

Deep Generative Models

Lecture 13

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2025, Autumn

Recap of Previous Lecture

Flow Matching (FM)

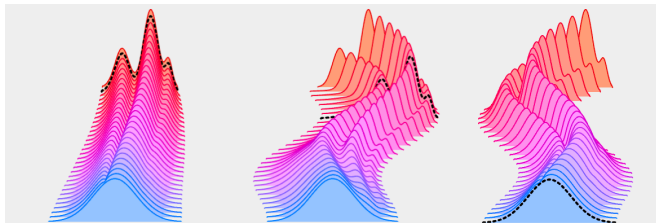
$$\mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{\mathbf{x} \sim p_t(\mathbf{x})} \|\mathbf{f}(\mathbf{x}, t) - \mathbf{f}_\theta(\mathbf{x}, t)\|^2 \rightarrow \min_{\theta}$$

Conditional Flow Matching (CFM)

$$\mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})} \mathbb{E}_{\mathbf{x} \sim p_t(\mathbf{x}|\mathbf{z})} \|\mathbf{f}(\mathbf{x}, \mathbf{z}, t) - \mathbf{f}_\theta(\mathbf{x}, \mathbf{z}, t)\|^2 \rightarrow \min_{\theta}$$

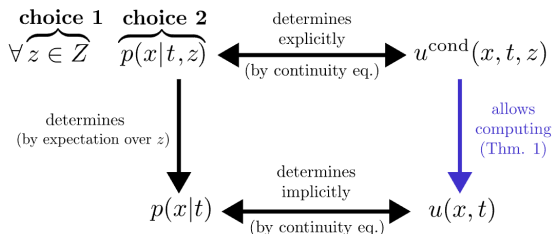
Theorem

If $\text{supp}(p_t(\mathbf{x})) = \mathbb{R}^m$, then the optimal value of the FM objective equals the optimum for CFM.



Tong A., et al. *Improving and Generalizing Flow-Based Generative Models with Minibatch Optimal Transport*, 2023

Recap of Previous Lecture



Constraints

$$p(\mathbf{x}) = \mathcal{N}(0, \mathbf{I}) = \mathbb{E}_{p(\mathbf{z})} p_0(\mathbf{x}|\mathbf{z}); \quad p_{\text{data}}(\mathbf{x}) = \mathbb{E}_{p(\mathbf{z})} p_1(\mathbf{x}|\mathbf{z}).$$

- ▶ How should we choose the conditioning latent variable \mathbf{z} ?
- ▶ How can we define $p_t(\mathbf{x}|\mathbf{z})$ so that it meets the constraints?

Gaussian Conditional Probability Path

$$p_t(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\mu_t(\mathbf{z}), \sigma_t^2(\mathbf{z}))$$

$$\mathbf{x}_t = \mu_t(\mathbf{z}) + \sigma_t(\mathbf{z}) \odot \mathbf{x}_0, \quad \mathbf{x}_0 \sim p_0(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$$

image credit: A Visual Dive into Conditional Flow Matching

Recap of Previous Lecture

Gaussian Conditional Probability Path

$$p_t(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\boldsymbol{\mu}_t(\mathbf{z}), \boldsymbol{\sigma}_t^2(\mathbf{z})); \quad \mathbf{x}_t = \boldsymbol{\mu}_t(\mathbf{z}) + \boldsymbol{\sigma}_t(\mathbf{z}) \odot \mathbf{x}_0$$

$$\mathbf{f}(\mathbf{x}, \mathbf{z}, t) = \boldsymbol{\mu}'_t(\mathbf{z}) + \frac{\boldsymbol{\sigma}'_t(\mathbf{z})}{\boldsymbol{\sigma}_t(\mathbf{z})} \odot (\mathbf{x} - \boldsymbol{\mu}_t(\mathbf{z}))$$

Conditioning Latent Variable

Let's choose $\mathbf{z} = \mathbf{x}_1$. Then $p(\mathbf{z}) = p_1(\mathbf{x}_1)$.

$$p_t(\mathbf{x}) = \int p_t(\mathbf{x}|\mathbf{x}_1)p_1(\mathbf{x}_1)d\mathbf{x}_1$$

We must ensure the boundary constraints:

$$\begin{cases} p(\mathbf{x}) = \mathbb{E}_{p(\mathbf{z})} p_0(\mathbf{x}|\mathbf{z}); (= \mathcal{N}(0, \mathbf{I})) \\ p_{\text{data}}(\mathbf{x}) = \mathbb{E}_{p(\mathbf{z})} p_1(\mathbf{x}|\mathbf{z}). \end{cases} \Rightarrow \begin{cases} p_0(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(0, \mathbf{I}); \\ p_1(\mathbf{x}|\mathbf{x}_1) = \delta(\mathbf{x} - \mathbf{x}_1). \end{cases}$$

Recap of Previous Lecture

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Let's consider straight conditional paths:

$$\begin{cases} \boldsymbol{\mu}_t(\mathbf{x}_1) = t\mathbf{x}_1; \\ \boldsymbol{\sigma}_t(\mathbf{x}_1) = 1 - t. \end{cases} \Rightarrow \begin{cases} p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(t\mathbf{x}_1, (1-t)^2\mathbf{I}); \\ \mathbf{x}_t = t\mathbf{x}_1 + (1-t)\mathbf{x}_0. \end{cases}$$

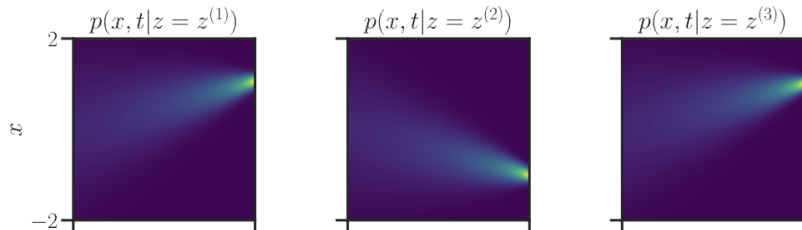


image credit: A Visual Dive into Conditional Flow Matching

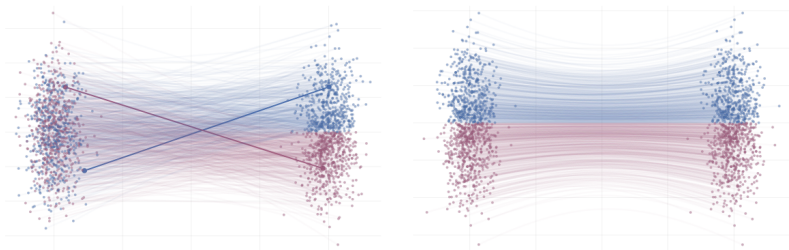
Recap of Previous Lecture

$$p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(t\mathbf{x}_1, (1-t)^2\mathbf{I}); \quad \mathbf{x}_t = t\mathbf{x}_1 + (1-t)\mathbf{x}_0$$

$$\mathbf{f}(\mathbf{x}_t, \mathbf{x}_1, t) = \frac{\mathbf{x}_1 - \mathbf{x}_t}{1-t} = \mathbf{x}_1 - \mathbf{x}_0$$

$$\begin{aligned} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})} \mathbb{E}_{\mathbf{x} \sim p_t(\mathbf{x}|\mathbf{z})} \|\mathbf{f}(\mathbf{x}, \mathbf{z}, t) - \mathbf{f}_\theta(\mathbf{x}, t)\|^2 &= \\ = \mathbb{E}_{\mathbf{x}_1 \sim p_{\text{data}}(\mathbf{x})} \mathbb{E}_{\mathbf{x}_0 \sim \mathcal{N}(0, \mathbf{I})} \|(\mathbf{x}_1 - \mathbf{x}_0) - \mathbf{f}_\theta(t\mathbf{x}_1 + (1-t)\mathbf{x}_0, t)\|^2 \end{aligned}$$

- ▶ $\mathbf{f}(\mathbf{x}_t, \mathbf{x}_1, t)$ defines straight lines between $p_{\text{data}}(\mathbf{x})$ and $\mathcal{N}(0, \mathbf{I})$.
- ▶ The **marginal** path $p_t(\mathbf{x})$ does not give straight lines.



Recap of Previous Lecture

$$\mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{\mathbf{x}_1 \sim p_{\text{data}}(\mathbf{x})} \mathbb{E}_{\mathbf{x}_0 \sim \mathcal{N}(0, \mathbf{I})} \|(\mathbf{x}_1 - \mathbf{x}_0) - \mathbf{f}_{\theta}(\mathbf{x}_t, t)\|^2 \rightarrow \min_{\theta}$$

Training

1. Sample $\mathbf{x}_1 \sim p_{\text{data}}(\mathbf{x})$.
2. Sample time $t \sim U[0, 1]$ and $\mathbf{x}_0 \sim \mathcal{N}(0, \mathbf{I})$.
3. Obtain the noisy image $\mathbf{x}_t = t\mathbf{x}_1 + (1 - t)\mathbf{x}_0$.
4. Compute the loss $\mathcal{L} = \|(\mathbf{x}_1 - \mathbf{x}_0) - \mathbf{f}_{\theta}(\mathbf{x}, t)\|^2$.

Sampling

1. Sample $\mathbf{x}_0 \sim \mathcal{N}(0, \mathbf{I})$.
2. Solve the ODE to obtain \mathbf{x}_1 :

$$\mathbf{x}_1 = \text{ODESolve}_f(\mathbf{x}_0, \theta, t_0 = 0, t_1 = 1)$$

Recap of Previous Lecture

Let us choose $\mathbf{z} = (\mathbf{x}_0, \mathbf{x}_1)$. Then $p(\mathbf{z}) = p(\mathbf{x}_0, \mathbf{x}_1) = p_0(\mathbf{x}_0)p_1(\mathbf{x}_1)$.

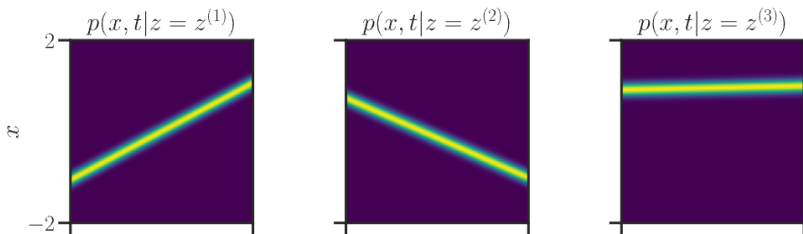
$$p_0(\mathbf{x}|\mathbf{x}_0, \mathbf{x}_1) = \delta(\mathbf{x} - \mathbf{x}_0); \quad p_1(\mathbf{x}|\mathbf{x}_0, \mathbf{x}_1) = \delta(\mathbf{x} - \mathbf{x}_1)$$

Gaussian Conditional Probability Path

$$p_t(\mathbf{x}|\mathbf{x}_0, \mathbf{x}_1) = \mathcal{N}(\boldsymbol{\mu}_t(\mathbf{x}_0, \mathbf{x}_1), \boldsymbol{\sigma}_t^2(\mathbf{x}_0, \mathbf{x}_1)); \quad \mathbf{x}_t = \boldsymbol{\mu}_t(\mathbf{x}_0, \mathbf{x}_1) + \boldsymbol{\sigma}_t(\mathbf{x}_0, \mathbf{x}_1) \odot \boldsymbol{\epsilon}$$

Let's consider straight conditional paths:

$$\boldsymbol{\mu}_t(\mathbf{x}_1) = t\mathbf{x}_1 + (1-t)\mathbf{x}_0 \quad \boldsymbol{\sigma}_t(\mathbf{x}_1) = \epsilon$$



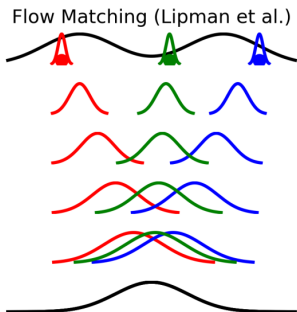
Recap of Previous Lecture

Endpoint conditioning

$$\mathbf{z} = \mathbf{x}_1$$

$$p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(t\mathbf{x}_1, (1-t)^2\mathbf{I})$$

$$\mathbf{x}_t = t\mathbf{x}_1 + (1-t)\mathbf{x}_0$$

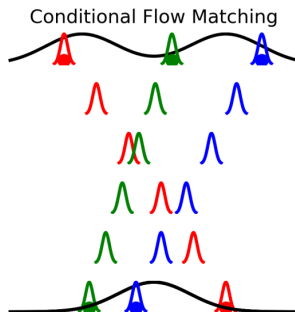


Pair conditioning

$$\mathbf{z} = (\mathbf{x}_0, \mathbf{x}_1)$$

$$p_t(\mathbf{x}|\mathbf{x}_0, \mathbf{x}_1) = \mathcal{N}(t\mathbf{x}_1 + (1-t)\mathbf{x}_0, \epsilon^2\mathbf{I})$$

$$\mathbf{x}_t = t\mathbf{x}_1 + (1-t)\mathbf{x}_0$$



Recap of Previous Lecture

- ▶ This conditioning allows us to transport any distribution $p_0(\mathbf{x})$ to any distribution $p_1(\mathbf{x})$.
- ▶ It's possible to apply this approach to paired tasks, e.g., style transfer.

Training Procedure

1. Sample $(\mathbf{x}_0, \mathbf{x}_1) \sim p(\mathbf{x}_0, \mathbf{x}_1)$.
2. Sample time $t \sim U[0, 1]$.
3. Compute the noisy image $\mathbf{x}_t = t\mathbf{x}_1 + (1 - t)\mathbf{x}_0$.
4. Compute the loss $\mathcal{L} = \|(\mathbf{x}_1 - \mathbf{x}_0) - \mathbf{f}_\theta(\mathbf{x}, t)\|^2$.

Sampling

1. Sample $\mathbf{x}_0 \sim p_0(\mathbf{x})$.
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Outline

1. Link between Flow Matching and Score-Based Models
2. Discrete Diffusion Models
 - Forward Discrete Process
 - Reverse Discrete Diffusion

Outline

1. Link between Flow Matching and Score-Based Models

2. Discrete Diffusion Models

Forward Discrete Process

Reverse Discrete Diffusion

Score-Based Generative Models through SDEs

Training

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x}(0))} \mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{q(\mathbf{x}(t)|\mathbf{x}(0))} \left\| \mathbf{s}_{\theta}(\mathbf{x}(t), t) - \nabla_{\mathbf{x}(t)} \log q(\mathbf{x}(t)|\mathbf{x}(0)) \right\|_2^2$$

Score-Based Generative Models through SDEs

Training

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x}(0))} \mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{q(\mathbf{x}(t)|\mathbf{x}(0))} \left\| \mathbf{s}_{\theta}(\mathbf{x}(t), t) - \nabla_{\mathbf{x}(t)} \log q(\mathbf{x}(t)|\mathbf{x}(0)) \right\|_2^2$$

Variance Exploding SDE (NCSN)

$$q(\mathbf{x}(t)|\mathbf{x}(0)) = \mathcal{N}(\mathbf{x}(0), [\sigma^2(t) - \sigma^2(0)] \cdot \mathbf{I}), \quad \sigma(0) = 0$$

Variance Preserving SDE (DDPM)

$$q(\mathbf{x}(t)|\mathbf{x}(0)) = \mathcal{N}(\mathbf{x}(0)\alpha(t), (1 - \alpha(t)^2) \cdot \mathbf{I}); \quad \alpha(t) = e^{-\frac{1}{2} \int_0^t \beta(s) ds}$$

Score-Based Generative Models through SDEs

Training

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x}(0))} \mathbb{E}_{t \sim U[0,1]} \mathbb{E}_{q(\mathbf{x}(t)|\mathbf{x}(0))} \left\| \mathbf{s}_{\theta}(\mathbf{x}(t), t) - \nabla_{\mathbf{x}(t)} \log q(\mathbf{x}(t)|\mathbf{x}(0)) \right\|_2^2$$

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Flow matching uses reverse time direction:

$$p_t(\mathbf{x}|\mathbf{x}_1) = q_{1-t}(\mathbf{x}|\mathbf{x}_0 = \mathbf{x}_1)$$

Score-Based Generative Models through SDEs

$$p_t(\mathbf{x}_t|\mathbf{x}_1) = q_{1-t}(\mathbf{x}_{1-t}|\mathbf{x}_0 = \mathbf{x}_1)$$

$$\mathbf{VE} \text{ (NCSN): } p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(\mathbf{x}_1, \sigma_{1-t}^2 \cdot \mathbf{I})$$

$$\mathbf{VP} \text{ (DDPM): } p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(\alpha_{1-t}\mathbf{x}_1, (1 - \alpha_{1-t}^2) \cdot \mathbf{I})$$

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Flow Matching Probability Path

$$p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(t\mathbf{x}_1, (1-t)^2\mathbf{I}); \quad \mathbf{f}(\mathbf{x}_t, \mathbf{x}_1, t) = \frac{\mathbf{x}_1 - \mathbf{x}_t}{1-t}$$

$$\frac{d\mathbf{x}_t}{dt} = \mathbf{f}(\mathbf{x}_t, \mathbf{x}_1, t) = \mu'_t(\mathbf{x}_1) + \frac{\sigma'_t(\mathbf{x}_1)}{\sigma_t(\mathbf{x}_1)} \odot (\mathbf{x}_t - \mu_t(\mathbf{x}_1))$$

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Let's derive the conditional vector fields for VE (NCSN) and VP (DDPM).

Flow Matching vs. Score-Based SDE Models

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Variance Exploding SDE Probability Path

$$p_t(\mathbf{x}|\mathbf{x}_1) = \mathcal{N}(\mathbf{x}_1, \sigma_{1-t}^2 \mathbf{I}) \quad \Rightarrow \quad \mathbf{f}(\mathbf{x}_t, \mathbf{x}_1, t) = -\frac{\sigma'_{1-t}}{\sigma_{1-t}}(\mathbf{x}_t - \mathbf{x}_1)$$

Flow Matching vs. Score-Based SDE Models

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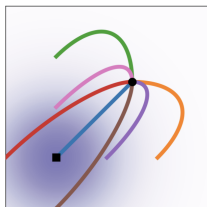
Variance Preserving SDE Probability Path

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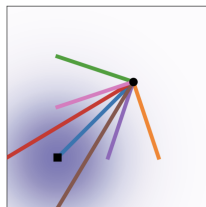
Thus, VE/VP SDE models correspond to particular choices of the Gaussian probability path within the flow matching framework.

Flow Matching vs. Score-Based SDE Models

Trajectories



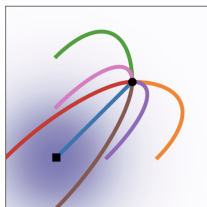
Diffusion



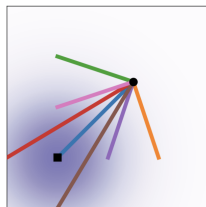
OT

Flow Matching vs. Score-Based SDE Models

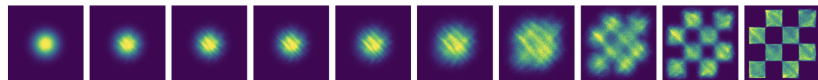
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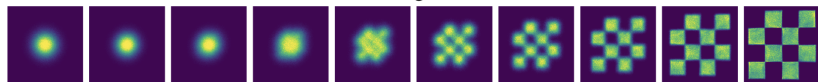
Diffusion



OT



Score matching ^{w/} Diffusion



Flow Matching ^{w/} OT

Outline

1. Link between Flow Matching and Score-Based Models

2. Discrete Diffusion Models

Forward Discrete Process

Reverse Discrete Diffusion

Discrete or Continuous Diffusion Models?

Reminder: Diffusion models define a forward corruption process and a reverse denoising process. Previously, we studied diffusion models with continuous states $\mathbf{x}(t) \in \mathbb{R}^m$.

Continuous state space

- ▶ **Discrete time** $t \in \{0, 1, \dots, T\} \Rightarrow$ **DDPM / NCSN**.
- ▶ **Continuous time** $t \in [0, 1] \Rightarrow$ **Score-based SDE models**.

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Now we turn to diffusion over discrete-value states $\mathbf{x}(t) \in \{1, \dots, K\}^m$.

Discrete state space

- ▶ **Discrete time** $t \in \{0, 1, \dots, T\}$.
- ▶ **Continuous time** $t \in [0, 1]$.

Let's discuss why we need discrete diffusion models.

Why Discrete Diffusion Models?

While autoregressive (AR) models dominate discrete-data domains (e.g., text or sequences), they have fundamental limitations.

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Key advantages of discrete diffusion

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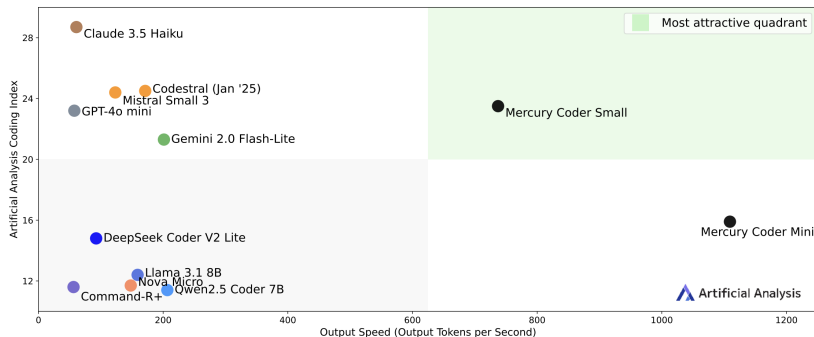
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- ▶ **Robustness:** diffusion avoids the "exposure bias" caused by teacher forcing in AR training.
- ▶ **Unified framework:** diffusion generalizes naturally to discrete domains that do not suit continuous Gaussian noise.

2025 – Big Bang of Discrete Diffusion Models

Coding Index vs. Output Speed: Smaller models

Artificial Analysis Coding Index (represents the average of LiveCodeBench & SciCode);
Output Speed: Output Tokens per Second; 1,000 Input Tokens; Coding focused workload



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Forward Discrete Process

Reverse Discrete Diffusion

Forward Discrete Process

Continuous Diffusion Markov Chain

In continuous diffusion, the forward Markov chain is defined by progressively corrupting data with Gaussian noise:

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I}).$$

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Discrete Diffusion Markov Chain

For discrete data, we instead define a Markov chain over categorical states:

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$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \text{Cat}(\mathbf{Q}_t \mathbf{x}_{t-1}),$$

- ▶ Each $\mathbf{x}_t \in \{0, 1\}^K$ is a **one-hot vector** encoding the categorical state (it is just one token).
- ▶ What is the transition matrix \mathbf{Q}_t ?

Forward Process over Time

Transition Matrix

$\mathbf{Q}_t \in [0, 1]^{K \times K}$ is a **transition matrix** where each column gives transition probabilities from one state to all others, and columns sum to 1:

$$[\mathbf{Q}_t]_{ij} = q(x_t = i | x_{t-1} = j), \quad \sum_{i=1}^K [\mathbf{Q}_t]_{ij} = 1.$$

Forward Process over Time

Transition Matrix

$\mathbf{Q}_t \in [0, 1]^{K \times K}$ is a **transition matrix** where each column gives transition probabilities from one state to all others, and columns sum to 1:

$$[\mathbf{Q}_t]_{ij} = q(x_t = i | x_{t-1} = j), \quad \sum_{i=1}^K [\mathbf{Q}_t]_{ij} = 1.$$

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- ▶ As $t \rightarrow T$, the process drives the data toward a stationary distribution.
- ▶ We design the transition matrices \mathbf{Q}_t to achieve this behavior.

Transition Matrix

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Common choices

- ▶ **Uniform diffusion**

$$\mathbf{Q}_t = (1 - \beta_t)\mathbf{I} + \beta_t\mathbf{U}, \quad \mathbf{U}_{ij} = \frac{1}{K}.$$

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- ▶ **Absorbing diffusion**

$$\mathbf{Q}_t = (1 - \beta_t)\mathbf{I} + \beta_t \mathbf{e}_m \mathbf{1}^\top.$$

Tokens are gradually replaced by a special mask m ; the stationary distribution is fully masked.

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$$q(\mathbf{x}_t | \mathbf{x}_0) = \text{Cat}(\mathbf{Q}_{1:t} \mathbf{x}_0), \quad \mathbf{Q}_{1:t} = \mathbf{Q}_t \mathbf{Q}_{t-1} \cdots \mathbf{Q}_1.$$

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- ▶ Each token retains its original value with prob. $\bar{\alpha}_t$.
- ▶ It becomes uniformly random with prob. $(1 - \bar{\alpha}_t)$.
- ▶ As $t \rightarrow T$, the process converges to the stationary uniform distribution.

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- ▶ Each token retains its original value with prob. $\bar{\alpha}_t$.
- ▶ It becomes \mathbf{e}_m with prob. $(1 - \bar{\alpha}_t)$.
- ▶ As $t \rightarrow T$, all tokens converge to the mask state:
 $q(\mathbf{x}_T) \approx \text{Cat}(\mathbf{e}_m)$.
- ▶ This makes the process analogous to **masked language modeling**.

Uniform vs. Absorbing Transition Matrix

Aspect	Uniform Diffusion	Absorbing Diffusion
\mathbf{Q}_t	$(1 - \beta_t)\mathbf{I} + \beta_t\mathbf{U}$	$(1 - \beta_t)\mathbf{I} + \beta_t\mathbf{e}_m\mathbf{1}^\top$
$\mathbf{Q}_{1:t}$	$\bar{\alpha}_t\mathbf{I} + (1 - \bar{\alpha}_t)\mathbf{U}$	$\bar{\alpha}_t\mathbf{I} + (1 - \bar{\alpha}_t)\mathbf{e}_m\mathbf{1}^\top$
$\mathbf{Q}_{1:\infty}$	\mathbf{U}	$\text{Cat}(\mathbf{e}_m)$
Interpretation	Random replacement	Gradual masking of tokens
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Observation

Both schemes gradually destroy information, but differ in their stationary limit. Absorbing diffusion bridges diffusion and masked-language-model objectives.

Outline

1. Link between Flow Matching and Score-Based Models

2. Discrete Diffusion Models

Forward Discrete Process

Reverse Discrete Diffusion

Posterior of the Forward Process

ELBO

$$\begin{aligned}\mathcal{L}_{\phi, \theta}(\mathbf{x}) = & \mathbb{E}_{q(\mathbf{x}_1|\mathbf{x}_0)} \log p_{\theta}(\mathbf{x}_0|\mathbf{x}_1) - \text{KL}(q(\mathbf{x}_T|\mathbf{x}_0) \| p(\mathbf{x}_T)) - \\ & - \sum_{t=2}^T \underbrace{\mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} \text{KL}(q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0) \| p_{\theta}(\mathbf{x}_{t-1}|\mathbf{x}_t))}_{\mathcal{L}_t}\end{aligned}$$

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- ▶ Conditioned reverse distribution $q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0)$ played crucial role in the continuous-state diffusion model.
- ▶ It shows the probability of a previous state given the noisy state \mathbf{x}_t and the original clean data \mathbf{x}_0 .

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Discrete conditioned reverse distribution

$$\begin{aligned}q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0) &= \frac{q(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{x}_0) q(\mathbf{x}_{t-1}|\mathbf{x}_0)}{q(\mathbf{x}_t|\mathbf{x}_0)} \\ &= \frac{\text{Cat}(\mathbf{Q}_t) \cdot \text{Cat}(\mathbf{Q}_{1:t-1})}{\text{Cat}(\mathbf{Q}_{1:t})}.\end{aligned}$$

Posterior of the Forward Process

Discrete conditioned reverse distribution

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Recall the ELBO term

$$\mathcal{L}_t = \mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} \text{KL}(q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0) \parallel p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t)),$$

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- ▶ Both $q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0)$ and $q(\mathbf{x}_t|\mathbf{x}_0)$ are known analytically from the forward process.
- ▶ The reverse process $p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t)$ is a learned categorical distribution:

$$p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) = \text{Cat}(\boldsymbol{\pi}_\theta(\mathbf{x}_t, t)),$$

where $\boldsymbol{\pi}_\theta$ is a neural network.

Discrete-time ELBO for Discrete Diffusion

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Categorical KL

$$\text{KL}(\text{Cat}(\mathbf{q}) \parallel \text{Cat}(\mathbf{p})) = \sum_{k=1}^K q_k \log \frac{q_k}{p_k} = H(\mathbf{q}, \mathbf{p}) - H(\mathbf{q}),$$

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Therefore, minimizing \mathcal{L}_t w.r.t. θ is equivalent to minimizing

$$\mathbb{E}_{q(\mathbf{x}_t|\mathbf{x}_0)} H(q(\mathbf{x}_{t-1}|\mathbf{x}_t, \mathbf{x}_0), p_{\theta}(\mathbf{x}_{t-1}|\mathbf{x}_t)).$$

Summary

- ▶ Diffusion and score-based models are special cases of the flow matching approach, but use curved trajectories.
- ▶ Diffusion approach has several key advantages over autoregressive approach.
- ▶ Forward discrete diffusion process defines Markov chain with discrete states.
- ▶ There are several ways to make it tractable (uniform / absorbing transitions).
- ▶ Reverse discrete diffusion process uses the variational approach to invert forward process.
- ▶ Discrete-state ELBO for discrete diffusion is a cross-entropy loss.