The University of New South Wales

Final Exam

Session 2, 2014

COMP9020

Foundations of Computer Science

Time allowed: **2 hour**
Total number of questions: **10**
Maximum number of marks: **50**

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper out of the exam.

Write your answers into the answer booklet provided.

Question 1 (5 marks)

A *leap year* is a year containing one additional day. Nowadays, that day, the 29th of February, is added on every year divisible by 4, unless it is also divisible by 100 but not by 400. For instance, the years 2000 and 2004 were leap years but 1900 was not.

For the purpose of this question let us pretend that this rule has always been followed since the year 0 and will always be followed.

For $f, \ell \in \mathbb{N}$ with $f \leq \ell$, devise a formula $leap(f, \ell)$ that expresses the exact number of leap years in the interval $[f, \ell]$ using at most the two arguments f and ℓ as well as the arithmetic operations floor, addition, subtraction, and division.

Example: applying the formula to $f = 1896$ and $\ell = 2004$ should yield $leap(1896, 2004) = 27$.

Answer:

$$\begin{array}{ll} \lfloor \ell/4 \rfloor - \lfloor (f-1)/4 \rfloor & \text{divisible by 4} \\ - (\lfloor \ell/100 \rfloor - \lfloor (f-1)/100 \rfloor) & \text{divisible by 100} \\ + \lfloor \ell/400 \rfloor - \lfloor (f-1)/400 \rfloor & \text{divisible by 400} \end{array}$$

Question 2 (5 marks)

Prove that $\sum_{i=0}^{n} k^i = \frac{k^{n+1}-1}{k-1}$ for all $n, k \in \mathbb{N}$ with $k > 1$.

Answer: by induction on n. Base case $\sum_{i=0}^{0} k^i = 1 = \frac{k-1}{k-1} = \frac{k^{0+1}-1}{k-1}$. Inductive case

$$\begin{aligned} \sum_{i=0}^{n+1} k^i &= k^{n+1} + \sum_{i=0}^{n} k^i \\ &= k^{n+1} + \frac{k^{n+1}-1}{k-1} && \text{by the ind. hyp.} \\ &= \frac{(k-1)k^{n+1} + k^{n+1} - 1}{k-1} \\ &= \frac{k^{n+2}-1}{k-1} \ . \end{aligned}$$

Question 3 (4 marks)

Prove that $(S \cap T) \times (U \cap V) = (S \times U) \cap (T \times V)$ holds for all sets S, T, U, and V.

Answer: the claimed equality holds.

$$\begin{aligned} (s, u) \in (S \cap T) \times (U \cap V) &\Leftrightarrow s \in (S \cap T) \wedge u \in (U \cap V) \\ &\Leftrightarrow s \in S \wedge s \in T \wedge u \in U \wedge u \in V \\ &\Leftrightarrow s \in S \wedge u \in U \wedge s \in T \wedge u \in V \\ &\Leftrightarrow (s, u) \in S \times U \wedge (s, u) \in T \times V \\ &\Leftrightarrow (s, u) \in (S \times U) \cap (T \times V) \end{aligned}$$

Question 4 (5 marks)

Consider the partial order $D_{45} = (\{ x \in \mathbb{N}_{>0} \mid x|45 \}, \sqsubseteq)$ defined on the positive divisors of 50 by $x \sqsubseteq y$ iff $x|y$. Give 3 different topological sorts of D_{45}.

Answer: $45 = 3 \cdot 3 \cdot 5$ with positive divisors $\{1, 3, 5, 9, 15, 45\}$.

$$1, 3, 5, 9, 15, 45$$
$$1, 5, 3, 9, 15, 45$$
$$1, 3, 5, 15, 9, 45$$
$$1, 5, 3, 15, 9, 45$$
$$1, 5, 3, 9, 15, 45$$

Question 5 (5 marks)

Consider the four functions

$$f_1(n) = n^3 \qquad\qquad f_2(n) = n^5$$
$$f_3(n) = \begin{cases} n^3 & \text{if } n \text{ is odd} \\ n^5 & \text{otherwise} \end{cases} \qquad\qquad f_4(n) = \begin{cases} n^3 & \text{if } n \text{ is prime} \\ n^5 & \text{otherwise} \end{cases}$$

For each pair $(i, k) \in \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 3)\}$, determine whether $f_i(n)$ is $\mathcal{O}(f_k(n))$. Either give values n_0 and c that prove the big-oh relationship, or assume that there are such values n_0 and c, and then derive a contradiction to prove that $f_i(n)$ is not $\mathcal{O}(f_k(n))$.

Answer:

$(1, 2)$: f_1 is $\mathcal{O}(f_2)$. Constants $n_0 = 0$ and $c = 1$ work as witnesses.

$(2, 1)$: f_2 is not $\mathcal{O}(f_1)$. Suppose otherwise for constants n_0 and c. Consider $m = \max(n_0+1, \sqrt{c})$. Then $f_2(m) = m^5 \geq m^3 c = cf_1(m)$.

$(2, 3)$: f_2 is not $\mathcal{O}(f_3)$. Suppose otherwise for constants n_0 and c. Consider the smallest odd number m at least $\max(n_0 + 1, \sqrt{c})$. Then $f_2(m) = m^5 \geq m^3 c = cf_3(m)$.

$(3, 4)$: f_3 is $\mathcal{O}(f_4)$. Constants $n_0 = 3$ and $c = 1$ work as witnesses since all primes after 2 are odd.

$(4, 3)$: f_4 is not $\mathcal{O}(f_3)$. Suppose otherwise for constants n_0 and c. Consider the smallest composite number m at least $\max(n_0 + 1, \sqrt{c})$. Then $f_4(m) = m^5 \geq m^3 c = cf_3(m)$.

Question 6 (8 marks)

For each of the following recurrences, what is $T(n)$'s order of growth?

$$T(n) = 2T(n-1) + 4n^2 + 3n + 2 \tag{1}$$
$$T(n) = 8T(\frac{n}{2}) + 4n^2 + 3n + 2 \tag{2}$$
$$T(n) = 8T(\frac{n}{2}) + 4n^3 + 3n^2 + 2n + 1 \tag{3}$$
$$T(n) = 2T(\frac{n}{2}) + 4n^2 + 3n + 2 \tag{4}$$

Answer: by the master theorem:

(a) $\mathcal{O}(2^n)$

(b) $\mathcal{O}(n^3)$

(c) $\mathcal{O}(n^3 \log n)$

(d) $\mathcal{O}(n^2)$

Question 7 (5 marks)

Suppose license plates are constructed using three letters picked (without replacing) from the word `SYDNEY` followed by two decimal digits picked (again without replacing) from the word `012345566778899`.

(a) How many license plates can be constructed if the letters and digits have to be different (e.g. `DYS54` is allowed but `YSY54` is not because `Y` occurs twice),

(b) How many license plates can be constructed if the letters and digits need not be different (e.g. `YSY54` is allowed and so is `END55` but not `SDD64`)?

Answer:

(a) remove `Y` from the first and `56789` from the second word to arrive at `SYDNE` and `0123456789`. Choosing three different letters from the first $= 5 \cdot 4 \cdot 3 = 60$ possibilities. Choosing two from the second word $= 10 \cdot 9 = 90$ possibilites. Together that's $60 * 90 = \mathbf{5400}$ different license plates.

(b) $6 \cdot 5 \cdot 4 = 120$ choices of three letters from the first word but in those we double-counted those with two `Y`s. There are 4 other letters and 3 positions for the other letter $= 12$ different words with two `Y`s. The two numbers can be chosen from 10 digits so that's $10^2 = 100$ possibilites. But we've accounted for certain combinations that we cannot have—the combinations of the form dd for $d \in \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}\}$. There are 5 of those. Altogether we have $(120 - 12) \cdot (100 - 5) = \mathbf{10260}$.

Question 8 (7 marks)

Let G_k be a graph on k vertices, consisting of a cycle of $k-1$ vertices plus a central node connected to all vertices on the cycle. Given $k > 3$, what is the chromatic number of G_k?

Answer: $\chi(G_k) \in \{3, 4\}$: the cycle may be odd or even, needing 2 or 3 colours, the central node takes one more.

Question 9 (5 marks)

You have applied to join a chess club and been told that to qualify you must play three games against X, winning two games in a row. "Who gets the white pieces?" you ask and are told you and X alternate and you get to decide whether to start with white or with black.

Knowing that the probability of beating X is better with the white pieces (first-move advantage) should you choose white or black for the first game?

Use probabilites to model the problem and prove your answer answer correct.

Answer: I should choose to start with black. Let $p, q \in (0,1)$ be the probability of me winning with black and white, respectively, so $p < q$. The probability of winning two games in a row when starting the first game with black is the sum of the probabilities of winning the first two, the last two, or all three games: $pq(1-p) + (1-p)qp + pqp = (2(1-p)+p)pq = (2-p)pq$ whereas the same calculation for beginning with white results in $qp(1-q) + (1-q)pq + qpq = (2(1-q)+q)qp = (2-q)qp$. Since $p < q$, $(2-p)pq > (2-q)pq$.

Question 10 (5 marks)

Suppose you are taking an exam consisting of 50 multiple choice questions. Each question has four possible answers exactly one of which is correct. A correct answer scores 2, an incorrect answer scores -1 and blank scores 0. You did not study at all, and decide to randomly guess all the answers and leave no blanks. What should you expect to score in the exam? Derive the correct answer to this question mathematically.

Answer: $2 * 1/4 * 50 - 1 * 3/4 * 50 = 25 - 37.5 = -12.5$