# VIO-hw3

VIO

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- 1.LM算法
  - 1.1 µ迭代
  - $1.2 y = ax^2 + bx + c$
  - 1.3 阻尼因子更新策略
- 2 公式推导 46/77
  - 2.1 f 15
  - 2.2 g\_12
- 3. 证明式9

### 作业

- 1 样例代码给出了使用 LM 算法来估计曲线  $y = \exp(ax^2 + bx + c)$  参数 a,b,c 的完整过程。
  - $oldsymbol{1}$  请绘制样例代码中 LM 阻尼因子  $\mu$  随着迭代变化的曲线图
  - ② 将曲线函数改成  $y = ax^2 + bx + c$ , 请修改样例代码中残差计算, 雅克比计算等函数, 完成曲线参数估计。
  - 3 如果有实现其他阻尼因子更新策略可加分(选做)。
- 2 公式推导,根据课程知识,完成  $\mathbf{F},\mathbf{G}$  中如下两项的推导过程:

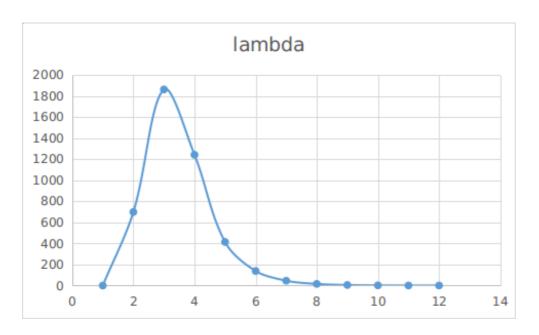
$$\mathbf{f}_{15} = \frac{\partial \boldsymbol{\alpha}_{b_i b_{k+1}}}{\partial \delta \mathbf{b}_k^g} = -\frac{1}{4} (\mathbf{R}_{b_i b_{k+1}} [(\mathbf{a}^{b_k} - \mathbf{b}_k^a)]_{\times} \delta t^2) (-\delta t)$$

$$\mathbf{g}_{12} = \frac{\partial \boldsymbol{\alpha}_{b_i b_{k+1}}}{\partial \mathbf{n}_k^g} = -\frac{1}{4} (\mathbf{R}_{b_i b_{k+1}} [(\mathbf{a}^{b_k} - \mathbf{b}_k^a)]_{\times} \delta t^2) (\frac{1}{2} \delta t)$$

3 证明式(9)。

# 1.LM算法

## 1.1 µ迭代



# $1.2 y = ax^2 + bx + c$

param	optimization parameters	ground truth
a	10.6107	10
b	19.6183	20
С	9.99517	10

#### 输出:

```
1.  10 20 10
2.  Test CurveFitting start...
3.  iter: 0 , chi= 61493.7 , Lambda= 0.001
4.  iter: 1 , chi= 91.3952 , Lambda= 0.000333333
5.  iter: 2 , chi= 91.395 , Lambda= 0.000222222
6.  problem solve cost: 3.28693 ms
7.  makeHessian cost: 2.33691 ms
8.  -----After optimization, we got these parameters:
9.  10.6107 19.6183 9.99517
10.  -----ground truth:
11.  10, 20, 10
```

### 1.3 阻尼因子更新策略

Cui, M., Zhao, Y., Xu, B., & Gao, X. W. . (2017). A new approach for determining damping factors in levenberg-marquardt algorithm for solving an inverse heat conduction problem. International Journal of Heat and Mass Transfer, 107, 747-754.

- 1.Initialize the weights and parameter  $\mu$ ,  $\beta$
- 2.Stop if the number of iteration exceeds the maximum iteration or F(w) is less than a desired error
- 3. After passing all training data, compute the sum of squared errors over all inputs, F(w)

$$F(w)$$
即 loss function. $w$ 即正常的自变量 $x$ 

- 4.Compute the Jacobian matrix J(w)
- 5.Solve Eq. (6) to obtain the weight change  $\Delta w$

Eq.(6)

$$egin{align} \Delta \mathbf{w} &= - \Big[ \mathbf{J}(\mathbf{w})^T \mathbf{J}(\mathbf{w}) + \mu \mathbf{I} \Big]^{-1} \mathbf{J}^T(\mathbf{w}) \mathbf{e}(\mathbf{w}) \ &= - [\mathbf{D}(\mathbf{w})]^{-1} \mathbf{g}(\mathbf{w}) \end{aligned}$$

6.Recompute the sum of squared errors  $F(w_{trial})$  using  $w_{trial} = w + \Delta w$  after passing all training data again, and judge

IF  $F(w_{trial}) < F(w)$  in step 3 THEN

$$wip = rac{\mathrm{dot}(\mathbf{w}_{\mathrm{trial}}, \mathbf{w})}{\|\mathbf{w}_{\mathrm{trial}}\| \|\mathbf{w}\|} \quad (-1 \leq wip \leq 1)$$

$$\mu = \mu \cdot eta^{wip} \qquad if \ F(\mathbf{w}_{ ext{trial}}) < F(\mathbf{w})$$

go back to step 2

**ELSE** 

if D(w) is positive definite

$$\mu = \mu/\beta$$

else

make D(w) diagonally dominant

$$\mathbf{D} = d_{ij}, \quad d_{ii} = \sum_{j 
eq i} \lvert d_{ij} 
vert \quad ext{for all } i$$

$$\mu = \min(d_{ii}) \quad ext{ for all } i$$

go back to step 2

# 2 公式推导 46/77

2.1 f<sub>\_15</sub>

$$\mathbf{f}_{15} = rac{\partial oldsymbol{lpha}_{b_i b_{k+1}}}{\partial \delta \mathbf{b}_k^g} = - \, rac{1}{4} \, \Big( \mathbf{R}_{b_i b_{k+1}} \, \Big[ \Big( \mathbf{a}^{b_k} - \mathbf{b}_k^a \Big) \Big] imes \delta t^2 \Big) (-\delta t)$$

解:

$$\mathbf{f}_{15} = rac{\partial (lpha_{b_ib_k} + eta_{b_ib_k} \delta t + rac{1}{2} \, \mathbf{a} \delta t^2)}{\partial \delta \mathbf{b}_k^g}$$

其中 ,  $(lpha_{b_ib_k}+eta_{b_ib_k}\delta t)$ 与导数无关。

$$\mathbf{f}_{15} = rac{\partial (rac{1}{2}\,\mathbf{a}\delta t^2)}{\partial \delta \mathbf{b}_k^g}$$

其中,
$$\mathbf{a}=rac{1}{2}\left(\mathbf{q}_{b_ib_k}ig(\mathbf{a}^{b_k}-\mathbf{b}_k^aig)+\mathbf{q}_{b_ib_{k+1}}ig(\mathbf{a}^{b_{k+1}}-\mathbf{b}_k^aig)
ight)$$

$$\mathbf{f}_{15} = rac{\partial (rac{1}{2}\,rac{1}{2}\left(\mathbf{q}_{b_ib_k}ig(\mathbf{a}^{b_k}-\mathbf{b}_k^aig)+\mathbf{q}_{b_ib_{k+1}}ig(\mathbf{a}^{b_{k+1}}-\mathbf{b}_k^aig)
ight)\delta t^2)}{\partial \delta \mathbf{b}_k^g}$$

其中, $\mathbf{q}_{b_ib_k} \left(\mathbf{a}^{b_k} - \mathbf{b}_k^a\right)$ 与导数无关。

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial {\left( {\mathbf{q}_{b_i b_{k+1}} \left( {{\mathbf{a}^{b_{k+1}}} - {\mathbf{b}_k^a}} 
ight)} 
ight)} \delta t^2}{\partial \delta {\mathbf{b}_k^g}}$$

其中,
$$\mathbf{q}_{b_ib_{k+1}} = \mathbf{q}_{b_ib_k} \otimes egin{bmatrix} 1 \ rac{1}{2} \, \omega \delta t \end{bmatrix} \ \omega = rac{1}{2} \left( \left( \omega^{b_k} - \mathbf{b}_k^g 
ight) + \left( \omega^{b_{k+1}} - \mathbf{b}_k^g 
ight) 
ight).$$

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial \left( \mathbf{q}_{b_i b_{k+1}} \left( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a 
ight) 
ight) \delta t^2}{\partial \delta \mathbf{b}_k^g}$$

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial \mathbf{q} b_i b_k \otimes igg[ rac{1}{rac{1}{2} \, \omega \delta t} igg] \otimes igg[ rac{1}{-rac{1}{2} \, \delta \mathbf{b}_k^g \delta t} igg] ig( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a ig) \delta t^2}{\partial \delta \mathbf{b}_k^g}$$

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial \mathbf{R}_{b_i b_{k+1}} \, \mathrm{exp} \Big( igl[ -\delta \mathbf{b}_k^g \delta t igr]_{ imes} \Big) igl( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a igr) \delta t^2}{\partial \delta \mathbf{b}_k^g}$$

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial \mathbf{R}_{b_i b_{k+1}} \Big( \mathbf{I} + igl[ -\delta \mathbf{b}_k^g \delta t igr]_{ imes} \Big) igl( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a igr) \delta t^2}{\partial \delta \mathbf{b}_k^g}$$

$$\mathbf{f}_{15} = rac{1}{4} \, rac{\partial - \mathbf{R}_{b_i b_{k+1}} \Big( ig[ ig( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a ig) \delta t^2 ig]_ imes \Big) ig( - \delta \mathbf{b}_k^g \delta t ig)}{\partial \delta \mathbf{b}_k^g}$$

$$\mathbf{f}_{15} = -\,rac{1}{4}\left(\mathbf{R}_{b_ib_{k+1}}\left[\left(\mathbf{a}^{b_{k+1}}\,-\,\mathbf{b}_k^a
ight)
ight]_ imes\delta t^2
ight)(-\delta t)$$

2.2 g\_12

$$egin{align*} \mathbf{g}_{12} &= rac{\partial oldsymbol{lpha}_{b_i b_{k+1}}}{\partial \mathbf{n}_k^g} = -rac{1}{4} \left( \mathbf{R}_{b_i b_{k+1}} \left[ \left( \mathbf{a}^{b_k} - \mathbf{b}_k^a 
ight) 
ight]_ imes \delta t^2 
ight) \left( rac{1}{2} \, \delta t 
ight) \ \mathbf{g}_{12} &= rac{\partial (lpha_{b_i b_k} + eta_{b_i b_k} \delta t + rac{1}{2} \, \mathbf{a} \delta t^2)}{\partial \delta \mathbf{n}_b^g} \end{split}$$

部分公式代换同上(其实是敲起来好麻烦... -\_-|)

$$egin{align*} \mathbf{g}_{12} &= rac{1}{4} rac{\partial \mathbf{q} b_i b_k \otimes \left[ rac{1}{rac{1}{2} \, \omega \delta t} 
ight] \otimes \left[ rac{1}{rac{1}{4} \, \delta \mathbf{n}_k^g \delta t} 
ight] (\mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a) \delta t^2}{\partial \delta \mathbf{n}_k^g} \ & \mathbf{g}_{12} &= rac{1}{4} rac{\partial \mathbf{R}_{b_i b_{k+1}} \, \exp \left( \left[ rac{1}{2} \, \delta \mathbf{n}_k^g \delta t 
ight]_{ imes} 
ight) (\mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a) \delta t^2}{\partial \delta \mathbf{n}_k^g} \ & \mathbf{g}_{12} &= rac{1}{4} rac{\partial \mathbf{R}_{b_i b_{k+1}} \left( \mathbf{I} + \left[ rac{1}{2} \, \delta \mathbf{n}_k^g \delta t 
ight]_{ imes} 
ight) (\mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a) \delta t^2}{\partial \delta \mathbf{n}_k^g} \ & \mathbf{g}_{12} &= rac{1}{4} rac{\partial - \mathbf{R}_{b_i b_{k+1}} \left( \left[ \left( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a 
ight) \delta t^2 
ight]_{ imes} 
ight) \left( rac{1}{2} \, \delta \mathbf{n}_k^g \delta t 
ight)}{\partial \delta \mathbf{n}_k^g} \ & \mathbf{g}_{12} &= -rac{1}{4} \left( \mathbf{R}_{b_i b_{k+1}} \left[ \left( \left[ \left( \mathbf{a}^{b_{k+1}} - \mathbf{b}_k^a 
ight) \delta t^2 
ight)_{ imes} \delta t^2 
ight) \left( rac{1}{2} \, \delta t 
ight) 
ight. \end{split}$$

## 3. 证明式9

需要注意以下几点:

$$\mathbf{f}(\mathbf{x}) = egin{bmatrix} f_1(\mathbf{x}) \ \dots \ f_m(\mathbf{x}) \end{bmatrix}_{m*1}$$

$$rac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{J} = egin{bmatrix} \mathbf{J}_1(\mathbf{x}) \ \dots \ \mathbf{J}_m(\mathbf{x}) \end{bmatrix}_{m*1}$$

 $\mathbf{f}(\mathbf{x})$ , $\mathbf{J}$ 是一个m\*1的列向量

所以 
$$F'(\mathbf{x}) = \left(\mathbf{J}^{\top}\mathbf{f}\right)^{\top} = \left(\mathbf{J}^{\top}\mathbf{f}\right)^{\top}_{1*m*m*1=1*1}$$
 是一个标量(所以在最后的答案里面可以放在分子的左边或者右边都是可以的)。-----式(1)

推导如下:

$$egin{aligned} \left(\mathbf{J}^{ op}\mathbf{J} + \mu\mathbf{I}
ight) \Delta\mathbf{x}_{ ext{lm}} &= -\mathbf{J}^{ op}\mathbf{f} \ & \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{ op} + \mu\mathbf{I}
ight) \Delta\mathbf{x}_{ ext{lm}} &= -\mathbf{J}^{ op}\mathbf{f} \end{aligned}$$
 $(2) => \left(\mathbf{V}(\mathbf{\Lambda} + \mu\mathbf{I})\mathbf{V}^{ op}\right) \Delta\mathbf{x}_{ ext{lm}} &= -\mathbf{J}^{ op}\mathbf{f} \end{aligned}$ 
 $(\mathbf{\Lambda} + \mu\mathbf{I})\mathbf{V}^{ op}\Delta\mathbf{x}_{ ext{lm}} &= -\mathbf{V}^{ op}\mathbf{J}^{ op}\mathbf{f} \end{aligned}$ 
 $(1) => \left(\mathbf{\Lambda} + \mu\mathbf{I}\right)\mathbf{V}^{ op}\Delta\mathbf{x}_{ ext{lm}} &= -\mathbf{V}^{ op}F'(\mathbf{x})^{ op} \end{aligned}$ 
 $\mathbf{V}^{ op}\Delta\mathbf{x}_{ ext{lm}} &= -(\mathbf{\Lambda} + \mu\mathbf{I})^{-1}\mathbf{V}^{ op}F'(\mathbf{x})^{ op} \end{aligned}$ 

$$\mathbf{V}^ op \Delta \mathbf{x}_ ext{lm} = - \left[ egin{array}{cccc} rac{1}{\lambda_1 + \mu} & 0 & 0 & \dots \ 0 & rac{1}{\lambda_2 + \mu} & 0 & \dots \ & \dots & rac{1}{\lambda_j + \mu} \end{array} 
ight] \mathbf{V}^ op F'(\mathbf{x})^ op$$

$$egin{aligned} \Delta \mathbf{x}_{ ext{lm}} = - [\mathbf{v}_1 \dots \mathbf{v}_2] \left[ egin{array}{cccc} rac{1}{\lambda_1 + \mu} & 0 & 0 & \dots \ 0 & rac{1}{\lambda_2 + \mu} & 0 & \dots \ & & rac{1}{\lambda_j + \mu} \end{array} 
ight] \left[ egin{array}{c} \mathbf{v}_1 \ \dots \ \mathbf{v}_2 \end{array} 
ight] F'(\mathbf{x})^ op \end{aligned}$$

$$\Delta \mathbf{x}_{ ext{lm}} = -\sum_{j=1}^n rac{\mathbf{v}_j^ op \mathbf{F}'^ op}{\lambda_j + \mu} \, \mathbf{v}_j$$