The periodogram is given by:

$$U_N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-\jmath \omega t} \quad , \quad \omega = \frac{2k\pi}{N} \tag{1}$$

The following holds: $U_N(\omega + 2\pi) = U_N(\omega)$, $U_N(-\omega) = \bar{U}_N(\omega)$.

The autocorrelation (autocovariance) function is given by:

$$R_s(\tau) = \lim_{n \to \infty} \frac{1}{N} \sum_{t=1}^{N} s(t) s(t - \tau) \tag{2}$$

For real sequences, $R_s(\tau)$ is symmetric and real. The crosscorrelation (crosscovariance) function for real sequences is given by:

$$R_{sw}(\tau) = \lim_{n \to \infty} \frac{1}{N} \sum_{t=1}^{N} s(t) w(t - \tau) \tag{3}$$

For real sequences is real. For periodic sequences with period M, the autocorrelation function is:

$$R_s(\tau) = \frac{1}{M} \sum_{t=1}^{M} s(t) s(t - \tau) \tag{4}$$

The spectrum is defined as:

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau) e^{-\jmath \omega \tau} \tag{5}$$

$\Phi_s(\omega)$ is real while $\Phi_{sw}(\omega)$ is complex in general.

Theorem:

$$\begin{aligned}
\text{Given:} \quad y(t) &= G(q)u(t) + H(q)e(t) && (6) \\
\text{The following holds:} \quad \Phi_y(\omega) &= \left|G(e^{\jmath\omega})\right|^2 \Phi_u(\omega) + \lambda \left|H(e^{\jmath\omega})\right|^2 && (7) \\
\Phi_{yu}(\omega) &= G(e^{\jmath\omega}) \Phi_u(\omega) && (8)
\end{aligned}$$

The smoothed estimate of the transfer function is given by:

$$\hat{G}_N(e^{\jmath\omega}) = \frac{\hat{\Phi}_{yu}^N(\omega)}{\hat{\Phi}_u^N(\omega)} \tag{9}$$

where:

$$\hat{\Phi}_s^N(\omega) = \sum_{\tau=-\gamma}^{\gamma} W_\gamma(\tau) \hat{R}_s(\tau) e^{-\jmath\omega\tau} \quad \text{(see next page)} \tag{10}$$

The disturbance spectrum is given by:

$$\hat{\Phi}_v^N(\omega) = \hat{\Phi}_y^N(\omega) - \frac{\left|\hat{\Phi}_{yu}^N(\omega)\right|^2}{\hat{\Phi}_u^N(\omega)} \tag{11}$$

The coherence spectrum is defined as:

$$\kappa = \sqrt{\frac{\left|\hat{\Phi}_{yu}^N(\omega)\right|^2}{\hat{\Phi}_y^N(\omega) \hat{\Phi}_u^N(\omega)}} \tag{12}$$

Algorithmically, the process is as follows:

1. Select γ and the window function W_γ.
2. Calculate the estimation of the correlation function using periodic shifting:

$$\hat{R}_u^N(\tau) = \frac{1}{N}\sum_{t=1}^{N} u(t)u(t-\tau) \ , \quad \tau \in [-\gamma, \gamma] \tag{13}$$

3. Calculate the estimation of the spectrum:

$$\hat{\Phi}_u^N(\omega) = \sum_{\tau=-\gamma}^{\gamma} W_\gamma(\tau)\hat{R}_u(\tau)e^{-\jmath\omega\tau} \tag{14}$$

   Avoid using FFT directly on the windowed sequence.

4. Calculate estimations for $\hat{R}_{yu}^N$, $\hat{\Phi}_{yu}^N$, $\hat{R}_y^N$, and $\hat{\Phi}_y^N$
5. Obtain the smoothed estimation using (9), the disturbance spectrum using (11), and the coherence spectrum using (12).