Game theory

a course for the
MSc in ICT for Internet and multimedia

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Signaling Games

What the interaction can tell

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Screening or signaling?

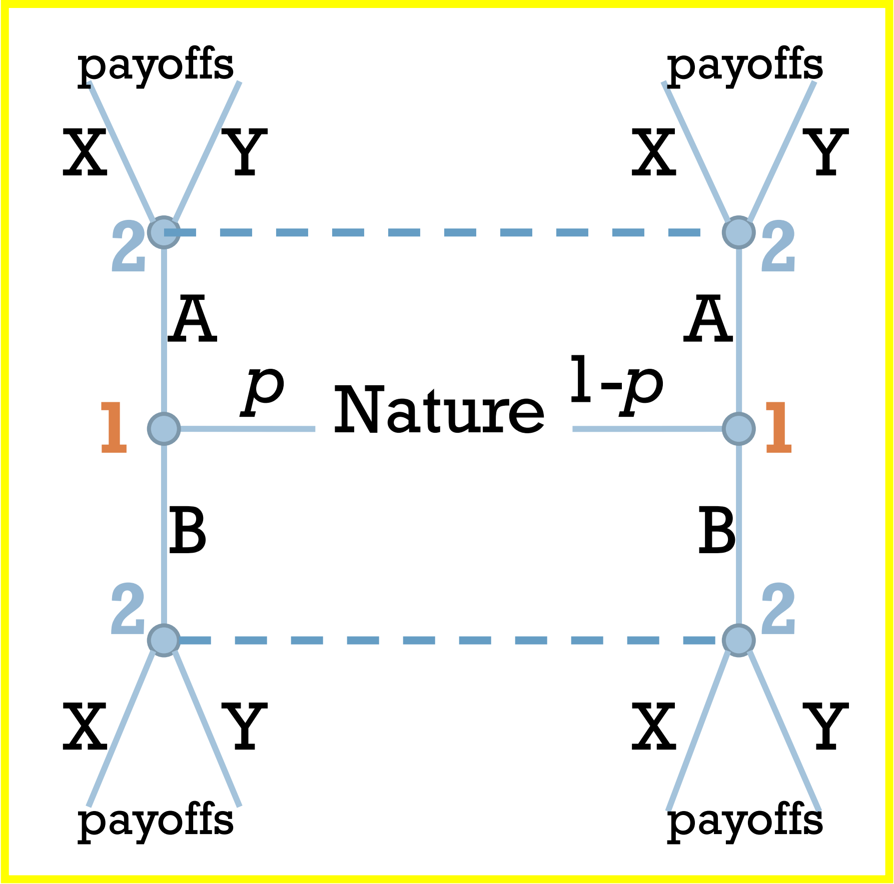
- We saw a 2-player entry game (1=entrant, 2=incumbent) made Bayesian in 2 ways
- This can be generalized to:
  - the typed player is 2: 1 has no type, 1 can only guess 2's reaction based on the prior; this is a **screening game** (think of quiz shows with a secret prize behind a screen), SPE is enough
  - the typed player is 1: we call it a signaling game since the action taken by 1 can be interpreted as a *signal* by 2 to guess 1's type

Signaling game: definition

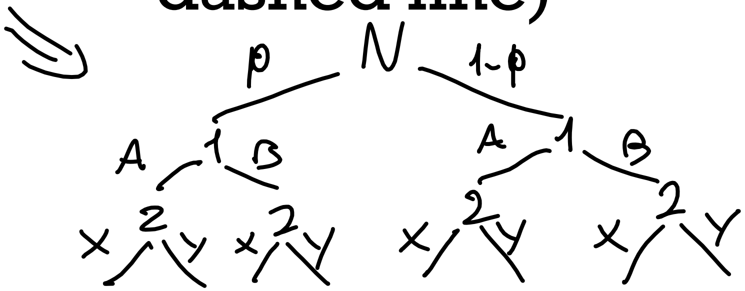
- A Bayesian dynamic game with 2 players, 1 (first to move) and 2 (observing 1's move):
  - 1 has one of many types (chosen by Nature)
  - 2 does not know 1's type but cares about it (i.e., the game is common values)
  - 1 has at least as many actions as types
  - 2 updates beliefs after 1's move
- This kind of games requires to use PBE

Graphical representation

- Binary case is often shown as a "butterfly"



- This structure is actually subject to some changes in certain case (esp. regarding the dashed line)



Equilibria of signaling games

- **Separating equilibria**: all types of 1 choose a different action, thus revealing the type to 2
- **Pooling equilibria**: all types of 1 choose the same action, thus no clue for 2 about 1's type
- **Intermediate cases**: called "hybrid" or "semi-separating" or "partially-pooling"
  - remember that 2 plays according to Bayes' rule only in those information sets that are reached with probability >0 (so, what is easy to solve?)

Signaling games

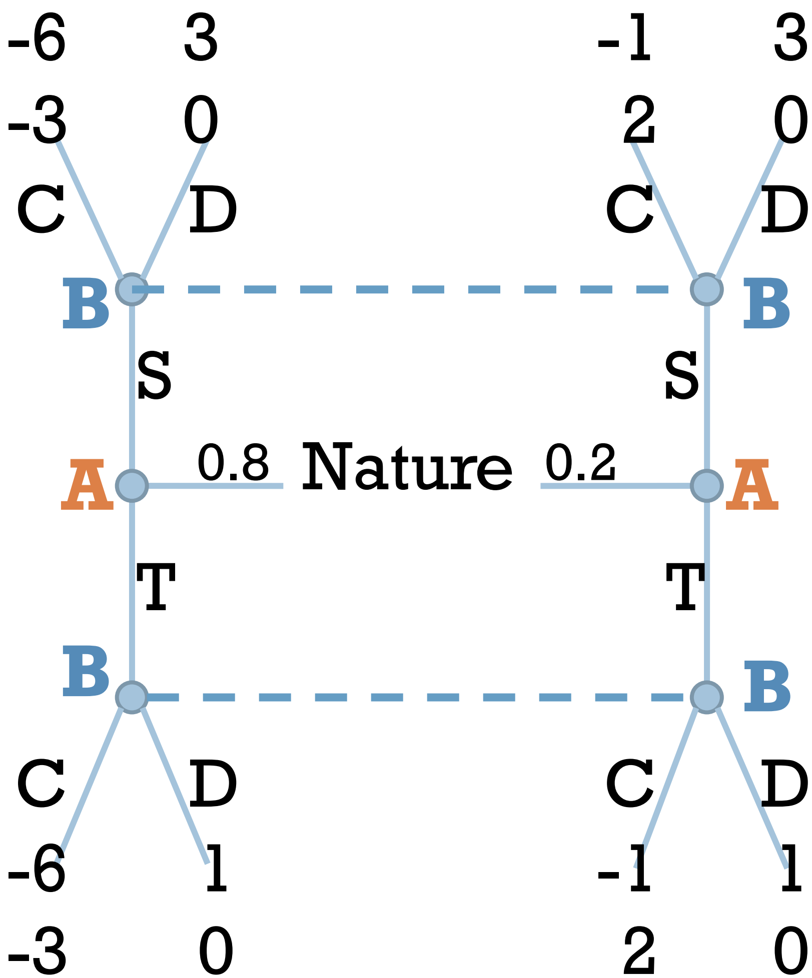
Step-by-step analysis on a case study

Example: a coffee for Brian

- Brian is invited by colleague Zöe to a coffee
- Ann is a typed player: her types are
  - Jealous with probability 0.8
  - Easygoing with probability 0.2

  (all of this is common knowledge)
- Ann can send a signal to either stay (S)ilent about this business or to (T)rash Zöe
- Brian observes the signal and can accept the (C)offee or kindly (D)ecline this offer

Extensive form of this game



- Payoffs explained
  - Jealous Ann is deeply hurt if Brian accepts
  - Easygoing Ann is just not-so-angry
  - Ann prefers to stay silent rather to trash Zöe
  - Brian likes to go if Ann is okay with it

How to solve this game?

- Both players have 4 strategies but for different reasons
  - Ann because she has a type: so her strategy is (what to do if jealous, what to do if easygoing)
  - Brian has no type but he observes Ann's move so (reaction to Ann's S, reaction to Ann's T)
  - e.g.: (TS, CD) means that Ann is vocal about her jealousy but is silent if easygoing (separating); Brian just "follows the signal" and declines if Ann is mad, if she is silent he accepts the offer

First part: find the NE

- Matrix form

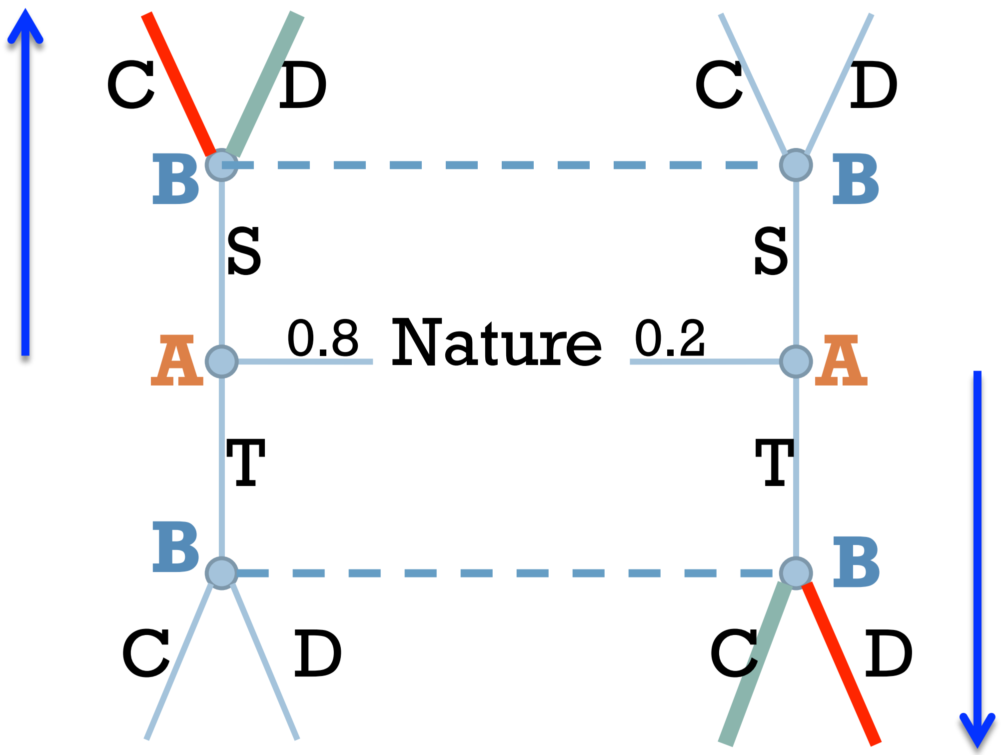
Brian

Ann	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2			
TS	-5, -2		swap!	
TT	-5, -2	1, 0	-5, -2	1, 0

- Caveat: A's pair is left/right but B's is **reaction to A**! So in the last row only the 2nd element counts
- Also check what happens in the swap! cell

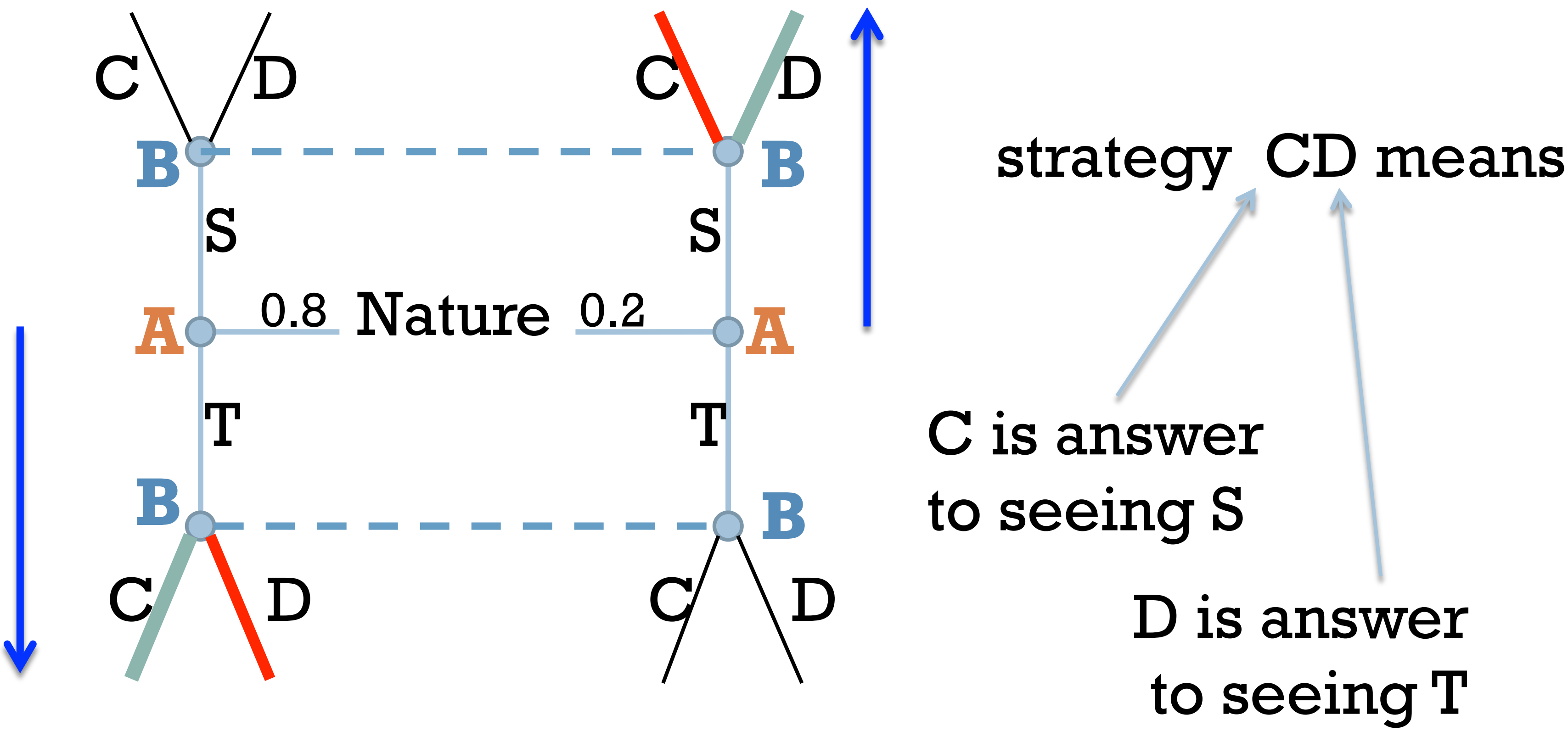
How to properly fill the matrix

- See e.g. (ST,CD) and (ST,DC)



How to properly fill the matrix

- But for (TS,CD) and (TS,DC) we need to swap!



First part: find the NE

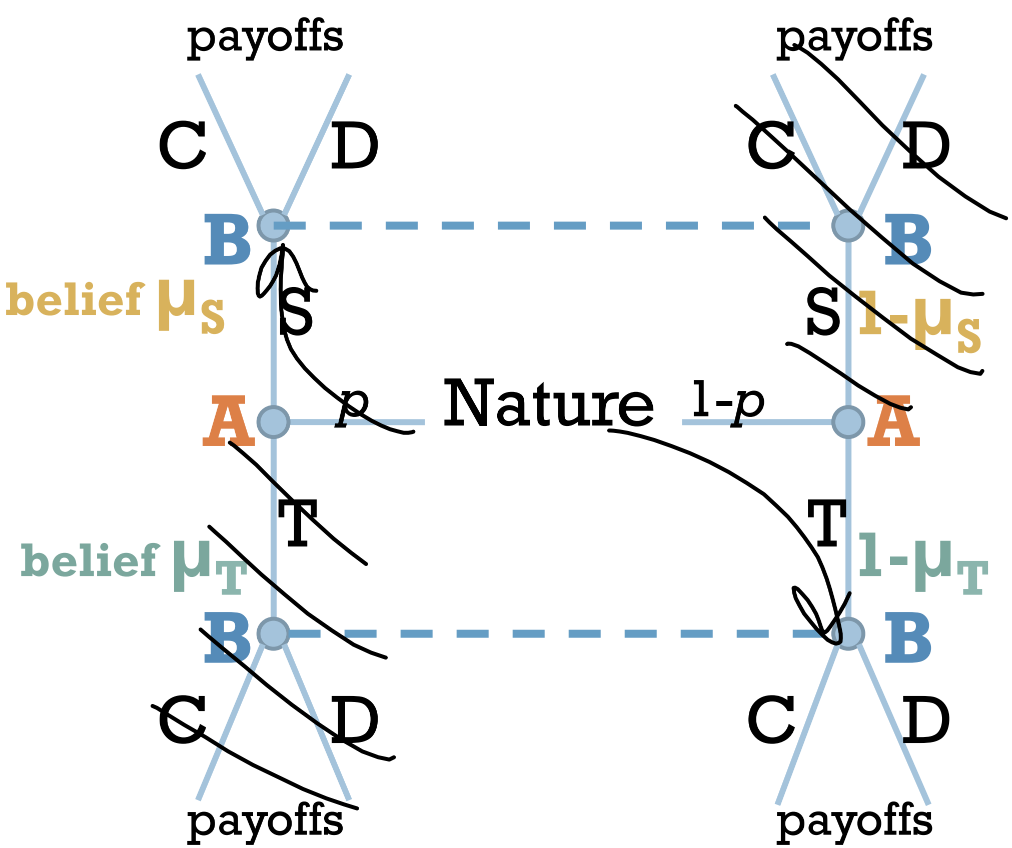
		Brian		
Ann	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2	-4.6, -2.4	2.2, 0.4	2.6, 0
TS	-5, -2	0.6, 1.6	-4.2, -2.4	1.4, 0
TT	-5, -2	1, 0	-5, -2	1, 0

- We find 5 NEs: 3 pure-strategy and 2 mixed-strategy in addition to what visible: **NE4:** (TT, ½ CD+½ DD) **NE5:** (1/6 SS + 5/6 TS, 2/9 CD + 7/9 DD)

Discussion of the NEs

- You are not done until you classify these NEs as perfect Bayesian equilibrium (if possible) and to do so, you have to check the beliefs
- In this game, the system of beliefs is a probability μ for Brian that Ann is **jealous** after seeing the signal (her move, S or T)
  - μ_S if she is silent, μ_T if she talks
- This is to mimic the probability of reaching a node in a dynamic game

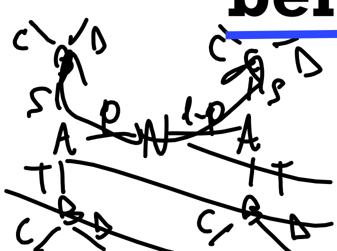
System of beliefs



- The system of beliefs is easy to compute for a **separating** PBE
- E.g. if A plays ST, B's belief is automatic:
  - $\mu_S = 1, \ \mu_T = 0$

System of beliefs

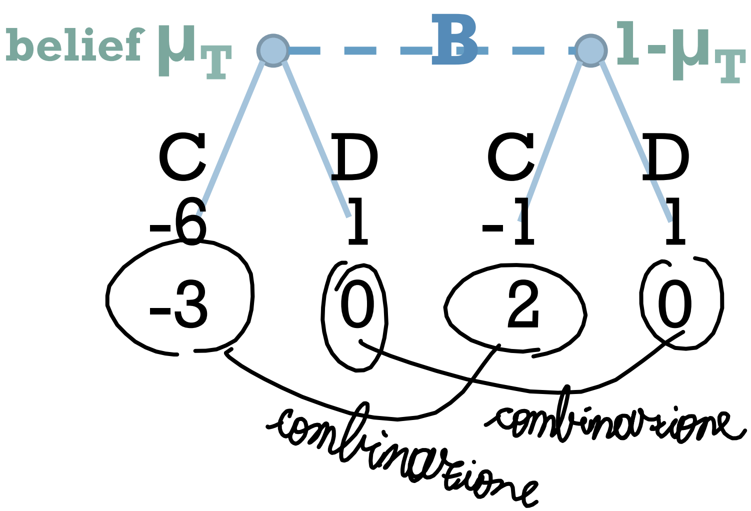
- Unfortunately, we do not have any PBE where A plays a separating strategy
- Pooling equilibria are more complex
- E.g., take a situation where Ann plays SS. What can Brian think of her jealousy?
  - If B sees S, no further info → A always plays S! Thus, B can only use the prior as the belief
  - And if B sees T? Never happens→**arbitrary beliefs**! But wait, they must imply rationality



μ_S?

Discussion of the PBEs

- **NE1: (SS, DD).** Brian is playing rationally (no clue, so uses the prior). Ann does not want to deviate: Brian is playing DD, no need to tell him that Zöe is a skunk
- What about B's response to T?



- Brian plays D, which is a best response only if $\boxed{0 \geq -3\,\mu_T + 2(1-\mu_T)}$
- Thus, the system of beliefs must be $\boxed{\mu_S = \underbrace{0.8}_{p},\ \mu_T \geq 0.4}$

Discussion of the PBEs

- **NE2: $\boxed{\textbf{(SS, DC)}}$**. It might seem strange, but Brian is still playing rationally, since he is still declining Zöe's invitation in reality
- But now, Brian plans to accept if A trashes her
  - Why? This must be supported by a different system of beliefs, namely $\boxed{\mu_S = 0.8, \mu_T \leq 0.4}$
- You see that PBE1 or PBE2 (full described) are NE1 or NE2 + the values of the beliefs!

Discussion of the PBEs

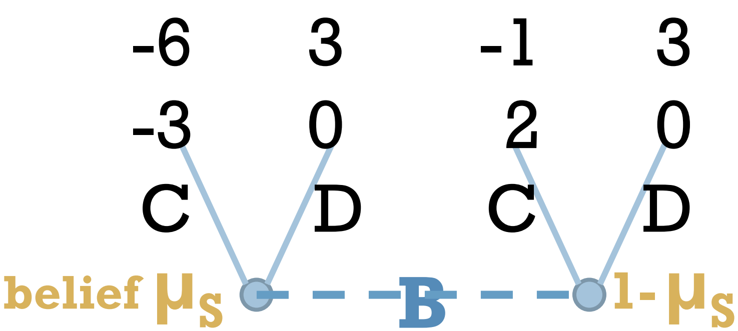
- **NE3: (TT, CD)**. Analogous to before but now $\mu_S \leq 0.4, \mu_T = 0.8$ (same numbers for B)
  - Still means that Brian declines, but if Ann becomes silent and if Brian believes that Ann is not likely to be jealous, he will accept
- **NE4: (TT, ½CD+½DD)**. Also pooling for A, but weirder: B indifferent between CD – DD
  - Follows from mixing these strategies

Ann		Brian: CD	Brian: DD
	TS	0.6, 1.6	1.4, 0
	TT	1, 0	1, 0

Discussion of the PBEs

- **NE4: (TT, ½CD+½DD)**: B's belief when signal is T is still $\boxed{\mu_T = 0.8 \text{ (the prior)}}$
  - While μ_S is off the equilibrium path, but we can manipulate the beliefs so that **½C+½D** is sustainable: indifference theorem!

azioni di B quando A sceglie S



- Payoff of D is $\boxed{0}$
- Payoff of C is $\boxed{-3\,\mu_S + 2(1-\mu_S)}$
- Hence $\mu_S = 0.4$

- So, $\boxed{\mu_S = 0.4,\ \mu_T = 0.8}$

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Discussion of the PBEs

- There is actually more to say for **PBE4**
  - one may get the wrong impression that a belief $\mu_S = 0.4$ sustains any mixture CD / DD
  - WRONG! See for example that (TT,DD) is not a BNE: incentive for A to deviate and be silent
- In reality, PBE4 = infinitely many PBEs where:
  - A chooses **T**
  - B uses the prior $\mu_T = 0.8$ and replies with **D**
  - off the equilibrium path, B believes $\mu_S = 0.4$ and plays a mixture q **C** + (1-q) **D** where $q \geq 0.5$

$q(-3 \cdot 0.4 + 2 \cdot 0.6) = (1-q) \cdot 0 \Rightarrow$ ~~zero~~ $\forall q$

$q(-6\mu_S - 1(1-\mu_S)) + (1-q) \cdot 3 = -3q + 3 - 3q = 3 - 6q > 0 \Rightarrow q < 0.5$

Discussion of the PBEs

- Finally, $\boxed{\textbf{NE5: } (1/6\,SS + 5/6\,TS,\ 2/9\,CD + 7/9\,DD)}$
- This can be a nice semi-separating PBE
  - A is always silent when easygoing but can become talkative when she is jealous
  - This is because she believes that B can sometimes choose C if she is silent too often
  - The description is very sensible but…what about the system of belief? It is actually more complex and requires Bayes' rule to be used non-trivially

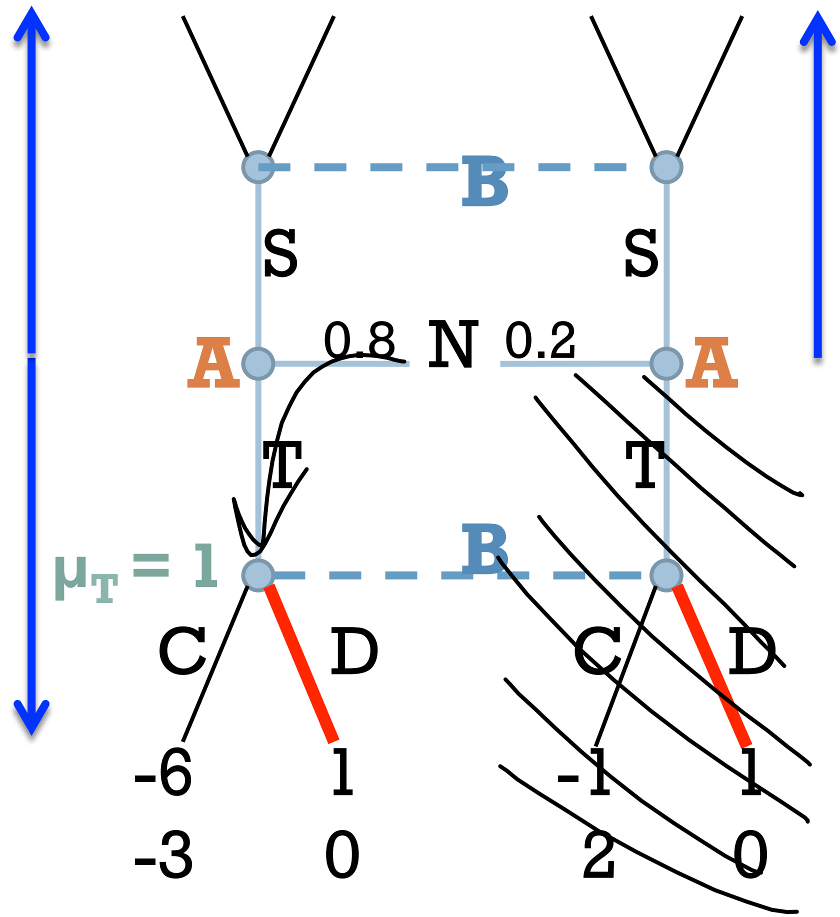
System of beliefs for PBE5

- Easy part: $\mu_T = 1$
  - since only jealous Ann plays T, to which Brian responds by playing D (highlighted in **red**)
  - note that Brian plays D even when easygoing Ann plays T (which she never does)

quando gelosa

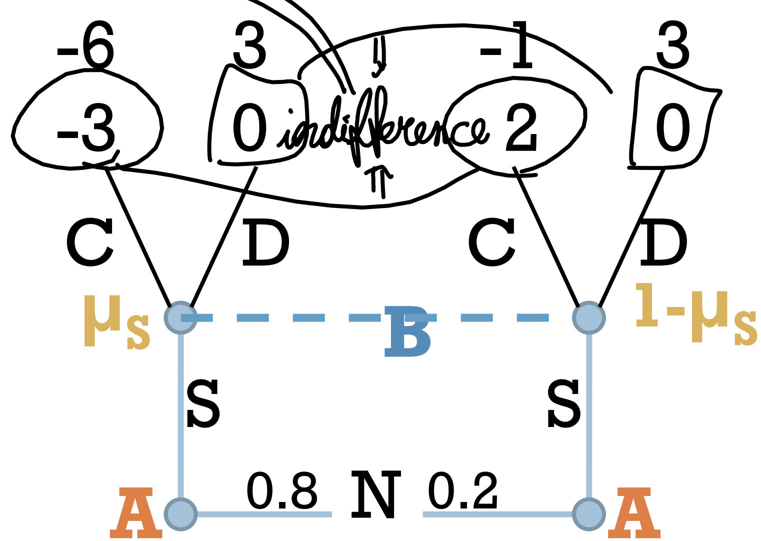
(1/6SS + 5/6 TS, 2/9[redacted]D + 7/9[redacted]D)

quando tranquilla



System of beliefs for PBE5

- What about μ_S? Depending on it, Brian may prefer C or D. And to play a mixed strategy, he must be **indifferent** between them!
- We know this needs $\mu_S = 0.4$ (not the prior!)
  - i.e. easygoing-A plays S more often than jealous-A
  - we actually know that easygoing-A always plays S
  - we check for the probability that jealous-A plays S: Bayes!



Applying Bayes' rule to PBE5

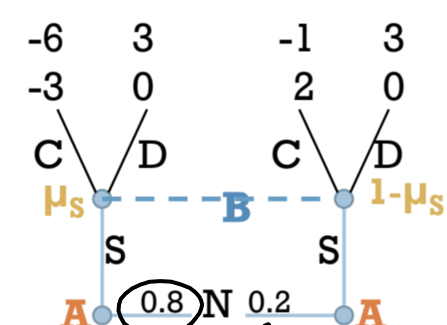
- Denote with $\boxed{q}$ (or 1-q) the probability that jealous-A plays S (or T, respectively)
- Remember:

$$\boxed{\mu_S = P[\text{jealous}\,|\,S] = \frac{P\,[S,\text{jealous}]}{P\,[S]} = \frac{p\,q}{p\,q + (1-p)}}$$

$P[S] = P[S,\text{jealous}] + P[S,\text{easygoing}]$

- Solving for $\boxed{p=0.8,\ \mu_S=0.4}$ gives: $\boxed{q = 1/6}$
  - consistent with what we found earlier

easygoing-A sceglie sempre S



S, jealous

Checking consistency for PBE5

- This justifies why A plays **1/6 SS + 5/6 TS**
- But why does B play **2/9 CD + 7/9 DD** ?
  - it means that B always chooses D after T, but takes a mixed stance after observing S
  - this is because it allows him to make jealous-A (who also plays a mixed strategy) to be indifferent between her options of S and T
  - with T, jealous-A gets 1 (since B responds with D)
  - with S, jealous-A gets -6 or 3 (for C or D, resp.)
  - hence $\boxed{-6\,P[C] + 3\,(1-P[C]) = 1 \;\rightarrow\; P[C] = 2/9}$

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