

第九章 电磁波的辐射和天线

电磁场的矢量势和标量势

- 存在电流源时的Maxwell方程

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

由第一个方程得 $\nabla \cdot \vec{B} = 0$

$$\vec{B} = \nabla \times \vec{A} \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

(A , ϕ) 通称为电磁势

对确定的电磁场 (E, B) 电磁势的取法不是唯一的

取 $\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla\Psi$

$$\vec{B}' = \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla\Psi) = \nabla \times \vec{A} = \vec{B}$$

$$\vec{E}' = -\nabla\phi - \frac{\partial}{\partial t}(\vec{A} + \nabla\Psi) = -\nabla\left(\phi + \frac{\partial\Psi}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t}$$

为使 E 也保持不变，需要具有以下形式的变化下 ϕ 保持不变。

$$\phi \rightarrow \phi' = \phi + \frac{\partial\Psi}{\partial t}$$

在变换

$$\vec{A}' = \vec{A} + \nabla\Psi, \quad \phi' = \phi + \frac{\partial\Psi}{\partial t}$$

条件下，电场和磁场保持不变。这种变换称为规范变换
在电磁理论中常用的规范变换有Lorenz变换和 库伦 变换

用电磁势表示的Maxwell方程

$$\begin{aligned}\vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla \phi - \frac{\partial \vec{A}}{\partial t}\end{aligned}$$

- 将电磁势代入 $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \times \mu_0$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} - \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \vec{J}$$

- 将电磁势代入 $\nabla \cdot \vec{E} = \rho / \varepsilon_0$

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon_0}$$

- 假设有下式成立：

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

电磁势方程：

Lorenz 条件

$$\begin{aligned}\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{j} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\end{aligned}$$

电磁势方程是有源的波动方程

- 如果 $\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \neq 0$, 则总能找到一个规范变换使得:

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

这种使上式成立的规范称为Lorenz规范, 故Lorenz条件也叫Lorenz规范条件。

- Lorenz规范条件下的电磁势方程:

$$\begin{aligned}\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{j} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\end{aligned}$$

- Lorenz条件的物理实质:

$$\nabla^2 \rightarrow \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

$$\nabla \cdot (\nabla^2 \vec{A}) + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla^2 \Phi) = 0$$

$$\begin{aligned} \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{j} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0} \end{aligned}$$

将波动方程代入上式得到

$$lhs := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) - \mu_0 \left(\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} \right)$$

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

电流连续性条件

- 库仑规范

找到一种规范条件，使得A满足

$$\nabla \cdot \vec{A} = 0$$

库仑规范条件下电磁势的方程

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2} \frac{\partial (\nabla \phi)}{\partial t} = -\mu_0 \vec{j}$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

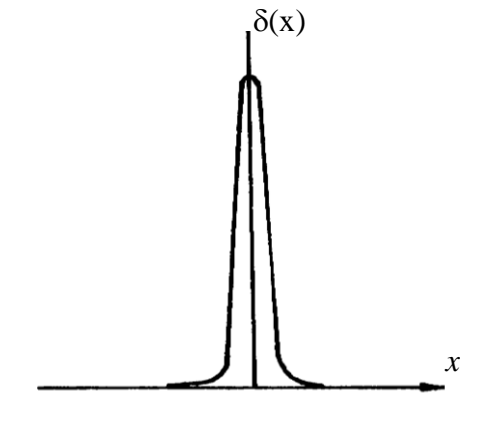
δ 函数

- δ 函数的定义

$$\delta(x) = \begin{cases} 0 & (x \neq 0) \\ \infty & (x = 0) \end{cases}$$

并且

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$



- δ 函数的解析函数表示

$$(1) \quad \delta(x) = \lim_{g \rightarrow \infty} \frac{\sin gx}{\pi x}$$

$$(2) \quad \delta(x) = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\pi (\alpha^2 + x^2)}$$

$$(3) \quad \delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jk(x-x_0)} dk$$

- δ 函数的性质

$$(1) \int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx &= \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\delta(x-x_0)dx \\ &= f(x_0) \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)dx = f(x_0) \end{aligned}$$

$$(2) \int_{-\infty}^{\infty} f(x) \frac{d}{dx} \delta(x-x_0)dx = -\left(\frac{df}{dx}\right)_{x=x_0}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \delta(x-x_0)dx \\ &= f(x)\delta(x-x_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-x_0) \frac{df}{dx} dx = -\frac{df}{dx}\Big|_{x=x_0} \end{aligned}$$

$$(3) \quad \int_a^b f(x) \frac{d^n}{dx^n} \delta(x - x_0) dx = \begin{cases} (-1)^n \left(\frac{d^n f}{dx^n} \right)_{x=x_0} & (a < x_0 < b) \\ 0 & (x_0 < a \text{ or } x_0 > b) \end{cases}$$

$$(4) \quad \int_{-\infty}^{\infty} g(x) \delta[f(x) - \alpha] dx = \left[\frac{g(x)}{df/dx} \right]_{f(x)=\alpha}$$

$$\begin{aligned} \int g(x) \delta[f(x) - \alpha] dx &= \int g(x) \delta[f(x) - \alpha] \frac{dx}{df} df \\ &= \int \frac{g(x)}{(df/dx)} \delta(f - \alpha) df = \left[\frac{g(x)}{(df/dx)} \right]_{f(x)=\alpha} \end{aligned}$$

$$(5) \quad \nabla^2 \frac{1}{r} = -4\pi\delta(r)$$

对上式两边积分

$$\begin{aligned} lhs &= \int_{\tau} \nabla^2 \left(\frac{1}{r} \right) d\tau = \int_{\tau} \nabla \cdot \nabla \left(\frac{1}{r} \right) d\tau = \oint_s \nabla \left(\frac{1}{r} \right) \cdot d\vec{s} \\ &= -\oint_s \frac{\vec{r}}{r^3} \cdot d\vec{s} = -\oint_s \frac{1}{r^2} r^2 d\Omega = -4\pi \end{aligned}$$

$$rhs = \int_{\tau} -4\pi\delta(r) d\tau = -4\pi$$

$$\therefore \quad \nabla^2 \frac{1}{r} = -4\pi\delta(r)$$

d'Alembert方程和推迟势

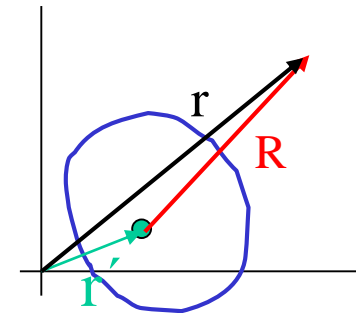
- 标量势和矢量势满足形式相似的线性偏微分方程：

d'Alembert方程

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon_0} \rho(r, t)$$

- 线性方程的迭加原理

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon_0} \sum_i q(\vec{r}, t) \delta(\vec{r} - \vec{r}_i')$$



- 点源的辐射场：在坐标原点中的解

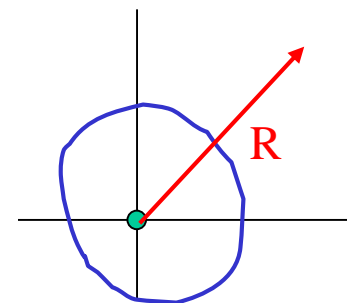
$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{q(\vec{r}, t)}{\epsilon_0} \delta(\vec{r} - \vec{r}_i')$$

$r_i \neq 0$ 时的齐次方程解

$$\frac{\partial^2 \phi'}{\partial R^2} + \frac{2}{R} \frac{\partial \phi'}{\partial R} - \frac{1}{c^2} \frac{\partial^2 \phi'}{\partial t^2} = 0$$

其中 $\phi' = \phi'(|\vec{r} - \vec{r}'|, t) = \phi'(R, t)$

作变换 $\phi'(R, t) = \frac{u(R, t)}{R}$



得到波动方程

$$\frac{\partial^2 u}{\partial R^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial R^2} \right) u = 0 \quad \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial R} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial R} \right) u = 0$$

令 $\xi = R + ct$; $\eta = R - ct$ 得

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \rightarrow u = f(\eta) + g(\xi)$$

其解的一般形式为

$$u(R, t) = f(t - R/c) + g(t + R/c)$$

点源产生的波只能取 f 的形式

$$\phi'(R, t) = \frac{f(t - R/c)}{R}$$

在 $R=0$ 处，上式应满足 d'Alembert 方程

$$\nabla^2 \phi' - \frac{1}{c^2} \frac{\partial^2 \phi'}{\partial t^2} = -\frac{q}{\epsilon_0} \delta(\vec{R})$$

在点源的球形邻域内积分

$$\int_{\tau} \left(f \nabla^2 \frac{1}{R} + 2 \nabla \frac{1}{R} \cdot \nabla f + \frac{1}{R} \nabla^2 f - \frac{1}{c^2 R} \frac{\partial^2 f}{\partial t^2} \right) d\tau = -\frac{q}{\epsilon_0}$$

$$LHS = f \int_{\tau} \nabla^2 \frac{1}{R} d\tau = f \oint_{\sigma} \nabla \frac{1}{R} \cdot d\vec{s} = -4\pi f$$

$$f = \frac{q}{4\pi\epsilon_0}$$

最终有

$$\phi' = \frac{1}{4\pi\epsilon_0} \frac{q(t - R/c)}{R}$$

迭加原理

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(r', t - R/c)}{R} d\tau'$$

相似性

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(r', t - R/c)}{R} d\tau'$$

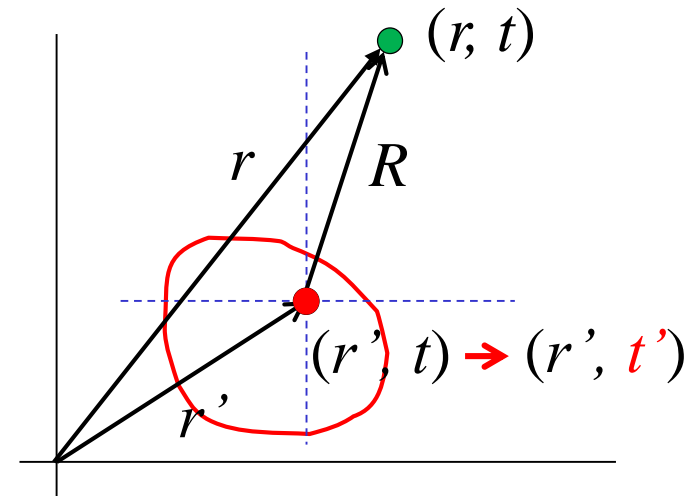
- 物理意义

(1) 电磁作用的传递不是瞬时的，而是以光速进行的。

(2) 点源对场点在 t 时刻、 r 处的电磁势贡献来源于 $(t - R/c)$ 时刻位于 r' 处的点源所激发的场。

(3) R/c 称为推迟时间

(4) 推迟效应， t 时刻 r 处的电磁场来源于不同时刻和不同地点处的所有点源贡献。离场点越远的点源 所需要推迟的时间越多。



辐射场

- 辐射场：源在远离源所在位置处产生的电磁场，辐射功率不随距离改变。
- 推迟势的化简

当 $r \gg r'$ 时 $R = |\vec{r} - \vec{r}'| \approx r - r' \cos \xi$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi r} \iiint \vec{j} \left(\vec{r}', t - \frac{(r - r' \cos \xi)}{c} \right) d\tau'$$

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0 r} \iiint \rho \left(\vec{r}', t - \frac{(r - r' \cos \xi)}{c} \right) d\tau'$$

- 例： $\vec{j}(\vec{r}', t) = \vec{j}(\vec{r}') e^{j\omega t}$

$$\begin{aligned}\vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi r} \iiint \vec{j}(\vec{r}') e^{j\omega \left(t - \frac{r - r' \cos \xi}{c} \right)} d\tau' \\ &= \frac{\mu_0 e^{j\omega t}}{4\pi r} \iiint \vec{j}(\vec{r}') e^{-jk(r - r' \cos \xi)} d\tau'\end{aligned}$$

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0 e^{-jkr}}{4\pi r} \iiint \vec{j}(\vec{r}') e^{jkr' \cos \xi} d\tau' \\ &= \frac{\mu_0}{4\pi r} e^{-jkr} f(\vec{r}', k)\end{aligned}$$

当只保留 r^{-1} 项时电磁场为

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}) = \frac{1}{\mu_0} \{0, jkA_\phi, -jkA_\theta\}$$

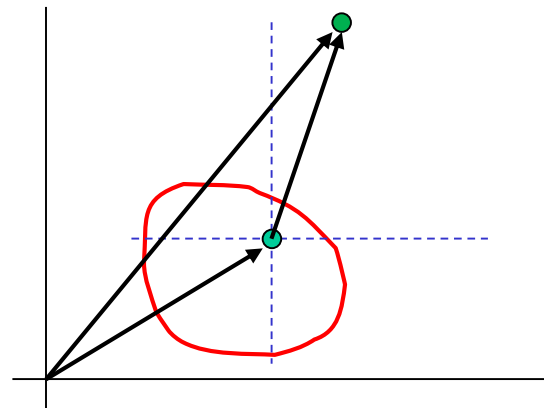
$$\vec{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \vec{H}$$

加速带电粒子的辐射场

- 带电粒子运动的推迟势

电流产生的矢量势

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}', t - R/c)}{R} d\tau' \\ &= \frac{\mu_0}{4\pi} \iint \frac{\vec{j}(\vec{r}', t')}{R} \delta[t' - (t - R/c)] d\tau' dt'\end{aligned}$$

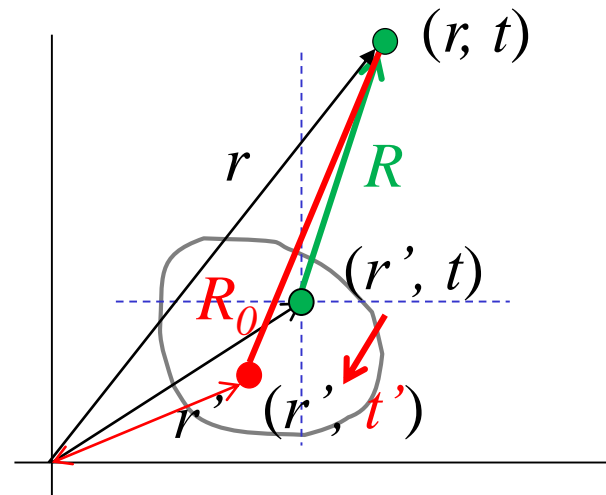


带电粒子的电荷密度 $\rho(\vec{r}', t) = q\delta(\vec{r} - \vec{r}'(t))$

带电粒子的电流密度 $\vec{j}(\vec{r}', t) = q\vec{v}(t)\delta(\vec{r} - \vec{r}'(t))$

$$\begin{aligned}
 \text{代入 } \vec{A} &= \frac{\mu_0 q}{4\pi} \int dt' \iiint \frac{\vec{v}(t') \delta(\vec{r} - \vec{r}'(t'))}{R} \delta\left(t' + \frac{R}{c} - t\right) dx' dy' dz' \\
 &= \frac{\mu_0 q}{4\pi} \int \frac{v(t')}{R_0} \delta\left(t' + \frac{R_0}{c} - t\right) dt'
 \end{aligned}$$

$$R = |\vec{r} - \vec{r}'|, \quad R_0 = |\vec{r} - \vec{r}'_0|$$



$$\text{应用} \quad \int g(x) \delta(f(x) - \alpha) dx = \frac{g(x)}{(df/dx)} \Big|_{f(x)=\alpha}$$

$$\left. \begin{aligned}
 g(t') &= v(t')/R_0 \\
 f(t') &= t' + R_0(t')/c \\
 \alpha &= t
 \end{aligned} \right\} \rightarrow \frac{df}{dt'} = 1 + \frac{1}{c} \frac{\partial R_0}{\partial t'}$$

而

$$\begin{aligned}\frac{\partial R}{\partial t'} &= \frac{\partial}{\partial t'} \left(\sqrt{(x-x_0')^2 + (y-y_0')^2 + (z-z_0')^2} \right) \\ &= -\frac{\hat{R} \cdot \vec{v}}{R} = -\hat{e}_R \cdot \vec{v}\end{aligned}$$

$$\frac{df}{dt'} = 1 - \frac{\hat{e}_R \cdot \vec{v}}{c}$$

从而

$$\begin{aligned}\vec{A} &= \frac{\mu_0 q}{4\pi} \left[\frac{\vec{v}}{R(1 - \hat{e}_R \cdot \vec{v}/c)} \right]^{ret} \\ \phi &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R(1 - \hat{e}_R \cdot \vec{v}/c)} \right]^{ret}\end{aligned}$$

再根据

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\text{令, } s^* = \left\{ R(1 - \hat{e}_R \cdot \vec{v}/c) \right\}^{ret}, \quad \vec{R}^* = \vec{r} - \left\{ \vec{r}^* \right\}^{ret}$$

令, 得到

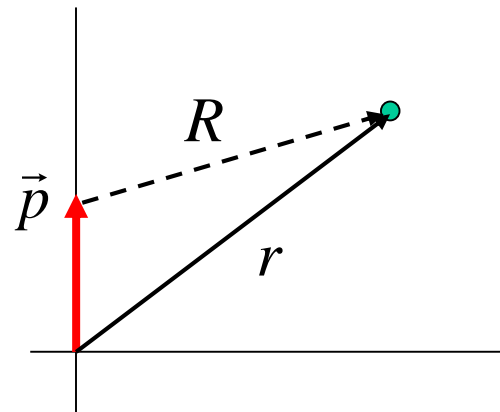
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{e}{s^{*3}} \left\{ \left(1 - \frac{v^{*2}}{c^2} \right) \left(\vec{R}^* - \frac{R^*}{c} \vec{v}^* \right) + \frac{1}{c^2} \vec{R}^* \times \left[\left(\vec{R}^* - \frac{R^*}{c} \vec{v}^* \right) \times \vec{a}^* \right] \right\}$$
$$\vec{B} = \frac{1}{c} \frac{\vec{R}^*}{R^*} \times \vec{E}$$

电偶极子的辐射场

- 负电荷静止不动，而正电荷以速度 v 相对负电荷运动，电偶极矩 $p=q \cdot l$

(1) $v \ll c$

(2)
$$\begin{aligned}\vec{R} &= \vec{r} - \vec{r}_0 \\ &= (x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z} \\ &\approx r\end{aligned}$$



$$\vec{A} = \frac{\mu_0}{4\pi} \left[\frac{q\vec{v}}{R(1 - \hat{e}_R \cdot \vec{v}/c)} \right]^{ret} \approx \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}^{ret}}{R} \approx \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}^{ret}}{r}$$

$$A_r = \frac{\mu_0 \dot{p}^{ret}}{4\pi r} \cos \theta, \quad A_\theta = -\frac{\mu_0 \dot{p}^{ret}}{4\pi r} \sin \theta, \quad A_\phi = 0$$

$$\begin{aligned} \phi &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R(1 - \hat{e}_R \cdot \vec{v}/c)} \right]^{ret} \approx \frac{q}{4\pi\epsilon_0 R} \left(1 + \frac{\hat{e}_R \cdot \vec{v}}{c} \right)^{ret} \\ &\approx \frac{q}{4\pi\epsilon_0 r} + \frac{\vec{r} \cdot \dot{\vec{p}}^{ret}}{4\pi\epsilon_0 cr^2} \end{aligned}$$

$$\phi = \phi^+ + \phi^- = \frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr} \cos \theta$$

$$\begin{aligned}
 \vec{B} = \nabla \times \vec{A} &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r \sin \theta A_\varphi \end{vmatrix} \\
 &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \frac{\mu_0 \dot{p}^{ret}}{4\pi r} \cos \theta & r \left(-\frac{\mu_0 \dot{p}^{ret}}{4\pi r} \sin \theta \right) & 0 \end{vmatrix}
 \end{aligned}$$

注意 $\dot{p}^{ret} = \dot{p}(r_0, t - R/c) \approx \dot{p}(r_0, t - r/c)$

$$\begin{aligned}
\vec{E} &= -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \\
&= -\nabla \left(\frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr} \cos \theta \right) - \frac{\partial}{\partial t} \left\{ \frac{\mu_0 \dot{p}^{ret}}{4\pi r} \cos \theta, -\frac{\mu_0 \dot{p}^{ret}}{4\pi r} \sin \theta, 0 \right\} \\
&= -\left\{ -\frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr^2} \cos \theta + \left(-\frac{\ddot{p}^{ret}}{4\pi\epsilon_0 c^2 r} \cos \theta \right), -\frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr} \sin \theta, 0 \right\} \\
&\quad - \left\{ \frac{\mu_0 \ddot{p}^{ret}}{4\pi r} \cos \theta, -\frac{\mu_0 \ddot{p}^{ret}}{4\pi r} \sin \theta, 0 \right\}
\end{aligned}$$

电磁场

$$\vec{B} = \left(\frac{\mu_0}{4\pi r^2} \dot{p}^{ret} \sin \theta + \frac{\mu_0}{4\pi cr} \ddot{p}^{ret} \sin \theta \right) \hat{\phi}$$
$$\vec{E} = \frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr^2} \cos \theta \hat{r} + \left(\frac{\dot{p}^{ret}}{4\pi\epsilon_0 cr^2} \sin \theta + \frac{\mu_0 \ddot{p}^{ret}}{4\pi r} \sin \theta \right) \hat{\theta}$$

忽略 r^2 项，得

$$\begin{cases} \vec{B} = \frac{\ddot{p}^{ret}}{4\pi\epsilon_0 c^3 r} \sin \theta \hat{\phi} \\ \vec{E} = \frac{\ddot{p}^{ret}}{4\pi\epsilon_0 c^2 r} \sin \theta \hat{\theta} \end{cases}$$

辐射功率

$$\vec{g} = \vec{E} \times \vec{H} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{(\ddot{p}^{ret})^2}{16\pi^2 \varepsilon_0 c^3 r^2} \sin^2 \theta \hat{r}$$

功率角分布

$$dP = \vec{g} \cdot d\vec{s} = \frac{(\ddot{p}^{ret})^2}{16\pi^2 \varepsilon_0 c^3} \sin^2 \theta d\Omega, \quad \frac{dP}{d\Omega} = \frac{(\ddot{p}^{ret})^2}{16\pi^2 \varepsilon_0 c^3} \sin^2 \theta$$

总功率

$$P = \frac{2}{3c^3} \frac{(\ddot{p}^{ret})^2}{4\pi\varepsilon_0} = \frac{(\ddot{p}^{ret})^2}{6\pi\varepsilon_0 c^3}$$

如 $p = p_0 e^{i\omega t}$, 则 $P \propto \omega^4$

偶极辐射场的矢量形式

电偶极辐射场:

$$\vec{B} = \frac{\mu_0 \ddot{\vec{p}}^{ret} \times \vec{r}}{4\pi r^2 c}; \quad \vec{E} = c\vec{B} \times \hat{r}$$

磁偶极辐射场:

$$\vec{E} = \frac{\mu_0 \vec{r} \times \ddot{\vec{M}}^{ret}}{4\pi r^2 c}; \quad \vec{B} = \frac{1}{c} \hat{r} \times \vec{E}$$

振子天线的辐射场

1. 基本振子天线：振荡电流元的电磁场

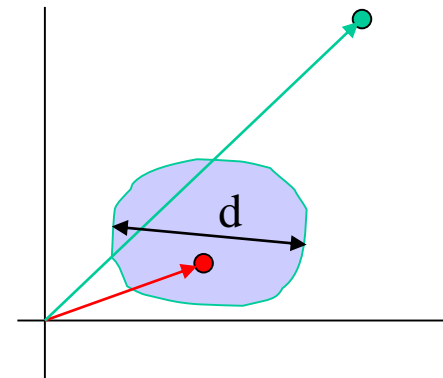
高频电流源为： $\vec{j}(\vec{r}, t) = j_0(\vec{r}) e^{j\omega t}$

$$\vec{A} = \frac{\mu}{4\pi} \int_{\tau'} \frac{\vec{j}(\vec{r}', t - R/v)}{R} d\tau' = \frac{\mu}{4\pi} \left(\int_{\tau'} \frac{j_0(\vec{r}') e^{-i\omega R/c}}{R} d\tau' \right) e^{i\omega t}$$

对远场问题 ($r \gg d$, $r \gg \lambda$, $r \gg d^2/\lambda$)

$$\frac{1}{R} = \frac{1}{r} + \frac{\vec{r}' \cdot \hat{r}}{r^2} + \dots$$

$$\frac{\omega R}{c} = k|\vec{r} - \vec{r}'| = kr - \vec{k} \cdot \vec{r}' - \frac{(\vec{k} \cdot \vec{r}')^2 + k^2 r'^2}{2kr} + \dots$$



保留到二阶项

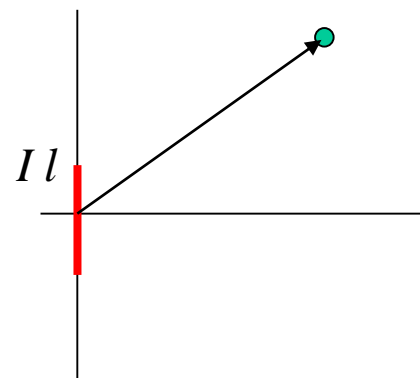
$$\begin{aligned}\vec{A} &= \frac{\mu}{4\pi} \left(\int_{\tau'} \frac{j_0(r') e^{-i\omega R/c}}{R} d\tau' \right) e^{i\omega t} = \frac{\mu}{4\pi} \left(\int_{\tau'} \frac{j_0(r') e^{-i(kr - \vec{k} \cdot \vec{r}')}}{r} d\tau' \right) e^{i\omega t} \\ &= \vec{A}_0(\theta, \varphi) \frac{e^{i(\omega t - kr)}}{r}\end{aligned}$$

其中

$$\vec{A}_0(\theta, \varphi) = \frac{\mu}{4\pi} \int_{\tau'} \vec{j}_0(\vec{r}') e^{i\vec{k} \cdot \vec{r}'} d\tau'$$

对极短的线电流源 $kr' \sim 0$,

$$A_z = \frac{\mu I l}{4\pi r} e^{-ikr}$$



在球坐标中

$$A_r = A_z \cos \theta = \frac{\mu I l}{4\pi r} e^{-ikr} \cos \theta$$

$$A_\theta = -A_z \sin \theta = -\frac{\mu I l}{4\pi r} e^{-ikr} \sin \theta$$

$$A_\phi = 0$$

电磁场

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}, \quad j\omega\epsilon_0 \vec{E} = \nabla \times \vec{H}$$

$$H_r = H_\theta = 0, H_\varphi = \frac{Il \sin \theta}{4\pi} \left(\frac{1}{r^2} + j \frac{k}{r} \right) e^{-ikr}$$

$$E_r = j \frac{2Il \cos \theta}{4\pi\omega\epsilon} \left(\frac{1}{r^3} + j \frac{k}{r^2} \right) e^{-ikr}$$

$$E_\theta = j \frac{Il \sin \theta}{4\pi\omega\epsilon} \left(-\frac{1}{r^3} + j \frac{k}{r^2} + \frac{k^2}{r} \right) e^{-ikr}$$

$$E_\varphi = 0$$

远区电磁场

$$E_\theta = j \frac{Il k^2 \sin \theta}{4\pi\omega\epsilon r} e^{-ikr}$$

$$H_\varphi = j \frac{Il k \sin \theta}{4\pi r} e^{-ikr}$$

$$Z_0 = \frac{E_\theta}{H_\varphi} = \sqrt{\frac{\mu}{\epsilon}}$$

设有任意两个交变量

$$\tilde{A} = A_0 \cos(\omega t + \alpha) = \operatorname{Re}(A e^{i\omega t}) = \operatorname{Re}(U)$$

$$\tilde{B} = B_0 \cos(\omega t + \beta) = \operatorname{Re}(B e^{i\omega t}) = \operatorname{Re}(V)$$

$$\tilde{A}\tilde{B} = \frac{1}{2} A_0 B_0 [\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)]$$

$$\langle \tilde{A}\tilde{B} \rangle = \frac{1}{2} A_0 B_0 \cos(\alpha - \beta)$$

$$UV^* = (A_0 e^{i\alpha} e^{i\omega t})(B_0 e^{i\beta} e^{i\omega t})^* = A_0 B_0 [\cos(\alpha - \beta) + i \sin(\alpha - \beta)]$$

$$\langle \tilde{A}\tilde{B} \rangle = \frac{1}{2} A_0 B_0 \cos(\alpha - \beta) = \frac{1}{2} \operatorname{Re}(UV^*)$$

平均辐射功率

$$\vec{g} = \vec{E} \times \vec{H}$$

$$\langle \vec{g} \rangle = \frac{1}{2} \text{Re}(\vec{E} \times \vec{H}^*) = \hat{r} \left(\frac{1}{2} E_{\theta} H_{\varphi}^* \right)$$

$$P = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} E_{\theta} H_{\varphi}^* r^2 \sin \theta d\theta d\varphi$$

代入表达式

$$P = \frac{40\pi^2 (I_0 l)^2}{\lambda^2} = I^2 \left[80\pi^2 \left(\frac{l}{\lambda} \right)^2 \right] = I^2 R_r$$

2. 对称振子天线

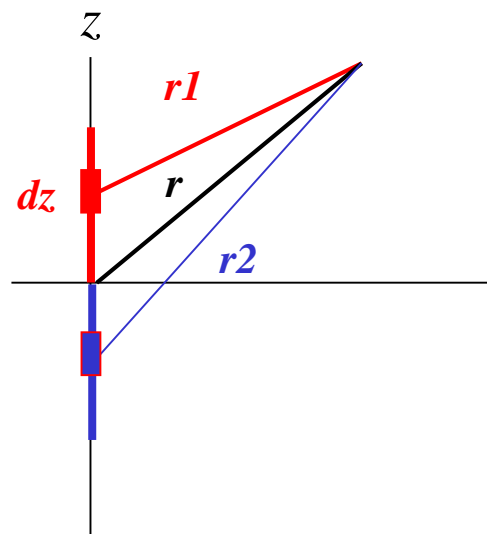
振子天线 的电流分布:

$$I_z = I_0 \sin[k(l/2 - |z|)]$$

短振子天线在自由空间的场

$$H_\phi = j \frac{Il}{2\lambda} \frac{\sin \theta}{r} e^{-jkr}$$

$$E_\theta = Z_0 H_\phi = j \frac{60\pi Il}{\lambda} \frac{\sin \theta}{r} e^{-jkr}$$



$$r_1 = r - z \cos \theta$$

$$r_2 = r + z \cos \theta$$

$$dE_{\theta}^{+} = \frac{60\pi(I_z dz)\sin\theta}{\lambda r_1} e^{-jkr_1} e^{j\omega t} = \frac{60\pi I_z}{\lambda r_1} e^{j(\omega t - kr + kz \cos\theta)} \sin\theta dz$$

$$dE_{\theta}^{-} = \frac{60\pi(I_z dz)\sin\theta}{\lambda r_2} e^{-jkr_2} e^{j\omega t} = \frac{60\pi I_z}{\lambda r_2} e^{j(\omega t - kr - kz \cos\theta)} \sin\theta dz$$

$$\begin{aligned} dE_{\theta} &= dE_{\theta}^{+} + dE_{\theta}^{-} \\ &= \frac{60\pi I_z \sin\theta}{\lambda} e^{j(\omega t - kr)} \left(\frac{e^{jkz \cos\theta}}{r_1} + \frac{e^{-jkz \cos\theta}}{r_2} \right) dz \end{aligned}$$

- 若 $r_1 \approx r_2 \approx r$, 对称电流元在场点产生合成电场

$$dE_{\theta} = dE_{\theta}^{+} + dE_{\theta}^{-} = \frac{120\pi \sin \theta}{\lambda r} e^{j(\omega t - kr)} [I_z \cos(kz \cos \theta) dz]$$

$$E_{\theta} = \int_0^{l/2} dE_{\theta} = \frac{60I_0}{r} e^{j(\omega t - kr)} F(\theta)$$

$$H_{\varphi} = E_{\theta} / Z_0$$

$$\text{其中 } F(\theta) = \frac{\cos(\pi l \cos \theta / \lambda) - \cos(\pi l / \lambda)}{\sin \theta}$$

方向性因子（函数）

功率角分布

$$\vec{g} = \frac{1}{2} \operatorname{Re} [\vec{E} \times \vec{H}^*] = \frac{1}{2Z_0} \left(\frac{60I_0 F(\theta)}{r} \right)^2 = \frac{15I_0^2}{\pi} \frac{F^2(\theta)}{r^2}$$

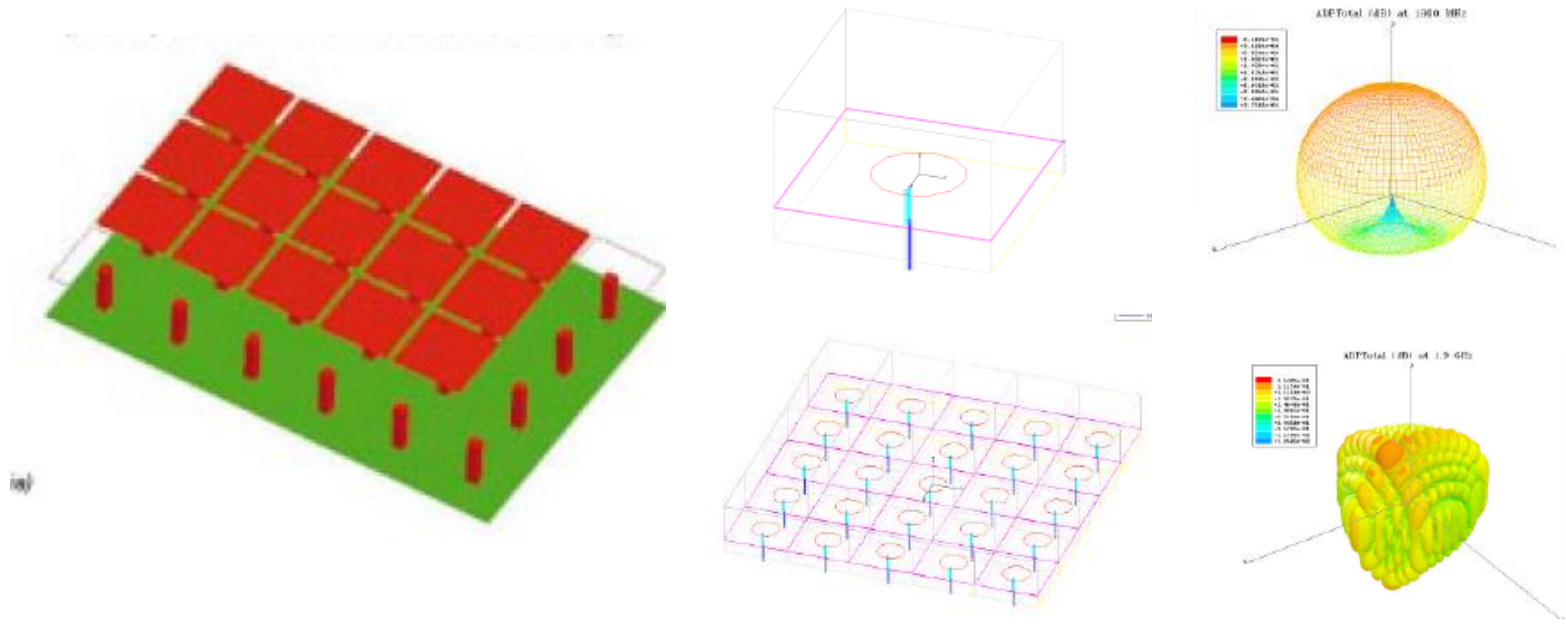
$$dP = \vec{g} \cdot d\vec{S} = g (r^2 \sin \theta d\theta d\varphi) = gr^2 d\Omega$$

$$\frac{dP}{d\Omega} = \frac{15I_0^2}{\pi} F^2(\theta) = \begin{cases} \frac{15I_0^2}{\pi} \left(\frac{l^4 \omega^4}{16c^4} \sin^2 \theta \right) & (kd \ll 1) \\ \frac{15I_0^2}{\pi} \left(\frac{\cos(\pi/2 \cos \theta)}{\sin \theta} \right)^2 & (kd = \pi) \end{cases}$$

半波振子

阵列天线的辐射场

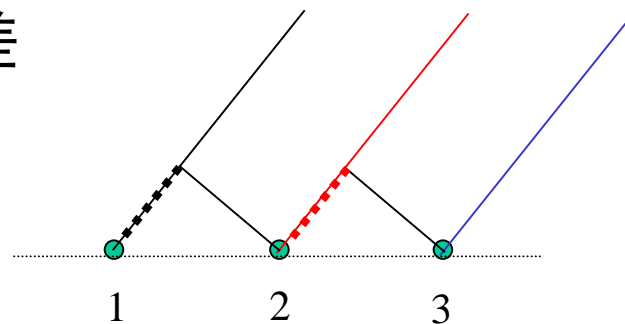
- 天线阵：把多个天线单元按一定规律排列组成的阵列，利用天线阵可以获得尖锐的功率角分布



- “相似天线元” 阵列：各天线元不仅形式相同，而且它们在空间放置的方向也相同。
- 均匀直线式天线阵
 1. 各天线元的电流大小相等，相位相差 Δ
 2. 各天线元间的距离为 d ，光程差是 $d \cdot \cos(\varphi)$
 3. 第1天线元的远区辐射场为 E_0 ，

相邻天线元间的总相位差

$$\psi = \beta d \cos \varphi + \Delta$$



合成电场

$$\vec{E} = \vec{E}_0 (1 + e^{j\psi} + e^{2j\psi} + \dots + e^{j(n-1)\psi}) = E_0 \frac{1 - e^{jn\psi}}{1 - e^{j\psi}}$$

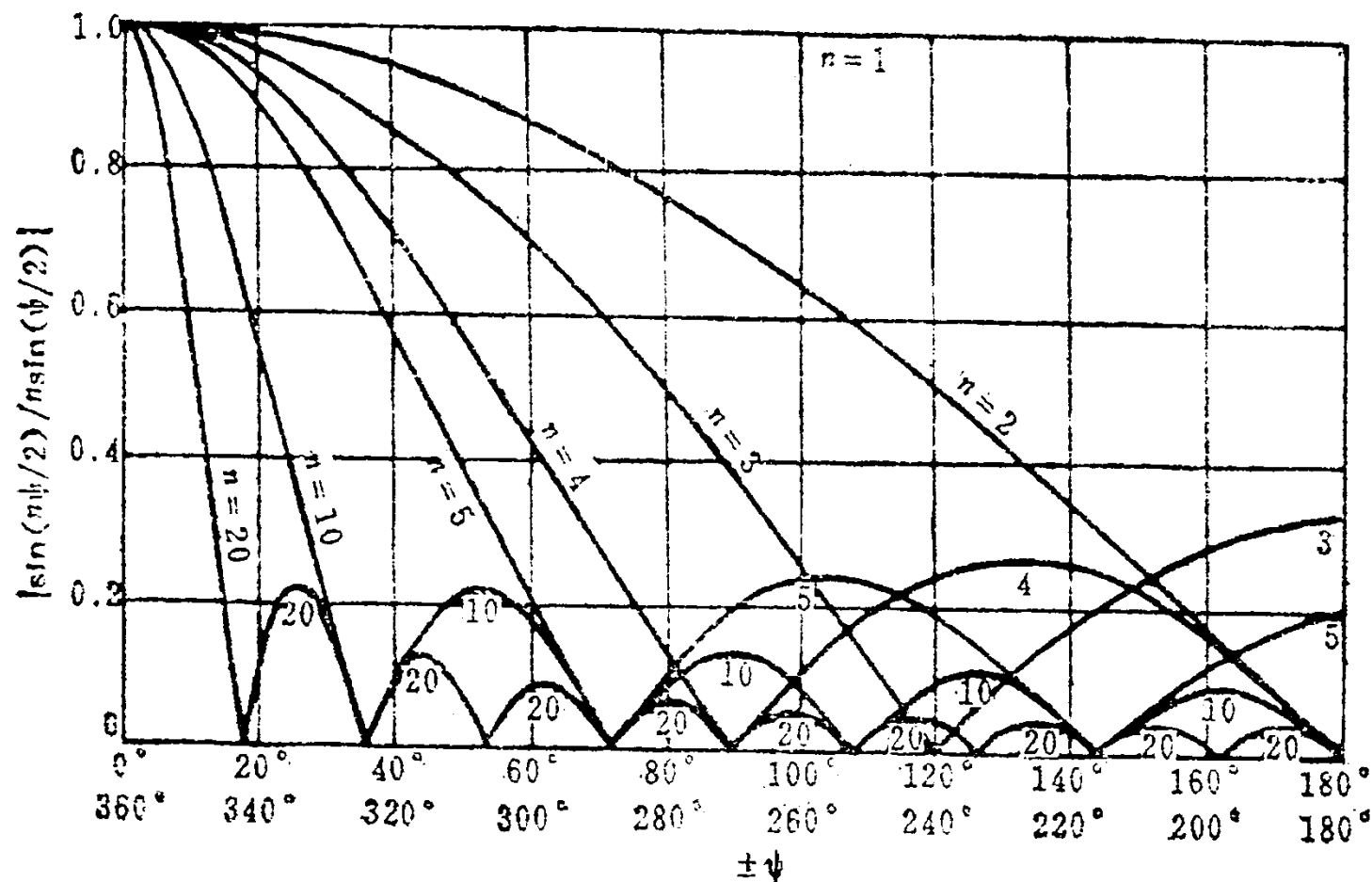
电场大小

$$|E| = |E_0| \left| \frac{1 - e^{jn\psi}}{1 - e^{j\psi}} \right| = |E_0| \frac{\sin\left(\frac{n\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} = |E_0| g_n$$

其中当 $\psi \rightarrow 0$ 时, g_n 有极大值

$$g_n = \lim_{\psi \rightarrow 0} \frac{\sin(n\psi/2)}{\sin(\psi/2)} = n$$

阵因子随相位差和天线个数的变化

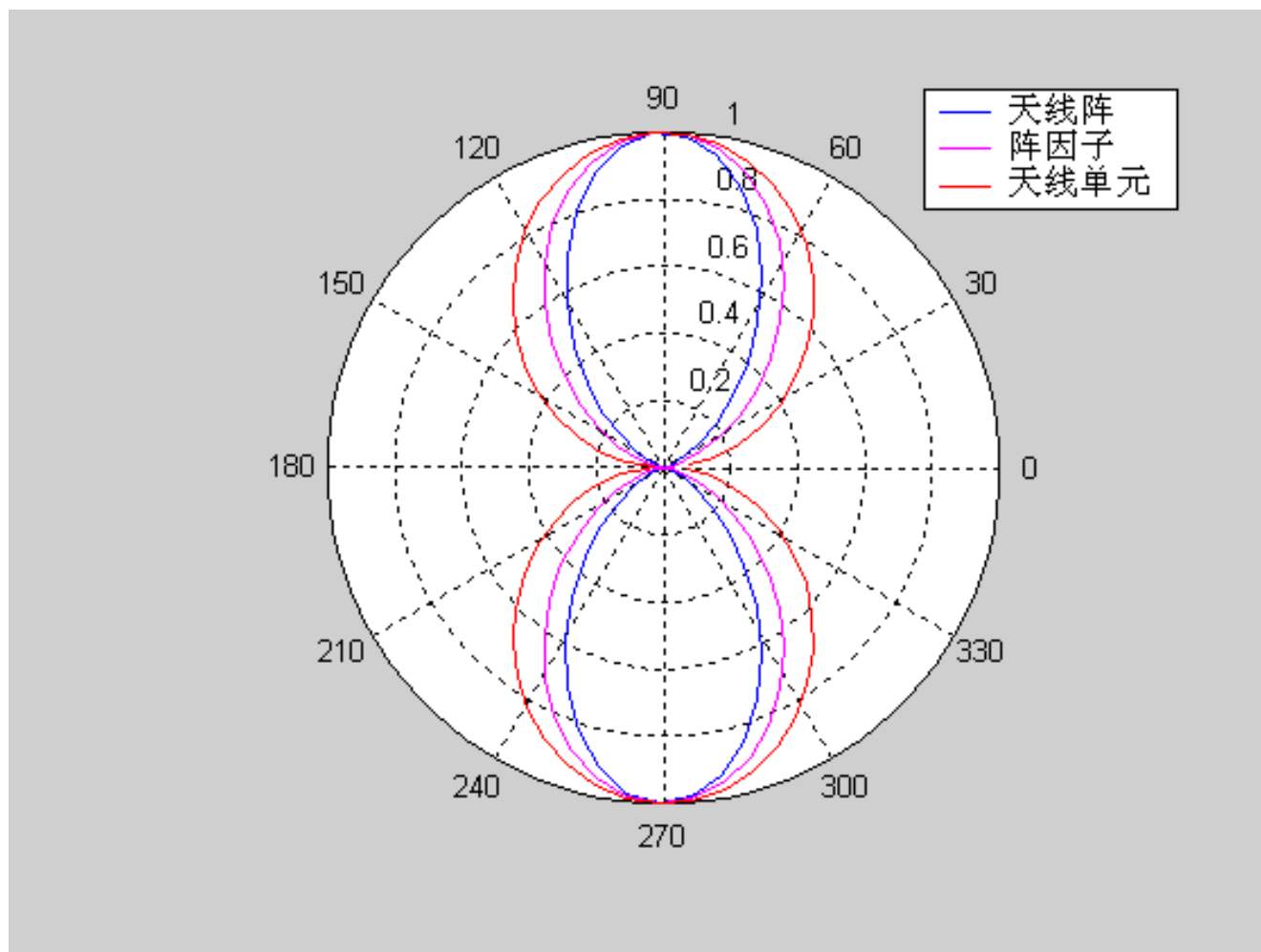


- 方向性图的乘法

$$|E| = |E_0| g_n$$

天线阵的合成电场可视为组成天线阵的天线的方向性图与天线阵的阵方向性图的乘积。

该方法仅限于：即组成天线阵的各天线元具有相同的型式和相同的电流分布规律。

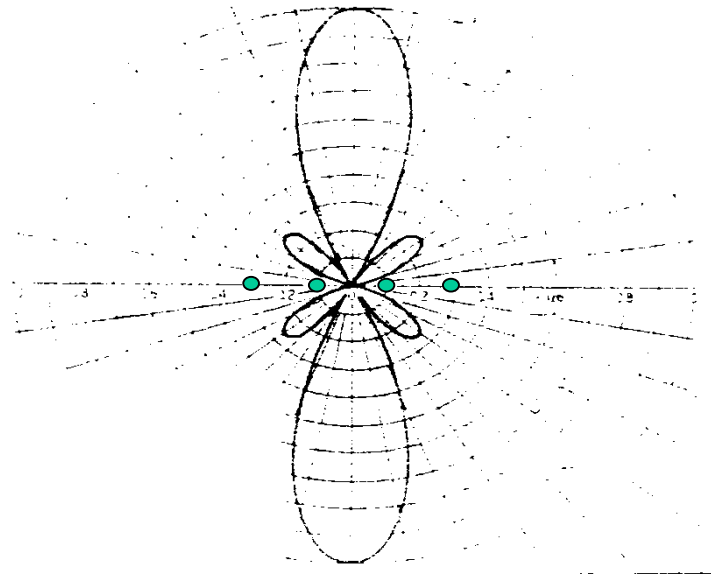


(1) 边射式天线阵

电流相位差 $\Delta=0$ ，由极大值条件 $\psi=0$ 得：

$$\varphi = (2m+1)\frac{\pi}{2} \quad (m=0,1,2,\dots)$$

间距为 $\lambda/4$ 的四元天线阵方向图

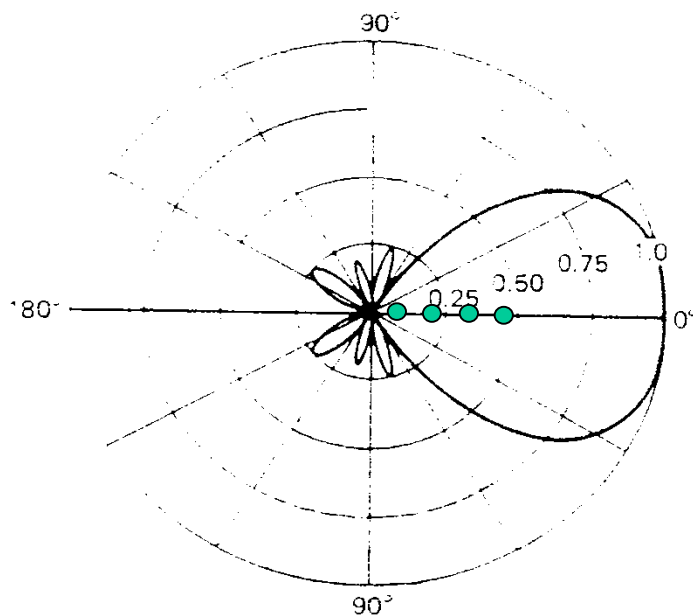


(2) 顶射式天线阵

相位角为 $\varphi=0$ 处，有极大值条件 $\psi=0$ 得：

$$\Delta = -\beta d$$

间距为 $\lambda/4$ 的八元天线阵的方向图



例题 两个相距的同相半波天线阵，半波天线的轴线与天线阵轴线重合，试利用方向性图乘法求出合成电场的方向性图。

解：半波天线的电场分布

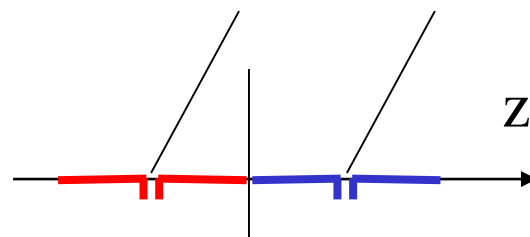
$$E_{\theta} = A F(\theta) = A \frac{\cos(\pi \cos \theta / 2)}{\sin \theta}$$

天线阵因子：

$$g_n = \frac{\sin(n\psi/2)}{\sin(\psi/2)} = \frac{\sin \psi}{\sin(\psi/2)} \sim \cos(\psi/2)$$

合成场分布：

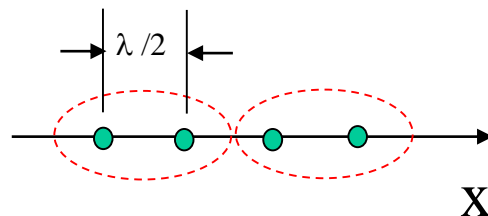
$$|E| = \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \times \cos\left(\frac{\pi}{2} \cos \theta\right) = \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta}$$



例题

四元边射式天线阵 ($\alpha=0$) 的方向性图

解:

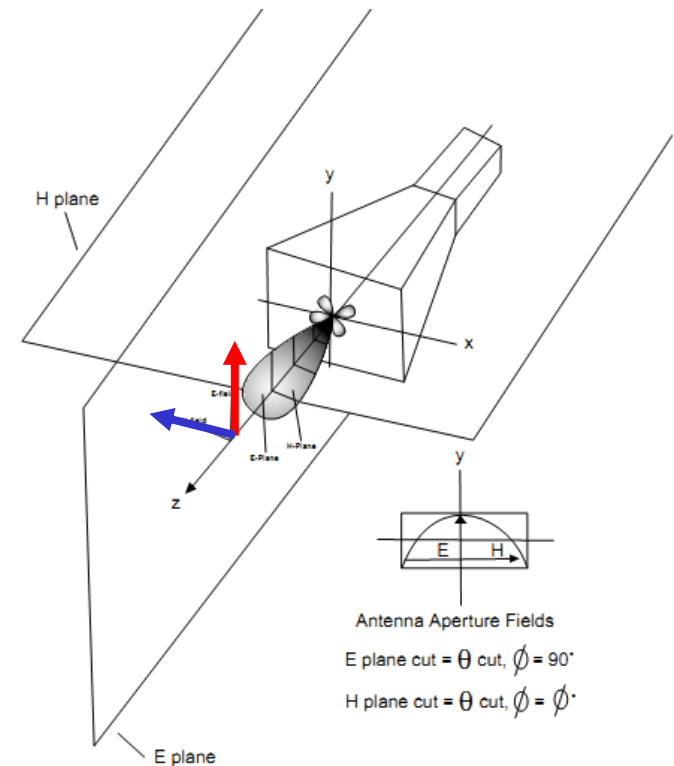


天线参数

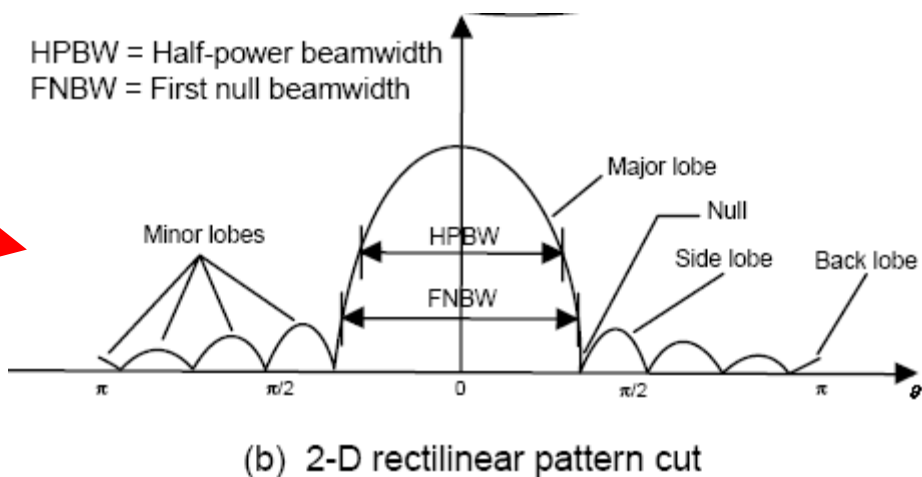
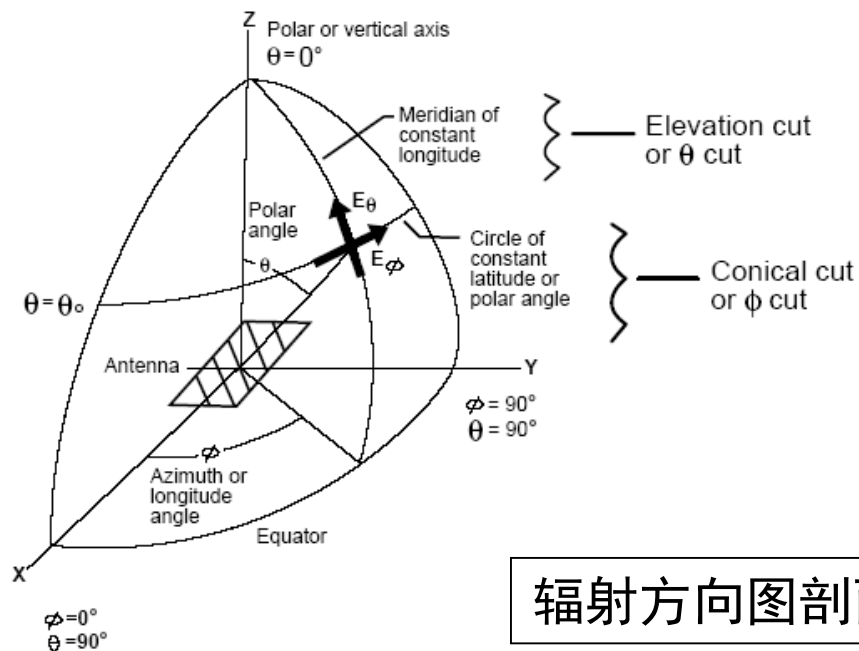
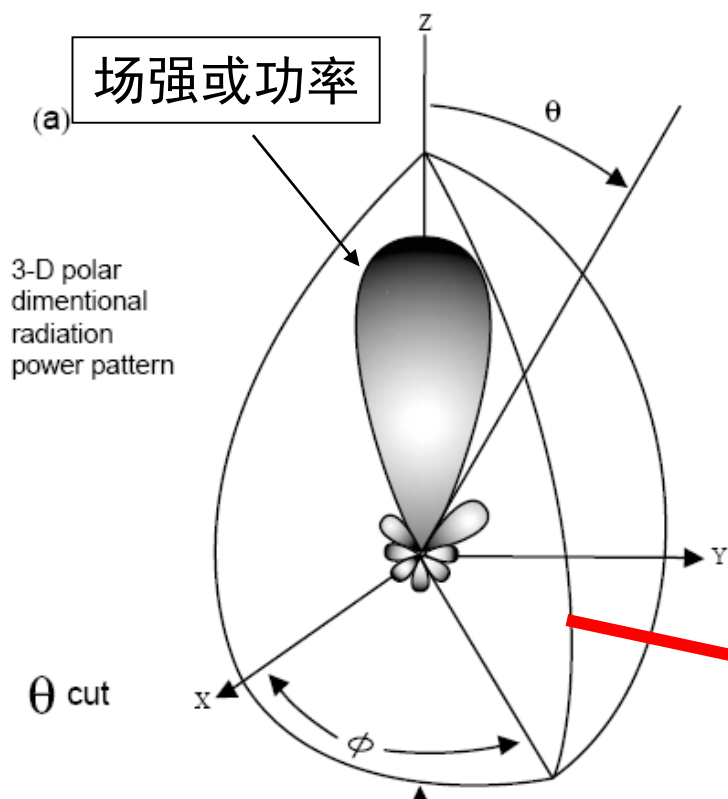
- 与天线相关的概念

E面和H面：

由通过原点和天线最大辐射方向的直线和电场矢量E构成的平面称为E面；由通过原点和天线最大辐射方向的直线和磁场矢量H构成的平面称为H面

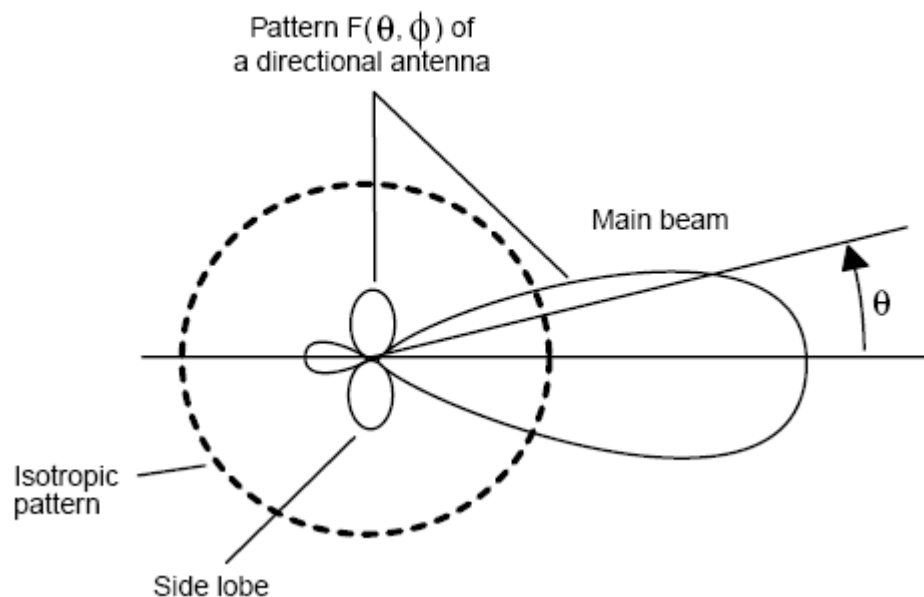


天线辐射方向图



各向同性辐射器(isotropic radiator): 电磁能量沿空间各个方向均匀地辐射出去。在三维空间辐射方向图表现为一个球,

全向天线(omnidirectional antenna): 在一个平面内表现出各向同性辐射, 而在与该平面正交平面内表现出定向辐射。



天线的近场和远场

天线周围的电磁场根据场分布的特点被分为近场（感应近场与辐射近场）和远场

- 感应近场 (reactive near field) :
- 辐射近场 [radiating near field (Fresnel)]：辐射方向图随距离变化
- 远场 [far field (Fraunhofer)]：辐射方向图不随距离变化

Reactive Near field/Fresnel Boundary: $R_{RNF} = 0.62 \cdot \sqrt{\frac{D^3}{\lambda}}$

Fresnel/Far Field Boundary: $R_{FF} = \frac{2D^2}{\lambda},$

- 天线参数

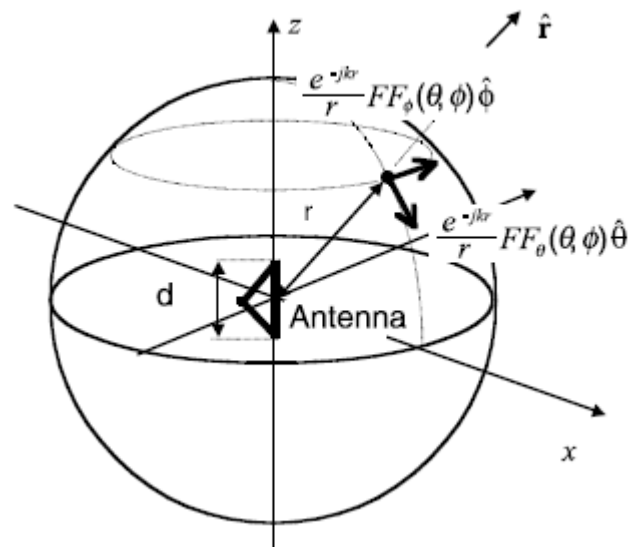
对于天线的远区辐射场，一般可以表示为

$$\mathbf{E}(\mathbf{r}) = \frac{e^{-jkr}}{r} \left[FF_{\phi}(\hat{\mathbf{k}}) \hat{\phi} + FF_{\theta}(\hat{\mathbf{k}}) \hat{\theta} \right]$$

远区场具有平面波性质，故有

$$\begin{aligned} \mathbf{H}(\mathbf{r}) &= \frac{1}{\eta} [\hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})] \\ &= \frac{1}{\eta} \frac{e^{-jkr}}{r} \left[FF_{\theta}(\hat{\mathbf{k}}) \hat{\phi} - FF_{\phi}(\hat{\mathbf{k}}) \hat{\theta} \right] \end{aligned}$$

$$S(\mathbf{r}) = \frac{1}{2\eta r^2} \left[\left| FF_{\phi}(\hat{\mathbf{k}}) \right|^2 + \left| FF_{\theta}(\hat{\mathbf{k}}) \right|^2 \right] \hat{\mathbf{r}}$$



辐射强度(radiation intensity)

$$U(\hat{\mathbf{k}}) = r^2 |\mathbf{S}(\mathbf{r})| = \frac{1}{2\eta} \left[\left| FF_\phi(\hat{\mathbf{k}}) \right|^2 + \left| FF_\theta(\hat{\mathbf{k}}) \right|^2 \right]$$

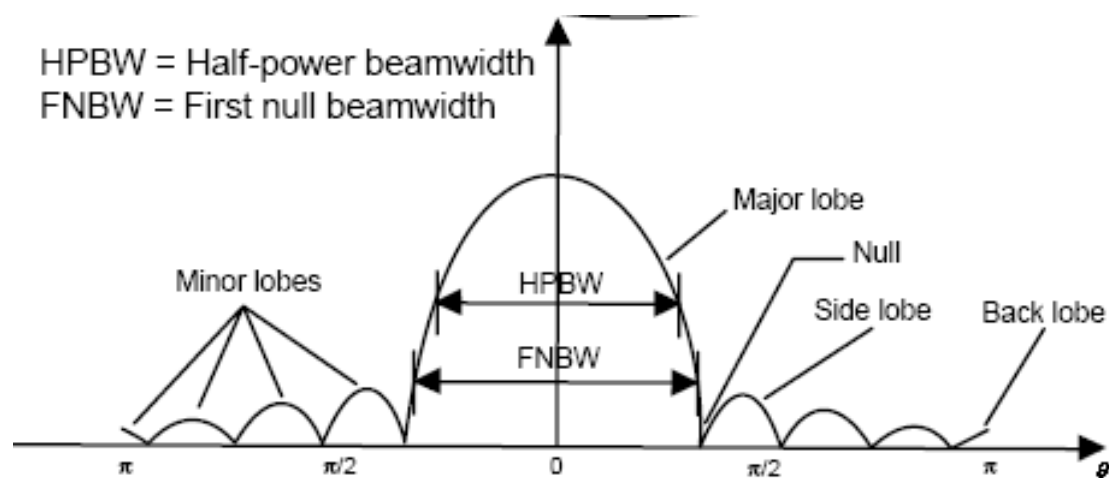
辐射功率

$$\begin{aligned} P_R &= \oiint_S \mathbf{S}(\mathbf{r}) \cdot d\mathbf{s} = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi U(\hat{\mathbf{k}}) \\ &= \oiint U(\hat{\mathbf{k}}) d\Omega \end{aligned}$$

平均辐射功率: $U_{ave} = P_R/4\pi$.

辐射方向图通常是U的图示, 即功率辐射方向图

1. 半功率点与主瓣宽度



(b) 2-D rectilinear pattern cut

基本振子天线 $F(\theta, \phi) = \sin \theta$

对称振子天线
$$F(\theta, \phi) = \frac{\cos(\pi l \cos \theta / \lambda) - \cos(\pi l / \lambda)}{\sin \theta}$$

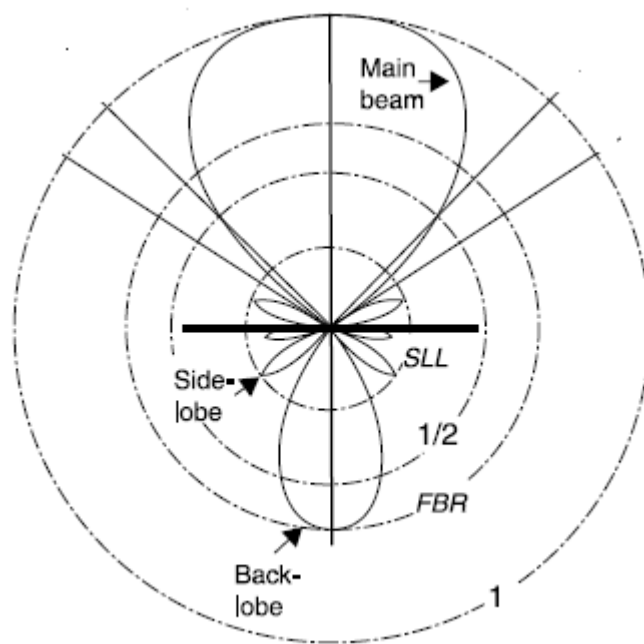
2. 方向性系数

天线方向性系数定义为：天线辐射强度和平均辐射强度之比

$$D(\hat{\mathbf{k}}) = \frac{U(\hat{\mathbf{k}})}{U_{ave}} = \frac{4\pi U(\hat{\mathbf{k}})}{P_t}$$

通常我们常用最大方向性系数来表示天线的方向性

$$D = D(\hat{\mathbf{k}}) \Big|_{\max} = \frac{U_{\max}}{U_{ave}}$$



$$E(\theta, \phi) = E_{\max} F(\theta, \phi)$$

$$P_r = \frac{1}{240\pi} \iint |E_{\max}|^2 |F(\theta, \phi)|^2 r^2 \sin \theta d\theta d\phi$$

$$D = \frac{P_{r0}}{P_0} = \frac{4\pi}{\iint |E_{\max}|^2 |F(\theta, \phi)|^2 r^2 \sin \theta d\theta d\phi}$$

基本振子天线 $F(\theta, \phi) = \sin \theta$

$$D = \frac{4\pi}{\iint \sin^3 \theta d\theta d\phi} = 1.5$$

3. 天线的增益

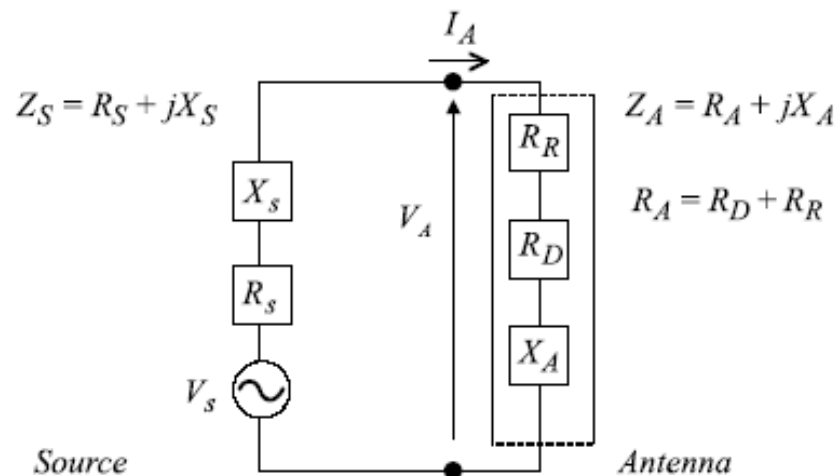
天线在相同的输入功率条件下，某天线在最大辐射方向的电场强度的平方与点源天线在同一处的电场强度的平方的比值称为该天线的增益。

$$V_A = \frac{V_S Z_A}{Z_S + Z_A} \quad I_A = \frac{V_S}{Z_S + Z_A}$$

$$P_A = \frac{1}{2} \operatorname{Re}[V_A I_A^*] = \frac{|V_S|^2 R_A}{2 |Z_S + Z_A|^2}$$

(1) 信号源和天线之间的匹配问题

(2) 天线本身的损耗



(1) 信号源和天线之间的匹配问题

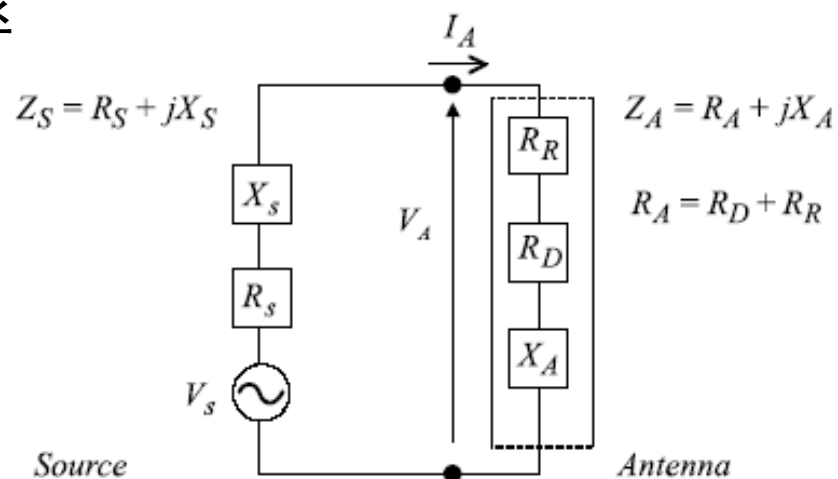
当信号源和天线之间不是共轭匹配时

$$P_A = qP_S \quad q = \frac{4R_A R_S}{|Z_S + Z_A|^2}$$

(2) 天线本身的损耗，天线效率

$$P_R = eP_A$$

$$e = \frac{R_R}{R_A} = \frac{R_R}{R_R + R_D}$$



天线增益表示为

$$G(\hat{k}) = \frac{4\pi U(\hat{k})}{P_A} = \frac{4\pi e U(\hat{k})}{P_R} = eD(\hat{k})$$

天线增益通常用最大天线增益表示

$$G = \frac{4\pi U_{\max}}{P_A} = eD$$

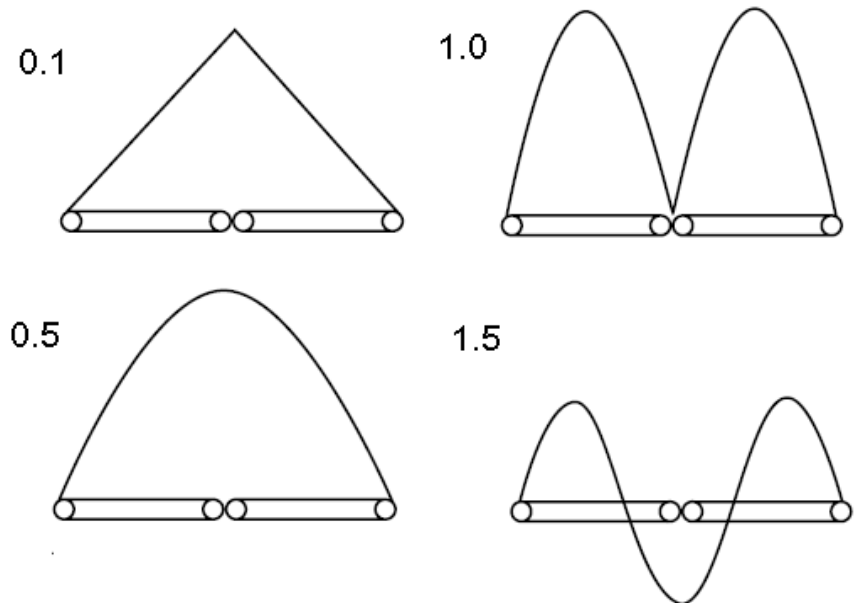
天线互易定理

- 互易定理
同一天线用作发射天线或接收天线时所有参数不变。
- 天线的电性能主要决定于它的外表形式及电尺寸，而与其应用方式无关。

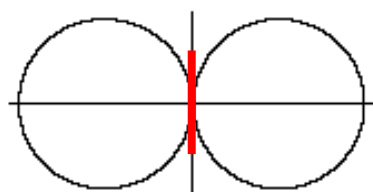
常见的天线

- 偶极子天线

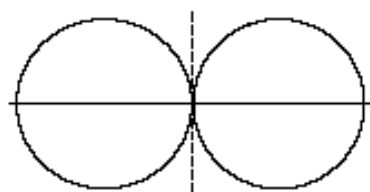
$$\mathbf{I}_{sd}(z) = I_0 \left(1 - \frac{2|z|}{L} \right) \hat{\mathbf{z}} \quad |z| < \frac{L}{2}$$
$$= 0 \quad \text{elsewhere}$$



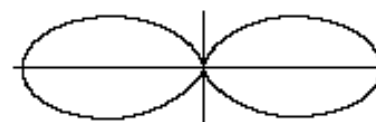
$$U(\hat{\mathbf{k}}) = I_o^2 \frac{\eta}{8\pi^2} \sin^2 \theta \left[\frac{\cos\left[\left(kL/2\right)\cos\theta\right] - \cos(kL/2)}{\sin^2 \theta} \right]^2$$



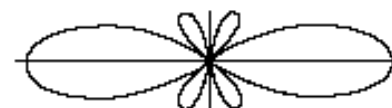
$L = 0.1\lambda$



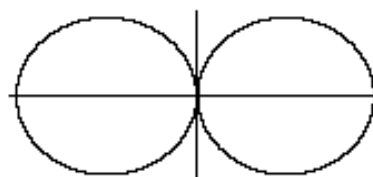
$L = 0.25\lambda$



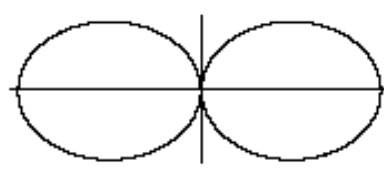
$L = 1.00\lambda$



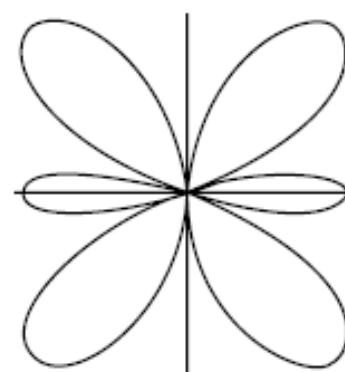
$L = 1.25\lambda$



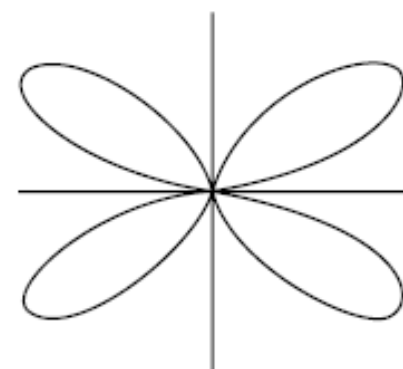
$L = 0.5\lambda$



$L = 0.75\lambda$

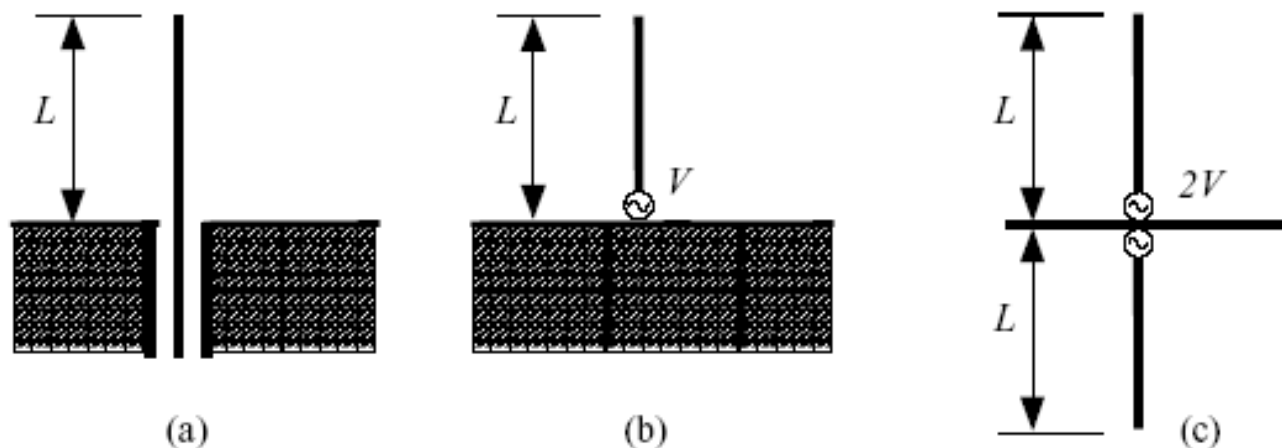


$L = 1.50\lambda$



$L = 2.00\lambda$

- 单极子天线



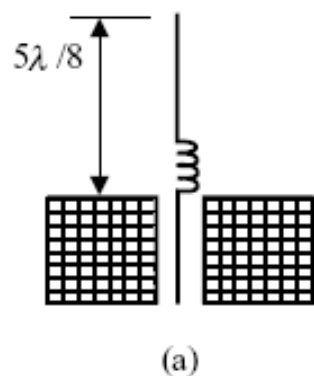
Monopole antennas. (a) Coax fed, (b) single half-space equivalent model, (c) image equivalent model.

单极子天线结构简单，易于实现，但天线增益较小

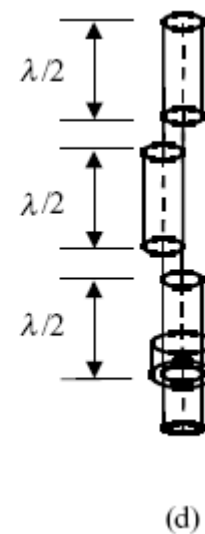
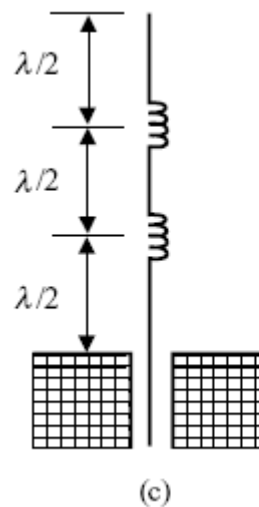
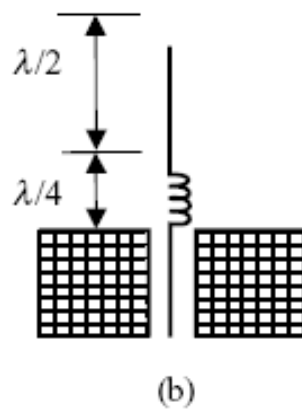
长的偶极子天线和单极子天线不太采用，因为不易匹配和其复杂的辐射方向图。

5/8波长的单极子天线和1.25波长的偶极子天线：

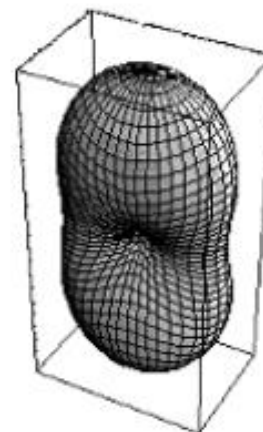
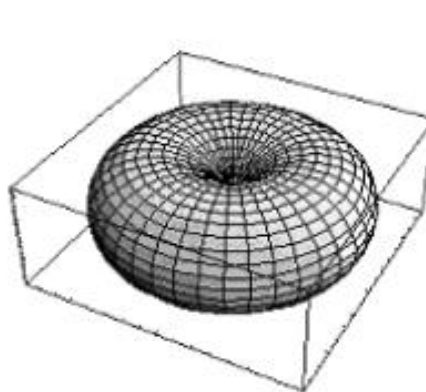
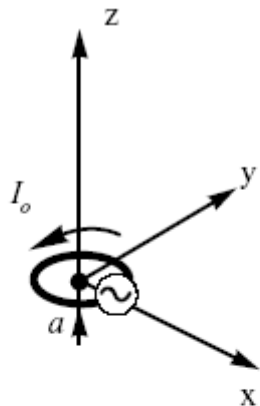
1. 天线具有宽的主瓣和小的副瓣
2. 方向性系数高，5/8波长达到6.6。
3. 加载电感后可以实现50Ω输入阻抗



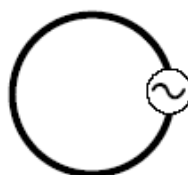
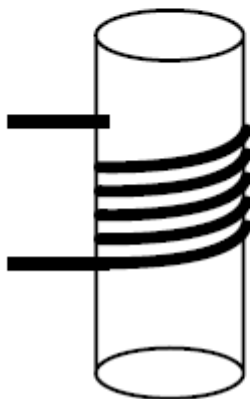
其它电感加载天线



• 环形天线



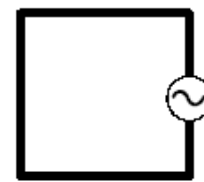
Radiation patterns of small loop and 1λ circumference loop



(a)



(b)



(c)



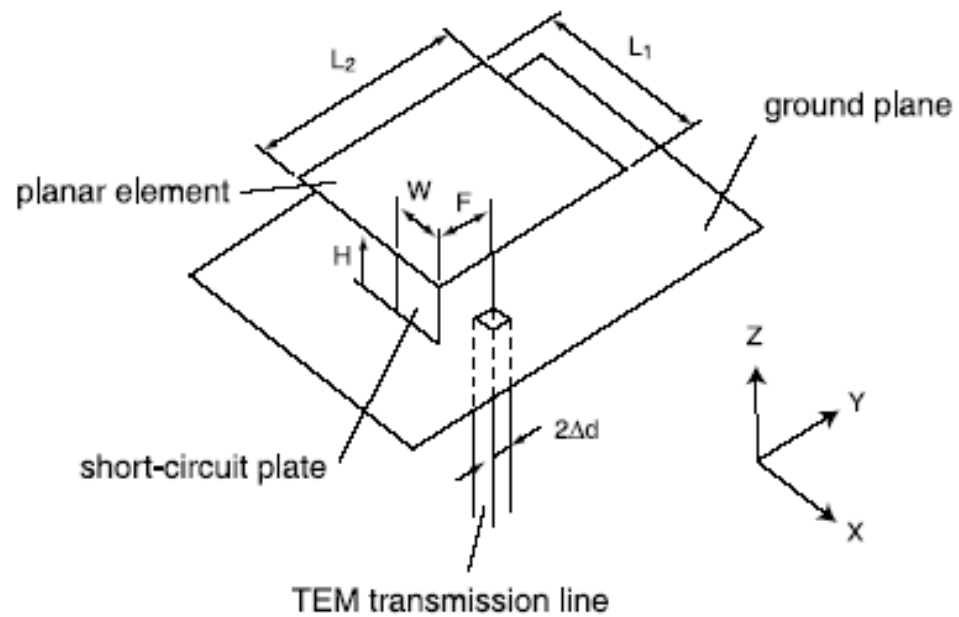
(d)



(e)



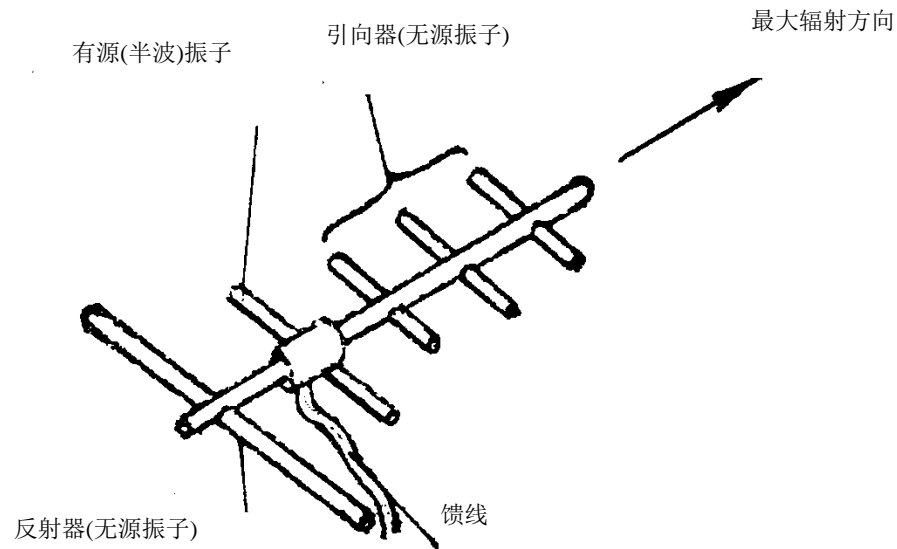
(f)



Structure of the planar inverted-F antenna.

八木天线与电视发射天线

- 八木天线是一种结构简单、增益高、适用于超高频范围的定向天线。由一个有源振子和一些无源振子所组成的天线阵



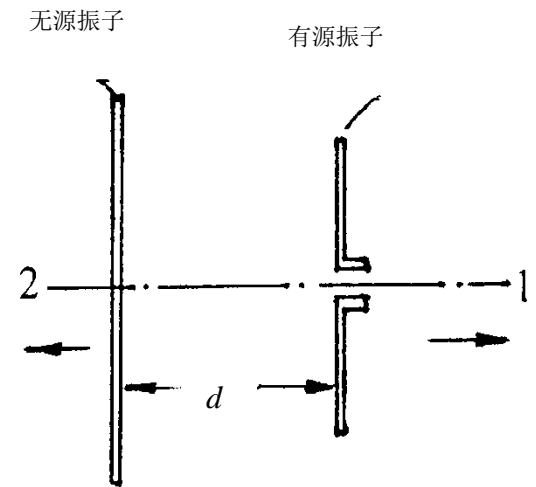
➤ 用途：通信、电视接受、雷达和导航

➤ 电场分布：

(1) 无源振子和有源振子的间距为 $\lambda/4$

(2) 无源振子长度大于半波长，其电流位落后有源振子 270° ，天线阵最辐射方向指向‘1’

(3) 无源振子长度小于半波长，其电流位落后有源振子 90° ，天线阵最辐射方向指向‘2’



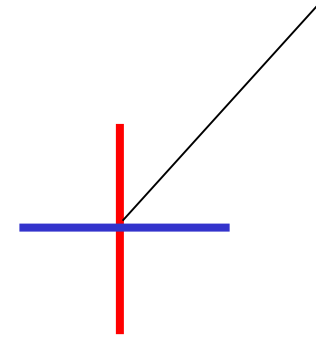
- 电视发射天线（旋转场天线）

它由两个互相垂直的振子组成，两振子的电流之间有90度的相位差。

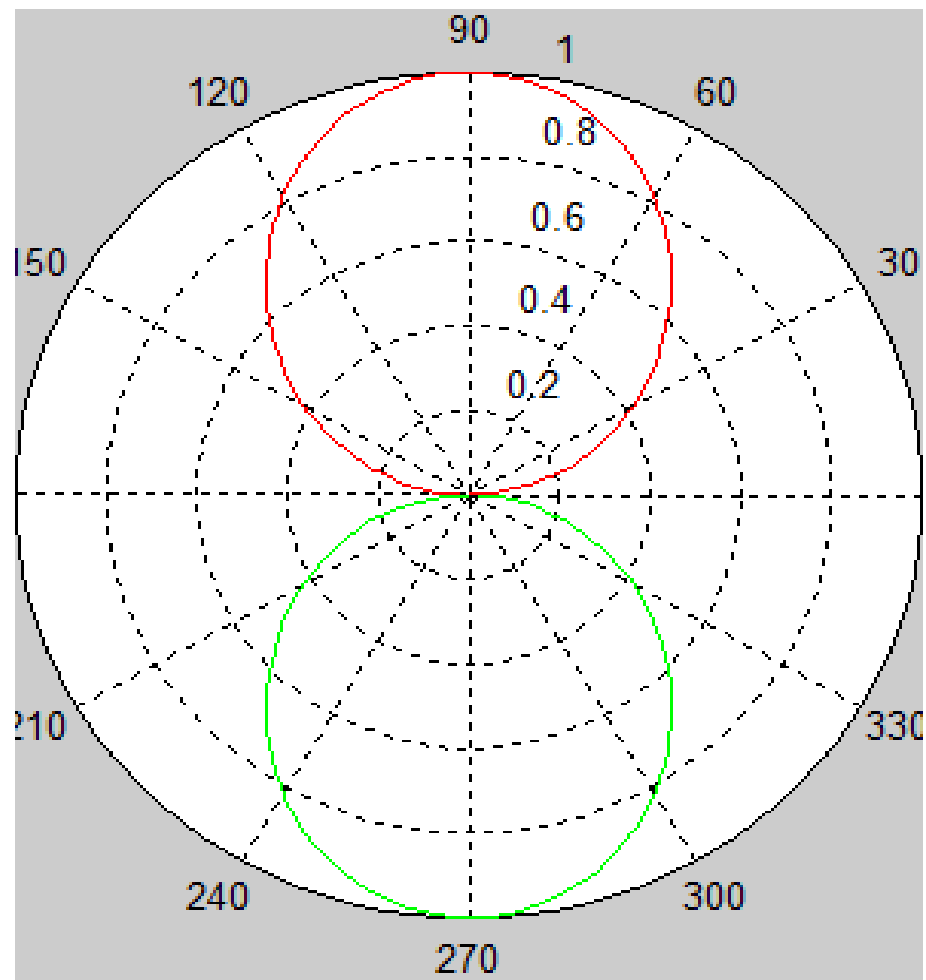
$$E_{\theta} = A_0 F(\theta) e^{j(\omega t - kr)}$$

$$E \sim F(\theta) e^{j(\omega t - kr)} + F(\pi/2 - \theta) e^{j(\omega t - kr + \pi/2)}$$

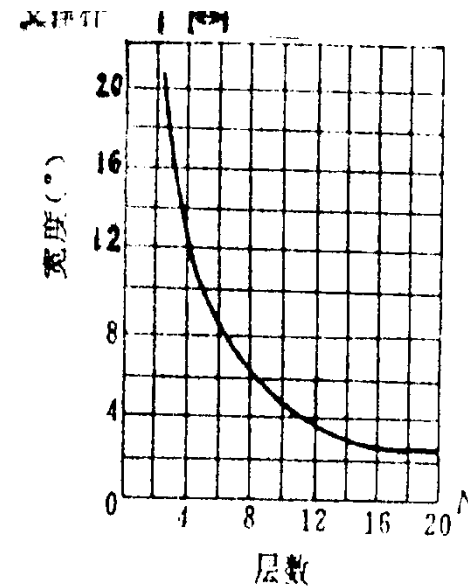
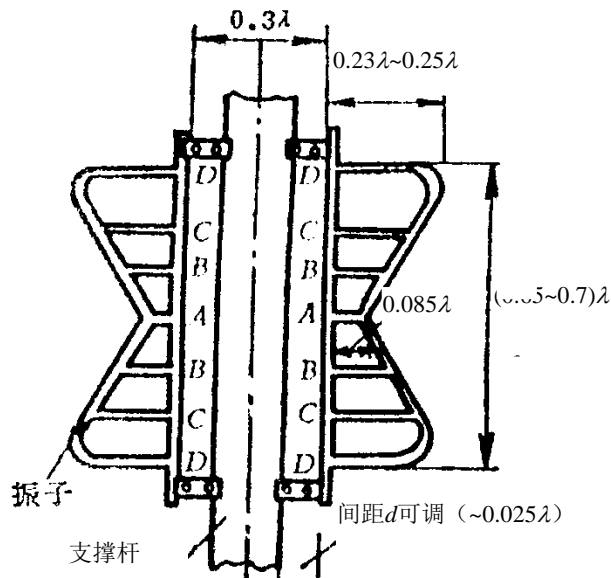
$$f(\theta, t) = \frac{\cos\left(\frac{\pi}{2} \cos\theta\right)}{\sin\theta} \cos\omega t + \frac{\cos\left(\frac{\pi}{2} \sin\theta\right)}{\cos\theta} \sin\omega t$$



这种十字形天线的8字形方向图是绕着垂直于天线平面的轴旋转的，因此称之为旋转场天线。

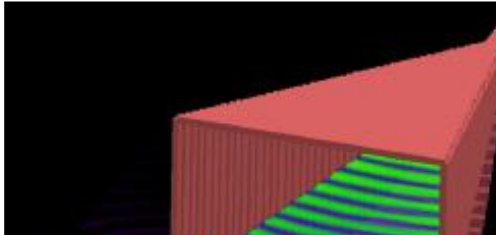


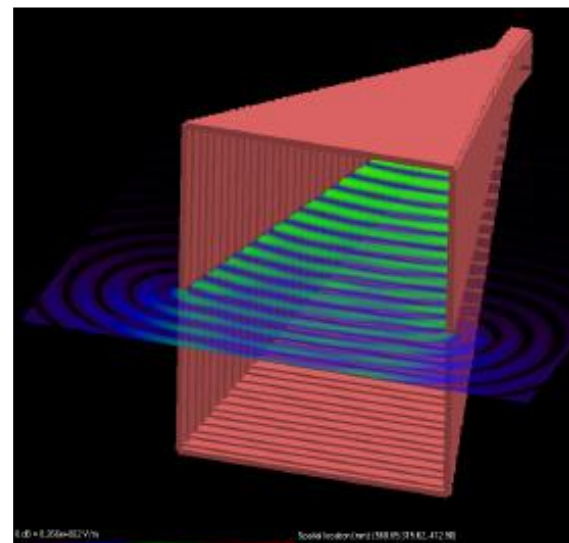
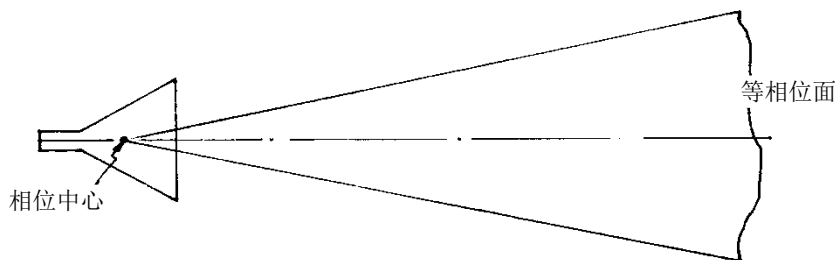
- 在电视广播中采用的蝙蝠翼天线也是一种旋转天线场。十字形蝙蝠翼天线的一个振子，在垂直于纸面方向，还有另一个形状完全相同但电流相位差的振子



喇叭天线、抛物面天线、卡塞格伦天线

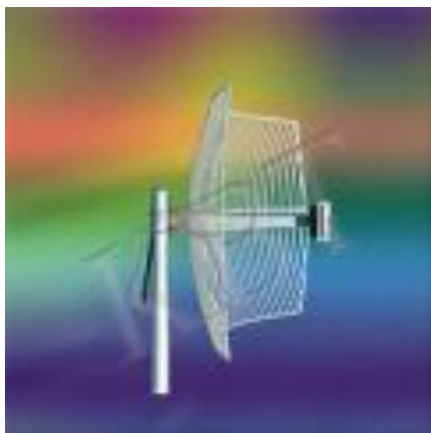
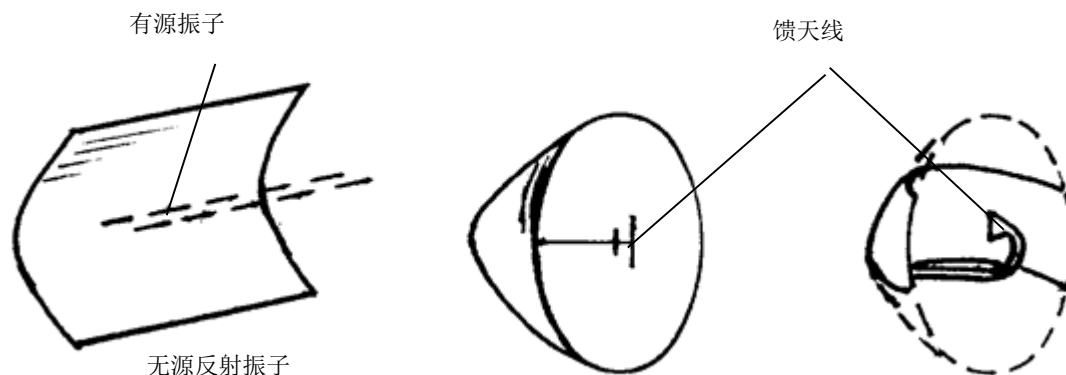
喇叭天线

- 结构简单调整容易，有较宽的频带特性，尺寸较大，口径上场的振幅和相位均无法调节。常用作馈源。
 - 相位中心的定义是：远区辐射场的等相位面与通过天线轴线的平面相交的曲线的曲率中心。
- 



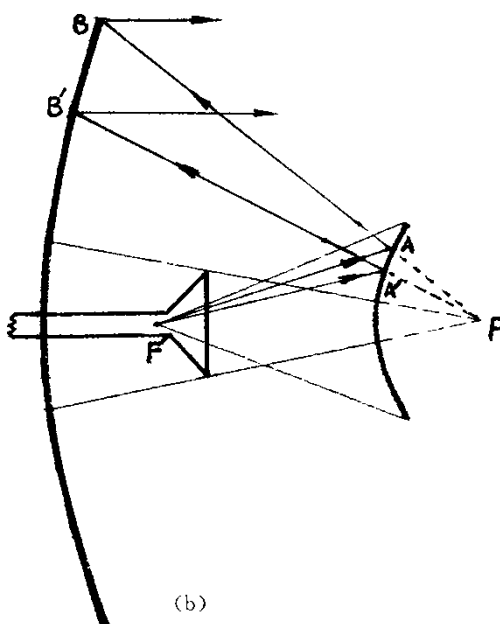
抛物面天线

由馈源与抛物面反射器两部分组成。

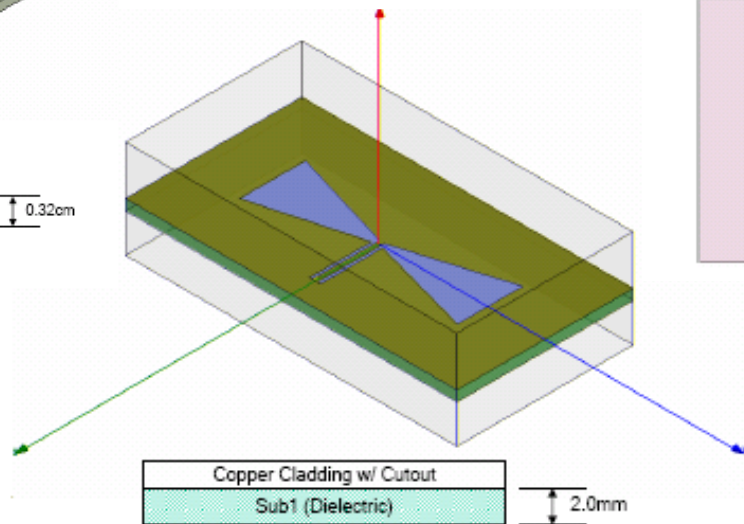
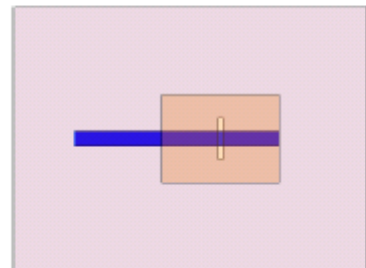
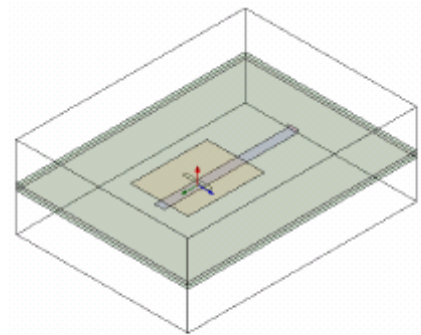
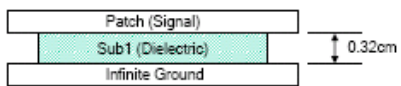
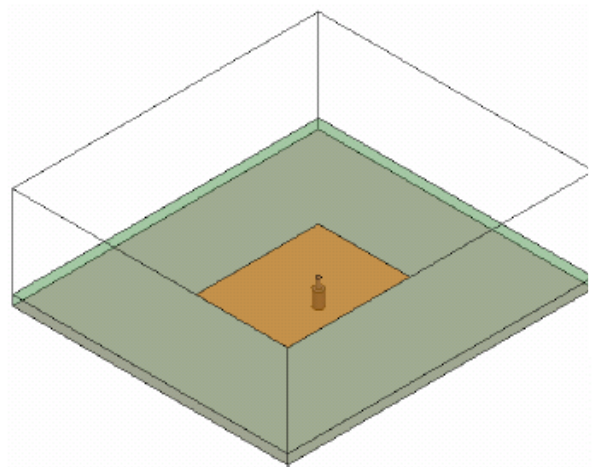


卡塞格伦天线

1. 主反射面由旋转抛物面构成；
2. 次反射面由旋转双曲面构成；
3. 馈源一般用喇叭天线构成



微带天线



- 微带天线有如下优点：
 - (1) 剖面薄、体积小、重量轻；
 - (2) 具有平面结构，可制成表面共形的结构；
 - (3) 馈电网络可与天线结构一起制成，适合用于印刷电路技术大批量生产；
 - (4) 能与有源器件和电路集成为单一的模件；
 - (5) 便于获得圆极化，容易实现双频段、双极化等多功能工作。

- 微带天线也有不足之处：
 - (1) 频带窄；
 - (2) 存在导体和介质损耗，并会激励表面波而导致辐射效率降低
 - (3) 功率容量较小，一般用于中、小功率场合；
 - (4) 性能受基片材料影响大