**
Let $\Omega :=]0,2[^2$ and $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x)\nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the scalar diffusion coefficient A^ϵ is given by

$$A^\epsilon(x_1,x_2) := 2 + \sin(2\pi \frac{x_1}{\epsilon}).$$

The exact solution is unknown.

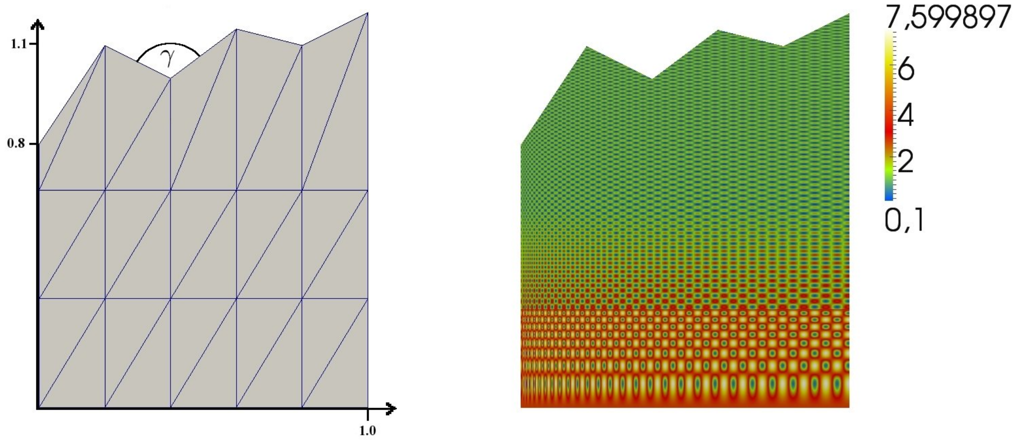
Model Problem 2 (TWO - *linear* - *heterogeneous*)**.**
Let Ω be as depicted in the figure below and let $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x)\nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the scalar diffusion coefficient A^ϵ is given by

$$A^\epsilon(x_1,x_2) :=:= \begin{cases} 4 + \frac{18}{5}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \leq 0.3 \\ 1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \geq 0.6 \\ (3 - \frac{10x_2}{3}) \cdot \Big(1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2)\Big) & \text{else.} \end{cases}$$

The exact solution is unknown.



In the left figure we see the computational domain Ω, as well as a corresponding initial triangulation (the coarsest macro grid). In the right figure we display the diffusion coefficient A^ϵ for $\epsilon = 0.05$. It rapidly takes values between 0.1 and 7.6.

Model Problem 3 (THREE - *nonlinear* - *heterogeneous*).
This is a nonlinear version of TWO. Let Ω be as depicted in the figure below and let $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x\nabla u^\epsilon(x))) = f(x) \quad & \text{in } \Omega \\ u^\epsilon(x) = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } x_2 \leq \frac{1}{10} \\ 1 & \text{else.} \end{cases}$$

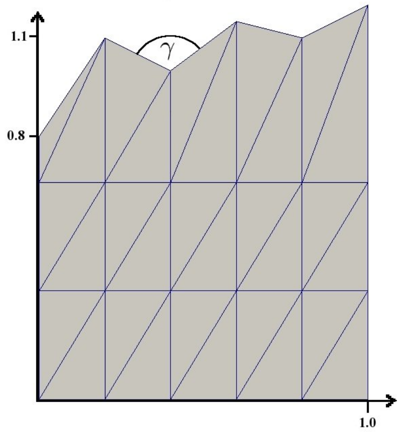
where the nonlinear diffusion operator A^ϵ is given by

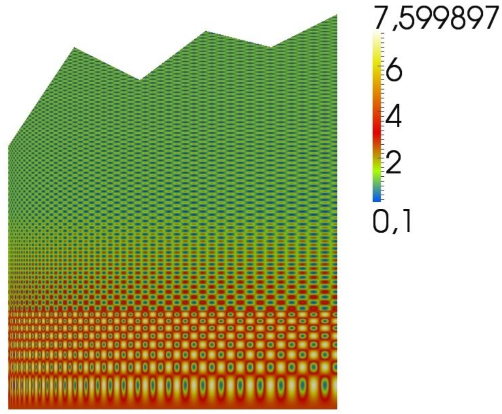
$$A^\epsilon(x,\xi) := c^\epsilon(x_1,x_2) \begin{pmatrix} \xi_1 + \frac{1}{3}\xi_1^3 \\ \xi_2 + \frac{1}{3}\xi_2^3 \end{pmatrix},$$

with

$$c^\epsilon(x_1,x_2) := \begin{cases} 4 + \frac{18}{5}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \leq 0.3 \\ 1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2) & \text{if } x_2 \geq 0.6 \\ (3 - \frac{10x_2}{3}) \cdot \left(1 + \frac{9}{10}\sin(2\frac{\pi}{\epsilon}\sqrt{|2x_1|})\sin(\frac{9}{2\epsilon}\pi x_2^2)\right) & \text{else.} \end{cases}$$

The exact solution is unknown.





In the left figure we see the computational domain Ω, as well as a corresponding initial triangulation (the coarsest macro grid). In the right figure we display the diffusion coefficient A^{ϵ} for $\epsilon = 0.05$. It rapidly takes values between 0.1 and 7.6.

Model Problem 4 (FOUR - *linear* - *periodic* - *stochastic perturbation*)**.**
Let $\Omega :=]0,1[^2 \backslash]\frac{1}{2},1[^2$ (L-shaped domain / corner singularity) and $\epsilon > 0$. Find u^{ϵ} with

$$\begin{aligned} -\nabla \cdot (A^{\epsilon}(x) \nabla u^{\epsilon}(x)) = f(x) \quad & \text{in } \ \Omega \\ u^{\epsilon}(x) = 0 \quad & \text{on } \ \partial\Omega, \end{aligned}$$

where (for $B_{\frac{1}{5}}(\frac{1}{2},\frac{1}{2})$ being the circle of radius $\frac{1}{5}$ around $(\frac{1}{2},\frac{1}{2})$)

$$f(x) = \begin{cases} 1 & \text{if } \ x \in B_{\frac{1}{5}}(\frac{1}{2},\frac{1}{2}) \\ \frac{1}{10} & \text{else} \end{cases}$$

and with the diffusion matrix A^{ϵ} given by

$$A^{\epsilon}(x_1,x_2) := \begin{pmatrix} \frac{1}{10} + \cos(2\pi \frac{x_1}{\epsilon})^2 + X(x) - E(X) & 0 \\ 0 & \frac{101}{1000} + \frac{1}{10}\sin(2\pi \frac{x_2}{\epsilon}) + X(x) - E(X) \end{pmatrix}$$

where X is a log-normal distributed random variable, with a variance σ and expected value $E(X) = e^{\frac{\sigma^2}{2}}$.

Model Problem 5 (FIVE - *nonlinear* - *periodic* - *stochastic perturbation*)**.**
This is a nonlinear version of FOUR. Let $\Omega :=]0,1[^2 \backslash]\frac{1}{2},1[^2$ (L-shaped domain / corner singularity) and $\epsilon > 0$. Find u^{ϵ} with

$$\begin{aligned} -\nabla \cdot A^{\epsilon}(x, \nabla u^{\epsilon}(x)) = f(x) \quad & \text{in } \ \Omega \\ u^{\epsilon}(x) = 0 \quad & \text{on } \ \partial\Omega, \end{aligned}$$

where (for $B_{\frac{1}{5}}(\frac{1}{2},\frac{1}{2})$ being the circle of radius $\frac{1}{5}$ around $(\frac{1}{2},\frac{1}{2})$)

$$f(x) = \begin{cases} 1 & \text{if } \ x \in B_{\frac{1}{5}}(\frac{1}{2},\frac{1}{2}) \\ \frac{1}{10} & \text{else} \end{cases}$$

and with the nonlinear diffusion operator $A^\epsilon(x,\xi)$ given by

$$A^\epsilon(x_1,x_2,\xi_1,\xi_2) := \begin{pmatrix} \left(\frac{1}{10} + \cos(2\pi\frac{x_1}{\epsilon})^2 + X(x) - E(X)\right)\left(\xi_1 + \frac{1}{3}\xi_1^3\right) \\ \left(\frac{101}{1000} + \frac{1}{10}\sin(2\pi\frac{x_2}{\epsilon}) + X(x) - E(X)\right)\left(\xi_2 + \frac{1}{3}\xi_2^3\right) \end{pmatrix}$$

where X is a log-normal distributed random variable, with a variance σ and expected value $E(X) = e^{\frac{\sigma^2}{2}}$.

Model Problem 6 (SIX - *linear* - *periodic* - *stochastic perturbation*)**.**
Let $\Omega :=]0,2[^2$ and $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot (A^\epsilon(x)\nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the diffusion coefficient A^ϵ is given by

$$A^\epsilon(x_1,x_2) := \frac{101}{100} + \cos(2\pi\frac{x_1}{\epsilon}) + X(x) - E(X)$$

where X is a log-normal distributed random variable, with a variance σ and expected value $E(X) = e^{\frac{\sigma^2}{2}}$.

Model Problem 7 (SEVEN - *nonlinear* - *periodic* - *stochastic perturbation*)**.**
This is a nonlinear version of SIX. Let $\Omega :=]0,2[^2$ and $\epsilon > 0$. Find u^ϵ with

$$\begin{aligned} -\nabla \cdot A^\epsilon(x,\nabla u^\epsilon(x)) &= 1 \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the nonlinear diffusion operator $A^\epsilon(x,\xi)$ is given by

$$A^\epsilon(x_1,x_2,\xi_1,\xi_2) := \left(\frac{101}{100} + \cos(2\pi\frac{x_1}{\epsilon}) + X(x) - E(X)\right)\begin{pmatrix} \xi_1 + \frac{1}{3}\xi_1^3 \\ \xi_2 + \frac{1}{3}\xi_2^3 \end{pmatrix},$$

where X is a log-normal distributed random variable, with a variance σ and expected value $E(X) = e^{\frac{\sigma^2}{2}}$.

Model Problem 8 (EIGHT - *nonlinear* - *disturbed periodic* - *exact solution*)**.** Let $\Omega :=]0,1[^2$ and ϵ such that $\epsilon^{-1} \in \mathbb{N}$. The problem reads: find u^ϵ with

$$\begin{aligned} -\nabla \cdot A^\epsilon(x,\nabla u^\epsilon(x)) &= f(x) \quad \text{in } \Omega \\ u^\epsilon(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with

$$f(x) := -\sum_{i,j=1,i\neq j}^{2} 2(x_i - x_j^2) - 12(2x_i-1)^2(x_j^2 - x_j)^3$$

and where the nonlinear diffusion operator A^ϵ is given by

$$A^\epsilon(x,\xi) := \begin{pmatrix} \xi_1 + (2+\sin(2\pi\frac{x_1+x_2}{\epsilon}))\xi_1^3 - d_{12}^\epsilon(x) \\ \xi_2 + (2+\sin(2\pi\frac{x_1+x_2}{\epsilon}))\xi_2^3 - d_{21}^\epsilon(x) \end{pmatrix},$$

with

$$\begin{aligned}
h^{\epsilon}_{ij}(x) &:= \Big(3(2x_i-1)(x_j^2-x_j)+3(x_i+x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big)\\
&\qquad\cdot(2x_i-1)(x_j^2-x_j)(x_i+x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon});\\
g^{\epsilon}_{ij}(x) &:= (2+\sin(2\pi\frac{x_i+x_j}{\epsilon}))\left(h^{\epsilon}_{ij}(x)+\Big((x_i+x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big)^3\right);\\
d^{\epsilon}_{ij}(x) &:= (x_i+x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})+\sin(2\pi\frac{x_i+x_j}{\epsilon})(2x_i-1)^3(x_j^2-x_j)^3+g^{\epsilon}_{ij}(x).
\end{aligned}$$

The solution u^{ϵ} of this problem has an asymptotic expansion $u^{\epsilon}(x)=u_0(x)+\epsilon u_1(x,\frac{x}{\epsilon})$ where

$$u_0(x)=-(x_1^2-x_1)(x_2^2-x_2) \quad \text{and} \quad u_1(x,y)=-(x_1+x_2)\sin(2\pi y_1)\sin(2\pi y_2).$$

Model Problem 9 (NINE - *linear - periodic - exact solution*)**.**
Let $\Omega :=]0,1[^2$ and ϵ such that $\epsilon^{-1}\in\mathbb{N}$. The problem reads: find u^{ϵ} with

$$\begin{aligned}
-\nabla\cdot(A^{\epsilon}(x)\nabla u^{\epsilon}(x)) = f^{\epsilon}(x) &\quad \text{in } \Omega\\
u^{\epsilon}(x)=0 &\quad \text{on } \partial\Omega,
\end{aligned}$$

where A^{ϵ} is given by

$$A^{\epsilon}(x_1,x_2) := \frac{1}{8\pi^2}\begin{pmatrix} 2(2+\cos(2\pi\frac{x_1}{\epsilon}))^{-1} & 0\\ 0 & 1+\frac{1}{2}\cos(2\pi\frac{x_1}{\epsilon})\end{pmatrix}$$

and f^{ϵ} by

$$f^{\epsilon}(x) := -\nabla\cdot(A^{\epsilon}(x)\nabla v^{\epsilon}(x)) \approx \sin(2\pi x_1)\sin(2\pi x_2)$$

with

$$v^{\epsilon}(x_1,x_2) := \sin(2\pi x_1)\sin(2\pi x_2)+\frac{\epsilon}{2}\cos(2\pi x_1)\sin(2\pi x_2)\sin(2\pi\frac{x_1}{\epsilon}).$$

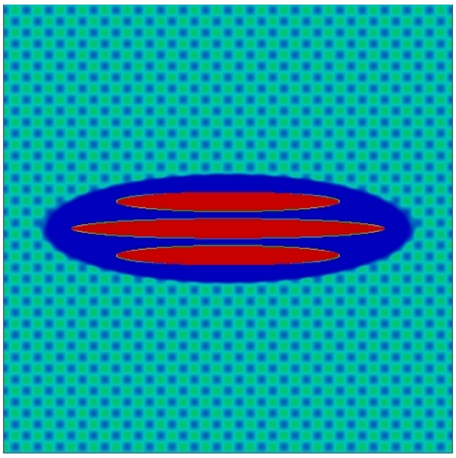
The exact solution is known and we have

$$u^{\epsilon}(x)=v^{\epsilon}(x).$$

Model Problem 10 (TEN - *linear - heterogeneous*)**.**
Let $\Omega :=]0,1[^2$ and $\epsilon>0$. The problem reads: find u^{ϵ} with

$$\begin{aligned}
-\nabla\cdot(A^{\epsilon}(x)\nabla u^{\epsilon}(x)) = 1 &\quad \text{in } \Omega\\
u^{\epsilon}(x)=0 &\quad \text{on } \partial\Omega,
\end{aligned}$$

Here, the synthetic scalar coefficient A^{ϵ} is depicted in the figure below for the special choice $\epsilon=5\cdot10^{-2}$. For small ϵ, A^{ϵ} is rapidly oscillating in an outer region. In an inner region, the the conductivity is very low ($5\cdot10^{-4}$) but still contains layers of constant high conductivity ($5\cdot10^{-2}$).



In the figure we can see a plot of the diffusion coefficient A^ϵ. The colorshading is from red (0.05) to blue (0.0005). The micro structure outside the inner patch is periodic and given by $(8\pi^2)^{-2}\left(1+2^{-1}\cos(2\pi\frac{x_0}{\epsilon})\sin(2\pi\frac{x_1}{\epsilon})\right)$ with $\epsilon = 5\cdot 10^{-2}$. The transition is smooth. The exact solution is unknown.