Problem 1

Operator ';' defined as

$$R_1; R_2 = \{(a,c) : \ there\ is\ a\ b\ with\ (a,b) \in R_1\ and\ (b,c) \in R_2\}$$

(a) For $(R_1; R_2); R_3$, we have 3 relations ①②③

$$(a,i) \in R_1$$
$$(i,j) \in R_2$$
$$(j,b) \in R_3$$

From ② and ③, we can have ④

$$(i,b) \in (R_2; R_3)$$

Then, combining ① with ④, we can obtain

$$(a,b) \in R_1; (R_2; R_3)$$

Therefore,

$$(R_1; R_2); R_3 = R_1; (R_2; R_3)$$

(b) From $I; R_1$, we have

$$I; R_1 = \{(a,c) : there\ is\ a\ b\ with\ (a,b) \in I\ and\ (b,c) \in R_1\}$$

And factor given is

$$I = \{(x,x) : x \in S\}$$

So $I; R_1$ turns into

$$I; R_1 = \{(a,c) : there\ is\ a\ b\ with\ (a,a) \in I\ and\ (a,c) \in R_1\} = R_1$$

Therefore, formula holds.

(c) does not holds followed by counterexample if

$$(a,i) \in R_1, (i,j) \in R_2$$

Then

$$(i,a) \in {R_1}^{\leftarrow}, (j,i) \in {R_2}^{\leftarrow}$$

Obviously, $R_1; R_2$ would be a $\emptyset$. Therefore, formula does not hold

(d) To prove this formula, we can just satisfy ③

$$(R_1 \cup R_2); R_3 = (R_1; R_2) \vee (R_1 \cup R_2); R_3 = (R_2; R_3)$$

We assume

$$(a,c) \in (R_1 \cup R_2); R_3$$

We can have its derivative ①②

$$(a,i) \in R_1 \vee (a,i) \in R_2$$
$$(i,c) \in R_3$$

Thus, we can observe that combining one of ① with ② could result in one of ③

(e) does not hold if we assume

$$(R_1; R_2) = (R_1; R_3) = (a,c)$$
$$R_1 = \{(a,i), (a,j)\}$$
$$R_2 = (i,c), R_3 = (j,c)$$

Obviously, we can observe that $(R_2 \cap R_3) = \emptyset$ so that formula does not hold.

Problem 2

Definition

$$R^0 := I = \{(x,x) : x \in S\}$$

$$R^{i+1} := R^i \cup \left(R; R^i\right) \mathit{for}\ i \geq 0$$

(a) For all $j \geq i$, $R^j = R^i$ when $i = j$. Then, for R^{j+1}, we have

$$R^{j+1} := R^j \cup \left(R; R^j\right)$$

Due to $R^j = R^i$, $R^{j+1} = R^{i+1}$. Because of $R^i = R^{i+1}$, $R^{j+1} = R^i$. Thus, for all $j \geq i$, formula holds.

(b) From (a), we have $k \geq i$. Then, $R^k = R^i$ and $R^k \sqsubseteq R^i$.From definition of R^{i+1}, we can get $R^k \sqsubseteq R^{k+1}\ \mathit{for}\ k \geq 0$ like $R^0 \sqsubseteq R^1 \sqsubseteq R^2 \sqsubseteq R^3 \sqsubseteq R^i$. Therefore, for domain$[0, i]$, $R^k \sqsubseteq R^i$.

(c) When $n = 0$, $P(0) \Longrightarrow R^0; R^m$. Due to $R^0 = I$, $R^0; R^m = R^m$.
If $R^k; R^m = R^{k+m}$, we have to prove $R^{k+1}; R^m = R^{k+m+1}$.
However, $R^{k+1}; R^m$ can transform to $R^k \cup (R; R^k); R^m$. Due to question d of problem1, it can also be turned to $(R^k; R^m) \cup \left((R; R^k); R^m\right)$ then same way from question a in problem1, it could be $(R^k; R^m) \cup \left(R; (R^k; R^m)\right)$. However, it again can be transformed into $R^{k+m} \cup (R; R^{k+m}) \Longrightarrow R^{k+m} \cup R^{k+m+1}$ due to $R^k; R^m = R^{k+m}$.
Thus, $R^{k+1}; R^m = R^{k+m} \cup R^{k+m+1} = R^{k+m+1}$.

(d)

(e) We can assume $(a, b) \in R^k, (b, c) \in R^k$ then we need to prove $(a, c) \in R^k$. Firstly, we can obtain $(a, c) \in R^{2k}$ because of question c in problem b. Secondly, from question d of problem 2 and $|S| = k$, we obtain $R^k = R^{k+1}$. In addition, from question a of problem 2, if $R^k = R^{k+1}$, then $R^k = R^i$ for all $k \geq i$. Due to this, for all $k \geq i$, $R^{2k} = R^k$, then $(a, c) \in R^k$.

(f) To show equivalence relation, we have to prove Reflexivity, Symmetry and Transitive.
For Reflexivity, from question b in problem 2, we have $I \sqsubseteq R^0 \sqsubseteq R^1 \sqsubseteq R^k$ which is $(x, x) \in R^k \Longrightarrow (x, x) \in (R \cup R^{\leftarrow})^k$so that Reflexivity holds.
For Symmetry, when k $= 0$, we have $R^0 = I$ which satisfy Symmetry. To prove Symmetry, we assume R^k satisfy Symmetry, we have to prove $R^{k+1} = \cup\ (R; R^k)$.
This means we need to show that $(R; R^k)$ also satisfy Symmetry

Problem 3

(a) Definition: A Binary Tree is either:
(B) an empty Tree, or
(R) an ordered pair (LeftChildTree, RightChildTree)

(b) count(T):

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if (T.isEmpty()):
    return 0
else:
    return 1 + count(T.left) + count(T.right)
```

(c) leaves(T):

```
if (T.isEmpty()):
    return 0
else:
    if (T.left.isEmpty() && T.Right.isEmpty()):
        return 1
    else:
        return leaves(T.left) + leaves(T.right)
```

(d) internal(T):

```
if (T.isEmpty):
    return 0
else:
    if (!T.left.isEmpty() && !T.right.isEmpty):
        return internal(T.left) + internal(T.right) + 1
    else:
        return internal(T.left) + internal(T.right)
```

(e) Every node has a parent node except root node so that a tree with N nodes owns N-1 edges. This means that from root to leaf, we obtain the sum of degree being $0 \times n_0 + 1 \times n_1 + 2 \times n_2$. Then $N - 1 = 0 \times n_0 + 1 \times n_1 + 2 \times n_2$ which can be transformed into $n_0 + n_1 + n_2 - 1 = 0 \times n_0 + 1 \times n_1 + 2 \times n_2$. This formula can simplify to

$$n_0 = n_2 + 1$$

Which is leaves(T) = internal(T) + 1.

Problem 4

(a)

(i) $\left(hi_{Alpha} \vee lo_{Alpha}\right) \wedge (hi_{Bravo} \vee lo_{Bravo}) \wedge (hi_{Charlie} \vee lo_{Charlie}) \wedge (hi_{Delta} \vee lo_{Delta})$

(ii) $\left(\left(hi_{Alpha} \wedge \neg lo_{Alpha}\right) \vee \left(lo_{Alpha} \wedge \neg hi_{Alpha}\right)\right) \wedge \left((hi_{Bravo} \wedge \neg lo_{Bravo}) \vee (lo_{Bravo} \wedge \neg hi_{Bravo})\right) \wedge \left((hi_{Charlie} \wedge \neg lo_{Charlie}) \vee (lo_{Charlie} \wedge \neg hi_{Charlie})\right) \wedge \left((hi_{Delta} \wedge \neg lo_{Delta}) \vee (lo_{Delta} \wedge \neg hi_{Delta})\right)$

(iii) $\left[\left(lo_{Alpha} \wedge hi_{Bravo}\right) \vee \left(hi_{Alpha} \wedge lo_{Bravo}\right)\right] \wedge [(lo_{Bravo} \wedge hi_{Charlie}) \vee (hi_{Bravo} \wedge lo_{Charlie})] \wedge [(lo_{Charlie} \wedge hi_{Delta}) \vee (hi_{Charlie} \wedge lo_{Delta})]$

(b)

(i) To satisfy $\varphi_1 \wedge \varphi_1 \wedge \varphi_1$, we have truth assignment below:

$hi_{Alpha} = 1 \quad hi_{Bravo} = 0 \quad hi_{Charlie} = 1 \quad hi_{Delta} = 0$

$lo_{Alpha} = 0 \quad lo_{Bravo} = 1 \quad lo_{Charlie} = 0 \quad lo_{Delta} = 1$

(ii) To avoid interference, the channels should be:

Alpha uses hi, Bravo uses lo, Charlie uses hi, Delta uses lo.