Quiz 1: Numbers, Sets and Alphabets

Q1 What is the value of $\lfloor -3.4 \rfloor$?

Answer: -4 is the largest integer which is less than or equal to -3.4.

Q2 Let A be the set $\{2, 4, 6, 8\}$ and B be the set $\{5, 6, 7, 8\}$. Which of the following sets is $A \oplus B$ (the symmetric difference of A and B)?

Answer: $A \oplus B = (A \cup B) \setminus (A \cap B) = \{2, 4, 5, 6, 7, 8\} \setminus \{6, 8\} = \{2, 4, 5, 7\}$

Q3 Let A be the set $\{b, a, n, a, n, a\}$ and B be the set $\{p, i, n, e, a, p, p, l, e\}$. Which of the following sets is $A \cap B$?

Answer: Simplifying, $A = \{a, n, b\}$ and $B = \{a, e, i, l, n, p\}$ so $A \cap B = \{a, n\}$

Q4 Let w be the word abb, and let v be the word bab. Which of the following is the word wv?

Answer: $wv = (abb)(bab) = abbbab$

Q5 Let w be the word abb and v be the word ba. What is the length of the word $vwvw$?

Answer: $vwvw = (ba)(abb)(ba)(abb) = baabbbaabb$ so $\text{length}(vwvw) = 10$.

Quiz 2: Functions and Relations I

Q1 Consider $f : \mathbb{P} \to \mathbb{P}$ given by $f(x) = 2x + 1$. What is the inverse image of $\{1, 2, 3\}$, i.e. what is $f^{\leftarrow}(\{1, 2, 3\})$?

Answer: The values of x for which $2x + 1 \in \{1, 2, 3\}$ are 0, $\frac{1}{2}$, and 1. Of these, only $1 \in \mathbb{P}$, the domain of f. So $f^{\leftarrow}(\{1, 2, 3\}) = \{1\}$.

Q2 Let $A = \{a, b, c\}$ and consider $g : \text{Pow}(A) \to \mathbb{N}$ given by $g(X) = |X|$. What is $\text{Im}(g)$?

Answer: $g(\emptyset) = 0$, $g(\{a\}) = g(\{b\}) = g(\{c\}) = 1$, $g(\{a, b\}) = g(\{b, c\}) = g(\{a, c\}) = 2$, and $g(\{a, b, c\}) = 3$ so $\text{Im}(g) = \{0, 1, 2, 3\}$.

Q3 Let $\Sigma = \{a, b\}$ and consider $f, g : \Sigma^* \to \Sigma^*$ given by

$$
\begin{aligned}
f(w) &= ww \\
g(w) &= awb
\end{aligned}
$$

What is $f \circ g(aba)$?

Answer: $f \circ g(aba) = f(g(aba)) = f(aabab) = aababaabab$

Q4 Suppose $f : S \to T$ and $g : T \to U$ are bijective. True or false: $g \circ f$ is always bijective.

Answer: If f and g are injective then: $g \circ f(x) = g \circ f(y)$ implies $g(f(x)) = g(f(y))$, which implies $f(x) = f(y)$ (because g is injective), which implies $x = y$ (because f is injective). So $g \circ f$ is injective.

If f and g are surjective then: for all $u \in U$ there is a $t \in T$ such that $g(t) = u$ and for all $t \in T$ there is an $s \in S$ such that $f(s) = t$. So, for all $u \in U$ there is an $s \in S$ such that $g \circ f(s) = g(f(s)) = u$. So $g \circ f$ is surjective.

Therefore, if f and g are bijective, then $g \circ f$ is (always) bijective.

Q5 Let $\Sigma = \{a, b\}$ and consider the relation $R \subseteq \Sigma^* \times \Sigma^*$ given by $(w, v) \in R$ if $\text{length}(wv)$ is even. Which of the properties Reflexivity (R) and Transitivity (T) does R have?

Answer: For all $w \in \Sigma^*$, $\text{length}(ww) = 2\text{length}(w)$ is even, so $(w, w) \in R$. So R is reflexive (R).

If $\text{length}(wv) = \text{length}(w) + \text{length}(v)$ is even and $\text{length}(vu) = \text{length}(v) + \text{length}(u)$ is even then $\text{length}(w)$, $\text{length}(v)$ and $\text{length}(u)$ are either all even, or all odd. In either case $\text{length}(w) + \text{length}(u) = \text{length}(wu)$ is even, so $(w, u) \in R$. So R is transitive (T).

Quiz 3: Relations and Functions II

Q1 Consider $R \subseteq \mathbb{N} \times \mathbb{N}$ given by $(x, y) \in R$ if $x - y \geq 7$. Which of the properties Reflexivity (R) and Transitivity (T) does R have?

Answer: $x - x = 0 < 7$ for all $x \in \mathbb{N}$ so $(x, x) \notin R$ for all $x \in \mathbb{N}$. Therefore R is antireflexive (so not reflexive).

If $(x, y) \in R$ and $(y, z) \in R$ then $x - y \geq 7$ and $y - z \geq 7$. So $x - z = (x - y) + (y - z) \geq 14 \geq 7$. So $(x, z) \in R$. Therefore R is transitive.

Q2 Suppose R is a partial order. True or false: $R \cup R^{\leftarrow}$ is an equivalence relation.

Answer: Consider the partial order $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (3, 2)\}$. We have $(1, 2) \in R$ and $(2, 3) \in R^{\leftarrow}$ so $(1, 2), (2, 3) \in R \cup R^{\leftarrow}$, however $(1, 3)$ is neither in R nor $R^{\leftarrow}$, so $(1, 3) \notin R \cup R^{\leftarrow}$. Hence $R \cup R^{\leftarrow}$ is not an equivalence relation.

Q3 Consider the poset $(\{1, 3, 5, 9, 15, 45\}, |)$. What is $\mathrm{glb}(15, 9)$?

Answer: The lower bounds of 15 and 9 are all the numbers in the set which divide both 15 and 9: $\{1, 3\}$. Of these, 3 is divisible by every element in $\{1, 3\}$ so it is the maximum element of the set of lower bounds. Hence $\mathrm{glb}(15, 9) = 3$.

Q4 Suppose R is a symmetric relation. True or false: $R = R^{\leftarrow}$?

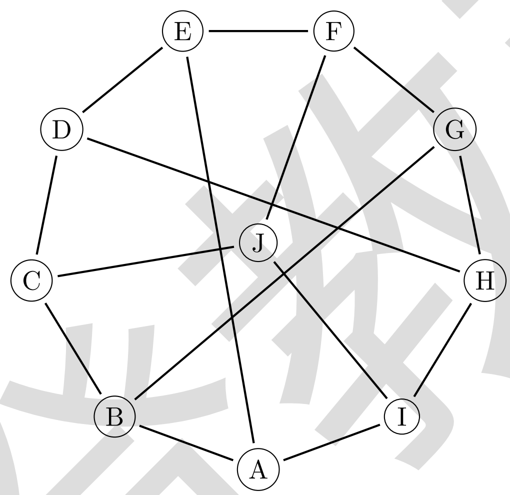
Answer: $(x, y) \in R$ if and only if $(y, x) \in R$ (because R is symmetric), and $(y, x) \in R$ if and only if $(x, y) \in R^{\leftarrow}$ (by the definition of converse). So $R = R^{\leftarrow}$.

Q5 Which of the following is the lexicographic ordering of: $01, 101, 1001, 11100, 01111, 0011$?

Answer: The lexicographic (i.e. dictionary) ordering is: 0011, 01, 01111, 1001, 101, 11100.

Quiz 4: Graphs

All questions refer to the following graph, G, which is the Petersen graph:



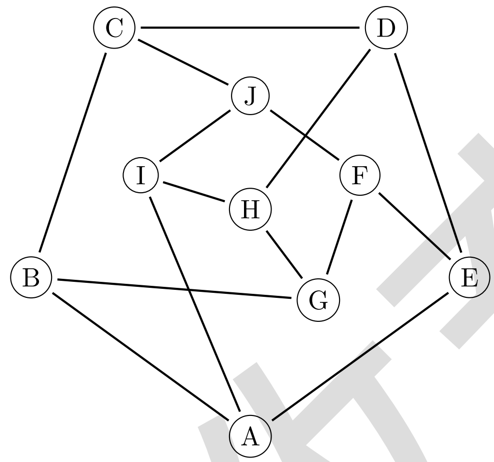
Q1 Which of the following is **not** isomorphic to G?

Answer: In G, every vertex has degree 3. In the graph specified by the adjacency matrix, the vertex in the fifth row has degree 4. So the adjacency matrix is not isomorphic to G. To see that the other representations are isomorphic to G:

Adjacency list: This is just the adjacency list of G.

A:	B, E, I
B:	A, C, G
C:	B, D, J
D:	C, E, H
E:	D, A, F
F:	E, G, J
G:	B, F, H
H:	D, G, I
I:	A, H, J
J:	C, F, I

Graph: Here is a labelling of the vertices showing the isomorphism.



Incidence matrix: Here is how we can label the columns and rows to get the incidence matrix of G:

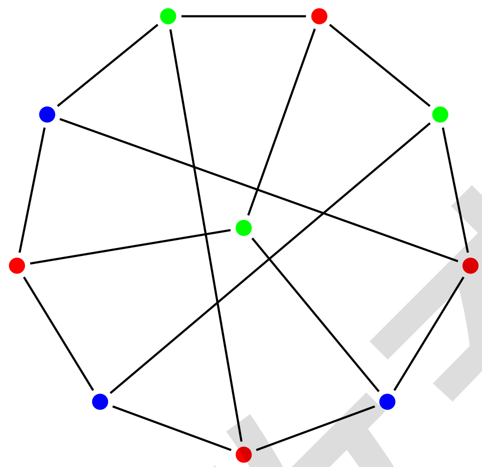
	AB	BC	CD	DE	EF	FG	GH	HI	AI	AE	BG	CJ	DH	FJ	IJ
A	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0
B	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0
C	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0
D	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0
E	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0
F	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0
G	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0
H	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0
I	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1
J	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

Q2 Does G have an Euler path, Hamiltonian path, both, or neither?

Answer: There are more than two vertices with odd degree, so G does not have an Euler path. A-B-C-D-E-F-G-H-I-J is an example of a Hamiltonian path.

Q3 What is the chromatic number, $\chi(G)$, of G?

Answer: G contains an odd-length cycle (A-B-C-D-E-F-G-H-I-A) so it requires at least 3 colours, i.e. $\chi(G) \geq 3$. On the other hand, here is a 3-colouring showing $\chi(G) \leq 3$:



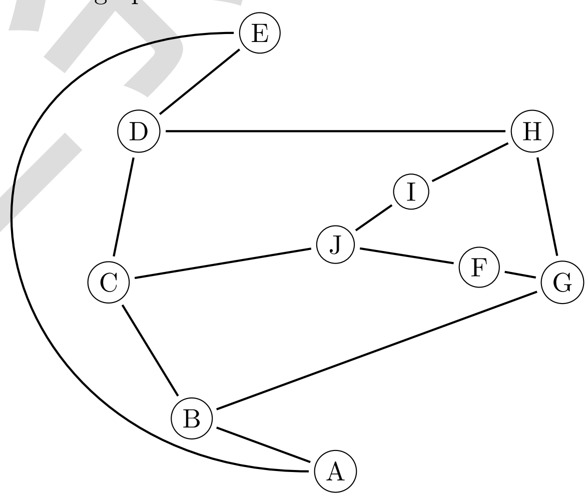
Q4 What is the clique number, $\kappa(G)$, of G?

Answer: There are edges so $\kappa(G) \geq 2$. There are no cycles with 3 vertices (i.e. 3-cliques), so $\kappa(G) \leq 2$.

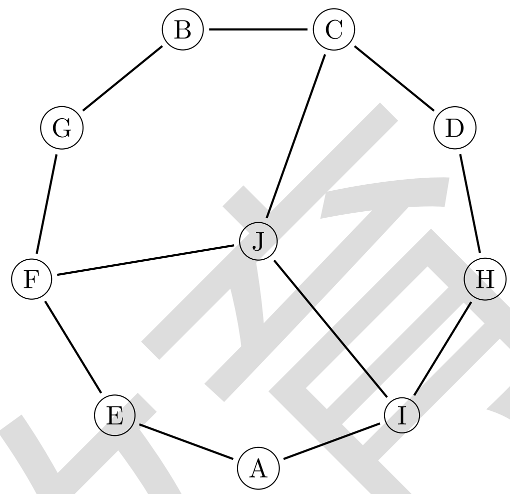
Q5 Which edges, when removed from G, result in a subdivision of $K_{3,3}$?

Answer: We can "untangle" three of the graphs as follows:

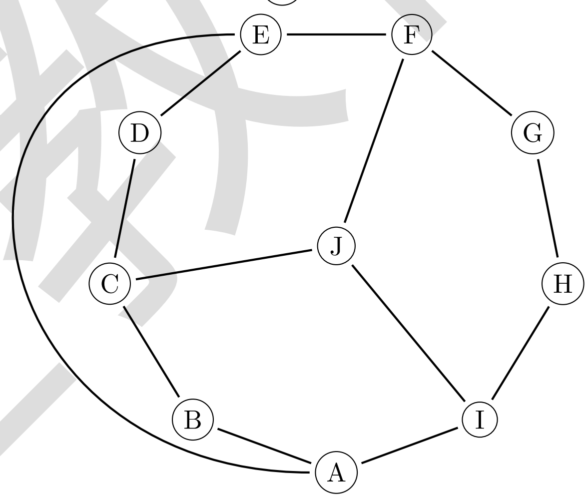
Remove $\{E, F\}$ and $\{A, I\}$



Remove $\{A, B\}, \{D, E\}, \{G, H\}$



Remove $\{B, G\}$ and $\{D, H\}$



This shows that these graphs are planar, so none of them can be a subdivision of $K_{3,3}$. For the remaining option we see that we can partition the six degree three vertices as $\{B, F, H\}$ and $\{D, G, I\}$ such that between each pair of vertices in separate partitions there is either an edge or a subdivided edge, and between each pair of vertices in the same partition there is neither an edge nor a subdivided edge.

1.

Which of the following is logically equivalent to: ¬(A ↔ B)

○	¬(A→B) ∧ ¬(B→A)
○	¬A ↔ ¬B
○	(A ∨ B) ∧ (¬A ∧ ¬B)
◉	(A ∧ ¬B) ∨ (¬A ∧ B)

Quiz 5: Logic I

Q1 Which of the following is logically equivalent to: $\varphi = \neg(A \leftrightarrow B)$:

Answer: Consider the valuation $v(A) = T$, $v(B) = F$. Then $v(\varphi) = T$. However $v(\neg(A \rightarrow B) \wedge \neg(B \rightarrow A)) = F$ and $v(\neg A \leftrightarrow \neg B) = F$ and $v((A \vee B) \wedge (\neg A \wedge \neg B)) = F$. Examination of the other three possible valuations shows that $(A \wedge \neg B) \vee (\neg A \wedge B)$ is logically equivalent to φ.

Q2 Which of the following is a well-formed propositional formula (allowing for conventional omissions):

- $p \leftrightarrow q \leftrightarrow r$
- $p \wedge q \leftrightarrow r \vee s$
- $p \rightarrow q \wedge q \rightarrow p$
- $p \vee q \vee r$

Answer: Because $\wedge$ and $\vee$ bind more tightly than $\leftrightarrow$ we can add parentheses as follows:

- $p \leftrightarrow q \leftrightarrow r$
- $(p \wedge q) \leftrightarrow (r \vee s)$
- $p \rightarrow (q \wedge q) \rightarrow p$
- $p \vee q \vee r$

Of these, only the second meets the definition of an unambiguous propositional formula.

Q3 Suppose φ and ψ are logically equivalent. Which of $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ are tautologies?

Answer: For any valuation v,

- if $v(\varphi) = T$ then $v(\psi) = T$ so $v(\varphi \rightarrow \psi) = v(\varphi \leftrightarrow \psi) = T$
- otherwise $v(\varphi) = F$, so $v(\psi) = F$ and hence $v(\varphi \rightarrow \psi) = v(\varphi \leftrightarrow \psi) = T$.

So both $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ evaluate to T for all valuations. So both are tautologies.

Q4 Suppose $\theta, \psi \models \neg\varphi$. Which of the following does NOT hold:

- $\theta, \varphi, \psi \models \bot$
- $\models \psi \to (\theta \to \neg\varphi)$
- $\theta, \varphi \models \neg\psi$
- $\theta \models \varphi \to \neg\psi$
- $\varphi \models \neg\theta \wedge \neg\psi$

Answer: Consider the following valuations:

	θ	ψ	φ	$\psi \to (\theta \to \neg\varphi)$	$\neg\psi$	$\varphi \to \neg\psi$	$\neg\theta \wedge \neg\psi$
v_1 :	F	F	F	T	T	T	T
v_2 :	F	F	T	T	T	T	T
v_3 :	F	T	F	T	F	T	F
v_4 :	F	T	T	T	F	F	F
v_5 :	T	F	F	T	T	T	F
v_6 :	T	F	T	T	T	T	F
v_7 :	T	T	F	T	F	T	F

Note that because $\theta, \psi \models \neg\varphi$ there is no valuation v that has $v(\theta) = v(\psi) = v(\varphi) = T$. In particular, even though there is no valuation v such that $v(\bot) = T$, it is the case that $v(\bot) = T$ for all valuations such that $v(\theta) = v(\psi) = v(\varphi) = T$ (because there are no such valuations). Thus $\theta, \varphi, \psi \models \bot$. For the other options, we see that

- for all valuations $v(\psi \to (\theta \to \neg\varphi)) = T$ so $\psi \to (\theta \to \neg\varphi)$ is a tautology;
- for $v \in \{v_6\}$: $v(\neg\psi) = T$, so $\theta, \varphi \models \neg\psi$;
- for $v \in \{v_5, v_6, v_7\}$: $v(\varphi \to \neg\psi) = T$, so $\theta \models \varphi \to \neg\psi$; and
- for $v = v_4$ (or $v = v_6$): $v(\neg\theta \wedge \neg\psi) = F$ but $v(\varphi) = T$, so $\varphi \not\models \neg\theta \wedge \neg\psi$.

Q5 True or false: $((p \to (q \to \bot)) \to \bot) \leftrightarrow (p \wedge q)$ is a tautology.

Answer: Observe that $p \to \bot$ is logically equivalent to $\neg p$, so the left-side is logically equivalent to $\neg(p \to \neg q)$ which is logically equivalent to $\neg(\neg p \vee \neg q)$ and this is logically equivalent to $(p \wedge q)$. It follows by the theorem given in lectures that $((p \to (q \to \bot)) \to \bot) \leftrightarrow (p \wedge q)$ is a tautology.

Quiz 6: Logic II

Q1 Let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and $B = \{1, 2, 4, 8\}$. Define:

- $a \vee b = \mathrm{lcm}(a, b)$
- $a \wedge b = \gcd(a, b)$
- $a' = 30/a$ for $a \in A$; $b' = 8/b$ for $b \in B$

Which of A and B equipped with these operations is a Boolean Algebra?

Answer: A is a Boolean algebra. This is seen easiest by associating each element of A with the set of its prime divisors – a subset of $\{2, 3, 5\}$. $\vee$ then corresponds to union (since the lcm of a and b will be the product of primes in the union of the sets of primes making up a and b), $\wedge$ corresponds to intersection, and $(\cdot)'$ corresponds to set complementation (w.r.t. $\{2, 3, 5\}$). In other words $(A, \vee, \wedge, ', 1, 30)$ is *isomorphic* to the Boolean Algebra $(\mathrm{Pow}(\{2, 3, 5\}), \cup, \cap, \cdot^c, \emptyset, \{2, 3, 5\})$.

B is not a Boolean algebra. This can be seen with a violation of the complementation law: $(2' \vee 2) = (4 \vee 2) = 4$ whereas $(1' \vee 1) = (8 \vee 1) = 8$. So there is no unique "1" element, so B cannot be a Boolean Algebra.

Q2 True or false: $\neg(p \to (q \wedge r)), (\neg r \vee p) \models \neg q$?

Answer: The valuation which maps $p \mapsto T$, $q \mapsto T$, $r \mapsto F$ will map $\neg(p \to (q \wedge r))$ to T and $(\neg r \vee p)$ to T but $\neg q$ to F. So there is a valuation which sets all the formulae on the left to T, but the formula on the right to F, so $\neg q$ is not a logical consequence of $\neg(p \to (q \wedge r)), (\neg r \vee p)$. So the answer is **false**.

Q3 The *parity* function, $\mathsf{parity}(x, y, z)$, is a 3-ary Boolean function that returns T if and only if an odd number of x, y, z are T. How many clauses are there in the canonical DNF for parity?

Answer: The canonical DNF can be read off the truth table for the function: each row which evaluates to T corresponds to a clause in the canonical DNF. For parity this is the truth table:

x	y	z	parity
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

and the resulting DNF will be:

$$\mathsf{parity}(x, y, z) = xy\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z}.$$

So there are **four** clauses in the canonical DNF.

Q4 How many clauses are there in an optimal DNF for parity (i.e. what is the minimum number of covering rectangles in a Karnaugh map for parity)?

Answer: The Karnaugh map for parity looks like:

	xy	$\overline{x}y$	$\overline{x}\,\overline{y}$	$x\overline{y}$
z	$+$		$+$	
$\overline{z}$		$+$		$+$

We cannot use any rectangle other than 1×1 rectangles to cover the $+$'s, so the minimum number of covering rectangles (and hence the minimum number of clauses in an optimal DNF) is **four**.

Q5 Which of the following propositional formulae is in CNF:

A: $\neg x \wedge y \wedge z$

B: $\neg x \vee y \vee z$

Answer: Recall: a formula is in CNF if it is a conjunction of CNF-clauses, where a CNF-clause is a disjunction of one or more literals, and a literal is a propositional variable or the negation of a propositional variable.

So, A is a formula in CNF with 3 clauses (one literal per clause) and B is a formula in CNF with 1 clause (containing three literals). So **both** A and B are in CNF.

We note that A and B are also formulae in DNF [A having one DNF-clause, B having three DNF-clauses].

Quiz 7: Induction and Recursion

Q1 Suppose you are trying to prove that the square of every natural number has remainder 0 or 1 on division by 4. If $P(n)$ is the proposition:

$$P(n):\ n^2 = 0 \text{ or } 1 \pmod 4$$

Which of the following would be sufficient to prove the result by induction:

(a) $P(0)$ and $P(1)$ and $\forall k \geq 0,\ P(k) \to P(k+2)$

(b) $P(0)$ and $\forall k \geq 0,\ P(k^2) \to P((k+1)^2)$

(c) $P(0)$ and $\forall k \geq 0,\ P(k) \to P(k+2)$ and $\forall k \geq 0,\ P(k+1) \to P(k+3)$

(d) $P(0)$ and $\forall k \geq 0,\ P(k^2) \to P(k^2+1)$

Answer: (a) This is the correct answer: From $P(0)$ and $P(k) \to P(k+2)$ we have that $P(n)$ holds for all even n; and from $P(1)$ and $P(k) \to P(k+2)$ we have that $P(n)$ holds for all odd n. So $P(n)$ holds for all natural numbers.

(b) This fails because from $P(0)$ and $P(k^2) \to P((k+1)^2)$, we can establish $P(1)$, $P(4)$, $P(9)$, etc. Although we can show that $P(n)$ holds for infinitely many natural numbers, there are some missing: for example, we cannot establish $P(2)$ holds.

(c) This fails because from $P(0)$ and $P(k) \to P(k+2)$ we can establish that $P(n)$ holds for all even n. $P(k+1) \to P(k+3)$ says that we could establish $P(n)$ for all odd n *if* we could establish $P(1)$, but we cannot establish that from $P(0)$ and the two inductive steps.

(d) This fails because from $P(0)$ and $P(k^2) \to P(k^2+1)$ we can establish $P(1)$ and $P(2)$ and that is all – there is no k other than $k = 0, 1$ such that $P(k^2)$ has been established.

Q2 The *dual* of a propositional formula is obtained by replacing $\top$ with $\bot$, $\bot$ with $\top$, $\wedge$ with $\vee$, and $\vee$ with $\wedge$. Recursively, if $\mathtt{PF}$ is the set of propositional formulas over Prop, we define $\mathsf{dual} : \mathtt{PF} \to \mathtt{PF}$ as follows:

- $\mathsf{dual}(\top) = \bot$; $\mathsf{dual}(\bot) = \top$;
- $\mathsf{dual}(p) = p$ for all $p \in \mathsf{Prop}$;
- $\mathsf{dual}(\neg\varphi) = \neg\mathsf{dual}(\varphi)$;
- $\mathsf{dual}(\varphi \wedge \psi) = \mathsf{dual}(\varphi) \vee \mathsf{dual}(\psi)$;

- $\mathsf{dual}(\varphi \vee \psi) = \mathsf{dual}(\varphi) \wedge \mathsf{dual}(\psi)$.

You may take it as given that if φ is logically equivalent to ψ, then $\mathsf{dual}(\varphi)$ is logically equivalent to $\mathsf{dual}(\psi)$. Which of the following is logically equivalent to $\mathsf{dual}(p \to q)$?

	p→q
	q→p
	¬(p→q)
	¬(q→p)

Answer: $p \to q$ is logically equivalent to $\neg p \vee q$. So

$$\begin{aligned}
\mathsf{dual}(p \to q) &\equiv \mathsf{dual}(\neg p \vee q) \\
&= \mathsf{dual}(\neg p) \wedge \mathsf{dual}(q) \\
&= \neg \mathsf{dual}(p) \wedge q \\
&= \neg p \wedge q \\
&\equiv \neg(\neg\neg p \vee \neg q) \\
&\equiv \neg(p \vee \neg q) \\
&\equiv \neg(q \to p).
\end{aligned}$$

Q3 As before, let $\mathtt{PF}$ be the set of well-formed formulas over Prop. Define $\mathsf{flip} : \mathtt{PF} \to \mathtt{PF}$ recursively as follows:

- $\mathsf{flip}(\top) = \top$; $\mathsf{flip}(\bot) = \bot$;
- $\mathsf{flip}(p) = \neg p$ for all $p \in \mathsf{Prop}$;
- $\mathsf{flip}(\neg\varphi) = \neg\mathsf{flip}(\varphi)$;
- $\mathsf{flip}(\varphi \wedge \psi) = \mathsf{flip}(\varphi) \wedge \mathsf{flip}(\psi)$;
- $\mathsf{flip}(\varphi \vee \psi) = \mathsf{flip}(\varphi) \vee \mathsf{flip}(\psi)$.

True or false: $\mathsf{dual}(\varphi)$ is logically equivalent to $\neg\mathsf{flip}(\varphi)$?

Answer: **True**. Proof is part of Assignment 3.

Q4 Suppose $T(n)$ is defined recursively as follows:

- $T(0) = 1$;
- $T(n) = T(n-1) + 2n$

Which of the following is a valid formula for $T(n)$?

(a) $T(n) = 2n + 1$
(b) $T(n) = 2^{n+1} - 1$
(c) $T(n) = n^2 + n + 1$
(d) $T(n) = n^3 - 2n^2 + 3n + 1$

Answer: We can compute the first few values of $T(n)$ to eliminate the possibilities. Observe that $T(0) = 1$, $T(1) = 3$, $T(2) = 7$ (eliminating (a)), and $T(3) = 13$ (eliminating (b) and (d)); so (c) is the only possibility by elimination. We can show that it is the case by unrolling the definition of $T(n)$, or, more formally, by induction.

Let $P(n)$ be the proposition that $T(n) = n^2 + n + 1$. We will show that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case. $T(0) = 1$ (by definition) and $0^2 + 0 + 1 = 1$ so $P(0)$ holds.

Inductive case. Assume $P(k)$ holds for $k \geq 0$. That is, $T(k) = k^2 + k + 1$. Then we have:

$$
\begin{array}{rcll}
T(k+1) & = & T(k) + 2(k+1) & \text{(definition of } T\text{)} \\
& = & (k^2 + k + 1) + 2(k+1) & \text{(IH)} \\
& = & k^2 + 3k + 3 & \\
& = & (k^2 + 2k + 1) + (k+1) + 1 & \\
& = & (k+1)^2 + (k+1) + 1 &
\end{array}
$$

So $P(k+1)$ holds. That is, $P(k) \to P(k+1)$.

So, by the principle of Mathematical Induction: $P(n)$ holds for all $n \in \mathbb{N}$.

Q5 Suppose $f, g : \{a, b\}^* \to \{a, b\}^*$ are defined recursively as follows:

- $f(\lambda) = a$
- $g(\lambda) = b$
- $f(aw) = f(w)g(w)$
- $f(bw) = g(w)f(w)$
- $g(aw) = g(bw) = f(w)$

What is $f(aab)$?

Answer:

$$
\begin{array}{rll}
f(aab) & = f(ab) \cdot g(ab) & \text{(Definition of } f(aw)\text{)} \\
& = f(b) \cdot g(b) \cdot g(ab) & \text{(Definition of } f(aw)\text{)} \\
& = g(\lambda) \cdot f(\lambda) \cdot g(b) \cdot g(ab) & \text{(Definition of } f(bw)\text{)} \\
& = b \cdot f(\lambda) \cdot g(b) \cdot g(ab) & \text{(Definition of } g(\lambda)\text{)} \\
& = b \cdot a \cdot g(b) \cdot g(ab) & \text{(Definition of } f(\lambda)\text{)} \\
& = b \cdot a \cdot f(\lambda) \cdot g(ab) & \text{(Definition of } g(bw)\text{)} \\
& = b \cdot a \cdot a \cdot g(ab) & \text{(Definition of } f(\lambda)\text{)} \\
& = b \cdot a \cdot a \cdot f(b) & \text{(Definition of } g(aw)\text{)} \\
& = b \cdot a \cdot a \cdot b \cdot a & \text{(Definition of } f(b)\text{, computed earlier)}
\end{array}
$$

Quiz 8: Big-Oh notation

Q1 Suppose $f, g, h, k : \mathbb{N} \to \mathbb{R}$ are such that $f(n) \in O(h(n))$ and $g(n) \in O(k(n))$. Which of the following are true:

A: $f(n)g(n) \in O(h(n)k(n))$

B: $f(n)/g(n) \in O(h(n)/k(n))$

Answer: $f(n) \in O(h(n))$ means that there exists c_1, n_1 such that $f(n) \leq c_1.h(n)$ for $n \geq n_1$; and $g(n) \in O(k(n))$ means that there exists c_2, n_2 such that $g(n) \leq c_2.k(n)$ for $n \geq n_2$. This means that for $n \geq \max\{n_1, n_2\}$ we have $f(n)g(n) \leq c_1c_2h(n)k(n)$, so $f(n)g(n) \in O(h(n)k(n))$.

On the other hand, we can't say $f(n)/g(n) \in O(h(n)/k(n))$: for example consider $f(n) = h(n) = n^3$, $g(n) = n$, and $k(n) = n^2$. We have $f(n) \in O(h(n))$ and $g(n) \in O(k(n))$ but $f(n)/g(n) = n^2$ and this is not $O(h(n)/k(n)) = O(n)$.

So **only A is true**.

Q2 Let F denote the set of all functions from $\mathbb{N}$ to $\mathbb{R}$. Define relations $R, S \subseteq F \times F$ as follows:

- $(f, g) \in R$ if $f \in O(g)$
- $(f, g) \in S$ if $f \in \Theta(g)$

Which of the following statements is true:

A: R is a partial order

B: S is an equivalence relation

Answer: R is reflexive and transitive but it is not anti-symmetric: for example $n \in O(2n)$ and $2n \in O(n)$ but $n \neq 2n$. S is reflexive, symmetric, and transitive so it is an equivalence relation. So **only B is true**.

Q3 Which of the following statements is always true:

A: For any graph G with vertices V, and edges E, $|E| \in O(|V|^2)$

B: For any tree T with vertices V, and edges E, $|E| \in O(|V|)$

Answer: In any (undirected) graph with n vertices, the number of edges is at most $n(n-1)/2 \in O(n^2)$. In any tree with n vertices, the number of edges is precisely $n-1 \in O(n)$. So **both A and B are true**.

Q4 Suppose $T(n)$ is defined as follows:

- $T(1) = 1$
- $T(n) = 8T(n/4) + 2n^2$

Which of the following provides the best upper bound for the asymptotic complexity of $T(n)$?

Answer: We can apply the Master Theorem here, with $d = 4$, $\alpha = \frac{3}{2}$, and $\beta = 2$. So we are in Case 3: meaning $T(n) \in O(n^\beta) = O(n^2)$.

Q5 Order the following functions in increasing asymptotic complexity:

- $2n.\log(n) + 3n^2$
- $\sqrt{7n^2 + 3n + 1}$
- $(2^{1.5})^{\log(n)}$
- $4n^{\log(\log(n))}$
- $n^2/\log(n)$

We have the following observations:

- $2n.\log(n) + 3n^2 \in \Theta(n^2)$
- $\sqrt{7n^2 + 3n + 1} = (7n^2 + 3n + 1)^{1/2} \in \Theta(n)$
- $(2^{1.5})^{\log(n)} = (2^{\log(n)})^{1.5} = n^{1.5} \in \Theta(n^{1.5})$
- $4n^{\log(\log(n))} \in \Omega(n^3)$ because for sufficiently large n, $\log(\log(n)) > 3$.
- For any constant c, for sufficiently large n, $n^2/\log(n) > c.n^{1.6}$ and $n^2/\log(n) < c.n^2$. So $n^2/\log(n) \in \Omega(n^{1.6})$ and $n^2 \notin O(n^2/\log(n))$.

This means that the ordering of the functions in increasing asymptotic complexity is:

1. $\sqrt{7n^2 + 3n + 1}$
2. $(2^{1.5})^{\log(n)}$
3. $n^2/\log(n)$
4. $2n.\log(n) + 3n^2$
5. $4n^{\log(\log(n))}$

Quiz 9: Counting

Q1 How many different 9 letter words can be made by the using the letters in PINEAPPLE once each?

Answer: There are two approaches to this.

First, following strategies used in the lectures, we can try placing the P's in 3 of the 9 places (in $\binom{9}{3}$ ways), then place the E's in 2 of the 6 remaining spots (in $\binom{6}{2}$ ways), then place the remaining four letters (in 4! ways). Thus the total number of ways is $\binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!2!}$.

A second approach is to first count the number of ways if we assume each of the P's and E's are distinguishable (9!) and then divide by the number of ways we have "overcounted": by assuming the P's are distinguishable, we have counted 3! duplicates, and by assuming the E's are distinguishable, we have counted 2! duplicates. So the total number of ways is $\frac{9!}{3!2!}$.

Q2 Let S be a set of size n. Which of the following gives the best asymptotic upper bound for the number of subsets of S of size k?

Answer: There are $\binom{n}{k}$ subsets of size k in a set of size n, so we are looking for an asymptotic bound for $\binom{n}{k}$. Note that $\binom{n}{k} \leq n^k$ because the number of ways of choosing k objects from n *without* replacement is going to be at most the number of ways of choosing k objects from n *with* replacement. In particular, $O(n^k)$ is going to be a better upper bound than $O(k^n)$ ($O(k^n)$ grows asymptotically faster than $O(n^k)$). Secondly, we know $\binom{n}{2} \in O(n^2)$, so $O(2^k)$ and $O(nk)$ are not suitable bounds even for $k = 2$. So the most appropriate upper bound for the number of subsets of S of size k is $O(n^k)$.

Q3 How many integers are there between 1 and 1000 which are divisible by 6 or 15 but not both?

Answer: Let $A_k = \{n \in [1, 1000] \ : \ k|n\}$. We are after $|A_6 \oplus A_{15}|$. Note that $A_6 \cap A_{15} = \{n \in [1, 1000] \ : \ 30|n\} = A_{30}$. From the lectures $|A_k| = \lfloor \frac{1000-1+1}{k} \rfloor$, so $|A_6| = 166$, $|A_{15}| = 66$, and $|A_{30}| = 33$. Now $A_6 \oplus A_{15} = (A_6 \cup A_{15}) \setminus (A_6 \cap A_{15})$, so

$$|A_6 \oplus A_{15}| \quad = \quad |A_6 \cup A_{15}| - |A_6 \cap A_{15}|$$

$$
\begin{aligned}
&= (|A_6| + |A_{15}| - |A_{30}|) - |A_{30}| \\
&= 166 + 66 - 33 - 33 \\
&= 166.
\end{aligned}
$$

Q4 How many sequences of $2n$ coin flips have exactly n heads and n tails?

Answer: We have $2n$ flips, and need to choose n of them to be heads. The remaining flips will be tails, so there are $\binom{2n}{n}$ possible sequences.

Q5 How many sequences of $2n$ coin flips contain no pair of consecutive heads (no HH) and no pair of consecutive tails (no TT)?

Answer: Note that a valid sequence is completely determined by the first flip: if it is heads then the sequence must proceed HTHTHT...; if it is tails then the sequence must be THTHTHT... Thus there are exactly 2 sequences that contain no pair of consecutive heads and no pair of consecutive tails.