NJU南京大学2020电磁场与微波技术2班

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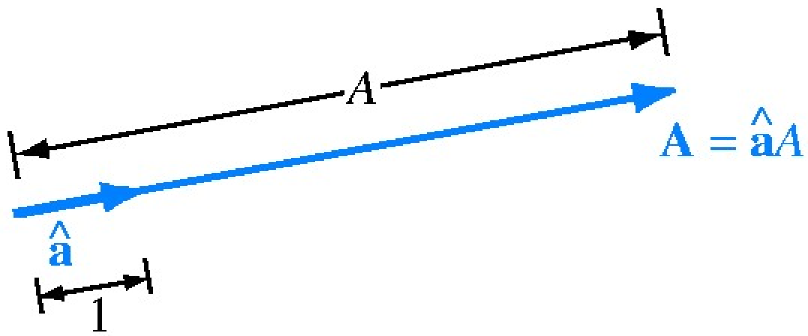
南京大学2020《电磁场与电磁波》2班学习交流群

第一章　电磁场理论的数学准备

1.1 矢量代数

- 标量
- 矢量　单位矢量
- 表示法

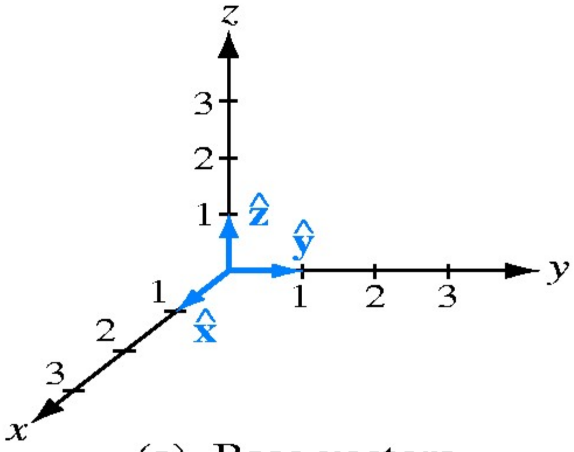


矢量在坐标系中的表示

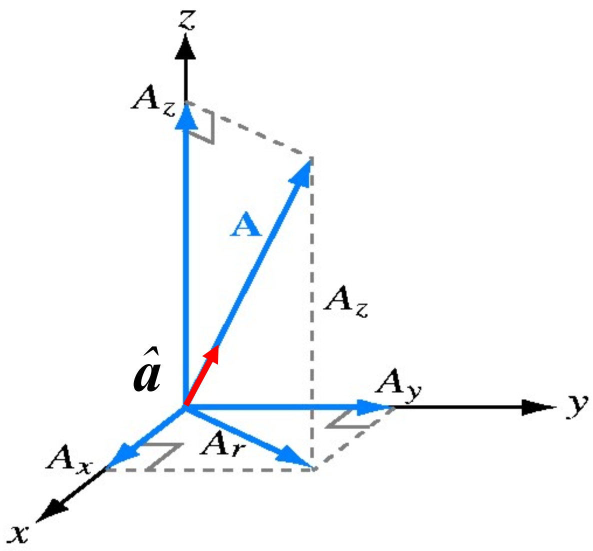
基矢量，坐标分量

$$\boldsymbol{A}=\boldsymbol{A}_r+\boldsymbol{A}_z=\boldsymbol{A}_x+\boldsymbol{A}_y+\boldsymbol{A}_z$$
$$=\mathrm{A}_x\,\hat{\boldsymbol{x}}+\mathrm{A}_y\,\hat{\boldsymbol{y}}+\mathrm{A}_z\,\hat{\boldsymbol{z}}$$

$$A=|\boldsymbol{A}|=\sqrt{A_x^{\ 2}+A_y^{\ 2}+A_z^{\ 2}}$$
$$\hat{\boldsymbol{a}}=\frac{\boldsymbol{A}}{A}=\frac{\mathrm{A}_x\,\hat{\boldsymbol{x}}+\mathrm{A}_y\,\hat{\boldsymbol{y}}+\mathrm{A}_z\,\hat{\boldsymbol{z}}}{\sqrt{A_x^{\ 2}+A_y^{\ 2}+A_z^{\ 2}}}$$



(a) Base vectors



(b) Components of **A**

Figure 3-2

矢量运算

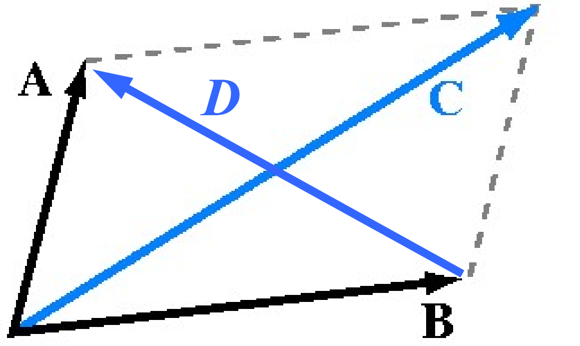
$$\boldsymbol{A} = \mathrm{A_x}\,\hat{\boldsymbol{x}} + \mathrm{A_y}\,\hat{\boldsymbol{y}} + \mathrm{A_z}\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = \mathrm{B_x}\,\hat{\boldsymbol{x}} + \mathrm{B_y}\,\hat{\boldsymbol{y}} + \mathrm{B_z}\,\hat{\boldsymbol{z}}$$

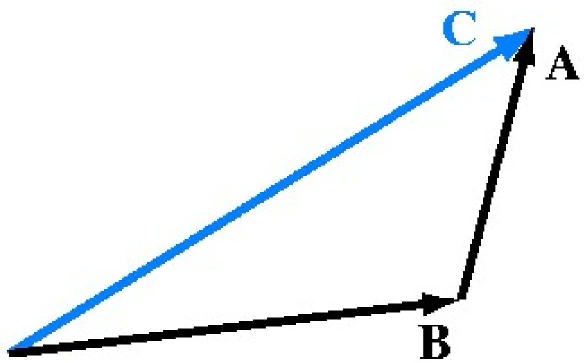
$$\begin{cases} \boldsymbol{a} = a_x \boldsymbol{e}_x + a_y \boldsymbol{e}_y + a_z \boldsymbol{e}_z \\ \boldsymbol{b} = b_x \boldsymbol{e}_x + b_y \boldsymbol{e}_y + b_z \boldsymbol{e}_z \end{cases}$$

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} \\ &= \left(\mathrm{A_x}\,\hat{\boldsymbol{x}} + \mathrm{A_y}\,\hat{\boldsymbol{y}} + \mathrm{A_z}\,\hat{\boldsymbol{z}}\right) + \left(\mathrm{B_x}\,\hat{\boldsymbol{x}} + \mathrm{B_y}\,\hat{\boldsymbol{y}} + \mathrm{B_z}\,\hat{\boldsymbol{z}}\right) \\ &= \left(\mathrm{A_x} + \mathrm{B_x}\right)\hat{\boldsymbol{x}} + \left(\mathrm{A}_y + \mathrm{B}_y\right)\hat{\boldsymbol{y}} + \left(\mathrm{A}_z + \mathrm{B}_z\right)\hat{\boldsymbol{z}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + \left(-\boldsymbol{B}\right) \\ &= \left(\mathrm{A_x}\,\hat{\boldsymbol{x}} + \mathrm{A_y}\,\hat{\boldsymbol{y}} + \mathrm{A_z}\,\hat{\boldsymbol{z}}\right) - \left(\mathrm{B_x}\,\hat{\boldsymbol{x}} + \mathrm{B_y}\,\hat{\boldsymbol{y}} + \mathrm{B_z}\,\hat{\boldsymbol{z}}\right) \\ &= \left(\mathrm{A_x} - \mathrm{B_x}\right)\hat{\boldsymbol{x}} + \left(\mathrm{A}_y - \mathrm{B}_y\right)\hat{\boldsymbol{y}} + \left(\mathrm{A}_z - \mathrm{B}_z\right)\hat{\boldsymbol{z}} \end{aligned}$$



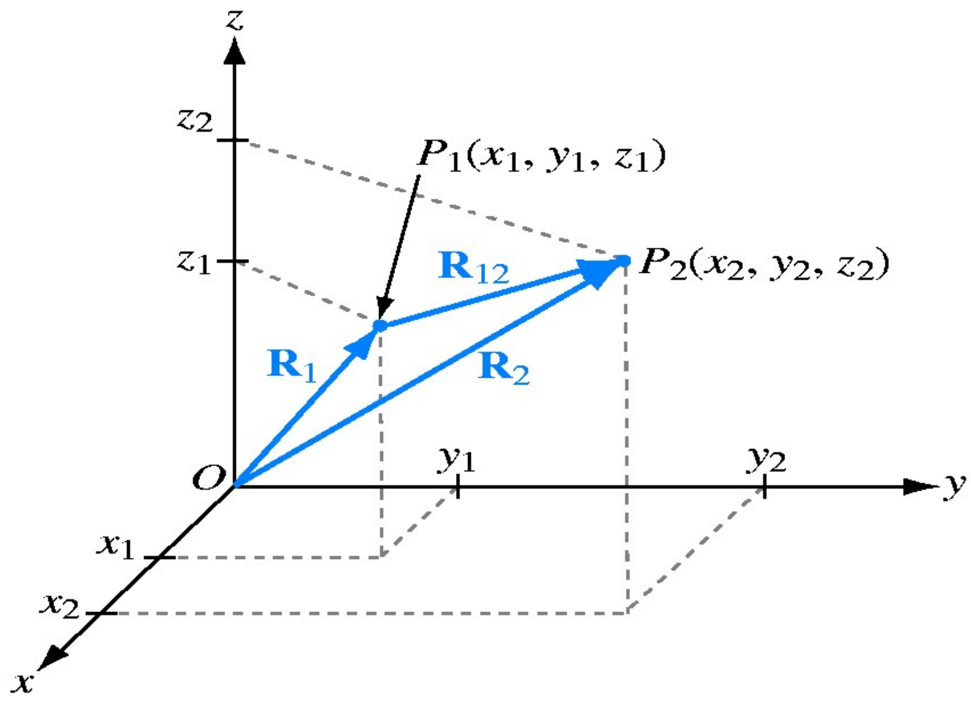
(a) Parallelogram rule



(b) Head-to-tail rule

位置矢量和距离矢量

$$\boldsymbol{R}_1 = x_1\hat{\boldsymbol{x}} + y_1\hat{\boldsymbol{y}} + z_1\hat{\boldsymbol{z}}$$
$$\boldsymbol{R}_2 = x_2\hat{\boldsymbol{x}} + y_2\hat{\boldsymbol{y}} + z_2\hat{\boldsymbol{z}}$$



$$\boldsymbol{R}_{12} = \overrightarrow{P_1P_2} = \boldsymbol{R}_2 - \boldsymbol{R}_1$$
$$= (x_2 - x_1)\hat{\boldsymbol{x}} + (y_2 - y_1)\hat{\boldsymbol{y}} + (z_2 - z_1)\hat{\boldsymbol{z}}$$

$$\boldsymbol{R}_{21} = \overrightarrow{P_2P_1} = \boldsymbol{R}_1 - \boldsymbol{R}_2$$
$$= (x_1 - x_2)\hat{\boldsymbol{x}} + (y_1 - y_2)\hat{\boldsymbol{y}} + (z_1 - z_2)\hat{\boldsymbol{z}}$$
$$= -\boldsymbol{R}_{12}$$

$$d = \left|\boldsymbol{R}_{12}\right|$$
$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

矢量乘法运算

标积或点乘

$$\boxed{\boldsymbol{A}\bullet\boldsymbol{B} = ABcos\theta_{AB}}$$

$$\hat{\boldsymbol{x}}\bullet\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\bullet\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\bullet\hat{\boldsymbol{z}} = 1$$

$$\hat{\boldsymbol{x}}\bullet\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}\bullet\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}\bullet\hat{\boldsymbol{x}} = 0$$

$$\boldsymbol{A}\bullet\boldsymbol{B} = \left(\mathrm{A_x}\,\hat{\boldsymbol{x}} + \mathrm{A_y}\,\hat{\boldsymbol{y}} + \mathrm{A_z}\,\hat{\boldsymbol{z}}\right)\bullet\left(\mathrm{B_x}\,\hat{\boldsymbol{x}} + \mathrm{B_y}\,\hat{\boldsymbol{y}} + \mathrm{B_z}\,\hat{\boldsymbol{z}}\right)$$
$$= \mathrm{A_x B_x} + \mathrm{A}_y\mathrm{B}_y + \mathrm{A}_z\mathrm{B}_z$$

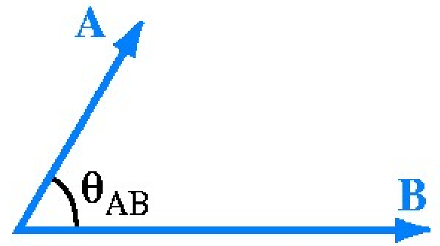
$$\boldsymbol{A}\bullet\boldsymbol{B} = \boldsymbol{B}\bullet\boldsymbol{A}$$ 交换律

$$\boldsymbol{A}\bullet\left(\boldsymbol{B}+\boldsymbol{C}\right) = \boldsymbol{A}\bullet\boldsymbol{B} + \boldsymbol{A}\bullet\boldsymbol{C}$$ 分配律

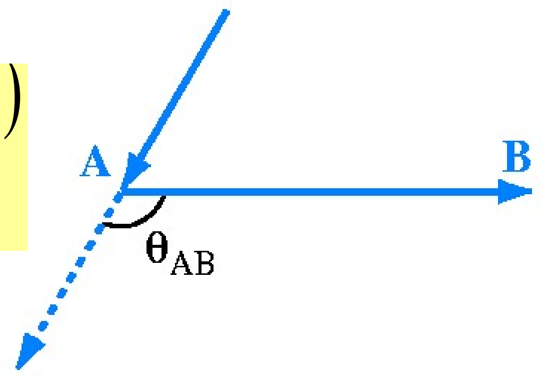
$$\boldsymbol{A}\bullet\boldsymbol{A} = \left|\boldsymbol{A}\right|^2 = A^2$$

$$A = \left|\boldsymbol{A}\right| = \sqrt{\boldsymbol{A}\bullet\boldsymbol{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\boldsymbol{A}\bullet\boldsymbol{B}}{\sqrt{\boldsymbol{A}\bullet\boldsymbol{A}}\sqrt{\boldsymbol{B}\bullet\boldsymbol{B}}}\right]$$



(a)



(b)

矢量乘法运算

矢积或叉乘

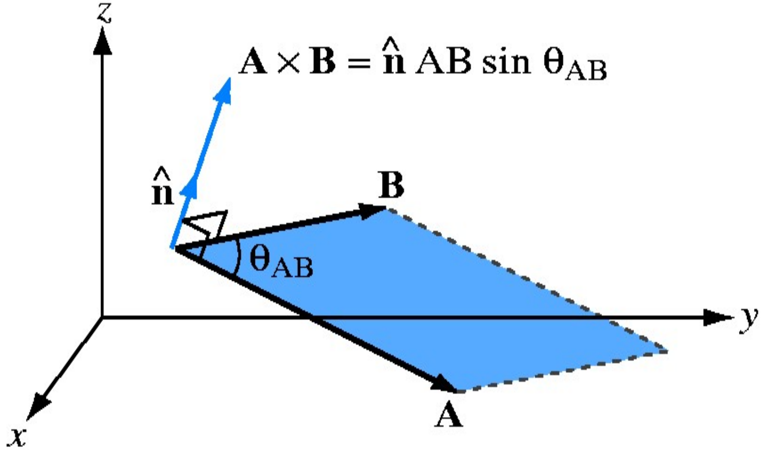
$$\boxed{\boldsymbol{A}\times\boldsymbol{B}=\hat{\boldsymbol{n}}AB\sin\theta_{AB}}$$

$$\hat{\boldsymbol{x}}\times\hat{\boldsymbol{x}}=\hat{\boldsymbol{y}}\times\hat{\boldsymbol{y}}=\hat{\boldsymbol{z}}\times\hat{\boldsymbol{z}}=0$$

$$\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}}=\hat{\boldsymbol{z}}\quad\hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}}=\hat{\boldsymbol{x}}\quad\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}=\hat{\boldsymbol{y}}$$

$$\boldsymbol{A}\times\boldsymbol{B}=\left(\mathrm{A_x}\,\hat{\boldsymbol{x}}+\mathrm{A_y}\,\hat{\boldsymbol{y}}+\mathrm{A_z}\,\hat{\boldsymbol{z}}\right)\times\left(\mathrm{B_x}\,\hat{\boldsymbol{x}}+\mathrm{B_y}\,\hat{\boldsymbol{y}}+\mathrm{B_z}\,\hat{\boldsymbol{z}}\right)$$
$$=\hat{\boldsymbol{x}}\left(\mathrm{A}_y\mathrm{B}_z-\mathrm{A}_z\mathrm{B}_y\right)+\hat{\boldsymbol{y}}\left(\mathrm{A}_z\mathrm{B}_x-\mathrm{A}_x\mathrm{B}_z\right)+\hat{\boldsymbol{z}}\left(\mathrm{A}_x\mathrm{B}_y-\mathrm{A}_y\mathrm{B}_x\right)$$

$$\boxed{\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \mathrm{A_x} & \mathrm{A_y} & \mathrm{A_z}\\ \mathrm{B_x} & \mathrm{B_y} & \mathrm{B_z}\end{vmatrix}}$$



(a) Cross product

(b) Right-hand rule

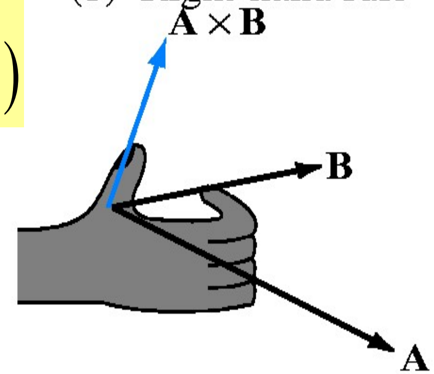


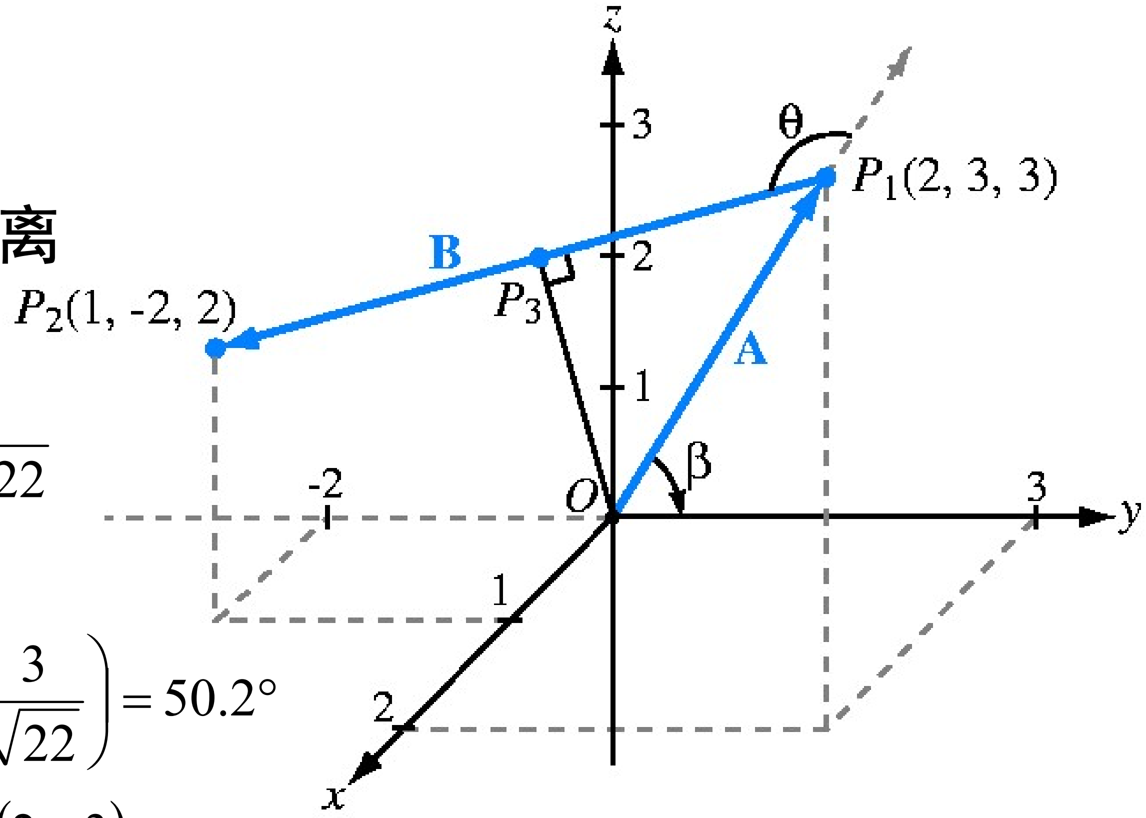
Figure 3-6

例题

1) A与y轴的夹角；

2) B；3) A, B的夹角；

4) 原点与B的垂直距离



$$\boldsymbol{A} = \hat{\boldsymbol{x}}2 + \hat{\boldsymbol{y}}3 + \hat{\boldsymbol{z}}3$$

$$A = |\boldsymbol{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$$

$$\boldsymbol{A} \bullet \hat{\boldsymbol{y}} = |\boldsymbol{A}| cos\beta$$

$$\beta = \cos^{-1}\left(\frac{\boldsymbol{A} \bullet \hat{\boldsymbol{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2°$$

$$\boldsymbol{B} = \hat{\boldsymbol{x}}(1-2) + \hat{\boldsymbol{y}}(-2-3) + \hat{\boldsymbol{z}}(2-3)$$

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{A} \bullet \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right) = \cos^{-1}\left(\frac{-2-15-3}{\sqrt{22}\sqrt{27}}\right) = 145.1°$$

$$|OP_3| = |\boldsymbol{A}|\sin(180° - \theta)$$

$$= \sqrt{22}\sin(180° - 145.1°) = 2.68$$

三角标积

$$\boldsymbol{A} \bullet (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \bullet (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \bullet (\boldsymbol{A} \times \boldsymbol{B})$$

$$\boldsymbol{A} \bullet \left(\boldsymbol{B} \times \boldsymbol{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

三角矢积

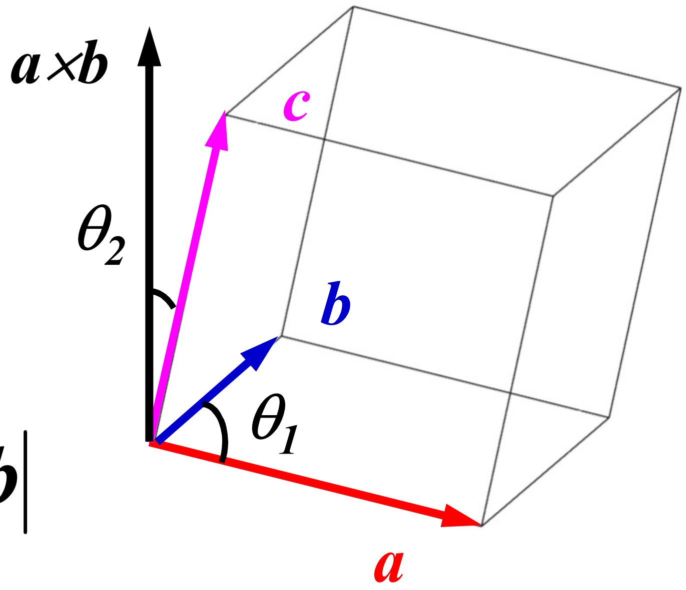
$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \bullet \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \bullet \boldsymbol{B})$$

如果$\boldsymbol{a}$与$\boldsymbol{b}$平行，则 $\boldsymbol{a} \times \boldsymbol{b} = 0$ 如果$\boldsymbol{a}$与$\boldsymbol{b}$垂直，则 $\boldsymbol{a} \bullet \boldsymbol{b} = 0$。反之也然。

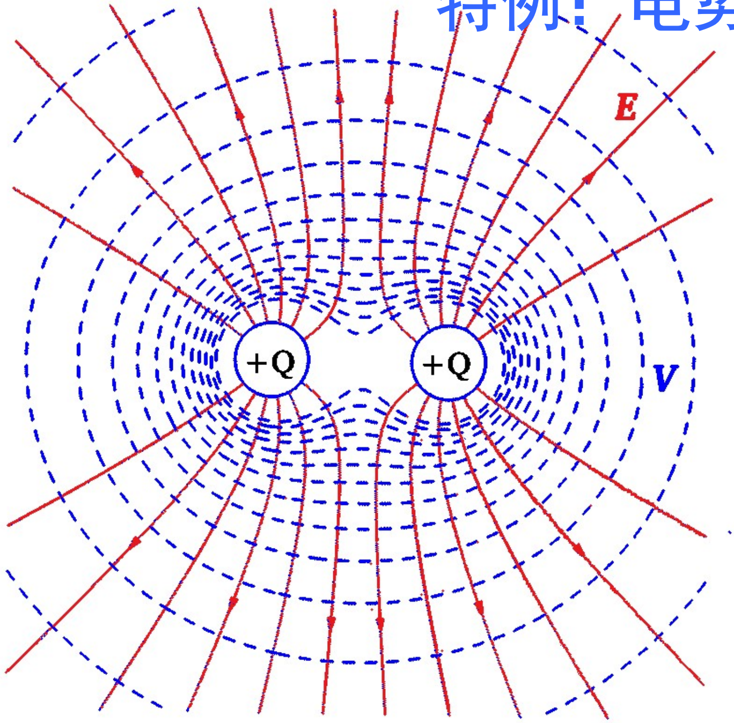
$$volume = |\mathbf{a}||\mathbf{b}||\mathbf{c}| sin\theta_1 cos\theta_2 = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$$

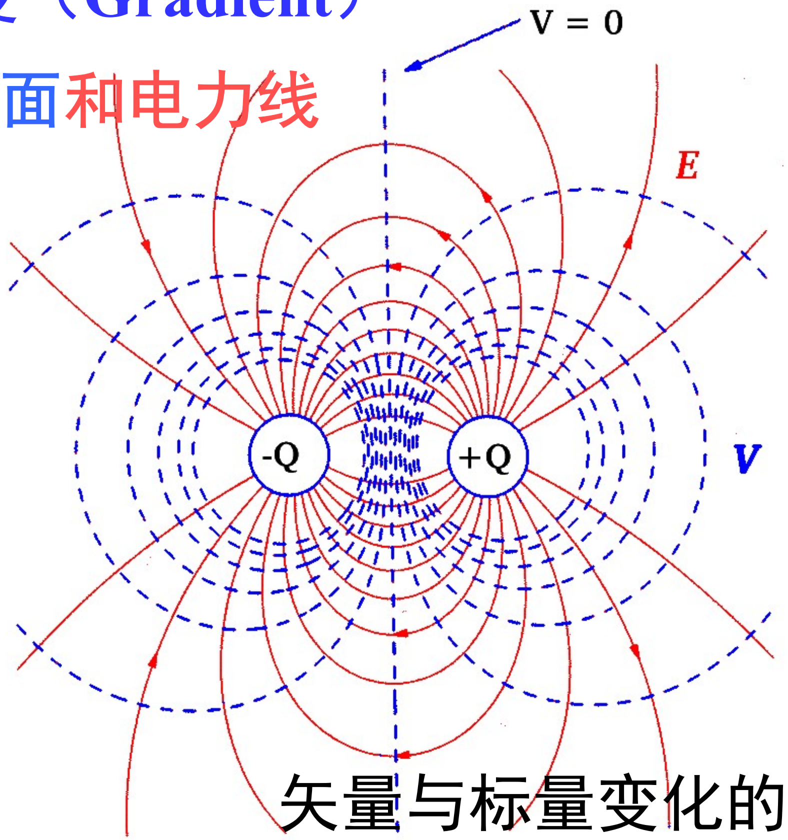


$$area = |\mathbf{b}||\mathbf{a}| sin\theta_1 = |\mathbf{a} \times \mathbf{b}|$$

1.2 标量场的梯度（Gradient）

特例：电势等势面和电力线

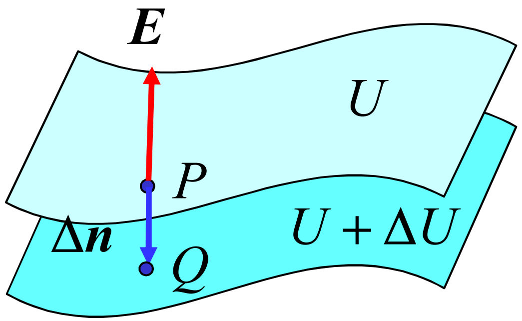




矢量与标量变化的关系

$$\Delta U = \left| \int_P^Q \boldsymbol{E} \cdot d\boldsymbol{l} \right| \approx E \Delta n \quad E = \left| \lim_{\Delta n \to 0} \frac{\Delta U}{\Delta n} \right| = \left| \frac{\partial U}{\partial n} \right|$$

$$\boldsymbol{E} = \nabla U = \boldsymbol{grad} U = -\hat{\boldsymbol{n}} \left| \frac{\partial U}{\partial n} \right|$$



$u(x,y,z)$

空间标量场的变化率

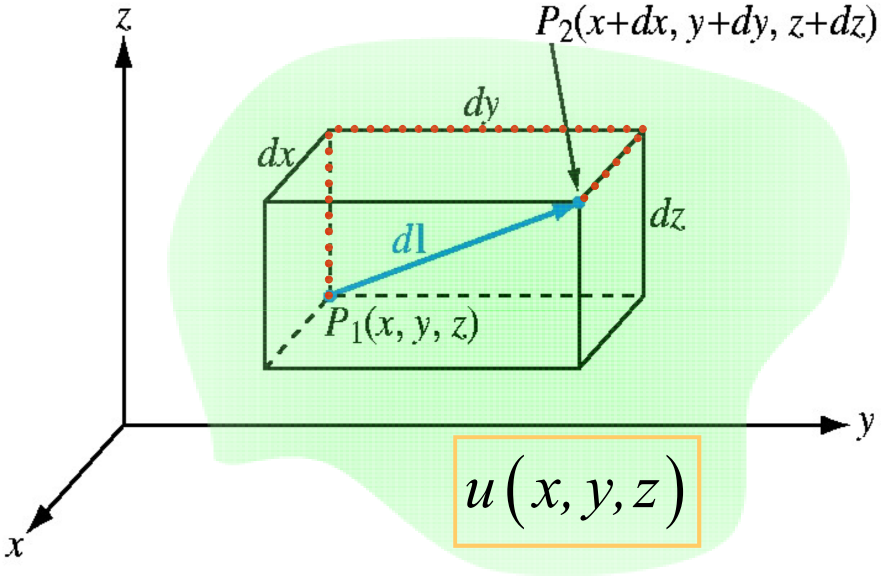


Figure 3-19

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}x\,\boldsymbol{e}_x + \mathrm{d}y\,\boldsymbol{e}_y + \mathrm{d}z\,\boldsymbol{e}_z$$

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z$$

$$= \left[\frac{\partial u}{\partial x}\boldsymbol{e}_x + \frac{\partial u}{\partial y}\boldsymbol{e}_y + \frac{\partial u}{\partial z}\boldsymbol{e}_z\right] \bullet d\boldsymbol{l}$$

$$\boldsymbol{grad}(u) = \nabla u = \frac{\partial u}{\partial x}\boldsymbol{e}_x + \frac{\partial u}{\partial y}\boldsymbol{e}_y + \frac{\partial u}{\partial z}\boldsymbol{e}_z$$

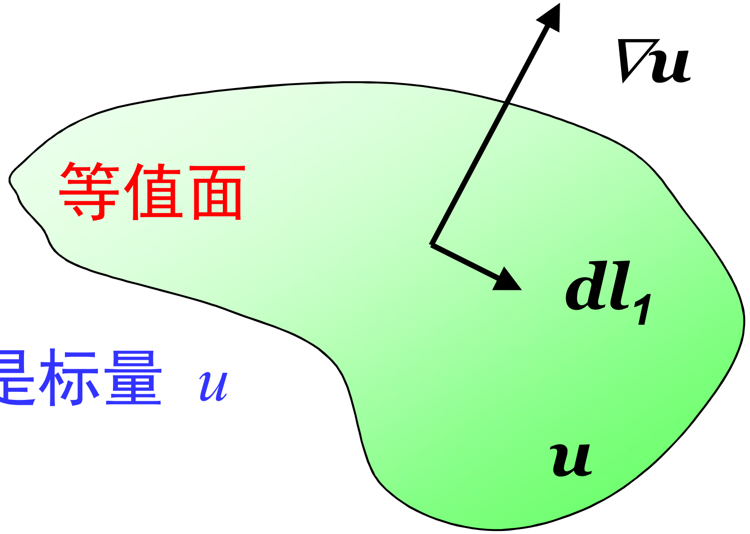
标量场的梯度

矢量微分算符

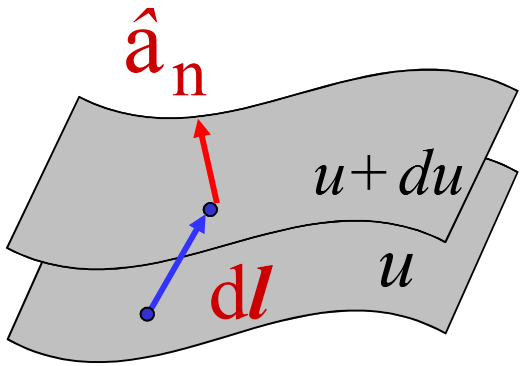
$$\nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z$$

$$du = \nabla \boldsymbol{u} \bullet \boldsymbol{dl}$$

$$du_1 = \boldsymbol{dl}_1 \bullet \nabla \boldsymbol{u} = 0$$



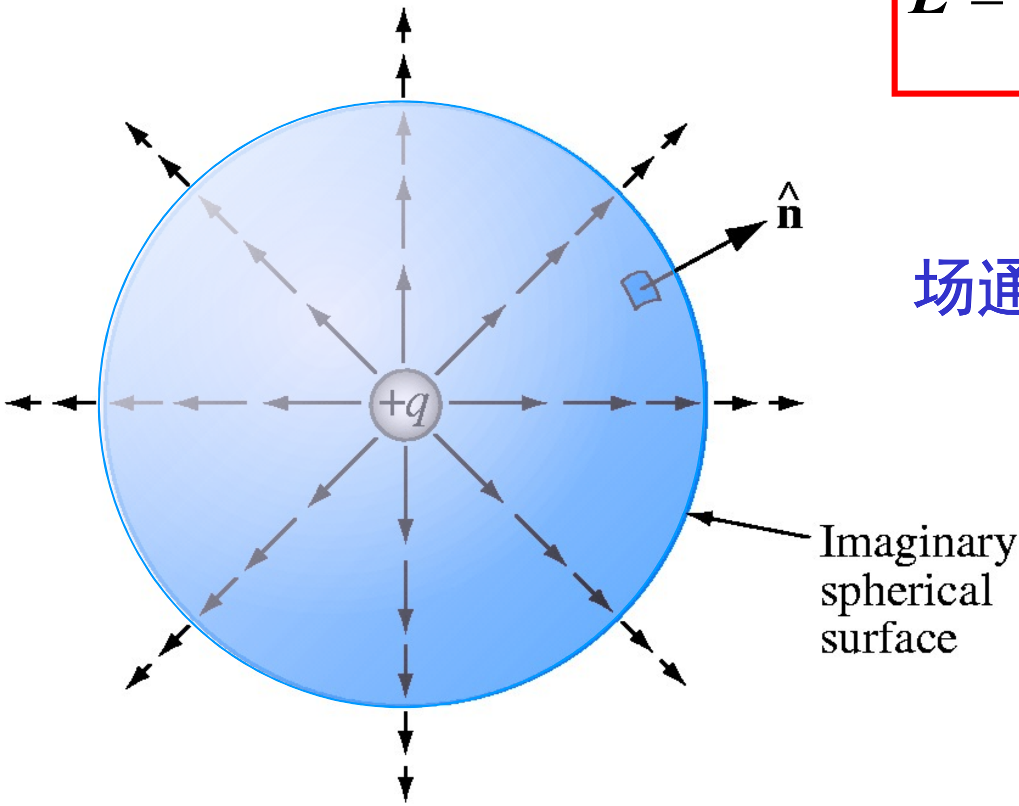
u 的梯度方向是等势面的法向，也是标量 u 有最大变化率的方向！



梯度运算法则

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = U\nabla V + V\nabla U$$
$$\nabla V^n = nV^{n-1}\nabla V$$

1.3　矢量场的散度和散度定理



$$\boxed{\boldsymbol{E} = \hat{\boldsymbol{R}}\frac{q}{4\pi\varepsilon_0 R^2} \quad (V/m)}$$

场通量　$\boldsymbol{E} \bullet d\boldsymbol{s} = \boldsymbol{E} \bullet \boldsymbol{n} ds$

总电场通量 $= \oint_S \boldsymbol{E} \bullet d\boldsymbol{s} = q/\varepsilon_0$

高斯定理

上顶场均值

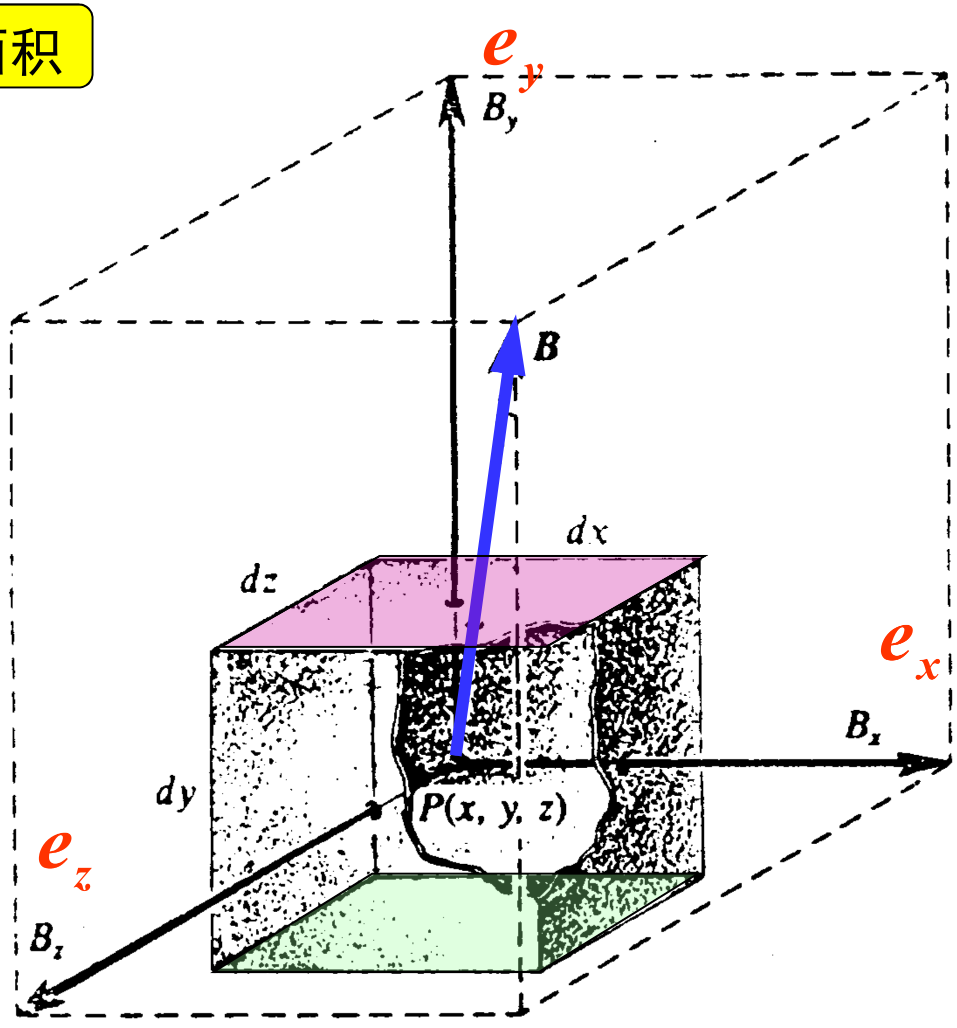
上顶面积

$$d\Phi_T = \left(B_y + \frac{\partial B_y}{\partial y}\frac{dy}{2}\right)dx\,dz$$

$$d\Phi_B = -\left(B_y - \frac{\partial B_y}{\partial y}\frac{dy}{2}\right)dx\,dz$$

$$d\Phi_T + d\Phi_B = \frac{\partial B_y}{\partial y}dx\,dy dz = \frac{\partial B_y}{\partial y}d\tau$$

$$d\Phi_{tot} = \boxed{\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)}d\tau$$

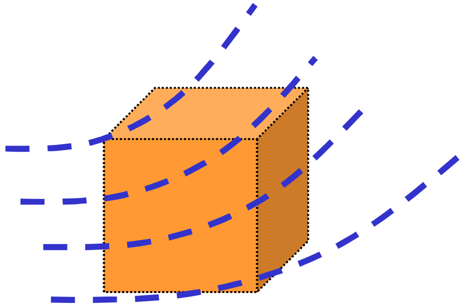


矢量场通量计算

$$\Phi_{tot} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \mathrm{d}\tau$$

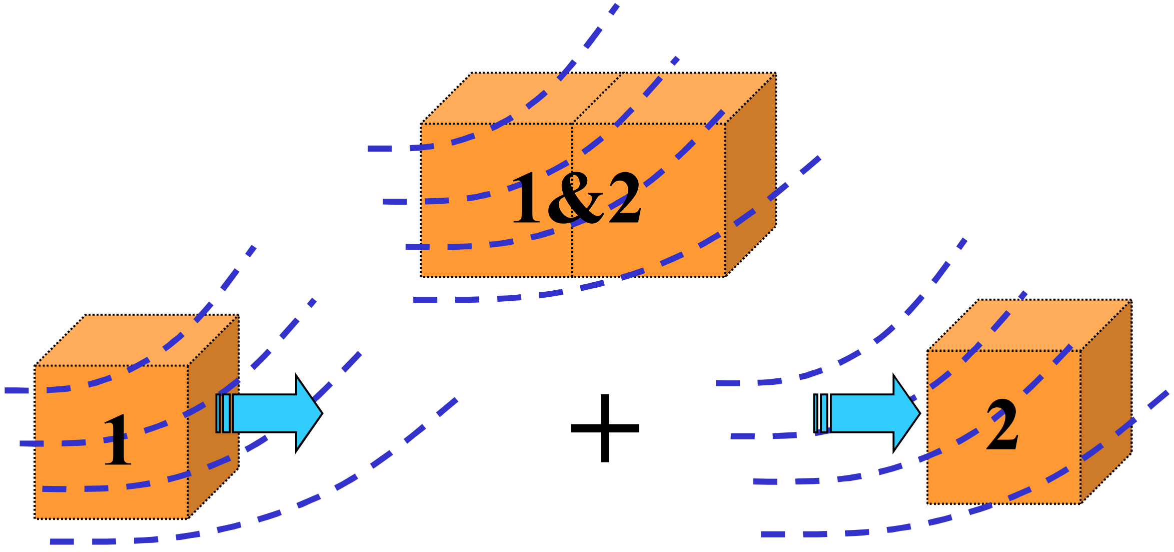
散度 ***Divergence***

$$div(\boldsymbol{B}) = \nabla \bullet \boldsymbol{B} = \boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}}$$



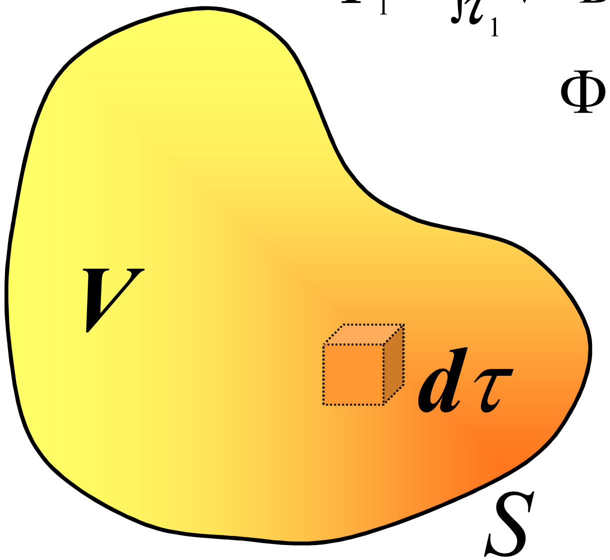
$$div(\boldsymbol{B}) = \lim_{\Delta v \to 0} \frac{\oint \mathrm{B} \bullet \mathrm{d}\mathrm{s}}{\Delta v}$$

$$\Phi_{tot} = \oint \boldsymbol{B} \bullet \mathrm{d}\boldsymbol{s} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \mathrm{d}\tau = (\nabla \bullet \boldsymbol{B}) \mathrm{d}\tau$$



$$\Phi_1 = \int_{\tau_1} \nabla \bullet \boldsymbol{B}\, \mathrm{d}\tau \quad + \quad \Phi_2 = \int_{\tau_2} \nabla \bullet \boldsymbol{B}\, \mathrm{d}\tau$$

$$\Phi_{1\&2} = \Phi_1 + \Phi_2 = \int_{\tau_1+\tau_2} \nabla \bullet \boldsymbol{B}\, \mathrm{d}\tau$$

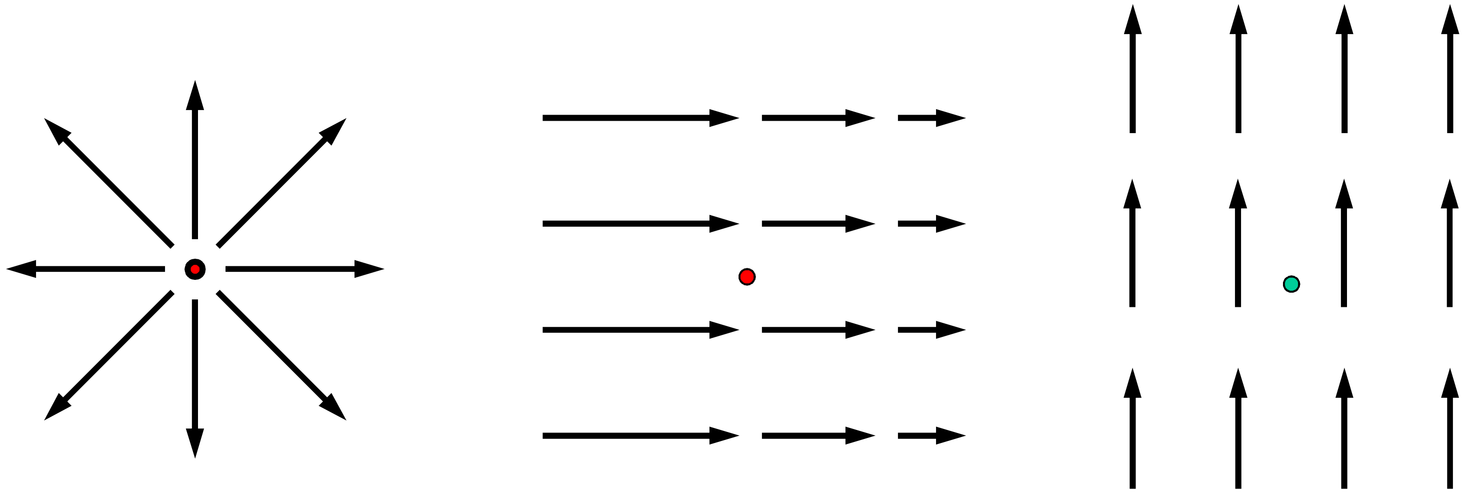


$$\boxed{\oint \boldsymbol{B} \bullet \mathrm{d}\boldsymbol{s} = \int_V \nabla \bullet \boldsymbol{B}\, \mathrm{d}\tau}$$

散度定理 divergence theorem

$$\boxed{div\,\boldsymbol{A} = \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{A} \bullet d\boldsymbol{s}}{\Delta v}}$$

$$\nabla \bullet (\boldsymbol{A}_1 + \boldsymbol{A}_2) = \nabla \bullet \boldsymbol{A}_1 + \nabla \bullet \boldsymbol{A}_2$$

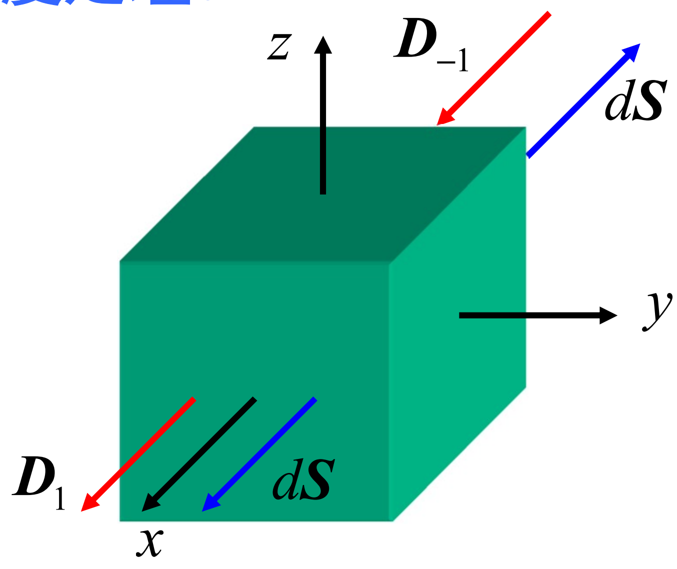


例题：已知 $\boldsymbol{D}=\left(10x^3/3\right)\boldsymbol{a}_x$，对以原点为中心、边长为2cm、各边长平行于坐标轴的立方体，验证散度定理。

$$\nabla\cdot\boldsymbol{D}=10x^2$$

$$\int_v(\nabla\cdot\boldsymbol{D})dv=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(10x^2\right)dxdydz$$

$$=\int_{-1}^{1}\int_{-1}^{1}\left[10\frac{x^3}{3}\right]_{-1}^{1}dydz=\frac{80}{3}$$



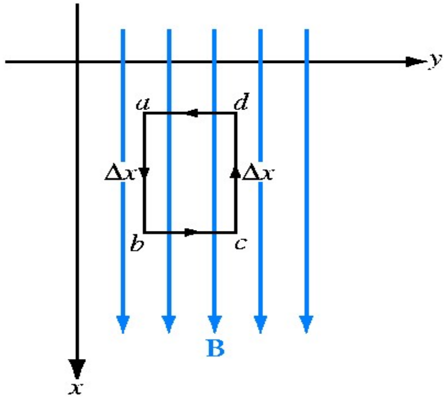
$$\oint\boldsymbol{D}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}\frac{10(1)}{3}\boldsymbol{a}_x\cdot dydz\boldsymbol{a}_x+\int_{-1}^{1}\int_{-1}^{1}\frac{10(-1)}{3}\boldsymbol{a}_x\cdot dydz\left(-\boldsymbol{a}_x\right)$$

$$=\frac{40}{3}+\frac{40}{3}=\frac{80}{3}$$

1.4 矢量场的旋度和斯托克斯定理

均匀磁场

$$\text{磁场环路积分} = \oint \boldsymbol{B} \bullet \mathrm{d}\boldsymbol{l} = 0$$



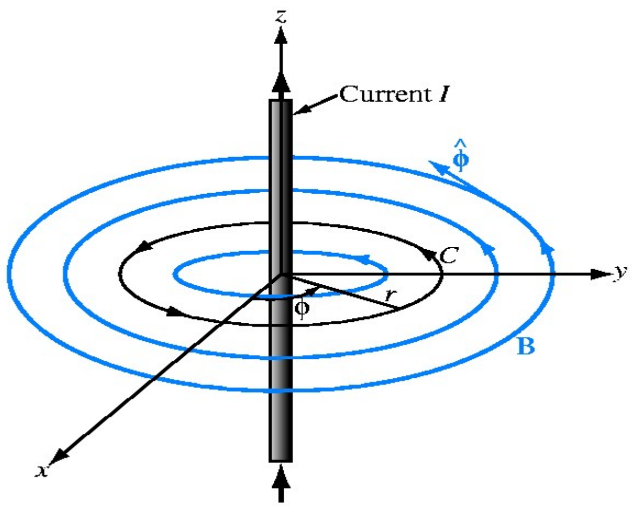
(a) Uniform field

线电流的磁场

$$\boldsymbol{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r}$$

$$\text{磁场环路积分} = \oint \boldsymbol{B} \bullet \mathrm{d}\boldsymbol{l}$$

$$= \int_0^{2\pi} \hat{\phi} \frac{\mu_0 I}{2\pi r} \bullet \hat{\phi}\, r d\phi = \mu_0 I$$



(b) Azimuthal field

Figure 3-22

任意矢量场的环路积分

考虑x−y平面上的闭合回路，对$\boldsymbol{B}$ 作线积分

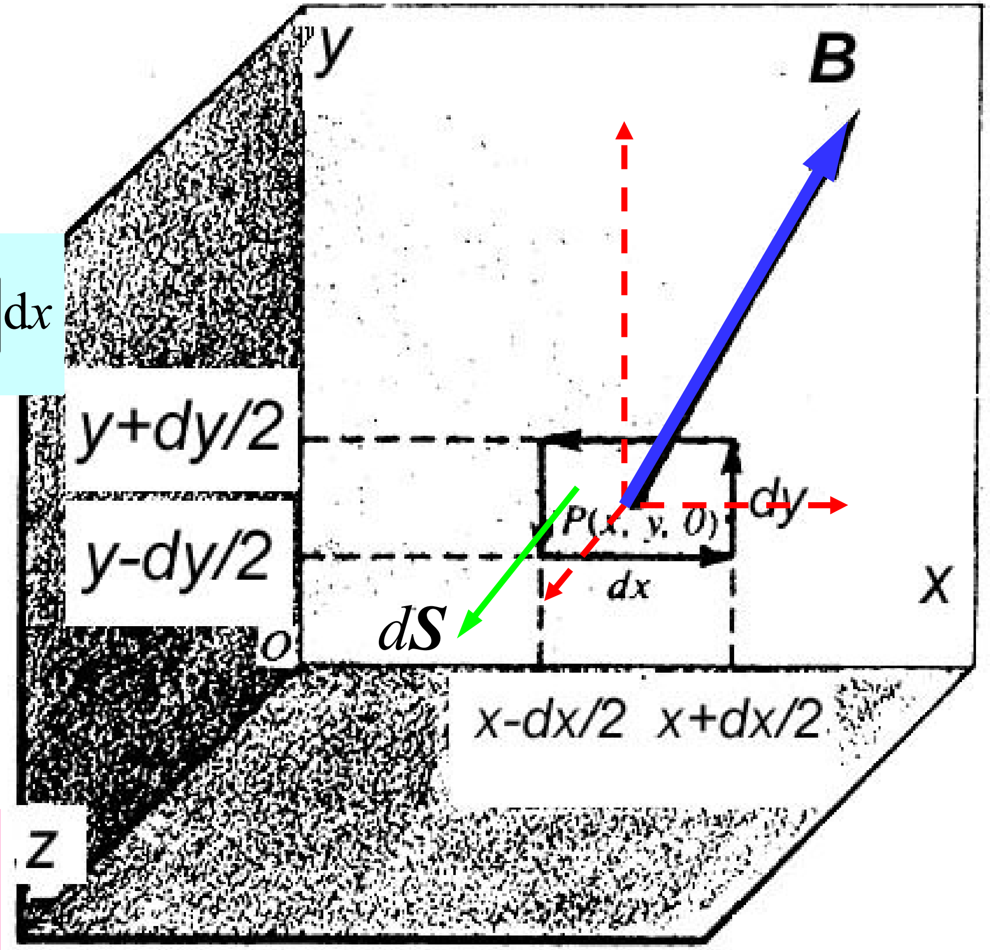
$$\oint \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l} = \oint B_x \mathrm{d}x + \oint B_y \mathrm{d}y$$

$$\oint B_x \mathrm{d}x = \left(B_x - \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} \right)\mathrm{d}x - \left(B_x + \frac{\partial B_x}{\partial y}\frac{\mathrm{d}y}{2} \right)\mathrm{d}x$$

$$\oint B_x \mathrm{d}x = -\frac{\partial B_x}{\partial y}\mathrm{d}x\mathrm{d}y$$

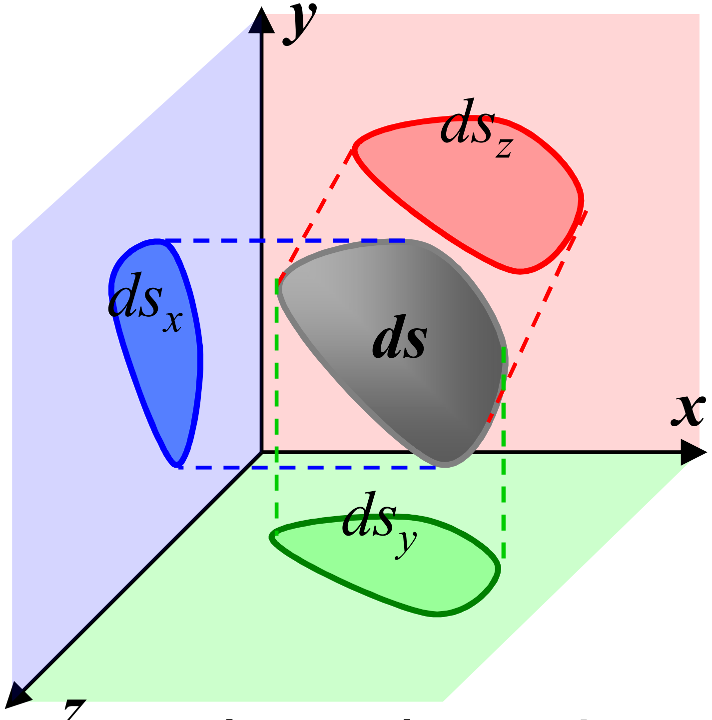
$$\oint B_y \mathrm{d}y = \frac{\partial B_y}{\partial x}\mathrm{d}x\mathrm{d}y$$

$$\oint \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)\mathrm{d}x\,\mathrm{d}y = g_z \mathrm{d}s_z$$



$$\oint B\bullet \mathrm{d}l = \underbrace{\boxed{\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\mathrm{d}x\mathrm{d}y}}_{g_z} + \underbrace{\boxed{\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\mathrm{d}z\mathrm{d}x}}_{g_y} + \underbrace{\boxed{\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\mathrm{d}y\mathrm{d}z}}_{g_x}$$

$$\boxed{\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)e_z} + \boxed{\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)e_y} + \boxed{\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)e_x} = \nabla\times \boldsymbol{B}$$



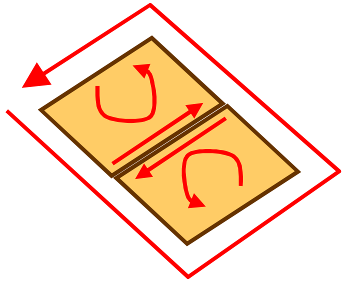
$$ds = \hat{x}ds_x + \hat{y}ds_y + \hat{z}ds_z$$

B的旋度

$$curl(\boldsymbol{B}) = \nabla\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

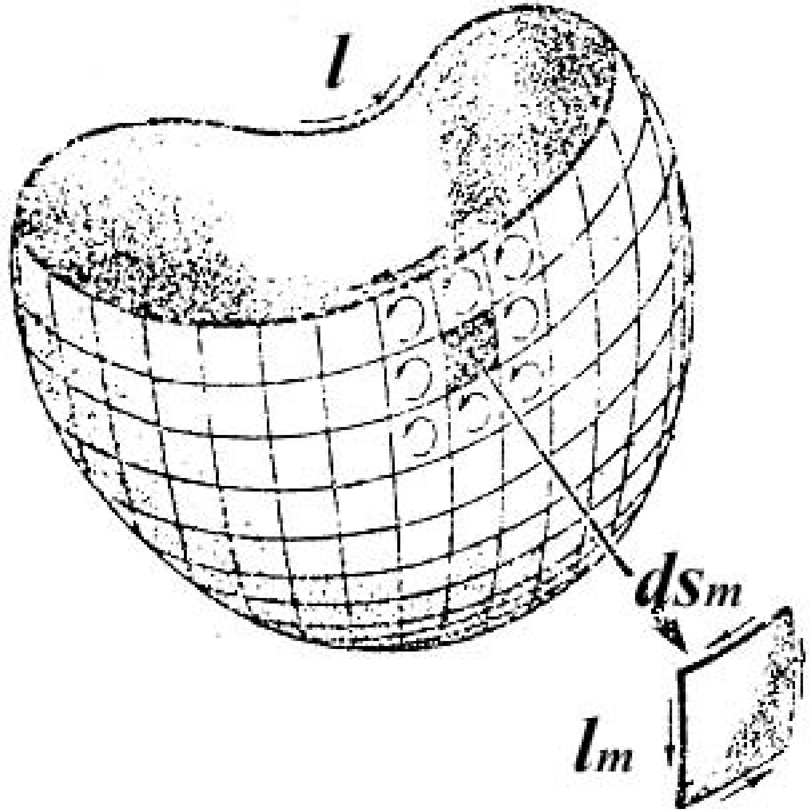
$$\oint \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l} = (\nabla\times\boldsymbol{B})\bullet \mathrm{ds}$$

可以应用到任意取向的小面积元：



$$(\nabla\times\boldsymbol{B})_{\mathrm{n}}=\lim_{s\to 0}\frac{1}{s}\oint \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l}$$

推广到任意曲面



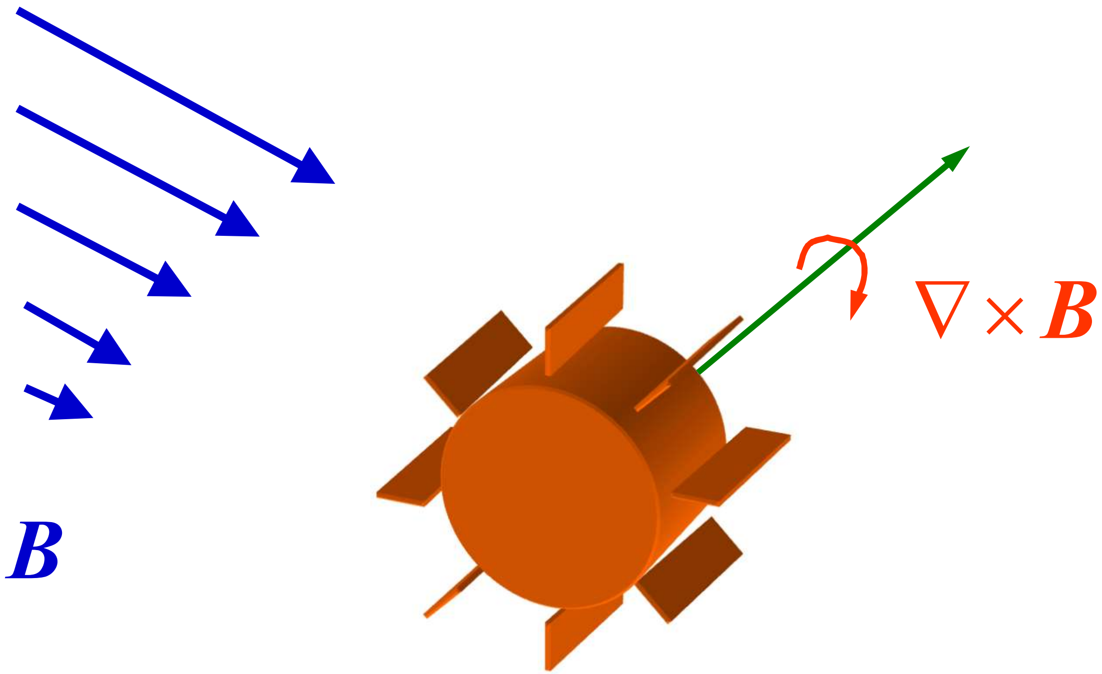
$$\oint_{l_1}\boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l}=(\nabla\times\boldsymbol{B})\bullet \mathrm{d}\boldsymbol{s}_1$$

$$\oint_{l_2}\boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l}=(\nabla\times\boldsymbol{B})\bullet \mathrm{d}\boldsymbol{s}_2$$

$$\vdots$$

$$\boxed{\oint \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{l}=\int_S(\nabla\times\boldsymbol{B})\bullet \mathrm{d}\boldsymbol{s}}$$

斯托克斯定理 Stokes’s theorem


$\nabla \times \boldsymbol{B}$
$\boldsymbol{B}$

$$\nabla\times\left(\boldsymbol{A}_1+\boldsymbol{A}_2\right)=\nabla\times\boldsymbol{A}_1+\nabla\times\boldsymbol{A}_2$$

两个矢量恒等式：

梯度的旋度恒等于零：

$$\boxed{\nabla\times(\nabla u)\equiv 0}$$

梯度无旋

旋度的散度也恒等于零：

$$\boxed{\nabla\bullet(\nabla\times\boldsymbol{A})\equiv 0}$$

旋度无散

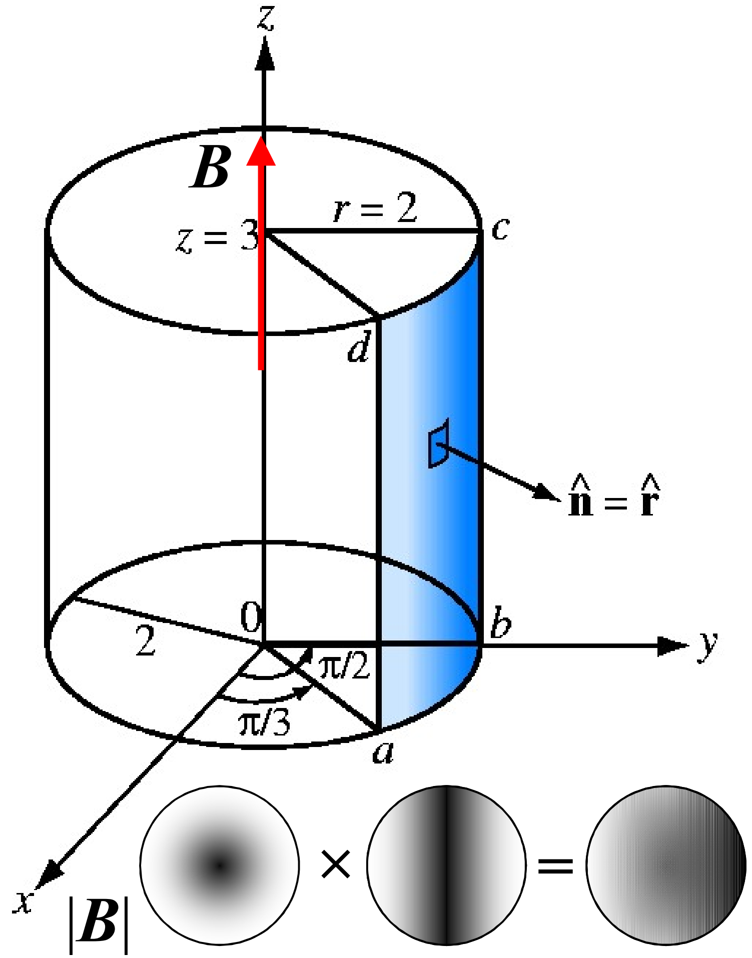
例题：验证Stokes定理.

矢量场 $B=\hat{z}\cos\phi/r$

$$\nabla\times\mathbf{B}=\hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial B_z}{\partial\phi}-\frac{\partial B_\phi}{\partial z}\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\phi)-\frac{\partial B_r}{\partial\phi}\right)$$
$$=\hat{\mathbf{r}}\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r}\right)-\hat{\boldsymbol{\phi}}\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right)$$
$$=-\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}.$$

S面积上的旋度积分为：

$$\int_S(\nabla\times\mathbf{B})\cdot d\mathbf{s}$$
$$=\int_{z=0}^{3}\int_{\phi=\pi/3}^{\pi/2}\left(-\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\cdot\hat{\mathbf{r}}r\,d\phi\,dz$$
$$=\int_0^3\int_{\pi/3}^{\pi/2}-\frac{\sin\phi}{r}\,d\phi\,dz=-\frac{3}{2r}=-\frac{3}{4},$$



$$\oint_C\mathbf{B}\cdot d\mathbf{l}=\int_a^b\mathbf{B}_{ab}\cdot d\mathbf{l}+\int_b^c\mathbf{B}_{bc}\cdot d\mathbf{l}+\int_c^d\mathbf{B}_{cd}\cdot d\mathbf{l}+\int_d^a\mathbf{B}_{da}\cdot d\mathbf{l},$$

$$\oint_C\mathbf{B}\cdot d\mathbf{l}=\int_d^a\left(\hat{\mathbf{z}}\frac{1}{4}\right)\cdot\hat{\mathbf{z}}\,dz=\int_3^0\frac{1}{4}\,dz=-\frac{3}{4},$$

例子

1.5 拉普拉斯算符

$$\nabla^2 u \equiv \nabla \bullet (\nabla u)$$

$$\nabla^2 u \equiv \nabla \bullet (\nabla u) = \nabla \bullet \left(\frac{\partial u}{\partial x} \boldsymbol{e}_x + \frac{\partial u}{\partial y} \boldsymbol{e}_y + \frac{\partial u}{\partial z} \boldsymbol{e}_z \right)$$
$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 \boldsymbol{B} \equiv \left(\boldsymbol{e}_x \nabla^2 B_x + \boldsymbol{e}_y \nabla^2 B_y + \boldsymbol{e}_z \nabla^2 B_z \right)$$

$$\nabla^2 \boldsymbol{B} \equiv \nabla \left(\nabla \bullet \boldsymbol{B} \right) - \nabla \times \left(\nabla \times \boldsymbol{B} \right)$$

1.6 正交坐标系

直角坐标系

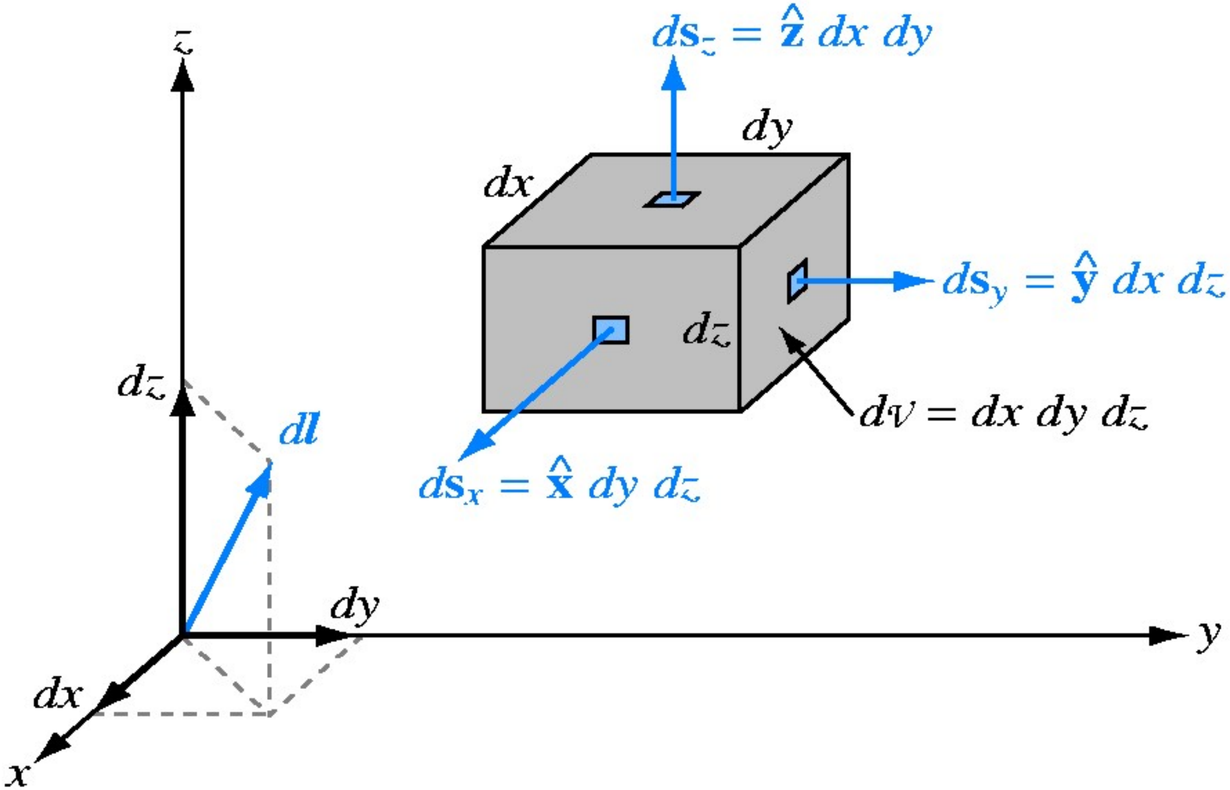


Figure 3-8

1.7 正交曲线坐标系

柱坐标系：

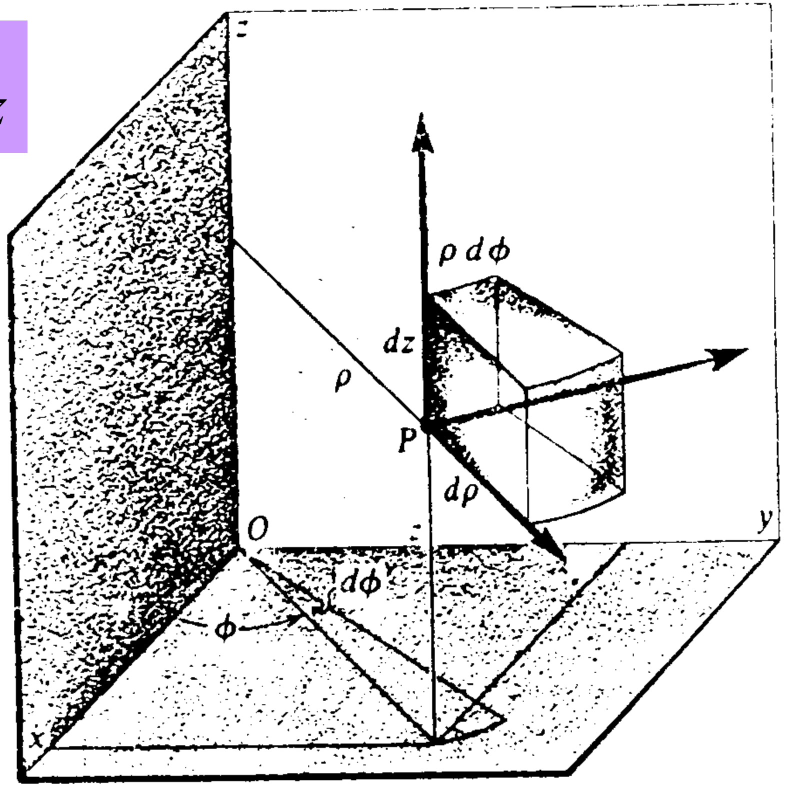
$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}},$$

微分长度的表示式

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}\rho\, \boldsymbol{e}_\rho + \rho \mathrm{d}\varphi\, \boldsymbol{e}_\varphi + \mathrm{d}z\, \boldsymbol{e}_z$$

微分面积的表示式

$$\left.\begin{aligned} (\mathrm{d}\boldsymbol{s})_\rho &= \rho \mathrm{d}\varphi \mathrm{d}z \\ (\mathrm{d}\boldsymbol{s})_\varphi &= \mathrm{d}\rho \mathrm{d}z \\ (\mathrm{d}\boldsymbol{s})_z &= \rho \mathrm{d}\varphi \mathrm{d}\rho \end{aligned}\right\}$$



微分体积的表示式

$$\mathrm{d}\tau = \rho \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}z$$

直角坐标系-柱坐标系相互变换

$$\rho = \sqrt{x^2 + y^2}$$

$$\varphi = tan^{-1}\left[\frac{y}{x}\right]$$

$$z = z$$

$$x = \rho cos\varphi$$

$$y = \rho sin\varphi$$

$$z = z$$

$$\boldsymbol{A} = A_\rho \hat{\boldsymbol{a}}_\rho + A_\varphi \hat{\boldsymbol{a}}_\phi + A_z \hat{\boldsymbol{a}}_z$$

$$\boldsymbol{A} \bullet \boldsymbol{B} = A_\rho B_\rho + A_\varphi B_\varphi + A_z B_z$$

$$|\boldsymbol{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\varphi & \hat{a}_z \\ A_\rho & A_\varphi & A_z \\ B_\rho & B_\varphi & B_z \end{vmatrix}$$

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

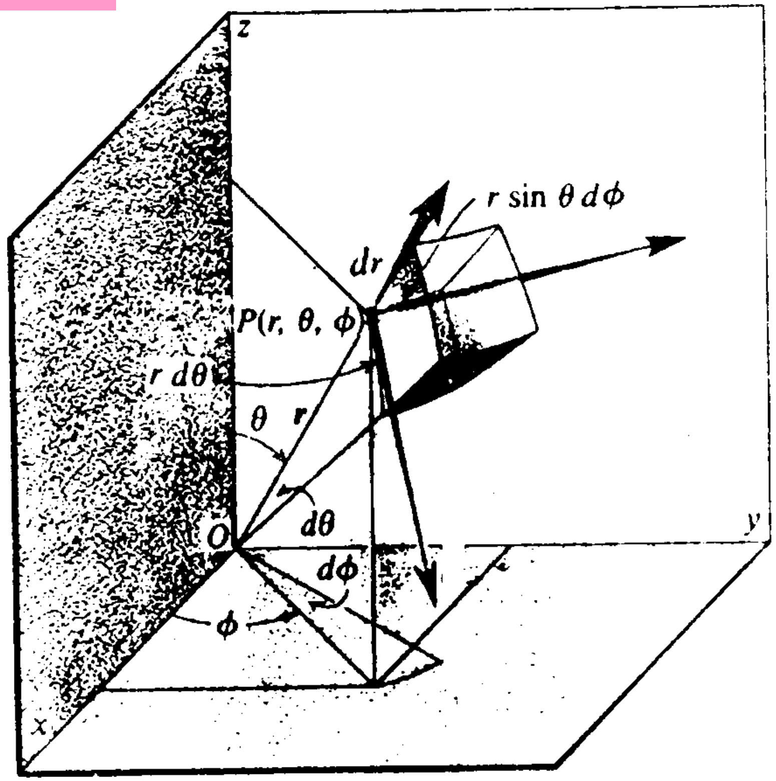
球坐标系：

微分长度的表示式

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\,\boldsymbol{e}_r + r\mathrm{d}\theta\,\boldsymbol{e}_\theta + r\sin\theta\,\mathrm{d}\varphi\,\boldsymbol{e}_\varphi$$

微分面积的表示式

$$\left.\begin{aligned}(\mathrm{d}s)_r &= r\mathrm{d}\theta\, r\sin\theta\,\mathrm{d}\varphi \\ (\mathrm{d}s)_\theta &= r\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r \\ (\mathrm{d}s)_\varphi &= r\mathrm{d}\theta\,\mathrm{d}r\end{aligned}\right\}$$



微分体积的表示式

$$\mathrm{d}\tau = r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r$$

直角坐标系-球坐标系

相互变换

$$r=\sqrt{x^2+y^2+z^2}$$

$$\theta=\cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\phi=\tan^{-1}\left[\frac{y}{x}\right]$$

$$x=r\sin\theta\cos\phi$$

$$y=r\sin\theta\sin\phi$$

$$z=r\cos\theta$$

$$\boldsymbol{A}=A_r\hat{\boldsymbol{\alpha}}_r+A_\theta\hat{\boldsymbol{\alpha}}_\theta+A_\phi\hat{\boldsymbol{\alpha}}_\phi$$

$$\boldsymbol{A}\bullet\boldsymbol{B}=A_rB_r+A_\theta B_\theta+A_\varphi B_\varphi$$

$$|\boldsymbol{A}|=\sqrt{A_r^2+A_\theta^2+A_\phi^2}$$

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\hat{\alpha}_r & \hat{\alpha}_\theta & \hat{\alpha}_\phi\\ A_r & A_\theta & A_\phi\\ B_r & B_\theta & B_\phi\end{vmatrix}$$

$$\nabla^2u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2u}{\partial\varphi^2}$$

三坐标系下的各种矢量表达和运算

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\|A\|$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, dv =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

矢量表达在三坐标系下的变换

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{x}\frac{\partial V}{\partial x} + \mathbf{y}\frac{\partial V}{\partial y} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \mathbf{r}\frac{\partial V}{\partial r} + \boldsymbol{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{r} & \boldsymbol{\phi} r & \mathbf{z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \mathbf{R}\frac{\partial V}{\partial R} + \boldsymbol{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \boldsymbol{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{R} & \boldsymbol{\theta} R & \boldsymbol{\phi} R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \mathbf{R}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \boldsymbol{\theta}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \boldsymbol{\phi}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Homework

- 1.1 and 1.2
- 1.3 证明以下矢量恒等式成立：

$$\nabla^2 \boldsymbol{B} \equiv \nabla(\nabla \bullet \boldsymbol{B}) - \nabla \times (\nabla \times \boldsymbol{B})$$

- 1.4 已知 $\boldsymbol{B} = \hat{\boldsymbol{r}} 10 e^{-2r} \cos\phi + \hat{\boldsymbol{z}} 10 \sin\phi$

  在(2, 0, 3)处计算 $\nabla \bullet \boldsymbol{B}$, $\nabla \times \boldsymbol{B}$

- 1.5 已知两矢量：$\boldsymbol{A} = \boldsymbol{e}_r z^2 \sin\phi + \boldsymbol{e}_\phi z^2 \cos\phi + \boldsymbol{e}_z 2rz \sin\phi$

$$\boldsymbol{B} = \boldsymbol{e}_x \left(3y^2 - 2x\right) + \boldsymbol{e}_y x^2 + \boldsymbol{e}_z 2z$$

1）那些矢量可由标量函数的梯度表示，那些矢量可由矢量函数的旋度表示？2）求出这些矢量的源分布。