

# Deep Generative Models

## Lecture 7

Roman Isachenko

Moscow Institute of Physics and Technology  
Yandex School of Data Analysis

2025, Autumn

## Recap of Previous Lecture

### Frechet Inception Distance (FID)

For normal distributions  $p_{\text{data}}(\mathbf{x}_1) = \mathcal{N}(\boldsymbol{\mu}_{\text{data}}, \boldsymbol{\Sigma}_{\text{data}})$ ,  
 $p(\mathbf{x}_2) = \mathcal{N}(\boldsymbol{\mu}_{\theta}, \boldsymbol{\Sigma}_{\theta})$ :

$$\begin{aligned}\text{FID}(p_{\text{data}}, p_{\theta}) &= W_2^2(p_{\text{data}}, p_{\theta}) = \inf_{\gamma \in \Gamma(p_{\text{data}}, p_{\theta})} \mathbb{E}_{(\mathbf{x}_1, \mathbf{x}_2) \sim \gamma} \|\mathbf{x}_1 - \mathbf{x}_2\|^2 \\ &= \|\boldsymbol{\mu}_{\text{data}} - \boldsymbol{\mu}_{\theta}\|^2 + \text{tr} \left[ \boldsymbol{\Sigma}_{\text{data}} + \boldsymbol{\Sigma}_{\theta} - 2 \left( \boldsymbol{\Sigma}_{\text{data}}^{1/2} \boldsymbol{\Sigma}_{\theta} \boldsymbol{\Sigma}_{\text{data}}^{1/2} \right)^{1/2} \right]\end{aligned}$$

- ▶ Requires a large sample size for evaluation
- ▶ FID computation is relatively slow
- ▶ Results are highly dependent on the chosen pretrained classifier
- ▶ Relies on the normality assumption

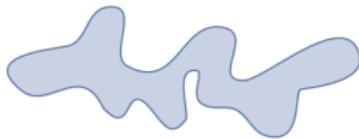
## Recap of Previous Lecture

- ▶  $\mathcal{S}_{\text{data}} = \{\mathbf{x}_i\}_{i=1}^n \sim p_{\text{data}}(\mathbf{x})$ : real samples
- ▶  $\mathcal{S}_{\theta} = \{\mathbf{x}_i\}_{i=1}^n \sim p_{\theta}(\mathbf{x})$ : generated samples

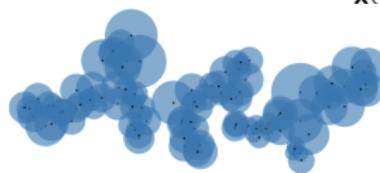
Define a binary indicator function:

$$\mathbb{I}(\mathbf{x}, \mathcal{S}) = \begin{cases} 1, & \text{if } \exists \mathbf{x}' \in \mathcal{S} : \|\mathbf{x} - \mathbf{x}'\|_2 \leq \|\mathbf{x}' - \text{NN}_k(\mathbf{x}', \mathcal{S})\|_2; \\ 0, & \text{otherwise} \end{cases}$$

$$\Pr(\mathcal{S}_{\text{data}}, \mathcal{S}_{\theta}) = \frac{1}{n} \sum_{\mathbf{x} \in \mathcal{S}_{\theta}} \mathbb{I}(\mathbf{x}, \mathcal{S}_{\text{data}}); \quad \text{Rec}(\mathcal{S}_{\text{data}}, \mathcal{S}_{\theta}) = \frac{1}{n} \sum_{\mathbf{x} \in \mathcal{S}_{\text{data}}} \mathbb{I}(\mathbf{x}, \mathcal{S}_{\theta}).$$



(a) True manifold



(b) Approx. manifold

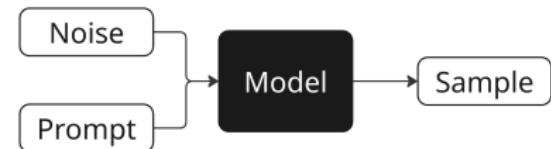
The samples are embedded using a pretrained network, as in FID evaluation.

# Recap of Previous Lecture

## Unconditional Model

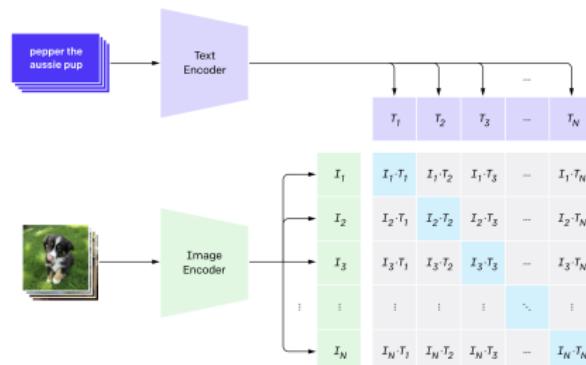


## Conditional Model



We require metrics that evaluate not only the quality of generated images, but also their relevance to the prompt.

## CLIP Score



# Recap of Previous Lecture

- ▶ No perfect automatic evaluation metric exists
- ▶ The most reliable assessment is via human evaluation
- ▶ It's important to evaluate a variety of model aspects

## Human Evaluation

Аспект	Yandex ART 2.0	Mj 6.1	Mj 6	Ideogram	Recraft	Google Imagen3	Dall-E 3	FLUX	SBER Kandi3.1
Релевантность	<b>0,59</b>	<b>0,58</b>	<b>0,63</b>	<b>0,45</b>	<b>0,51</b>	<b>0,50</b>	<b>0,50</b>	<b>0,54</b>	<b>0,75</b>
Эстетика	<b>0,49</b>	<b>0,55</b>	<b>0,55</b>	<b>0,51</b>	<b>0,51</b>	<b>0,61</b>	<b>0,61</b>	<b>0,54</b>	<b>0,59</b>
Комплексность	<b>0,44</b>	<b>0,73</b>	<b>0,70</b>	<b>0,68</b>	<b>0,76</b>	<b>0,75</b>	<b>0,75</b>	<b>0,71</b>	<b>0,74</b>
Дефектность	<b>0,69</b>	<b>0,57</b>	<b>0,68</b>	<b>0,55</b>	<b>0,59</b>	<b>0,63</b>	<b>0,63</b>	<b>0,50</b>	<b>0,75</b>
Предпочтение	<b>0,66</b>	<b>0,60</b>	<b>0,69</b>	<b>0,49</b>	<b>0,54</b>	<b>0,63</b>	<b>0,63</b>	<b>0,51</b>	<b>0,84</b>

# Recap of Previous Lecture

## Langevin Dynamics

Let  $\mathbf{x}_0$  be a random vector. Under mild regularity conditions, samples from the following dynamics will eventually follow  $p_\theta(\mathbf{x})$  (for sufficiently small  $\eta$  and large  $I$ ):

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p_\theta(\mathbf{x}_I) + \sqrt{\eta} \cdot \boldsymbol{\epsilon}_I, \quad \boldsymbol{\epsilon}_I \sim \mathcal{N}(0, \mathbf{I}).$$

- ▶ The density  $p_\theta(\mathbf{x})$  is the **stationary** distribution of the Markov chain.
- ▶ The gradient is taken with respect to  $\mathbf{x}$ , not  $\theta$ .
- ▶  $\nabla_{\mathbf{x}} \log p_\theta(\mathbf{x})$  defines a vector field.

## Fisher Divergence

$$D_F(p_{\text{data}}, p_\theta) = \frac{1}{2} \mathbb{E}_\pi \| \nabla_{\mathbf{x}} \log p_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x}) \|_2^2 \rightarrow \min_{\theta}$$

# Recap of Previous Lecture

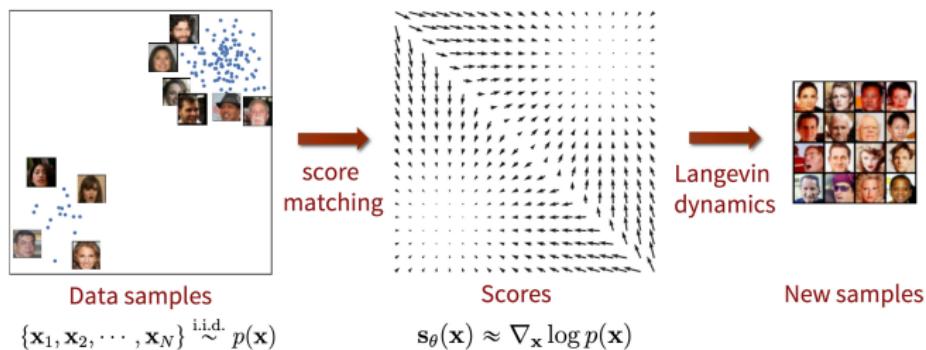
Define **score function**  $\mathbf{s}_\theta(\mathbf{x}) = \nabla_{\mathbf{x}} \log p_\theta(\mathbf{x})$ .

## Training (Score matching)

$$D_F(p_{\text{data}}, p_\theta) = \frac{1}{2} \mathbb{E}_\pi \| \mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x}) \|_2^2 \rightarrow \min_{\theta}$$

## Sampling (Langevin Dynamics)

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \mathbf{s}_\theta(\mathbf{x}) + \sqrt{\eta} \cdot \boldsymbol{\epsilon}_I, \quad \boldsymbol{\epsilon}_I \sim \mathcal{N}(0, \mathbf{I})$$



## Recap of Previous Lecture

Let us perturb the original data with Gaussian noise  
 $q(\mathbf{x}_\sigma | \mathbf{x}) = \mathcal{N}(\mathbf{x}, \sigma^2 \cdot \mathbf{I})$ .

$$q(\mathbf{x}_\sigma) = \int q(\mathbf{x}_\sigma | \mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x}.$$

Then the solution of

$$\frac{1}{2} \mathbb{E}_{q(\mathbf{x}_\sigma)} \| \mathbf{s}_{\theta, \sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma) \|_2^2 \rightarrow \min_{\theta}$$

satisfies  $\mathbf{s}_{\theta, \sigma}(\mathbf{x}_\sigma) \approx \mathbf{s}_{\theta, 0}(\mathbf{x}_0) = \mathbf{s}_\theta(\mathbf{x})$  if  $\sigma$  is sufficiently small.

**Theorem (Denoising Score Matching)**

$$\begin{aligned} & \mathbb{E}_{q(\mathbf{x}_\sigma)} \| \mathbf{s}_{\theta, \sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma) \|_2^2 = \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma | \mathbf{x})} \| \mathbf{s}_{\theta, \sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma | \mathbf{x}) \|_2^2 + \text{const}(\theta) \end{aligned}$$

Here,  $\nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma | \mathbf{x}) = -\frac{\mathbf{x}_\sigma - \mathbf{x}}{\sigma^2} = -\frac{\epsilon}{\sigma}$ .  $\mathbf{s}_{\theta, \sigma}(\mathbf{x}_\sigma)$  attempts to **denoise** a corrupted sample.

# Outline

## 1. Score Matching

Denoising Score Matching (continued)

Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

# Outline

## 1. Score Matching

Denoising Score Matching (continued)

Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

# Outline

## 1. Score Matching

Denoising Score Matching (continued)

Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

# Denoising Score Matching

## Theorem

$$\begin{aligned} & \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} = \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) = \int q(\mathbf{x}_\sigma) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) &= \int q(\mathbf{x}_\sigma) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \\ &= \int \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma) \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) &= \int q(\mathbf{x}_\sigma) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \\ &= \int \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma) \\ \mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) &= \int q(\mathbf{x}_\sigma) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \\ &= \int \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) h(\mathbf{x}_\sigma) d\mathbf{x}_\sigma = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma) \\ \mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ &= \mathbb{E}_{q(\mathbf{x}_\sigma)} \left[ \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma)\|^2 + \underbrace{\|\nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{\text{const}(\theta)} - 2\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma) \right] \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta)\end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)]$$

# Denoising Score Matching

## Theorem

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta)\end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] = \int q(\mathbf{x}_\sigma) \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \frac{\nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma)}{q(\mathbf{x}_\sigma)} \right] d\mathbf{x}_\sigma$$

# Denoising Score Matching

## Theorem

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta)\end{aligned}$$

## Proof (Continued)

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] &= \int q(\mathbf{x}_\sigma) \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \frac{\nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma)}{q(\mathbf{x}_\sigma)} \right] d\mathbf{x}_\sigma = \\ &= \int \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) \right] d\mathbf{x}_\sigma\end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta)\end{aligned}$$

## Proof (Continued)

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] &= \int q(\mathbf{x}_\sigma) \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \frac{\nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma)}{q(\mathbf{x}_\sigma)} \right] d\mathbf{x}_\sigma = \\ &= \int \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) \right] d\mathbf{x}_\sigma = \\ &= \int \int p_{\text{data}}(\mathbf{x}) [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma|\mathbf{x})] d\mathbf{x}_\sigma d\mathbf{x}\end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta)\end{aligned}$$

## Proof (Continued)

$$\begin{aligned}\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] &= \int q(\mathbf{x}_\sigma) \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \frac{\nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma)}{q(\mathbf{x}_\sigma)} \right] d\mathbf{x}_\sigma = \\ &= \int \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) \right] d\mathbf{x}_\sigma = \\ &= \int \int p_{\text{data}}(\mathbf{x}) [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma|\mathbf{x})] d\mathbf{x}_\sigma d\mathbf{x} = \\ &= \int \int p_{\text{data}}(\mathbf{x}) q(\mathbf{x}_\sigma|\mathbf{x}) [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})] d\mathbf{x}_\sigma d\mathbf{x}\end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof (Continued)

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] &= \int q(\mathbf{x}_\sigma) \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \frac{\nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma)}{q(\mathbf{x}_\sigma)} \right] d\mathbf{x}_\sigma = \\ &= \int \left[ \mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \left( \int q(\mathbf{x}_\sigma|\mathbf{x}) p_{\text{data}}(\mathbf{x}) d\mathbf{x} \right) \right] d\mathbf{x}_\sigma = \\ &= \int \int p_{\text{data}}(\mathbf{x}) [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} q(\mathbf{x}_\sigma|\mathbf{x})] d\mathbf{x}_\sigma d\mathbf{x} = \\ &= \int \int p_{\text{data}}(\mathbf{x}) q(\mathbf{x}_\sigma|\mathbf{x}) [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})] d\mathbf{x}_\sigma d\mathbf{x} = \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})] \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma)$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma)$$

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})]$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma)$$

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})]$$

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma)\|^2 - 2\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})] + \text{const}(\theta) \end{aligned}$$

# Denoising Score Matching

## Theorem

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \underbrace{\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2}_{h(\mathbf{x}_\sigma)} &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

## Proof (Continued)

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} h(\mathbf{x}_\sigma) = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} h(\mathbf{x}_\sigma)$$

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)] = \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})]$$

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} [\|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma)\|^2 - 2\mathbf{s}_{\theta,\sigma}^T(\mathbf{x}_\sigma) \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})] + \text{const}(\theta) = \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

# Denoising Score Matching

Original objective:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \|\mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

# Denoising Score Matching

Original objective:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \|\mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Noisy objective:

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}} \log q(\mathbf{x}_\sigma)\|_2^2 \rightarrow \min_{\theta}$$

# Denoising Score Matching

Original objective:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \|\mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Noisy objective:

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}} \log q(\mathbf{x}_\sigma)\|_2^2 \rightarrow \min_{\theta}$$

This is equivalent to a denoising task:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

# Denoising Score Matching

Original objective:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \|\mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Noisy objective:

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}} \log q(\mathbf{x}_\sigma)\|_2^2 \rightarrow \min_{\theta}$$

This is equivalent to a denoising task:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{\mathcal{N}(0, \mathbf{I})} \left\| \mathbf{s}_{\theta,\sigma}(\mathbf{x} + \sigma\epsilon) + \frac{\epsilon}{\sigma} \right\|_2^2 \rightarrow \min_{\theta}$$

# Denoising Score Matching

Original objective:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \|\mathbf{s}_\theta(\mathbf{x}) - \nabla_{\mathbf{x}} \log p_{\text{data}}(\mathbf{x})\|_2^2 \rightarrow \min_\theta$$

Noisy objective:

$$\mathbb{E}_{q(\mathbf{x}_\sigma)} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}} \log q(\mathbf{x}_\sigma)\|_2^2 \rightarrow \min_\theta$$

This is equivalent to a denoising task:

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_\sigma|\mathbf{x})} \|\mathbf{s}_{\theta,\sigma}(\mathbf{x}_\sigma) - \nabla_{\mathbf{x}_\sigma} \log q(\mathbf{x}_\sigma|\mathbf{x})\|_2^2 \rightarrow \min_\theta$$

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{\mathcal{N}(0, \mathbf{I})} \left\| \mathbf{s}_{\theta,\sigma}(\mathbf{x} + \sigma\epsilon) + \frac{\epsilon}{\sigma} \right\|_2^2 \rightarrow \min_\theta$$

## Langevin Dynamics

$$\mathbf{x}_{l+1} = \mathbf{x}_l + \frac{\eta}{2} \cdot \mathbf{s}_{\theta,\sigma}(\mathbf{x}_l) + \sqrt{\eta} \cdot \boldsymbol{\epsilon}_l, \quad \boldsymbol{\epsilon}_l \sim \mathcal{N}(0, \mathbf{I})$$

# Outline

## 1. Score Matching

Denoising Score Matching (continued)

Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

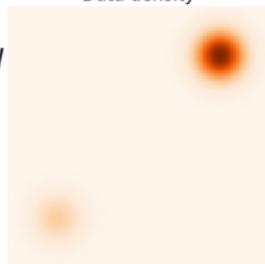
# Denoising Score Matching

$$\mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{\mathcal{N}(0, \mathbf{I})} \left\| \mathbf{s}_{\theta, \sigma}(\mathbf{x} + \sigma \epsilon) + \frac{\epsilon}{\sigma} \right\|_2^2 \rightarrow \min_{\theta}$$

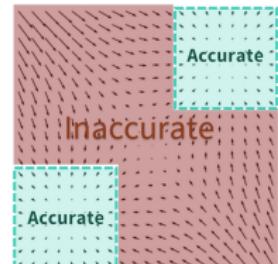
$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \mathbf{s}_{\theta, \sigma}(\mathbf{x}_I) + \sqrt{\eta} \cdot \epsilon_I$$

- ▶ For **small**  $\sigma$ ,  $\mathbf{s}_{\theta, \sigma}(\mathbf{x})$  becomes inaccurate and Langevin dynamics fails to traverse modes
- ▶ For **large**  $\sigma$ , robustness in low-density regions is achieved, but the model learns a distribution that is overly corrupted

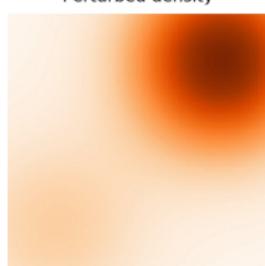
Data density



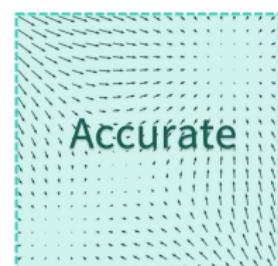
Estimated scores



Perturbed density



Estimated scores



## Noise-Conditioned Score Network (NCSN)

- ▶ Specify a sequence of noise levels:  $\sigma_1 < \sigma_2 < \dots < \sigma_T$
- ▶ Perturb each data point with different noise levels:  
 $\mathbf{x}_t = \mathbf{x} + \sigma_t \boldsymbol{\epsilon}$ , so  $\mathbf{x}_t \sim q(\mathbf{x}_t)$
- ▶ Choose  $\sigma_1, \sigma_T$  such that:

$$q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

# Noise-Conditioned Score Network (NCSN)

- ▶ Specify a sequence of noise levels:  $\sigma_1 < \sigma_2 < \dots < \sigma_T$
- ▶ Perturb each data point with different noise levels:  
 $\mathbf{x}_t = \mathbf{x} + \sigma_t \epsilon$ , so  $\mathbf{x}_t \sim q(\mathbf{x}_t)$
- ▶ Choose  $\sigma_1, \sigma_T$  such that:

$$q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

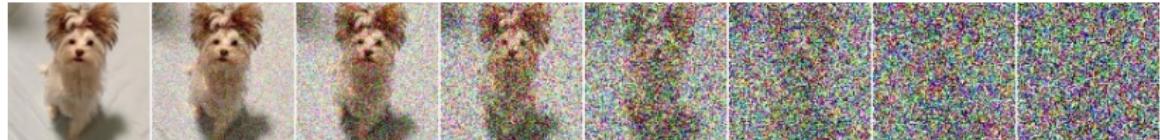
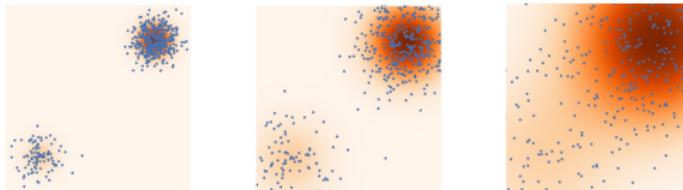
$$\sigma_1$$

$$<$$

$$\sigma_2$$

$$<$$

$$\sigma_3$$



## Noise-Conditioned Score Network (NCSN)

Train the denoising score function  $s_{\theta, \sigma_t}(\mathbf{x}_t)$  for each noise level  $\sigma_t$  using a unified weighted objective:

$$\sum_{t=1}^T \sigma_t^2 \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|s_{\theta, \sigma_t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

## Noise-Conditioned Score Network (NCSN)

Train the denoising score function  $s_{\theta, \sigma_t}(\mathbf{x}_t)$  for each noise level  $\sigma_t$  using a unified weighted objective:

$$\sum_{t=1}^T \sigma_t^2 \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|s_{\theta, \sigma_t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Here,  $\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2} = -\frac{\epsilon}{\sigma_t}$

# Noise-Conditioned Score Network (NCSN)

Train the denoising score function  $s_{\theta, \sigma_t}(\mathbf{x}_t)$  for each noise level  $\sigma_t$  using a unified weighted objective:

$$\sum_{t=1}^T \sigma_t^2 \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|s_{\theta, \sigma_t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Here,  $\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2} = -\frac{\epsilon}{\sigma_t}$

## Training

1. Sample  $\mathbf{x}_0 \sim p_{\text{data}}(\mathbf{x})$
2. Sample  $t \sim U\{1, T\}$  and  $\epsilon \sim \mathcal{N}(0, \mathbf{I})$
3. Construct noisy image  $\mathbf{x}_t = \mathbf{x}_0 + \sigma_t \epsilon$
4. Evaluate loss  $\mathcal{L} = \sigma_t^2 \left\| s_{\theta, \sigma_t}(\mathbf{x}_t) + \frac{\epsilon}{\sigma_t} \right\|^2$

# Noise-Conditioned Score Network (NCSN)

Train the denoising score function  $s_{\theta, \sigma_t}(\mathbf{x}_t)$  for each noise level  $\sigma_t$  using a unified weighted objective:

$$\sum_{t=1}^T \sigma_t^2 \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|s_{\theta, \sigma_t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 \rightarrow \min_{\theta}$$

Here,  $\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2} = -\frac{\epsilon}{\sigma_t}$

## Training

1. Sample  $\mathbf{x}_0 \sim p_{\text{data}}(\mathbf{x})$
2. Sample  $t \sim U\{1, T\}$  and  $\epsilon \sim \mathcal{N}(0, \mathbf{I})$
3. Construct noisy image  $\mathbf{x}_t = \mathbf{x}_0 + \sigma_t \epsilon$
4. Evaluate loss  $\mathcal{L} = \sigma_t^2 \left\| s_{\theta, \sigma_t}(\mathbf{x}_t) + \frac{\epsilon}{\sigma_t} \right\|^2$

How do we sample from such a model?

# Noise-Conditioned Score Network (NCSN)

## Sampling (Annealed Langevin Dynamics)

- ▶ Sample initial point  $\mathbf{x}_0 \sim \mathcal{N}(0, \sigma_T^2 \mathbf{I}) \approx q(\mathbf{x}_T)$
- ▶ At each noise level, apply  $L$  steps of Langevin dynamics:

$$\mathbf{x}_l = \mathbf{x}_{l-1} + \frac{\eta_t}{2} \mathbf{s}_{\theta, \sigma_t}(\mathbf{x}_{l-1}) + \sqrt{\eta_t} \boldsymbol{\epsilon}_l,$$

- ▶ Update  $\mathbf{x}_0 := \mathbf{x}_L$  and reduce to the next lower  $\sigma_t$

# Noise-Conditioned Score Network (NCSN)

## Sampling (Annealed Langevin Dynamics)

- ▶ Sample initial point  $\mathbf{x}_0 \sim \mathcal{N}(0, \sigma_T^2 \mathbf{I}) \approx q(\mathbf{x}_T)$
- ▶ At each noise level, apply  $L$  steps of Langevin dynamics:

$$\mathbf{x}_l = \mathbf{x}_{l-1} + \frac{\eta_t}{2} \mathbf{s}_{\theta, \sigma_t}(\mathbf{x}_{l-1}) + \sqrt{\eta_t} \boldsymbol{\epsilon}_l,$$

- ▶ Update  $\mathbf{x}_0 := \mathbf{x}_L$  and reduce to the next lower  $\sigma_t$



# Outline

## 1. Score Matching

Denoising Score Matching (continued)  
Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

## Forward Gaussian Diffusion Process

Let  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$ ,  $\beta_t \ll 1$ . Define a Markov chain:

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

# Forward Gaussian Diffusion Process

Let  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$ ,  $\beta_t \ll 1$ . Define a Markov chain:

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

# Forward Gaussian Diffusion Process

Let  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$ ,  $\beta_t \ll 1$ . Define a Markov chain:

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Langevin Dynamics

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p_{\theta}(\mathbf{x}_I) + \sqrt{\eta} \boldsymbol{\epsilon}_I, \quad \boldsymbol{\epsilon}_I \sim \mathcal{N}(0, \mathbf{I})$$

# Forward Gaussian Diffusion Process

Let  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$ ,  $\beta_t \ll 1$ . Define a Markov chain:

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Langevin Dynamics

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p_{\theta}(\mathbf{x}_I) + \sqrt{\eta} \boldsymbol{\epsilon}_I, \quad \boldsymbol{\epsilon}_I \sim \mathcal{N}(0, \mathbf{I})$$

$$\begin{aligned} \mathbf{x}_t &= \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t \approx \left(1 - \frac{\beta_t}{2}\right) \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t = \\ &= \mathbf{x}_{t-1} + \frac{\beta_t}{2} (-\mathbf{x}_{t-1}) + \sqrt{\beta_t} \boldsymbol{\epsilon}_t \end{aligned}$$

# Forward Gaussian Diffusion Process

Let  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$ ,  $\beta_t \ll 1$ . Define a Markov chain:

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Langevin Dynamics

$$\mathbf{x}_{I+1} = \mathbf{x}_I + \frac{\eta}{2} \cdot \nabla_{\mathbf{x}_I} \log p_{\theta}(\mathbf{x}_I) + \sqrt{\eta} \boldsymbol{\epsilon}_I, \quad \boldsymbol{\epsilon}_I \sim \mathcal{N}(0, \mathbf{I})$$

$$\begin{aligned} \mathbf{x}_t &= \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t \approx \left(1 - \frac{\beta_t}{2}\right) \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t = \\ &= \mathbf{x}_{t-1} + \frac{\beta_t}{2} (-\mathbf{x}_{t-1}) + \sqrt{\beta_t} \boldsymbol{\epsilon}_t \end{aligned}$$

- ▶  $\beta_t = \eta$
- ▶  $\nabla_{\mathbf{x}_{t-1}} \log p(\mathbf{x}_{t-1} | \theta) = -\mathbf{x}_{t-1} = \nabla_{\mathbf{x}_{t-1}} \log \mathcal{N}(0, \mathbf{I})$

## Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

Thus, samples at any timestep  $t$  can be generated directly from  $\mathbf{x}_0$

$$\mathbf{x}_t = \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

Thus, samples at any timestep  $t$  can be generated directly from  $\mathbf{x}_0$

$$\begin{aligned}\mathbf{x}_t &= \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t} (\sqrt{\alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_{t-1}} \boldsymbol{\epsilon}_{t-1}) + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t\end{aligned}$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

Thus, samples at any timestep  $t$  can be generated directly from  $\mathbf{x}_0$

$$\begin{aligned}\mathbf{x}_t &= \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t} (\sqrt{\alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_{t-1}} \boldsymbol{\epsilon}_{t-1}) + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t \alpha_{t-1}} \mathbf{x}_{t-2} + (\sqrt{\alpha_t (1 - \alpha_{t-1})} \boldsymbol{\epsilon}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t)\end{aligned}$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

Thus, samples at any timestep  $t$  can be generated directly from  $\mathbf{x}_0$

$$\begin{aligned}\mathbf{x}_t &= \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t} (\sqrt{\alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_{t-1}} \boldsymbol{\epsilon}_{t-1}) + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t \alpha_{t-1}} \mathbf{x}_{t-2} + (\sqrt{\alpha_t (1 - \alpha_{t-1})} \boldsymbol{\epsilon}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t) = \\ &= \sqrt{\alpha_t \alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_t \alpha_{t-1}} \boldsymbol{\epsilon}'_t\end{aligned}$$

# Forward Gaussian Diffusion Process

$$\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \mathbf{I})$$

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I})$$

## Statement 1

Let  $\alpha_t = 1 - \beta_t$  and  $\bar{\alpha}_t = \prod_{s=1}^t \alpha_s = \prod_{s=1}^t (1 - \beta_s)$ . Then

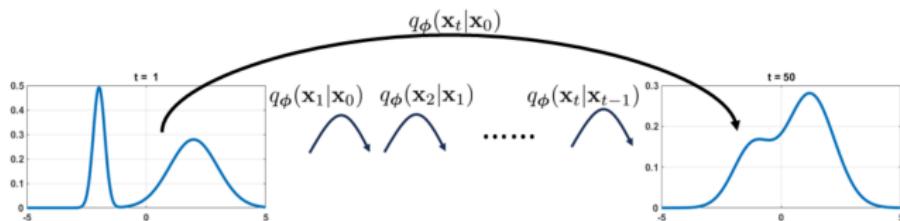
$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I})$$

Thus, samples at any timestep  $t$  can be generated directly from  $\mathbf{x}_0$

$$\begin{aligned}\mathbf{x}_t &= \sqrt{\alpha_t} \mathbf{x}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t} (\sqrt{\alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_{t-1}} \boldsymbol{\epsilon}_{t-1}) + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t = \\ &= \sqrt{\alpha_t \alpha_{t-1}} \mathbf{x}_{t-2} + (\sqrt{\alpha_t (1 - \alpha_{t-1})} \boldsymbol{\epsilon}_{t-1} + \sqrt{1 - \alpha_t} \boldsymbol{\epsilon}_t) = \\ &= \sqrt{\alpha_t \alpha_{t-1}} \mathbf{x}_{t-2} + \sqrt{1 - \alpha_t \alpha_{t-1}} \boldsymbol{\epsilon}'_t = \\ &= \dots = \sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I})\end{aligned}$$

# Forward Gaussian Diffusion Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I} \right); \quad q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N} \left( \sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I} \right)$$

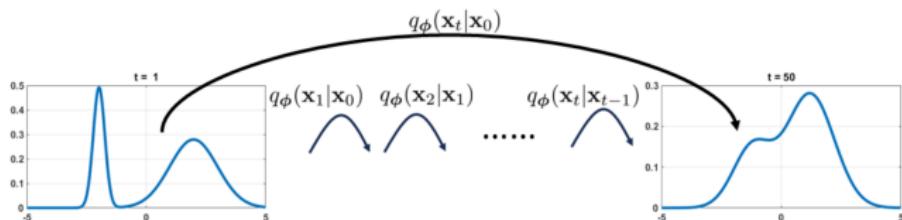


Chan S. Tutorial on Diffusion Models for Imaging and Vision, 2024

Sohl-Dickstein J. Deep Unsupervised Learning using Nonequilibrium Thermodynamics, 2015

# Forward Gaussian Diffusion Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I} \right); \quad q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N} \left( \sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I} \right)$$



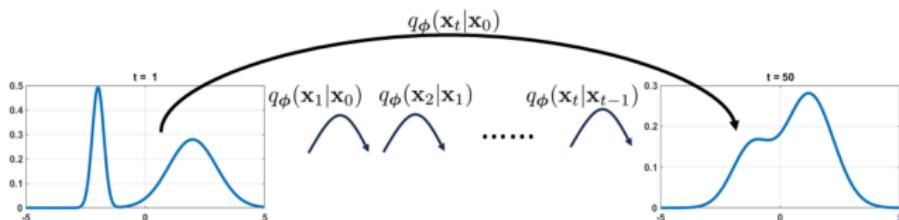
## Statement 2

Applying the Markov chain to any distribution  $p_{\text{data}}(\mathbf{x})$  yields  $\mathbf{x}_\infty \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$ , the **stationary** (limiting) distribution:

$$p_\infty(\mathbf{x}) = \int q(\mathbf{x} | \mathbf{x}') p_\infty(\mathbf{x}') d\mathbf{x}'$$

# Forward Gaussian Diffusion Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I} \right); \quad q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N} \left( \sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I} \right)$$



## Statement 2

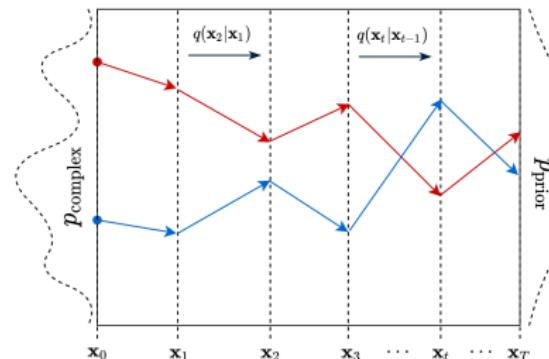
Applying the Markov chain to any distribution  $p_{\text{data}}(\mathbf{x})$  yields  $\mathbf{x}_\infty \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$ , the **stationary** (limiting) distribution:

$$p_\infty(\mathbf{x}) = \int q(\mathbf{x} | \mathbf{x}') p_\infty(\mathbf{x}') d\mathbf{x}'$$

$$p_\infty(\mathbf{x}) = \int q(\mathbf{x}_\infty | \mathbf{x}_0) p_{\text{data}}(\mathbf{x}_0) d\mathbf{x}_0 \approx \mathcal{N}(0, \mathbf{I}) \int p_{\text{data}}(\mathbf{x}_0) d\mathbf{x}_0 = \mathcal{N}(0, \mathbf{I})$$

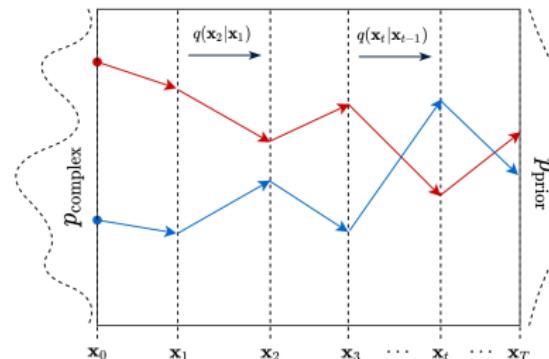
# Forward Gaussian Diffusion Process

**Diffusion** describes the migration of particles from regions of high density to those of low density.



# Forward Gaussian Diffusion Process

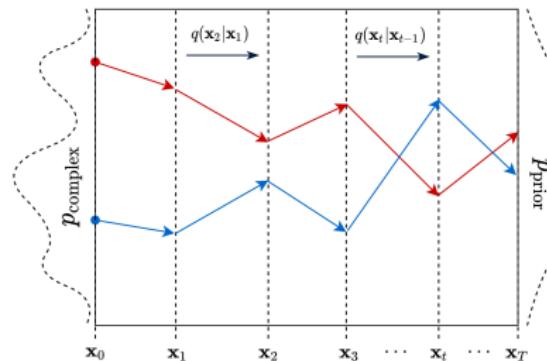
**Diffusion** describes the migration of particles from regions of high density to those of low density.



1.  $x_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$
2.  $\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}), t \geq 1$
3. After  $T \gg 1$  steps:  $\mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$

# Forward Gaussian Diffusion Process

**Diffusion** describes the migration of particles from regions of high density to those of low density.



1.  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$
2.  $\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{I}), t \geq 1$
3. After  $T \gg 1$  steps:  $\mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$

If this process can be reversed, we can sample from  $p_{\text{data}}(\mathbf{x})$  by starting from noise  $p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$ .

Our goal now becomes inverting this diffusion.

# Outline

## 1. Score Matching

- Denoising Score Matching (continued)
- Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

# Denoising Score Matching

## NCSN

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0, \sigma_t^2 \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2}$$

# Denoising Score Matching

## NCSN

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0, \sigma_t^2 \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2}$$

## Gaussian Diffusion

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}_0) = -\frac{\mathbf{x}_t - \sqrt{\bar{\alpha}_t} \mathbf{x}_0}{1 - \bar{\alpha}_t}$$

# Denoising Score Matching

## NCSN

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0, \sigma_t^2 \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2}$$

## Gaussian Diffusion

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}_0) = -\frac{\mathbf{x}_t - \sqrt{\bar{\alpha}_t} \mathbf{x}_0}{1 - \bar{\alpha}_t}$$

## Theorem (Denoising Score Matching)

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_t)} \|\mathbf{s}_{\theta,t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|\mathbf{s}_{\theta,t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

# Denoising Score Matching

## NCSN

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0, \sigma_t^2 \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \sigma_T^2 \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}) = -\frac{\mathbf{x}_t - \mathbf{x}}{\sigma_t^2}$$

## Gaussian Diffusion

$$q(\mathbf{x}_t | \mathbf{x}_0) = \mathcal{N}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0, (1 - \bar{\alpha}_t) \mathbf{I}), \quad q(\mathbf{x}_1) \approx p_{\text{data}}(\mathbf{x}), \quad q(\mathbf{x}_T) \approx \mathcal{N}(0, \mathbf{I})$$

$$\nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x}_0) = -\frac{\mathbf{x}_t - \sqrt{\bar{\alpha}_t} \mathbf{x}_0}{1 - \bar{\alpha}_t}$$

## Theorem (Denoising Score Matching)

$$\begin{aligned} \mathbb{E}_{q(\mathbf{x}_t)} \|\mathbf{s}_{\theta,t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t)\|_2^2 &= \\ &= \mathbb{E}_{p_{\text{data}}(\mathbf{x})} \mathbb{E}_{q(\mathbf{x}_t | \mathbf{x})} \|\mathbf{s}_{\theta,t}(\mathbf{x}_t) - \nabla_{\mathbf{x}_t} \log q(\mathbf{x}_t | \mathbf{x})\|_2^2 + \text{const}(\theta) \end{aligned}$$

**Note:** This enables applying the NCSN approach with annealed Langevin dynamics to diffusion-based denoising models.

# Outline

## 1. Score Matching

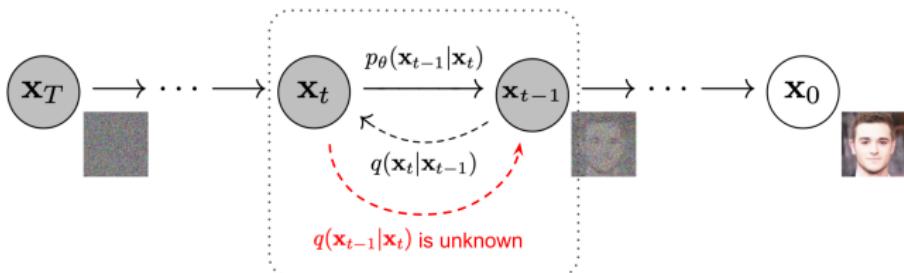
- Denoising Score Matching (continued)
- Noise-Conditioned Score Network

## 2. Forward Gaussian Diffusion Process

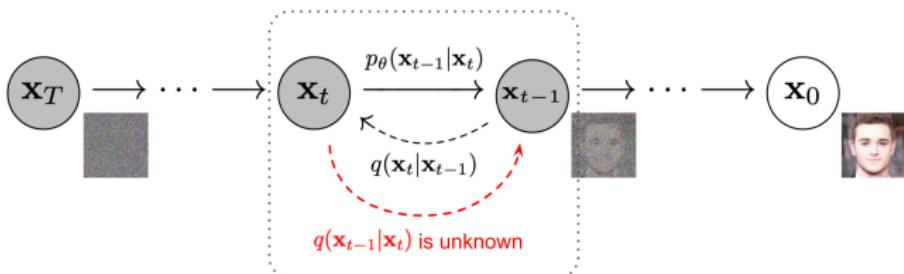
## 3. Denoising Score Matching for Diffusion

## 4. Reverse Gaussian Diffusion Process

# Reverse Gaussian Diffusion Process



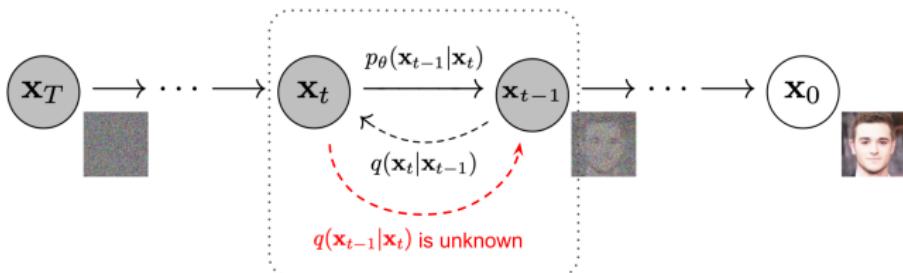
# Reverse Gaussian Diffusion Process



Forward Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I} \right)$$

# Reverse Gaussian Diffusion Process



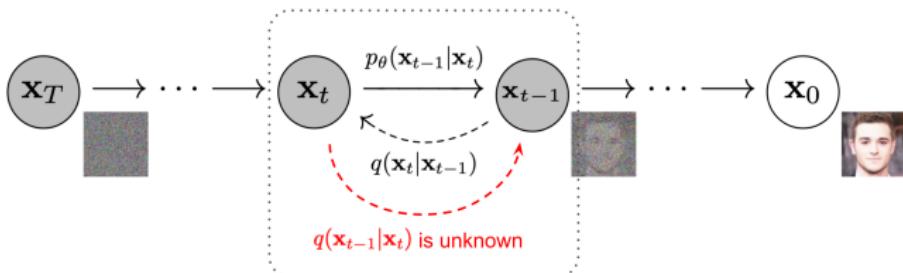
Forward Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N} \left( \sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I} \right)$$

Reverse Process

$$q(\mathbf{x}_{t-1} | \mathbf{x}_t) = \frac{q(\mathbf{x}_t | \mathbf{x}_{t-1}) q(\mathbf{x}_{t-1})}{q(\mathbf{x}_t)} \approx p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)$$

# Reverse Gaussian Diffusion Process



Forward Process

$$q(\mathbf{x}_t | \mathbf{x}_{t-1}) = \mathcal{N}\left(\sqrt{1 - \beta_t} \mathbf{x}_{t-1}, \beta_t \mathbf{I}\right)$$

Reverse Process

$$q(\mathbf{x}_{t-1} | \mathbf{x}_t) = \frac{q(\mathbf{x}_t | \mathbf{x}_{t-1}) q(\mathbf{x}_{t-1})}{q(\mathbf{x}_t)} \approx p_\theta(\mathbf{x}_{t-1} | \mathbf{x}_t)$$

$q(\mathbf{x}_{t-1})$  and  $q(\mathbf{x}_t)$  are intractable:

$$q(\mathbf{x}_t) = \int q(\mathbf{x}_t | \mathbf{x}_0) p_{\text{data}}(\mathbf{x}_0) d\mathbf{x}_0$$

# Reverse Gaussian Diffusion Process

$$q(\mathbf{x}_{t-1} | \mathbf{x}_t) = \frac{q(\mathbf{x}_t | \mathbf{x}_{t-1}) q(\mathbf{x}_{t-1})}{q(\mathbf{x}_t)}$$

Theorem (Feller, 1949)

If  $\beta_t$  is sufficiently small,  $q(\mathbf{x}_{t-1} | \mathbf{x}_t)$  is Gaussian (thus, diffusion requires  $T \approx 1000$  steps for convergence)

---

Feller W. *On the theory of stochastic processes, with particular reference to applications*, 1949

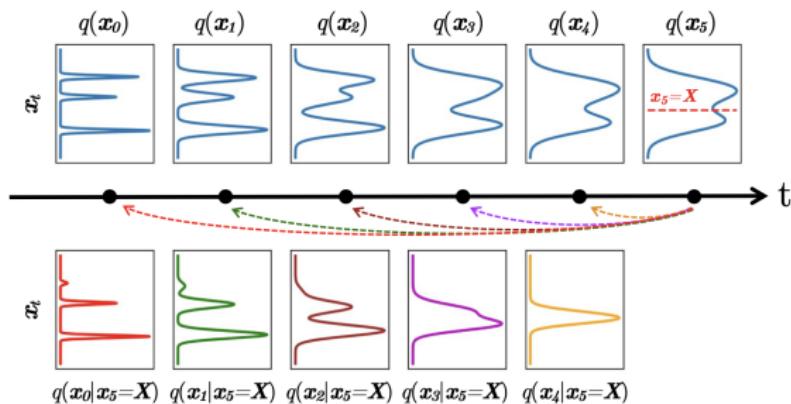
Xiao Z., Kreis K., Vahdat A. *Tackling the generative learning trilemma with denoising diffusion GANs*, 2021

# Reverse Gaussian Diffusion Process

$$q(\mathbf{x}_{t-1}|\mathbf{x}_t) = \frac{q(\mathbf{x}_t|\mathbf{x}_{t-1})q(\mathbf{x}_{t-1})}{q(\mathbf{x}_t)}$$

Theorem (Feller, 1949)

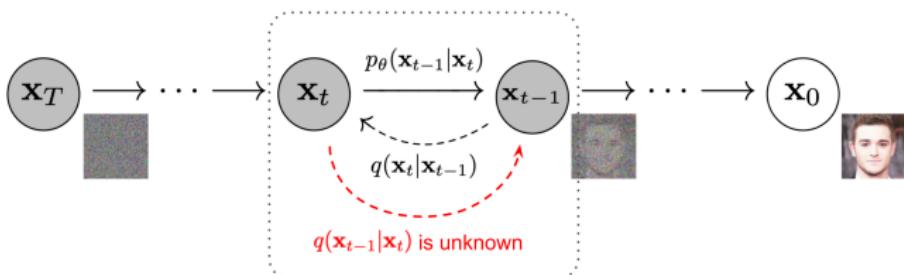
If  $\beta_t$  is sufficiently small,  $q(\mathbf{x}_{t-1}|\mathbf{x}_t)$  is Gaussian (thus, diffusion requires  $T \approx 1000$  steps for convergence)



Feller W. On the theory of stochastic processes, with particular reference to applications, 1949

Xiao Z., Kreis K., Vahdat A. Tackling the generative learning trilemma with denoising diffusion GANs, 2021

# Reverse Gaussian Diffusion Process (Ancestral Sampling)

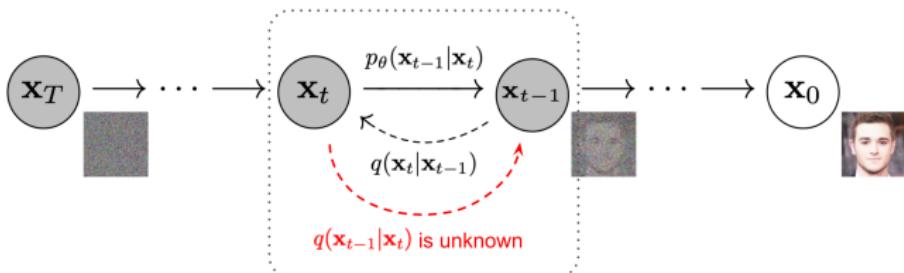


Define the reverse process as:

$$q(\mathbf{x}_{t-1}|\mathbf{x}_t) \approx p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) = \mathcal{N}(\boldsymbol{\mu}_{\theta,t}(\mathbf{x}_t), \sigma_{\theta,t}^2(\mathbf{x}_t))$$

Feller's theorem justifies this Gaussian assumption.

# Reverse Gaussian Diffusion Process (Ancestral Sampling)



Define the reverse process as:

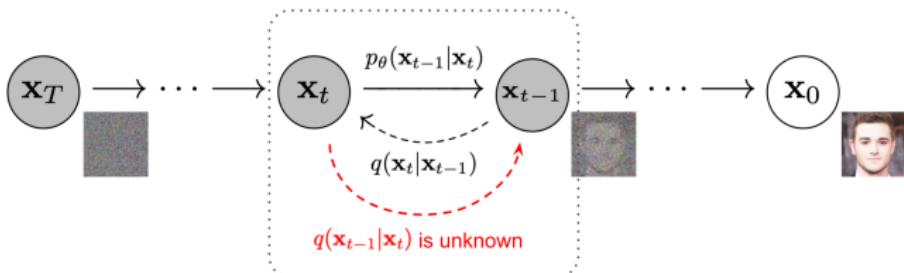
$$q(\mathbf{x}_{t-1}|\mathbf{x}_t) \approx p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) = \mathcal{N}(\boldsymbol{\mu}_{\theta,t}(\mathbf{x}_t), \sigma_{\theta,t}^2(\mathbf{x}_t))$$

Feller's theorem justifies this Gaussian assumption.

## Forward Process

1.  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$
2.  $\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}$
3.  $\mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$

# Reverse Gaussian Diffusion Process (Ancestral Sampling)



Define the reverse process as:

$$q(\mathbf{x}_{t-1}|\mathbf{x}_t) \approx p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) = \mathcal{N}(\mu_{\theta,t}(\mathbf{x}_t), \sigma_{\theta,t}^2(\mathbf{x}_t))$$

Feller's theorem justifies this Gaussian assumption.

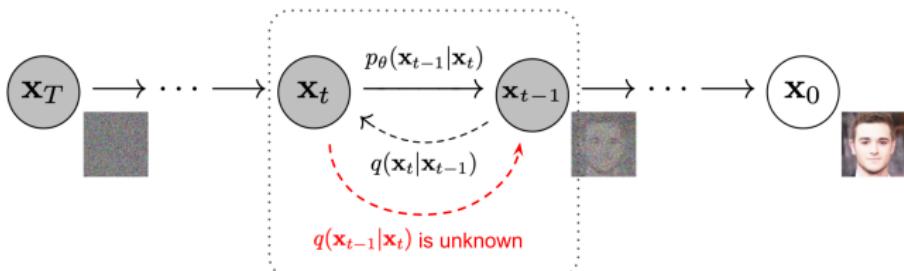
## Forward Process

1.  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$
2.  $\mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}$
3.  $\mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$

## Reverse Process

1.  $\mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$
2.  $\mathbf{x}_{t-1} = \sigma_{\theta,t}(\mathbf{x}_t) \boldsymbol{\epsilon} + \mu_{\theta,t}(\mathbf{x}_t)$
3.  $\mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$

# Reverse Gaussian Diffusion Process (Ancestral Sampling)



Define the reverse process as:

$$q(\mathbf{x}_{t-1}|\mathbf{x}_t) \approx p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) = \mathcal{N}(\mu_{\theta,t}(\mathbf{x}_t), \sigma_{\theta,t}^2(\mathbf{x}_t))$$

Feller's theorem justifies this Gaussian assumption.

## Forward Process

$$1. \quad \mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$$

$$2. \quad \mathbf{x}_t = \sqrt{1 - \beta_t} \mathbf{x}_{t-1} + \sqrt{\beta_t} \boldsymbol{\epsilon}$$

$$3. \quad \mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$$

## Reverse Process

$$1. \quad \mathbf{x}_T \sim p_\infty(\mathbf{x}) = \mathcal{N}(0, \mathbf{I})$$

$$2. \quad \mathbf{x}_{t-1} = \sigma_{\theta,t}(\mathbf{x}_t) \boldsymbol{\epsilon} + \mu_{\theta,t}(\mathbf{x}_t)$$

$$3. \quad \mathbf{x}_0 = \mathbf{x} \sim p_{\text{data}}(\mathbf{x})$$

**Note:** The forward process is non-learnable, i.e., it does not involve trainable parameters

## Summary

- ▶ Denoising score matching minimizes the Fisher divergence on corrupted samples, making the divergence estimable via sampling
- ▶ The noise-conditioned score network leverages a range of noise levels and annealed Langevin dynamics to learn the score function and enable sampling
- ▶ The Gaussian diffusion process is a Markov chain that incrementally corrupts data with carefully structured Gaussian noise
- ▶ Denoising score matching, together with Langevin dynamics, can be applied to the Gaussian diffusion process
- ▶ The reverse process reconstructs data from noise samples, although its precise form is intractable