The University of New South Wales

Final Exam

Session 2, 2016

COMP9020

Foundations of Computer Science

Time allowed: **2 hours + 10 minutes reading time**
Total number of questions: **10**
Maximum number of marks: **100**

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper and your 2 A4 sheets out of the exam.

Write your answers into the answer booklet provided.

Number of pages in this exam paper: 4

Question 1 (10 marks)

Prove or disprove $(A \wedge B \Rightarrow C) \wedge (A \wedge \neg B \Rightarrow C) \Leftrightarrow (A \Rightarrow C)$.

Answer: 1 mark for the right claim, up to 9 marks for the proof.

This is valid. Proof by truth table is ok.

A	B	C	$A \wedge B \Rightarrow C$	$A \wedge \neg B \Rightarrow C$	$A \Rightarrow C$
F	**F**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**
T	**F**	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**T**	**F**
T	**T**	**T**	**T**	**T**	**T**

The fourth and fifth columns are both labeled **T** iff the last one is, which proves the claimed equivalence.

Alternative proofs include:

$$\begin{aligned}
(A \wedge B \Rightarrow C) \wedge (A \wedge \neg B \Rightarrow C) &\Leftrightarrow (\neg(A \wedge B) \vee C) \wedge (\neg(A \wedge \neg B) \vee C) && \text{impl.}\\
&\Leftrightarrow (\neg(A \wedge B) \wedge \neg(A \wedge \neg B)) \vee C && \text{distrib.}\\
&\Leftrightarrow ((\neg A \vee \neg B) \wedge (\neg A \vee \neg\neg B)) \vee C && \text{de Morgan}\\
&\Leftrightarrow ((\neg A \vee \neg B) \wedge (\neg A \vee B)) \vee C && \text{double neg.}\\
&\Leftrightarrow (\neg A \vee (\neg B \wedge B)) \vee C && \text{distrib.}\\
&\Leftrightarrow (\neg A \vee \mathbf{F}) \vee C && \text{comm. \& contrad.}\\
&\Leftrightarrow \neg A \vee C && \text{ident.}\\
&\Leftrightarrow A \Rightarrow C && \text{impl.}
\end{aligned}$$

Question 2 (10 marks)

Prove that $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$, for all $n \in \mathbb{N}_{>0}$.

Answer: Proof by induction on n. Base case: $\sum_{i=1}^{1} i(i+1) = 1(1+1) = \frac{1(1+1)(1+2)}{3}$.

Inductive step:

$$\begin{aligned}
\sum_{i=1}^{n+1} i(i+1) &= \sum_{i=1}^{n} i(i+1) + (n+1)(n+2)\\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) && \text{by the ind. hyp.}\\
&= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3}\\
&= \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}$$

Alternatives/variations: the claim actually holds for $n = 0$, so one could've started with that as base case: $\sum_{i=1}^{0} i(i+1) = 0 = \frac{0(0+1)(0+2)}{3}$. One could've use the WOP with roughly the same steps.

A direct proof goes as follows. Consider

$$\begin{aligned}\sum_{i=1}^{n}(i+1)^3 &= \sum_{i=1}^{n}(i^3 + 3(i+1)i + 1)\\ &= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n}(i+1)i + n\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=1}^{n}(i+1)i &= \frac{1}{3}\left(\sum_{i=1}^{n}(i+1)^3 - \sum_{i=1}^{n} i^3 - n\right)\\ &= \frac{1}{3}\left(\sum_{i=2}^{n+1} i^3 - \sum_{i=1}^{n} i^3 - n\right)\\ &= \frac{(n+1)^3 - 1 - n}{3}\\ &= \frac{n^3 + 3n^2 + 2n}{3}\\ &= \frac{n(n+1)(n+2)}{3}\end{aligned}$$

Question 3 (10 marks)

Let S be a finite set. Suppose a subset $U \subseteq S$ with $|U| \geq 2$ is put in sequence to form a permutation of U.

Translate each of the assertions (a)–(e) into predicate formulas with S as the domain of discourse. The only predicates you may use are

- equality (as in "$s_1 = s_2$") and
- $B(s_1, s_2)$, meaning that "both s_1 and s_2 occur in the sequence, and s_1 occurs somewhere to the left of s_2 in the sequence."

You are of course allowed to use other names than s_1 and s_2. For example, in the sequence 132, both $B(1, 2)$ and $B(3, 2)$ are true but $B(3, 1)$ is not.

Once you have defined a formula for a predicate, say P, you may use the abbreviation P in further formulas.

(a) Both s_1 and s_2 occur in the sequence, and s_1 occurs somewhere to the right of s_2 in the sequence.

(b) Element $s \in S$ occurs in the sequence. (Careful, you cannot use $\ldots \in U$ here.)

(c) Element $s \in S$ is the last in the sequence.

(d) Element s is immediately to the left of element t in the sequence.

(e) Element s is the second to last element of the sequence.

Example: $S = [0..13]$, $U = \{0, 2, 4, 6, 8, 10\}$ and the sequence is $\langle 2, 8, 10, 6, 0, 4\rangle$. For instance, 10 is in the sequence but 7 is not; 4 is last, 8 is immediately to the left of 10, and 0 is second to last.

Answer:

(a) $A(s_1, s_2) \stackrel{\text{def}}{=} B(s_2, s_1)$

(b) $s \in U \stackrel{\text{def}}{=} \exists t\,(B(s, t) \vee A(s, t))$

(c) $L(s) \stackrel{\text{def}}{=} s \in U \wedge \neg\exists t\,(B(s, t))$

(d) $Z(s, t) \stackrel{\text{def}}{=} B(s, t) \wedge \neg\exists u\,(B(s, u) \wedge B(u, t))$

(e) $\exists t\,(L(t) \wedge Z(s, t))$

Without re-using previous formulas this becomes marginally more tedious but is possible:

(a) $B(s_2, s_1)$

(b) $\exists t\,(B(s, t) \vee B(t, s))$

(c) $\exists t\,(B(t, s)) \wedge \neg\exists t\,(B(s, t))$

(d) $B(s, t) \wedge \neg\exists u\,(B(s, u) \wedge B(u, t))$

(e) $\exists t\,(B(s, t) \wedge \forall u\,(B(s, u) \Rightarrow u = t))$

Question 4 (10 marks)

There is a bucket containing more blue balls than red balls. As long as there are more blues than reds, any one of the following rules may be applied to add and/or remove balls from the bucket:

(i) Add a red ball.

(ii) Remove a blue ball.

(iii) Add two reds and one blue.

(iv) Remove two blues and one red.

Example: with (x, y) denoting that there are x reds and y blues, a possible maximal sequence of rule applications is this:

$$(1, 4) \xrightarrow{\text{(i)}} (2, 4) \xrightarrow{\text{(iii)}} (4, 5) \xrightarrow{\text{(ii)}} (4, 4)$$

(a) Starting with 10 reds and 16 blues, what is the largest number of balls the bucket will contain by applying these rules?

(b) Let b be the number of blue balls and r be the number of red balls in the bucket at any given time. Prove that $b - r \geq 0$ is a preserved invariant of the process of adding and removing balls according to rules (i)–(iv).

Answer: This could be helpfully modeled as a state machine $M = (S, s_0, \rightarrow)$ where $S = \mathbb{N}^2$, $s_0 = (10, 16)$ and $(r, b) \rightarrow (r', b')$ iff $b > r$ and either of the following is true:

- $(r', b') = (r + 1, b)$
- $(r', b') = (r, b - 1)$

- $(r', b') = (r+2, b+1)$
- $r > 0 \wedge (r', b') = (r-1, b-2)$

Observe that each transition decreases $b - r$ by 1. This implies that every maximal sequence of transitions has exactly six transitions.

(a) 44—obtained by $(10, 16) \xrightarrow{\text{(iii)}}^{6} (22, 22)$—is optimal because transition (iii) is the best for adding balls and we have a fixed number, 6, of transitions at our disposal.

(b) $b - r = 6 \geq 0$ in s_0. Each transition preserves the invariant because each transition decreases $b - r$ by only 1. None of the transitions is enabled when $b \leq r$, so we're bound to hit $b = r$ without overshooting it.

Question 5 (12 marks)

(a) Prove that if n is not divisible by 3, then $n^2 \equiv 1 \pmod 3$.

(b) Show that if n is odd, then $n^2 \equiv 1 \pmod 8$.

(c) Conclude from the previous two results that if $p > 3$ is prime, then $24 | (p^2 - 1)$.

Answer:

(a) Let $a \in \{1, 2\}$ such that $n \equiv a \pmod 3$. Hence $sn^2 \equiv a^2 \pmod 3$. We have $a^2 \in \{1, 4\}$ and hence $a^2 \equiv 1 \pmod 3$.

(b) $n = 2k+1$: First observe that either one of k and $k+1$ is even, and hence $k^2 + k = k(k+1)$ is even, too. We conlcude that $(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1 = 8\frac{k^2+k}{2} + 1 \equiv 1 \pmod 8$.

(c) As a prime, $p > 3$ is divisible neither by 2 nor by 3. We have by (a) that $p^2 \equiv 1 \pmod 3$ and by (b) that $p^2 \equiv 1 \pmod 8$. Thus $p^2 - 1$ is divisible by both 3 and 8. The claim follows by observing that $\mathrm{lcm}(3, 8) = 24$.

Alternatives:

(a) i) There are only two cases to consider.

Case $n = 3k + 1$: $(3k+1)^2 = 9k^2 + 6k + 1 \equiv 1 \pmod 3$.

Case $n = 3k + 2$: $(3k+2)^2 = 9k^2 + 12k + 4 \equiv 1 \pmod 3$.

ii) $n^2 \bmod 3 = (\sum_{i=1}^{n} n) \bmod 3 = (\sum_{i=1}^{n} (n \bmod 3)) \bmod 3 = (n(n \bmod 3)) \bmod 3$. Since $3 \not| n$ we have that $n \bmod 3 \in \{1, 2\}$.
Case $n \bmod 3 = 1$: $(n(n \bmod 3)) \bmod 3 = n \bmod 3 = 1$.
Case $n \bmod 3 = 2$: $(n(n \bmod 3)) \bmod 3 = (n \cdot 2) \bmod 3 = (2(n \bmod 3) \bmod 3) = 4 \bmod 3 = 1$.

Question 6 (8 marks)

Let A, B, and C be sets. Let $r \subseteq A \times B$, $s \subseteq A \times C$, and $t \subseteq B \times C$ be binary relations.

We define a new operator "$\rightsquigarrow$" on relations by

$$r \rightsquigarrow s \stackrel{\text{def}}{=} \{ (b, c) \in B \times C \mid \forall a \in A \, ((a, b) \in r \Rightarrow (a, c) \in s) \} \quad . \tag{1}$$

Prove that $t \subseteq r \rightsquigarrow s$ iff $t \circ r \subseteq s$.

Answer: this is probably best dealt with by going to predicate calculus.

$$\begin{aligned} t \circ r \subseteq s &\Leftrightarrow \forall \sigma, \tau\, (\exists \rho\, ((\sigma,\rho) \in r \wedge (\rho,\tau) \in t) \Rightarrow (\sigma,\tau) \in s) && \text{Def. } \circ \\ &\Leftrightarrow \forall \rho, \tau\, ((\rho,\tau) \in t) \Rightarrow \forall \sigma\, ((\sigma,\rho) \in r \Rightarrow (\sigma,\tau) \in s) && \text{pred. calc.} \\ &\Leftrightarrow t \subseteq r \rightsquigarrow s && (1) \end{aligned}$$

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Question 7 (8 marks)

Among connected simple graphs whose sum of vertex degrees is 18:

(a) what is the largest possible number of vertices?

(b) what is the smallest possible number of vertices?

Justify your answers briefly.

Answer: The sum of vertex degrees is equal to the number of edges times two. So we're looking a simple graphs with 9 edges.

(a) With 9 edges we can connect at most 10 nodes, so the answer is **10**.

Examples: a line graph with 10 nodes has two vertices of degree 1 and 8 of degree 2; a star graph with 10 nodes has one node of degree 9 and 9 nodes of degree 1.

(b) Complete graphs are most efficient in terms of the ratio vertices/edges. K_4 has only 6 edges, hence 4 nodes are not enough. K_5 has 10 edges, so remove one edge to obtain a graph with 9 edges. The answer is thus **5**.

Question 8 (8 marks)

Suppose we have a recurrence

$$\begin{aligned} &T(1) = a \\ &T(n) = T(n-1) + g(n) \ \text{, for } n > 1 \end{aligned}$$

Give tight big-oh upper bounds on the solution if $g(n)$ is

(a) $n^2 + 3n$

(b) 2^n

Answer:

(a) $g(n) = n^2 + 3n$ The solution is the same as without the $3n$ term as summing of the term $3n$ leads to a lower order expression than summing n^2.

$$T(n) \sim \sum_{1}^{n} (j^2 + 3j) = \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2} \sim \frac{n^3}{3} \in \mathcal{O}(\mathbf{n^3})$$

(b) $g(n) = 2^n$

$$T(n) = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^2 = 2 \cdot 2^n - 4 + a \in \mathcal{O}(\mathbf{2^n})$$

Question 9 (12 marks)

`myplates.com.au` licenses and sells personalised number plates. One of their offers in the mid-price section is for car plates that have numbers before letters. The allowed formats are

- three digits before three letters
- two digits before four letters
- two digits before three letters

where digits are in the set $\{\texttt{0}, \ldots, \texttt{9}\}$ and letters in $\{\texttt{A}, \ldots, \texttt{Z}\}$.

(a) How many license plates can be constructed if the letters and digits have to be different (e.g. `678DIY` is allowed but `678DYY` is not because `Y` occurs twice)?

(b) How many license plates can be constructed if the letters and digits need not be different (e.g. `55FIVE` is allowed) but a simple profanity filter is in place? The simple profanity filter rules out

- a certain three-letter-word, and
- six four-letter-words, suffices to say that neither of them contains the three-letter-word as a substring.

(c) Suppose we randomly generate license plates according to the second of the three formats (i.e., two digits before four letters) so that each plate is equally likely. What is the probability of having `HI` on the number plate?

There's no need to calculate the final numbers—answers featuring sums (as in $98+45$), products (as in $98 \cdot 45$), fractions on lowest terms (as in $\frac{98}{45}$), powers (as in 17^{45}), or factorials (as in $17!$) are admissible.

Answer:

(a) We add up the three formats

$$
\begin{array}{r}
10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 = 11232000 \\
10 \cdot 9 \cdot 26 \cdot 25 \cdot 24 \cdot 23 = 32292000 \\
10 \cdot 9 \cdot 26 \cdot 25 \cdot 24 = 1404000 \\
\hline
= \mathbf{44928000}
\end{array}
$$

Other ways to write this $\frac{10!}{7!} \cdot \frac{26!}{23!} + \frac{10!}{8!} \cdot \frac{26!}{22!} + \frac{10!}{8!} \cdot \frac{26!}{23!}$ and $\frac{32 \cdot 10! \cdot 26!}{8! \cdot 23!}$.

(b) Again, we add up the three formats

$$
\begin{array}{rl}
10^3 \cdot (26^3 - 1) = 17575000 & \\
10^2 \cdot (26^4 - 6 - 2 \cdot 26) = 45691800 & \quad 2 \cdot 26 \text{ 4-letter-words with } \texttt{SEX} \\
10^2 \cdot (26^3 - 1) = 1757500 & \\
\hline
= \mathbf{65024300} &
\end{array}
$$

(c) We calculate $\frac{|\text{plates with } \texttt{HI}|}{|\text{plates}|}$. We can safely ignore the number part of plates for this as it only adds a constant multiplier of 10^2 to both set sizes. We have $26^4 = 456976$ possible 4-letter plates of which $3 \cdot 26^2 - 1 = 2027$ contain `HI` (and 1 even `HIHI` which is why we have the -1 to account for it being counted twice). The answer is thus $\frac{\mathbf{3 \cdot 26^2 - 1}}{\mathbf{26^4}} = \frac{\mathbf{2027}}{\mathbf{456976}} \simeq \mathbf{0.0044}$.

Question 10 (12 marks)

Bruce Lee, on a movie that didn't go public, is practicing by breaking 5 boards with his fists. He is able to break a board with probability 0.8—he is practicing with his left fist, that's why it's not 1—and he breaks each board independently.

(a) What is the probability that Bruce breaks exactly 2 out of the 5 boards that are placed before him?

(b) What is the probability that Bruce breaks at most 3 out of the 5 boards that are placed before him?

(c) What is the expected number of boards Bruce will break?

Explain your working. There's no need to compute fractions; answers on lowest terms are accepted.

Answer:

(a) We add up the probabilities along five attempts at breaking a board of which precisely two succeed. Each success has probability 0.8. Along each such path we have probability $(0.8)^2 \cdot (1-0.8)^3$. There are $\binom{5}{2}$ such paths. The answer is thus $\binom{5}{2} \cdot (0.8)^2 \cdot (1-0.8)^3 = \mathbf{0.0512}$.

(b) $\Pr(\text{Bruce breaks at most 3}) = 1 - \Pr(\text{Bruce breaks 4 or 5}) = 1 - (\Pr(\text{Bruce breaks 4}) + \Pr(\text{Bruce breaks 5}))$. We can re-apply the same style of computation as in the previous part to arrive at $1 - (\binom{5}{4} \cdot (0.8)^4 \cdot (1-0.8) + \binom{5}{1} \cdot (0.8)^5) = 1 - (5 \cdot (0.8)^4 \cdot (0.2) + (0.8)^5) = 1 - ((0.8)^4 + (0.8)^5) \simeq \mathbf{0.2627}$.

(c) The attempts are independent so the expectation is additive. Each attempt yields an expected number of 0.8 broken boards so the answer is $5 \cdot 0.8 = \mathbf{4}$.

— END OF EXAM PAPER —