Due: 26th of August 2018 at 11:59pm

COMP 9020 – Assignment 1

Note: In your assignment, *how* you arrived at your solution is as important (if not more so) than the solution itself and will be assessed accordingly. There may be more than one way to find a solution, and your approach should contain enough detail to justify its correctness. Lecture content can be assumed to be common knowledge.

1. (a) Compute $\gcd(132, 84)$.
   (b) Suppose $a, b \in \mathbb{N}$ are co-prime. What is $\gcd(a, a+b)$?

2. For sets A and B, define $A * B$ to be $(A \cup B)^c$ (the complement of $A \cup B$).
   (a) Simplify $(A * B) * (A * B)$. Justify your answer (e.g. using a Venn diagram or some other technique).
   (b) Express A^c using A and $*$. Justify your answer.
   (c) Express $A \cap B$ using A, B, and $*$. Justify your answer.

3. (a) List all possible functions $f : \{a, b, c\} \to \{0, 1\}$
   (b) Describe a connection between your answer for (a) and $\mathrm{Pow}(\{a, b, c\})$.
   (c) In general, if $\mathrm{card}(A) = m$ and $\mathrm{card}(B) = n$, how many:
      (i) functions are there from A to B?
      (ii) relations are there between A and B?

4. Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* : 3|\mathrm{length}(w)\}$.
   (a) List the elements of $L^{\leq 3}$ in lexicographic order.

   Define $R \subseteq \Sigma^* \times \Sigma^*$ as follows: $(w, w') \in R$ if there is a $v \in \Sigma^*$ such that: either $wv \in L$ and $w'v \notin L$, or $wv \notin L$ and $w'v \in L$. For example $(a, bbb) \in R$ because for $v = \lambda$, $av = a \notin L$ and $bbbv = bbb \in L$. On the other hand, $(a, b) \notin R$ because for any $v \in \Sigma^*$, $\mathrm{length}(av) = \mathrm{length}(bv)$; so whenever $av \in L$, $bv \in L$ and vice-versa.

   (b) Which of the following are elements of R:
      (i) $(abab, baba)$?
      (ii) $(ab, abab)$?
      (iii) (λ, b)?
      (iv) (λ, bb)?
      (v) (λ, bbb)?

Now define $S \subseteq \Sigma^* \times \Sigma^*$ as the complement of R. That is $(w, w') \in S$ if $(w, w') \notin R$.

(b) State a simple rule for determining if $(w, w') \in S$.

(c) Show that S is an equivalence relation. That is, show that S is reflexive, symmetric, and transitive.

(d) How many equivalence classes does S have?

Due: 16th of September 2018 at 11:59pm

COMP 9020 – Assignment 2

Note: In your assignment, *how* you arrived at your solution is as important (if not more so) than the solution itself and will be assessed accordingly. There may be more than one way to find a solution, and your approach should contain enough detail to justify its correctness. Lecture content can be assumed to be common knowledge.

1. If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ are binary relations, the *composition* of R_1 and R_2 is the relation $R_1; R_2$ defined as:

   $R_1; R_2 := \{(a, c) : \text{There exists } b \in T \text{ such that } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$

   (a) If $f : S \to T$ and $g : T \to U$ are functions is $f; g$ a function?

   (b) If $R \subseteq S \times S$ is transitive, show that $R = R \cup (R; R)$. (*Hint: One way to show $A = B$ is to show $A \subseteq B$ and $B \subseteq A$. One of these directions is trivial.*)

   Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

   $$\begin{aligned} R^0 &:= R, \text{ and} \\ R^{i+1} &:= R^i \cup (R^i; R) \text{ for } i \geq 0 \end{aligned}$$

   (c) Prove that if $R^i = R^{i+1}$ for some i, then $R^i = R^j$ for all $j \geq i$.

   (d) Prove that if $R^i = R^{i+1}$ for some i, then $R^k \subseteq R^i$ for all $k \geq 0$.

   (e) If $|S| = n$, explain why $R^n = R^{n+1}$. (*Hint: Show that if $(a, b) \in R^{n+1}$ then $(a, b) \in R^i$ for some $i < n + 1$.*)

   In the above sequence, R^n is defined to be the *transitive closure* of R, denoted R^* (closely related to the * operator used to describe the set of all words over an alphabet).

   (f) Show that R^* is transitive.

   (20 marks)

2. The following table describes several subjects and the students taking them:

Potions	**Charms**	**Herbology**	**Astronomy**	**Transfiguration**
Harry	Ron	Harry	Hermione	Hermione
Ron	Luna	George	Neville	Fred
Malfoy	Ginny	Neville	Seamus	Luna

You have been tasked to create an examination timetable for these subjects, and your goal is to find the *smallest number* of timeslots needed so that all subjects can be examined, without any conflicts occurring (i.e. no students having to take two or more exams at the same time).

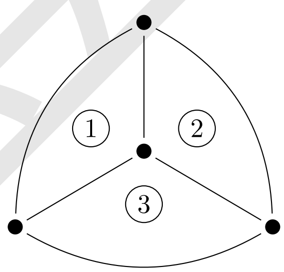
(a) Explain how this can be formulated as a graph-based problem. That is, describe what the vertices and edges would be, and how to relate the given problem to a common graph problem.

(b) For this problem in particular determine the minimum number of timeslots required.

(c) Suppose instead your goal was to determine the *largest number* of subjects that can be examined at the same time without conflicts. How do your answers to (a) and (b) change?

(10 marks)

3. Given a plane-drawing (i.e. no crossing edges) of a *connected* planar graph G, a *face* is a region that is enclosed by edges. For example, the following plane-drawing of K_4 has 3 faces (labelled 1,2,3):



(a) How many edges must a connected graph with n vertices and 1 face have?

(b) By examining several planar graphs, come up with an equation that relates the number of vertices (n), the number of edges (m) and the number of faces (f) of a plane-drawing of a planar graph.

(c) Prove, by induction on f or otherwise, that your formula is correct. *Hint: What happens if you delete an edge of a plane-drawing that doesn't disconnect the graph?*

(10 marks)

4. Extend the syntactical definition of propositional formulae to include the connective $\circ$:

   - If φ and ψ are propositional formulae, then $(\varphi \circ \psi)$ is a propositional formula.

   Given a truth valuation $v : Prop \to \mathbb{B}$, define the semantics for $\circ$ as

$$v(\varphi \circ \psi) \; = !\big(v(\varphi) \,\&\, v(\psi)\big)$$

   (a) Draw the truth table for $(p \circ q) \circ (p \circ q)$. Give a logically equivalent formula.

   (b) For each of the following formulae, give a logically equivalent formula that only uses $\circ$ and propositional variables. Justify your answer.

      i. $\neg p$

      ii. $p \vee q$

      iii. $p \to q$

      iv. $p \leftrightarrow q$

(10 marks)

Due: 14th of October 2018 at 11:59pm

COMP 9020 – Assignment 3

Note: In your assignment, *how* you arrived at your solution is as important (if not more so) than the solution itself and will be assessed accordingly. There may be more than one way to find a solution, and your approach should contain enough detail to justify its correctness. Lecture content can be assumed to be common knowledge.

1. Let $(T, \wedge, \vee, ', 0, 1)$ be a Boolean Algebra.

   Define $* : T \times T \to T$ and $\circ : T \times T \to T$ as follows:

   $$x * y := (x \vee y)' \qquad\qquad x \circ y := (x \wedge y)'$$

   (a) Show, using the laws of Boolean Algebra, how to define $x * y$ using only x, y, $\circ$ and parentheses.

   (b) Show, using the laws of Boolean Algebra, how to define $x \circ y$ using only x, y, $*$ and parentheses.

   (10 marks)

   Define $R \subseteq T \times T$ as follows:

   $$(x, y) \in R \text{ if, and only if, } (x \wedge y) \vee (x' \wedge y') = 1$$

   (c) Show, using the laws of Boolean Algebra, that R is an equivalence relation. *Hint: You may want to use the observation that if $A = B = 1$ then $A \wedge B \wedge C = A \wedge B$ implies $C = 1$ (why?)*

   (10 marks)

2. Let PF denote the set of well-formed propositional formulas made up of propositional variables, $\top$, $\bot$, and the connectives $\neg$, $\wedge$, and $\vee$. Recall from Quiz 7 the definitions of dual and flip as functions from PF to PF:

   - $\mathsf{dual}(p) = p$
   - $\mathsf{dual}(\top) = \bot$; $\mathsf{dual}(\bot) = \top$
   - $\mathsf{dual}(\neg\varphi) = \neg\mathsf{dual}(\varphi)$
   - $\mathsf{dual}(\varphi \wedge \psi) = \mathsf{dual}(\varphi) \vee \mathsf{dual}(\psi)$
   - $\mathsf{dual}(\varphi \vee \psi) = \mathsf{dual}(\varphi) \wedge \mathsf{dual}(\psi)$
   - $\mathsf{flip}(p) = \neg p$
   - $\mathsf{flip}(\top) = \top$; $\mathsf{flip}(\bot) = \bot$
   - $\mathsf{flip}(\neg\varphi) = \neg\mathsf{flip}(\varphi)$
   - $\mathsf{flip}(\varphi \wedge \psi) = \mathsf{flip}(\varphi) \wedge \mathsf{flip}(\psi)$
   - $\mathsf{flip}(\varphi \vee \psi) = \mathsf{flip}(\varphi) \vee \mathsf{flip}(\psi)$

   (a) For the formula $\varphi = p \vee (q \wedge \neg r)$:

   (i) What is $\mathsf{dual}(\varphi)$?

   (ii) What is $\mathsf{flip}(\varphi)$?

   (b) Prove that for all $\varphi \in PF$: $\mathsf{flip}(\varphi)$ is logically equivalent to $\neg\mathsf{dual}(\varphi)$.

   (10 marks)

3. Let $P(n)$ be the proposition that: for all k, with $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(a) Prove that $P(n)$ holds for all $n \geq 1$. (*Note: it is possible to do this without using induction*)

(10 marks)

We can compute $\binom{n}{k}$ from the formula given in lectures, however this can often require computing unnecessarily large numbers. For example, $\binom{100}{15} = 253338471349988640$ which can be expressed as a 64-bit integer, but 100! is larger than a 512-bit integer. We can, however, make use of the equation above to compute $\binom{n}{k}$ more efficiently. Here are two algorithms for doing this:

```
chooseRec(n, k) :
  if k = 0 or k = n : return 1
  else :
    x := chooseRec(n − 1, k − 1)
    y := chooseRec(n − 1, k)
    return x + y
```

```
chooseIter(n, k) :
  Let C be a n × n array
  for m = 1 to n :
    C[m][0]=C[m][m]=1
    for j = 1 to m − 1 :
      C[m][j]=C[m − 1][j − 1]
               + C[m − 1][j]
  return C[n][k]
```

Let $t_{\mathsf{rec}}(n,k)$ be the running time for $\mathsf{chooseRec}(n,k)$, and let $t_{\mathsf{iter}}(n)$ be the running time for $\mathsf{chooseIter}(n,k)$. Let $T_{\mathsf{rec}}(n) = \max_{0 \leq k \leq n} t_{\mathsf{rec}}(n,k)$ and $T_{\mathsf{iter}}(n) = \max_{0 \leq k \leq n} t_{\mathsf{iter}}(n,k)$ (so $T_{\mathsf{rec}}(n) \geq t_{\mathsf{rec}}(n,k)$ for all k, and likewise for $T_{\mathsf{iter}}(n)$).

(b) Give an asymptotic upper bound for $T_{\mathsf{rec}}(n)$. Justify your answer.

(c) Give an asymptotic upper bound for $T_{\mathsf{iter}}(n)$. Justify your answer.

(10 marks)