Game theory

a course for the
MSc in ICT for Internet and multimedia

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Lotteries ⇒ randomness in giochi

How to introduce randomness

di base, no ⇒ modi per rimuoverla
in certi casi reali, può esserci

Random outcomes

- Assume of our payoff involves random parts
  - At the canteen, the "soup" is different every day (and there is no pattern). How do we tell if ravioli are preferable?
- Rational players do not like this randomness
  - They mess with preference order
  - and also with knowledge of the system (rationality also means ability to infer consequences)

Random outcomes

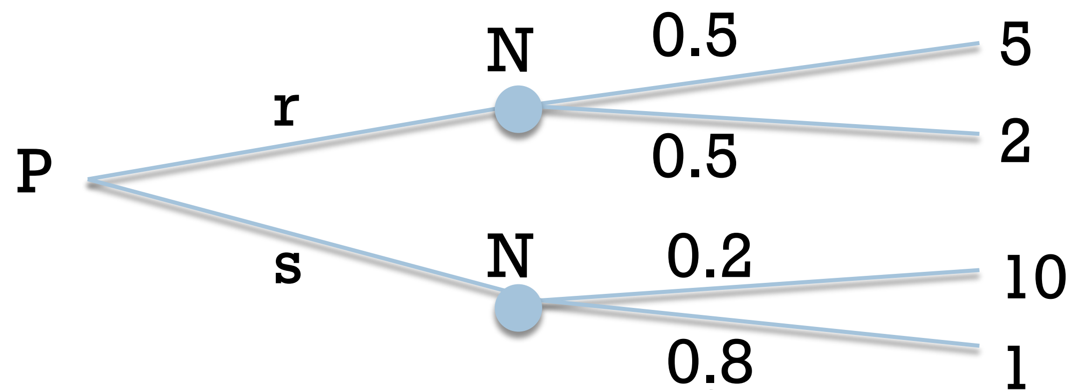
- Example
  - Ravioli give u(r) = 5 only 50% of the time; otherwise, they give u(r) = 2
  - Soup gives u(s) = 1 most of the time (80%); sometimes, it gives u(s) = 10
- We can model the choice between r and s as a choice between two **lotteries**
  - (r): utility is 5 or 2 according to a coin toss
  - (s): utility is 1 or 10 with probabilities 0.8 or 0.2

Random outcomes

- A **lottery** over outcomes $X = \{x_1, x_2, \ldots x_n\}$ is defined as a probability distribution p over X
  - this means that $p = \{ p(x_1), p(x_2), \ldots p(x_n) \}$ where $p(x_k) \geq 0$ for all k, and $\sum_{k=1..n} p(x_k) = 1$
- If actions are involved, p is conditional
  - for an action $a \in A$, we consider $p(x_k \mid a)$
- The case with certain outcomes can be seen as a **degenerate** lottery where $p(x_k \mid a) = 1$ for a given k, and 0 for all other options

Nature

- In the language of Game Theory, random events are the consequences of the choices of another player, called "Nature"
  - Nature (N) chooses within the lottery p
  - This can be represented in the decision tree



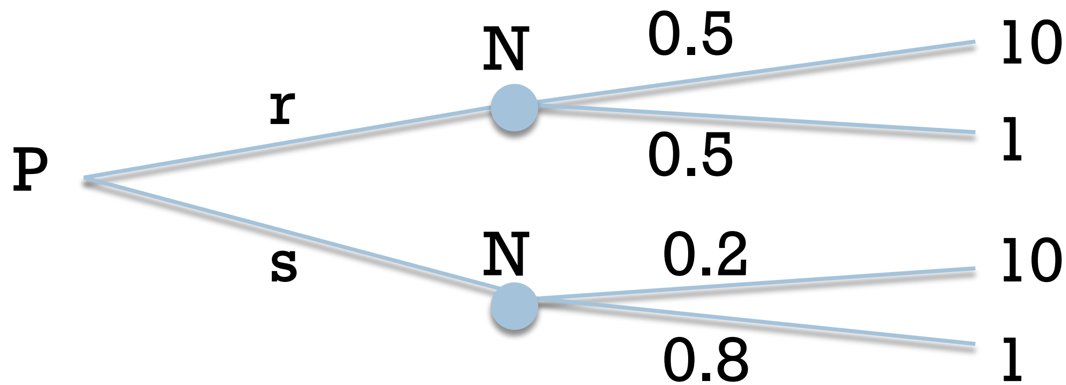
non azioni, ma probabilità

Continuous lotteries

- Lotteries can also describe probabilities over a continuous space of events
  - A specific outcome has probability 0 though
  - Probability densities replace distributions under this setup
  - Representation within the decision framework is still valid, but more cumbersome (e.g., no decision trees)

Evaluating random outcomes

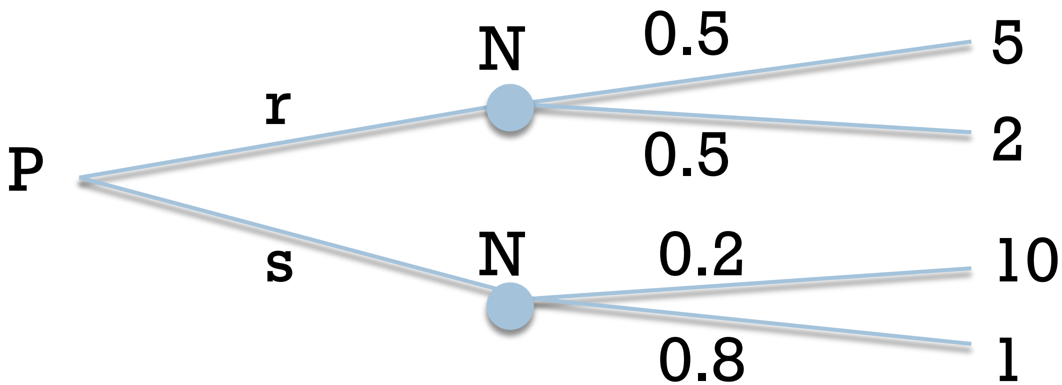
- Assume that ravioli and soup can only be "tasty" or "not tasty" giving u=10 or u=1



- We may assume that a rational user prefers r, since it has higher chances of getting 10

Evaluating random outcomes

- However, with different numbers the result is not so clear. What is better? r or s?



- A fair coin toss between 5 and 2, or a chance of getting 10 with a likely risk of getting 1?

Expected utility

- The usual methodology to compare random outcomes is to take expectations
  - also works to compare lotteries with certainties
  - "Expected utility theory" developed by von Neumann and Morgenstern
  - Intuition behind this: if you try $N \to \infty$ times, you will eventually get average payoff = expectation
- Expected payoff from lottery p
  - $\mathbb{E}[u(x) \mid p] = \sum_{k=1..n} p(x_k)\, u(x_k)$

Expected utility

- Expected utility theory relate expectations with preference relations
- Assume we want to define $\succcurlyeq$ among **lotteries** and we seek for a utility u representing $\succcurlyeq$
  - i.e. we replace A with set $P(A)$ of lotteries over A
- von Neumann & Morgenstern proposed a framework (vN-M utilities) where $\succcurlyeq$ satisfies
  - Rationality (completeness and transitivity)
  - Continuity axiom
  - Independence axiom

Continuity axiom

- For p, q, r $\in P(A)$, it must hold that sets
  - $\{a \in [0,1] : ap + (1-a)\, q \succcurlyeq r\}$
  - $\{a \in [0,1] : r \succcurlyeq ap + (1-a)\, q\}$

  are closed.
- That is, arbitrarily small variations in the gamble does not change preferred lotteries
  - Example: I prefer a 100% safe walk in the park over staying home. I have the same preference if I have a very small probability of being mugged when choosing the walk in the park

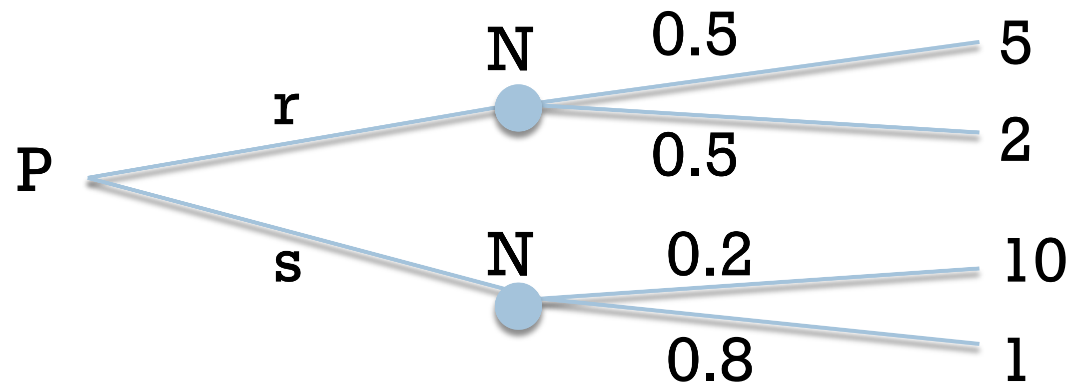
Independence axiom

- For p, q, r $\in P(A)$, it holds that $\forall a \in [0,1]$:
  - if $p \succcurlyeq q$ then: $ap + (1-a)\, r \succcurlyeq aq + (1-a)\, r$
- This axiom means that when mixing gambles we preserve the preference order not counting other alternatives
  - I prefer betting on football than horse races. Then I also prefer after flipping a coin to do "heads: bet on football, tails: play roulette" over "heads: bet on horse races, tails: play roulette"

vN-M utility theorem

- If $\succcurlyeq$ satisfies the four axioms, it can be represented by $\boxed{u(\cdot)}$ such that $\forall p, q \in P(A)$ $p \succcurlyeq q$ implies $\mathbb{E}[u(x)|p] \geq \mathbb{E}[u(x)|q]$
  - Such a function u is called vN-M utility
- Theorem can be proved after many lemmas
  - E.g.: u represents $\succcurlyeq$ with expected utility form only if it is a linear map from $P(A)$ to $\mathbb{R}$
  - **Proof**: $p \in P(A)$ = a combination of degenerate lotteries $p = p_1(1, 0, 0, ..) + p_2(0, 1, 0\ ..) + ..$
- Any affine (linear) transformation of u works

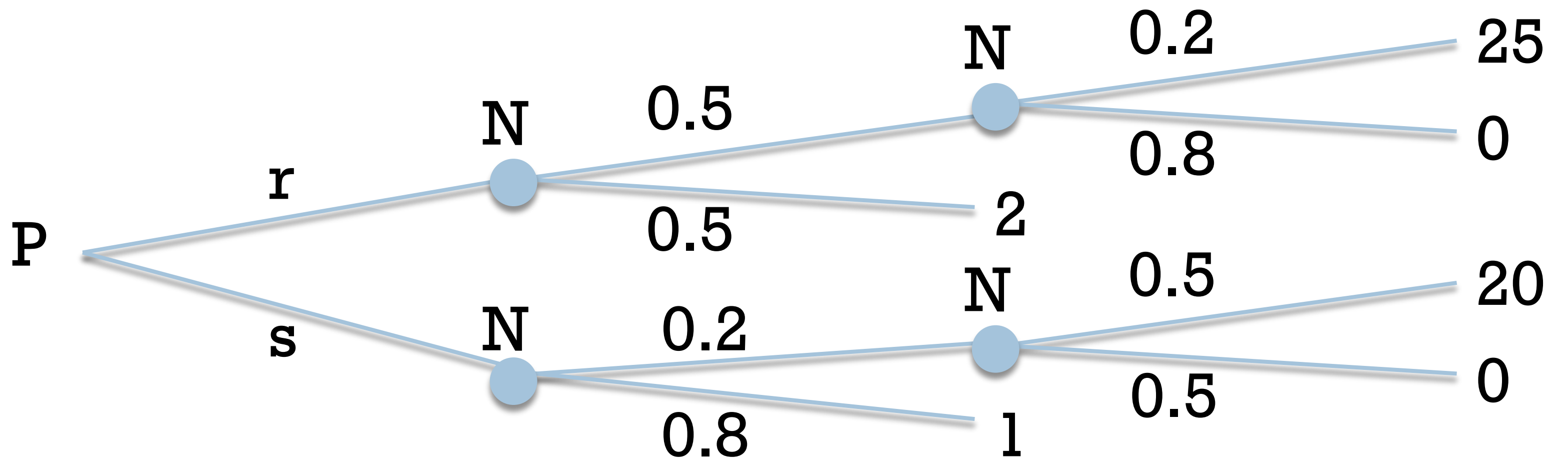
Expected utility



- Now we have a way to compare r and s
  - $\mathbb{E}[u(x)\,|\,r] = 0.5 \times 5 + 0.5 \times 2 = 3.5$
  - $\mathbb{E}[u(x)\,|\,s] = 0.2 \times 10 + 0.8 \times 1 = 2.8$
- So it seems that r is rationally preferable

Compound lotteries

- If Nature has more subsequent choices…



- we just take compound expectations
  - in this case, r and s lotteries are same as before
  - (implying: independent Nature choices)

Continuous case

- Identical application to continuous cases
  - only the graphical formulation is harder
- E.g.: dig a well, select how deep (d meters)
  - this is a continuous action $0 \leq d$ ($\leq$ Earth radius)
  - effort: $d^2/2$; water extracted: $W(d) \sim u[0, 20d]$
  - utility u for digging the well: water – effort
- $\mathbb{E}[u|d] = \mathbb{E}[W(d) - d^2] = 10\,d - d^2/2$
  - the utility of digging 3.2 meters is 26.88
  - rational best choice is $d = 10.0$ giving $u = 50.0$

Ordinal vs. absolute value

dipende: dobbiamo confrontare expected utilities

- Random setup: absolute utilities **do matter**!
- Replace u(s)=10 in the "tasty" case with 100
  - Same order but a different absolute value
  - The equivalence of utilities and preference relationship no longer hold in the uncertain case
- "a $\succcurlyeq$ b" is not enough: also, how much?
  - It holds for other cases with uncertainties and probabilities (mixed strategies) as well

Risk attitude

- Consider three possible outcomes of getting
  $x_1 = 0, \qquad x_2 = 1 \text{ euro}, \qquad x_3 = 20 \text{ euro}$
  and lotteries $p_A = (0, 1, 0)$, $p_B = (0.95, 0, 0.05)$
  the expected outcome is always the same,
  but A is a degenerate lottery
- Expected **utility** is
  $\mathbb{E}[u \,|\, A] = u(x_2)$, $\mathbb{E}[u \,|\, B] = 0.95\, u(x_1) + 0.05\, u(x_3)$
- It depends! On how the rational player values the payoff of getting X euros

Risk attitude

- A **risk neutral** player sees A and B as perfect substitute choices
  - They do not see any difference in lotteries as long as the expected outcome is the same
- A **risk averse** player always prefers a degenerate lottery (the sure thing) to one with same expected outcome ($A \succcurlyeq B$)
- A **risk loving** player does the opposite

non confondere outcome e utility

Risk attitude

- Definition based on outcomes, not on utilities
- Actually, utilities can serve to the same end:
  - Linear u (e.g., $u(x) = x$) → risk neutral
  - Concave u → risk averse
  - Convex u → risk loving

- Monotonic utilities such as $u(x) = x, x^2, \log x$, do not change preference of the user, but they change the risk attitude

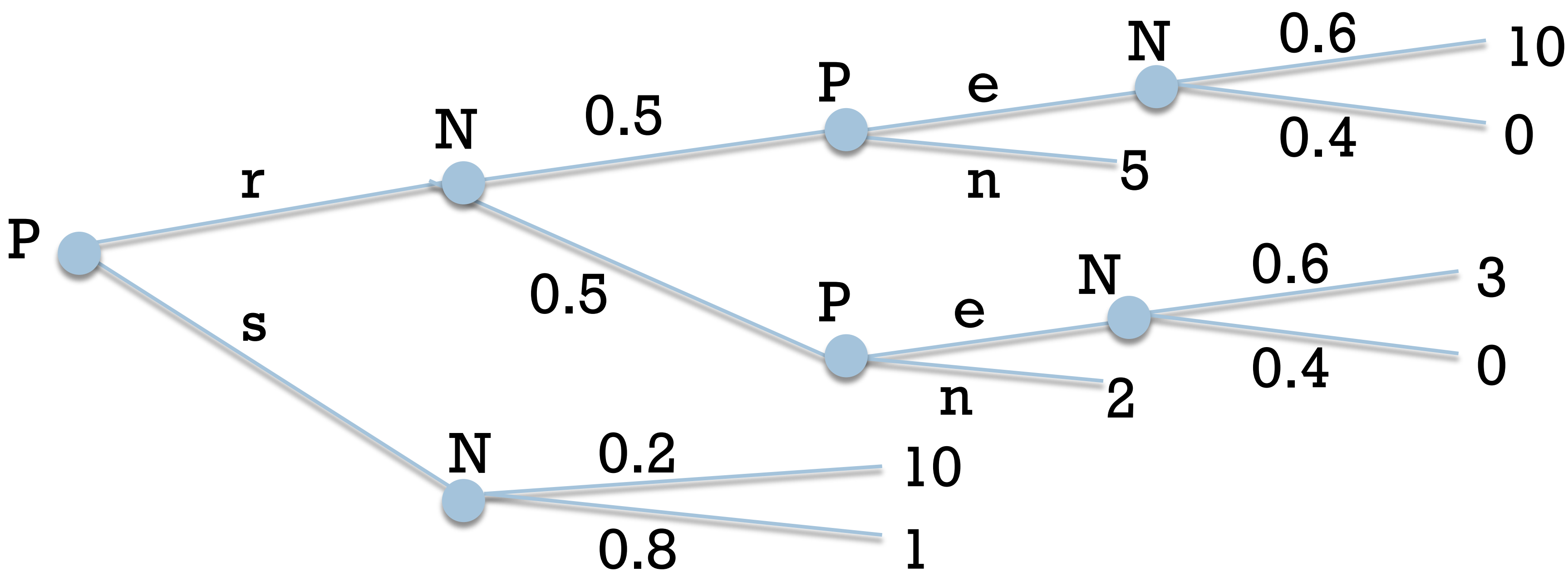
Risk attitude

- So be careful: expected utility theory does not say that it is the same to get 1 euro or to gamble 2 euros with 50/50 probability
- It actually says that if your utility function of outcome x is $u(x)=x$ then you are indifferent between these two lotteries
- But you may prefer either of them depending on your risk attitude and therefore on your u

Decisions over time

- Actions of player and Nature may alternate
  - E.g.: assume the canteen problem as before, with same choice between ravioli and soup
  - Ravioli can be had with (e)xtra cheese on top
  - Cheese makes ravioli even tastier, but there is a chance that you do not like the cheese served
  - Assume cheese is good with 0.6 probability
  - Good cheese increases u: 10 for tasty ravioli, 3 for bland ravioli. Bad cheese always give 0.

Decisions over time

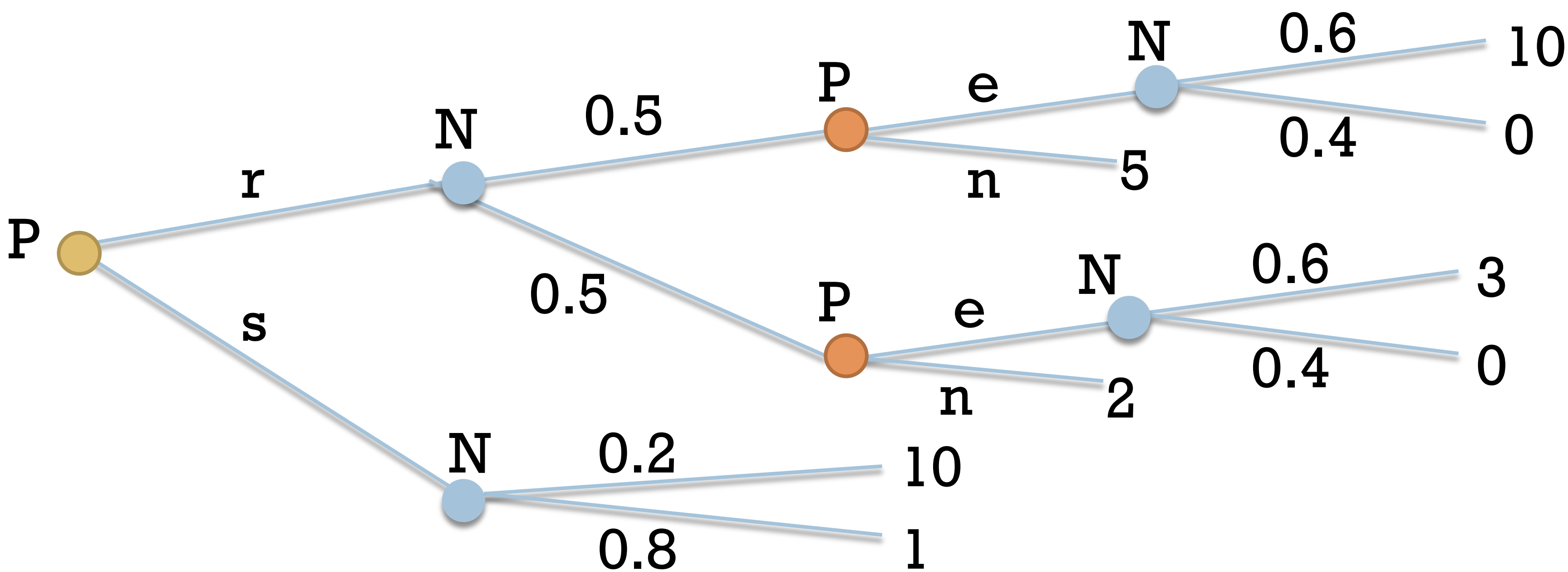


- How do we "solve" this decision tree?
- Principle known as **Backward Induction** (or **Dynamic Programming**)

Backward induction

- Classify all nodes with P's action into groups
  - Group 1 includes all nodes after which no further action is possible in any case; that is, only final outcomes or Nature's moves follow
  - Group k includes all nodes that are followed only by at least one Group k-1 node, without any higher-order node
  - In the previous examples we have just 2 groups

Backward induction

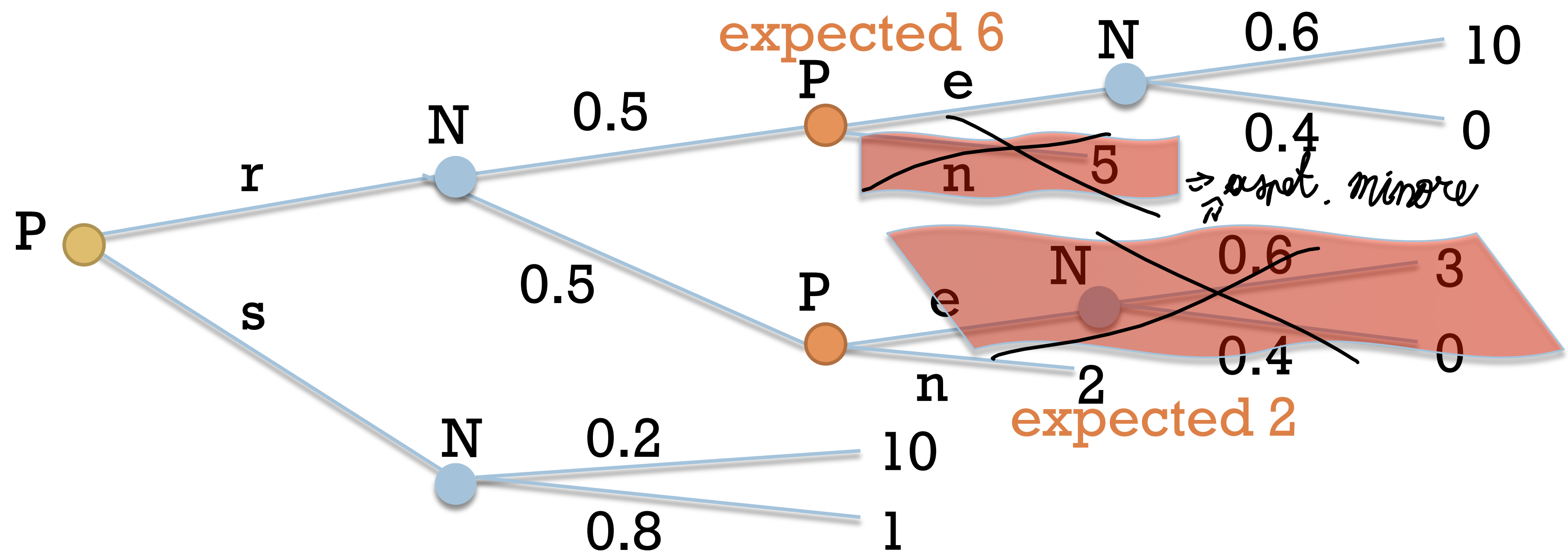


- Orange: group 1, Yellow: group 2
  - Note that the root of the decision tree belongs to group 2 in spite of the lower branch having no further choice (but the upper branch does)

Backward induction

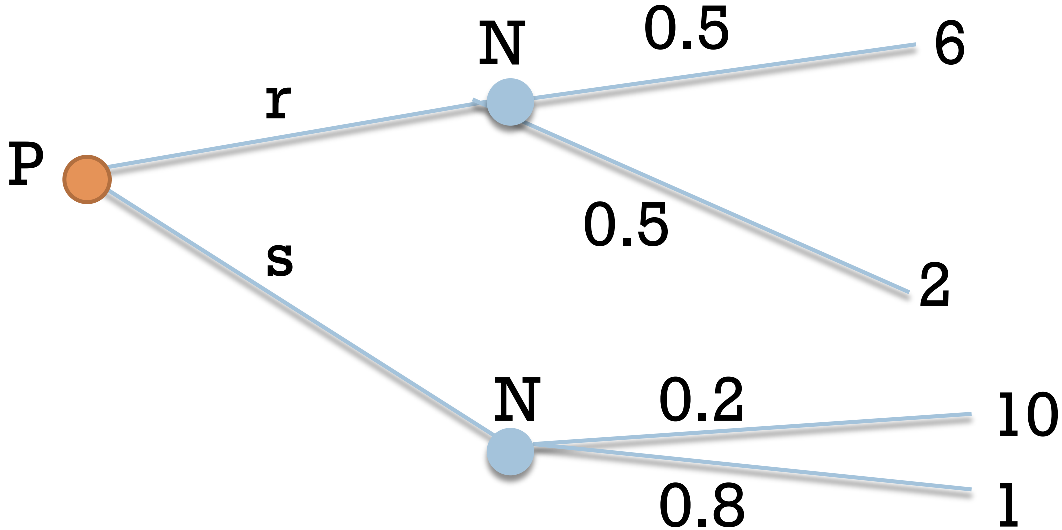
- P knows what to do if at Group 1-nodes
  - Rational P will maximize its own expected utility!
  - We can identify transform these intermediate points into final outcomes with maximal u
- After doing so, no more Group 1-nodes and all Group k-nodes are now Group (k-1)
  - Iterate the procedure ad lib
  - It should be evident why "backward induction"

Backward induction



- Now the problem is reduced to P with one decision to make at the root node
  - (root node is now Group 1, it was Group 2)

Backward induction

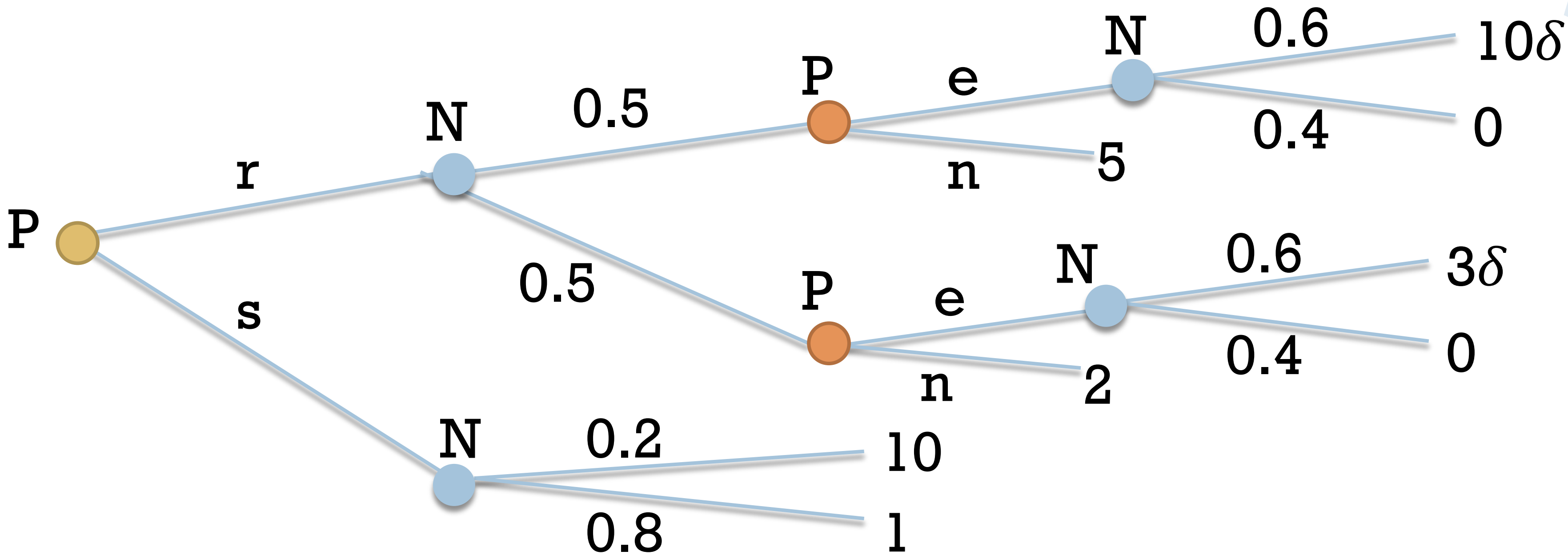


- In the pruned tree, r is preferred over s

$$\mathbb{E}[u \mid r] = 4, \qquad \mathbb{E}[u \mid s] = 2.8$$

Discounts for future payoffs

- If P's multiple decisions are made far apart, we may include a discount factor $\boxed{\delta,\ 0 < \delta < 1}$

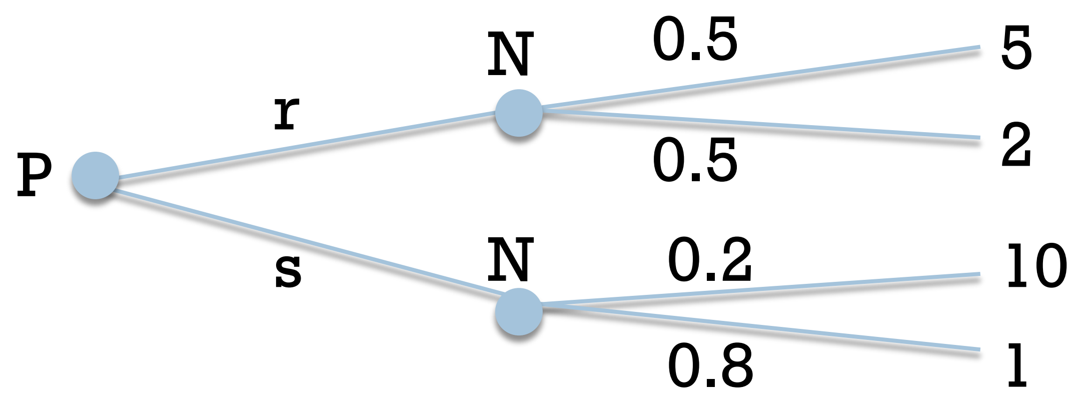


- Clearly the end result depends on δ

The value of information

- Expected utility implies that a rational player chooses its actions so as to make the right choice **on average**
- But if Nature's choice is known in advance, P might have chosen differently
- So, assume we have a chance of seeing Nature's choice ahead: is this information valuable? How much is it worth?

The value of information

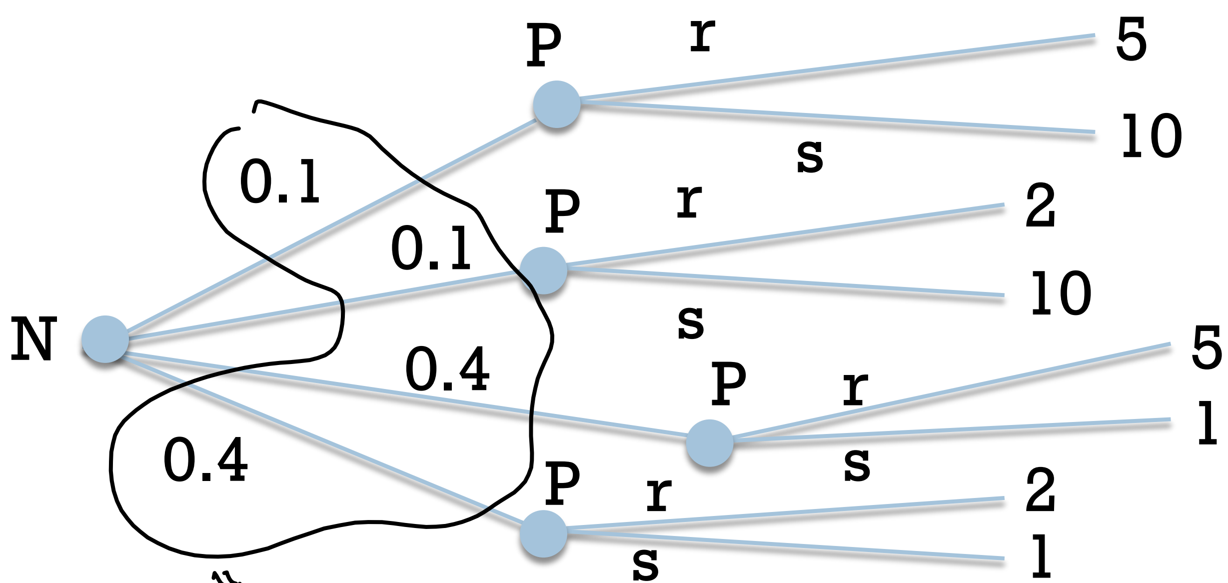


- Assume a friend of P knows how good is the food at the canteen today, and is willing to notify P about this (for a return)

The value of information

- If the friend is willing to tell, P is able to anticipate the expected payoff with the friend's advice and compare it to the one without the friend's advice
- The possible outcomes are unchanged, but their order changes!
- Basically, we need to account for P moving after Nature's choice is known, thus the order of movement is reversed

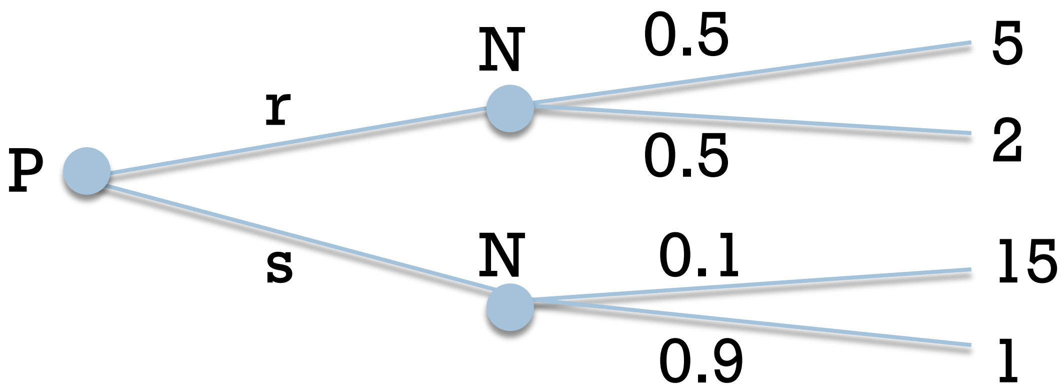
The value of information



ravioli buoni/cattivi e zuppa buoni/cattivi

- In this setup, P is always able to select the best outcome without any gambling
- utility = 1 + 1 + 2 + 0.8 = 4.8

The value of information



- $\mathbb{E}[u \mid \text{knowledge}] = 4.8$
- The expected utility without knowing N's choice was 3.5 (because r was selected)
- Thus, knowing Nature's choice is worth 1.3