The University of New South Wales

Final Exam

Session 2, 2016

COMP9020

Foundations of Computer Science

Time allowed: **2 hours + 10 minutes reading time**
Total number of questions: **10**
Maximum number of marks: **100**

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper and your 2 A4 sheets out of the exam.

Write your answers into the answer booklet provided.

Number of pages in this exam paper: 4

Question 1 (10 marks)

Prove or disprove $(A \wedge B \Rightarrow C) \wedge (A \wedge \neg B \Rightarrow C) \Leftrightarrow (A \Rightarrow C)$.

Question 2 (10 marks)

Prove that $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$, for all $n \in \mathbb{N}_{>0}$.

Question 3 (10 marks)

Let S be a finite set. Suppose a subset $U \subseteq S$ with $|U| \geq 2$ is put in sequence to form a permutation of U.

Translate each of the assertions (a)–(e) into predicate formulas with S as the domain of discourse. The only predicates you may use are

- equality (as in "$s_1 = s_2$") and
- $B(s_1, s_2)$, meaning that "both s_1 and s_2 occur in the sequence, and s_1 occurs somewhere to the left of s_2 in the sequence."

You are of course allowed to use other names than s_1 and s_2. For example, in the sequence 132, both $B(1, 2)$ and $B(3, 2)$ are true but $B(3, 1)$ is not.

Once you have defined a formula for a predicate, say P, you may use the abbreviation P in further formulas.

(a) Both s_1 and s_2 occur in the sequence, and s_1 occurs somewhere to the right of s_2 in the sequence.

(b) Element $s \in S$ occurs in the sequence. (Careful, you cannot use $\ldots \in U$ here.)

(c) Element $s \in S$ is the last in the sequence.

(d) Element s is immediately to the left of element t in the sequence.

(e) Element s is the second to last element of the sequence.

Example: $S = [0..13]$, $U = \{0, 2, 4, 6, 8, 10\}$ and the sequence is $\langle 2, 8, 10, 6, 0, 4 \rangle$. For instance, 10 is in the sequence but 7 is not; 4 is last, 8 is immediately to the left of 10, and 0 is second to last.

Question 4 (10 marks)

There is a bucket containing more blue balls than red balls. As long as there are more blues than reds, any one of the following rules may be applied to add and/or remove balls from the bucket:

(i) Add a red ball.

(ii) Remove a blue ball.

(iii) Add two reds and one blue.

(iv) Remove two blues and one red.

Example: with (x, y) denoting that there are x reds and y blues, a possible maximal sequence of rule applications is this:

$$(1,4) \xrightarrow{\text{(i)}} (2,4) \xrightarrow{\text{(iii)}} (4,5) \xrightarrow{\text{(ii)}} (4,4)$$

(a) Starting with 10 reds and 16 blues, what is the largest number of balls the bucket will contain by applying these rules?

(b) Let b be the number of blue balls and r be the number of red balls in the bucket at any given time. Prove that $b - r \geq 0$ is a preserved invariant of the process of adding and removing balls according to rules (i)–(iv).

Question 5 (12 marks)

(a) Prove that if n is not divisible by 3, then $n^2 \equiv 1 \pmod 3$.

(b) Show that if n is odd, then $n^2 \equiv 1 \pmod 8$.

(c) Conclude from the previous two results that if $p > 3$ is prime, then $24 | (p^2 - 1)$.

Question 6 (8 marks)

Let A, B, and C be sets. Let $r \subseteq A \times B$, $s \subseteq A \times C$, and $t \subseteq B \times C$ be binary relations.

We define a new operator "$\rightsquigarrow$" on relations by

$$r \rightsquigarrow s \stackrel{\text{def}}{=} \{ (b,c) \in B \times C \mid \forall a \in A\,((a,b) \in r \Rightarrow (a,c) \in s) \} \quad . \tag{1}$$

Prove that $t \subseteq r \rightsquigarrow s$ iff $t \circ r \subseteq s$.

Question 7 (8 marks)

Among connected simple graphs whose sum of vertex degrees is 18:

(a) what is the largest possible number of vertices?

(b) what is the smallest possible number of vertices?

Justify your answers briefly.

Question 8 (8 marks)

Suppose we have a recurrence

$$\begin{aligned} T(1) &= a \\ T(n) &= T(n-1) + g(n) \ , \text{ for } n > 1 \end{aligned}$$

Give tight big-oh upper bounds on the solution if $g(n)$ is

(a) $n^2 + 3n$

(b) 2^n

Question 9 (12 marks)

`myplates.com.au` licenses and sells personalised number plates. One of their offers in the mid-price section is for car plates that have numbers before letters. The allowed formats are

- three digits before three letters
- two digits before four letters
- two digits before three letters

where digits are in the set $\{0, \ldots, 9\}$ and letters in $\{A, \ldots, Z\}$.

(a) How many license plates can be constructed if the letters and digits have to be different (e.g. `678DIY` is allowed but `678DYY` is not because `Y` occurs twice)?

(b) How many license plates can be constructed if the letters and digits need not be different (e.g. `55FIVE` is allowed) but a simple profanity filter is in place? The simple profanity filter rules out

- a certain three-letter-word, and
- six four-letter-words, suffices to say that neither of them contains the three-letter-word as a substring.

(c) Suppose we randomly generate license plates according to the second of the three formats (i.e., two digits before four letters) so that each plate is equally likely. What is the probability of having `HI` on the number plate?

There's no need to calculate the final numbers—answers featuring sums (as in $98 + 45$), products (as in $98 \cdot 45$), fractions on lowest terms (as in $\frac{98}{45}$), powers (as in 17^{45}), or factorials (as in $17!$) are admissible.

Question 10 (12 marks)

Bruce Lee, on a movie that didn't go public, is practicing by breaking 5 boards with his fists. He is able to break a board with probability 0.8—he is practicing with his left fist, that's why it's not 1—and he breaks each board independently.

(a) What is the probability that Bruce breaks exactly 2 out of the 5 boards that are placed before him?

(b) What is the probability that Bruce breaks at most 3 out of the 5 boards that are placed before him?

(c) What is the expected number of boards Bruce will break?

Explain your working. There's no need to compute fractions; answers on lowest terms are accepted.

— END OF EXAM PAPER —