Game theory

a course for the
MSc in ICT for Internet and multimedia

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Dynamic bargain

Negotiation of resource sharing

Bargain

- Assume two players need to split a given amount of resources (for simplicity, = 1)
  - Player 1 gets x, player 2 gets 1-x
  - This is like saying they split one (1) pie
- Two approaches
  - Nash bargaining (axiomatic, static)
  - Seen as a dynamic game (the one seen here): modeled as alternate stage where players 1 and 2 exchange proposer/responder roles

Dynamic bargain

- In stage 1, (P)roposer is 1, (R)esponder is 2
  - 1 offers shares (x, 1-x), 2 can accept (the game ends) or refuse (the game goes on, stage 2)
- In stage 2, (P)roposer is 2, (R)esponder is 1
  - roles swapped from stage 1; this time P (which is player 2) makes the offer; R refuses⇒stage 3
- In stage t, P=1 if t is odd, otherwise P=2
  - R accepts⇒game ends; R refuses⇒stage $t+1$
- Assume that, if disagreement persists after a deadline (T stages), both 1 and 2 get nothing

Dynamic bargain

- When the game ends before the deadline, the players receive discounted payoffs
  - Because "time is money," so that the entire pie had value 1 at time 1, but every further round, a fraction $(1-\delta)$ is wasted
- If the game ends at stage 1, $u_1 = x$, $u_2 = 1-x$
- If the game ends at a later stage $t > 1$, compute discounted payoffs with discount $\delta<1$: $u_1 = \delta^{t-1} x$, $u_2 = \delta^{t-1} (1-x)$

Dynamic bargain

- If the deadline $T = 1$ → Ultimatum game
  - All solutions with P proposing x, 1-x and R accepting everything up to 1-x are NEs
  - However, only one SPE: x=1 (that is, the proposer keeps everything)
- Assume the deadline is at an odd T
  - Then player 1 is the last proposer; at round T, player 2 will accept everything,

Dynamic bargain

- Assume the deadline is at an odd T
  - 1 is the last proposer; at round T, 2 will accept everything, so 1 proposes x=1: $u_1 = \delta^{t-1}$, $u_2 = 0$
  - at round T-1, 2 can avoid going at round T, where he/she gets nothing, by offering $x \geq \delta$ this way, 1 will accept and $u_1 \geq \delta \times \delta^{t-2}$, $u_2 \geq 0$ (actually, 2 will simply offer $x = \delta$ then)
  - by iterating the reasoning, we see that 1 can start the game by offering something, 2 accepts:

$$u_1 = \frac{1 + \delta^T}{1 + \delta} \qquad u_2 = \frac{\delta - \delta^T}{1 + \delta}$$

Dynamic bargain

- Proposition. Any SPE must have the players reaching agreement in the first round.
  - Simply a consequence of backward induction
  - Iterating the game: (i) wastes reward, because of the discount (ii) sends the game to another (symmetric) round of proposer-responder, which rational players generally want to avoid
- Note that this is not a "repeated" game because of the termination option

Dynamic bargain

- Interestingly, this reasoning applies even to infinite horizon (even though backward induction does not work, but reason (i) does)
- For $T \to \infty$, $\quad u_1 = \dfrac{1}{1+\delta} \qquad u_2 = \dfrac{\delta}{1+\delta}$
  - that for $\delta \to 1$ tends to equal split
- For infinite horizon we can similarly prove that an agreement must be reached in the first stage in order to have an SPE

Dynamic bargain

- However, we need to prove the SPE is unique
  - without resorting to backward induction!
- Assume that there is more than one SPE: thus, 1 can get a best $\mathbf{v}_1$ and a worst $\mathbf{w}_1$ SPE payoff
- 2 gets what 1 gives up: thus, 2 can get a best $\mathbf{v}_2=1-\mathbf{w}_1$ and a worst $\mathbf{w}_2=1-\mathbf{v}_1$ SPE payoff
- Because the game is iterated, if stage 2 is reached, 2 can get either $\mathbf{v}_2=\delta\mathbf{v}_1$ or $\mathbf{w}_2=\delta\mathbf{w}_1$
- All of this implies $\mathbf{v}_1=\mathbf{w}_1=(1+\delta)^{-1}$

Dynamic duopolies

Dynamic games in the duopoly theory

Stackelberg duopoly

- A dominant (leader, 1) firm moves first and a subordinate (follower, 2) firm moves second
- Assume, for example, they decide quantities as per Cournot
  - **Recall.** The cost to produce q is $C(q) = cq$ (with constant c)
  - The market price is $P(Q) = a - Q$ (with constant $a > c$)
- 1 knows that 2 will play a best response

Stackelberg duopoly

- The profit of 2 is $u_2(q_1, q_2) = q_2(a - q_1 - q_2 - c)$, so q_2 maximizing u_2 is a best response to q_1, called $R_2(q_1)$

$$R_2(q_1) = (a - q_1 - c) / 2$$

- Note that $R_2(q_1) = (a - q_1 - c) / 2$ appeared also in Cournot's monopoly, when we figured out what is the best the duopolist can do
  - There this was a hypothesis, here it is real

Stackelberg duopoly

- Knowing all of this, the leader can choose q_1 so as to

$$\max\ u_1(q_1, R_2(q_1)) = q_1(a - q_1 - R_2(q_1) - c)$$
$$= q_1(a - q_1 - c)/2$$

- We obtain $q_1^* = (a - c)/2$, $q_2^* = (a - c)/4$
- Recall Cournot: $q_1^* = q_2^* = (a - c)/3$
- The leader exploits the advantage of moving first

Stackelberg duopoly

- **Remark 1.** What if follower poses a threat?
- Like, "Choose the Cournot quantity or I'll choose a high quantity"
  - This is, as usual, just a virtual threat (something the leader can just imagine)
  - In any event, the leader is not scared, as this is a <u>non-credible</u> threat
  - Such a behavior is irrational, as the follower would be hurt too

Stackelberg duopoly

- **Remark 2.** In multi-decision problems, more information can make one player worse off (it is not so in single-decision)
  - Player 1 <u>knows</u> 2 will have more information
  - So, 2 may have better awareness but fewer choices, as 1 does not let them available
  - This leads to "first-mover advantage" (not necessarily a disadvantage for the second player, but this time the game is competitive)

Stackelberg duopoly

- Having more knowledge when moving (and the other player knows it) is indeed harmful
- Assume 2 plays after 1, <u>without knowing</u> q_1 (we should know what happens there)
  - 2 may assume a Stackelberg $q_1^* = (a - c)/2$
  - So $q_2 = (a - c)/4$
  - 1 knows q_2 and chooses a better $q_1 = 3(a - c)/8$
  - Now, 2's best answer changes again
  - In the end, this is Cournot: $q_1^* = q_2^* = (a - c)/3$

Cournot duopoly + collusion

- As previously seen the Cournot duopoly the NE is, for both firms, $q_C = (a - c)/3$
- The aggregate production is higher than the monopoly $q_m = (a - c)/2$: lower profit
  - This is not happening due to lack of trust
- However, according to Friedman's Theorem, there should be a way to build trust if the game is repeated infinitely

Cournot duopoly + collusion

- In repeated games, we built cooperation with a "Grim Trigger" strategy
- As per Repeated Prisoner's Dilemma, GrT is:
  - At t=1 produce $q_m/2$ (half of monopoly quantity)
  - At t >1, produce $q_m/2$ if in every stage $u < t$ production was $q_m/2$ for both firms; otherwise produce q_C forever after
- We expect this GrT work if the discount factor δ is close to 1

Cournot duopoly + collusion

- GrT is a NE for subgames where one deviated
- Analogously to the Prisoner's Dilemma, we need to compute the best response of firm 2 at the first stage
- Assume player 1 chooses $q_1 = q_m/2 = (a - c)/4$
  - Myopic strategy is $\mathrm{argmax}_{q_2}\ q_2 (a - q_2 - q_m/2 - c)$.
  - Solution is $3(a - c)/8$, profit $u_D = 9(a - c)^2/64$.
  - Or, 2 keeps cooperating at $q_m/2$, $u_m/2 = (a - c)^2/8$

Cournot duopoly + collusion

- Myopic strategy: u_D at first stage, then u_C
  Present value is $u_D + \delta\, u_C / (1-\delta)$.
- Collaborative strategy: $u_m/2$ at every stage.
  Present value is $(u_m/2)/(1-\delta)$.
- Recall $u_m/2=(a-c)^2/8$, $u_C=(a-c)^2/9$,
  $$u_D = 9(a-c)^2/64$$
- Collaboration can be triggered if $\delta \geq 9/17$

Cournot duopoly + collusion

- What if $\delta < 9/17$? GrT is no longer a SPE
- Still we can do better than always playing the Cournot value q_C !
- Take a less ambitious GrT' with objective q^* in $[q_C, q_m/2]$.
  - GrT' is: "Start at q^*; after any deviation stay at q_C forever"
- When both firms play q^*, they have utility $\boldsymbol{u^*} = q^*(a - 2q^* - c)$

Cournot duopoly + collusion

- Also this GrT' has a myopic response which looks only at the immediate payoff, i.e., trying to $\max_{q_i} q_i\,(a - q_i - q^* - c)$
- This "deviation" solution is $q_D = (a - q^* - c)/2$, yielding payoff $u_D = (a - q^* - c)^2/4$ which is better than u^* so one is tempted to betray

Cournot duopoly + collusion

- Again, TS' is better if

$$u^*/(1-\delta) \geq u_D + \delta\, u_C/(1-\delta)$$

$$\underline{q^*(a-2q^*-c)}/(1-\delta) \geq \underline{(a-q^*-c)^2/4}$$
$$+\ \delta\, \underline{((a-c)^2/9)}/(1-\delta)$$

- Take equality for minimum q^* (i.e. max u^*) which can be achieved by a given δ
- Solving,

$$q^* = (a-c)(9-5\delta)/(3(9-\delta))$$

between $(a-c)/3 = q_C$ and $(a-c)/4 = q_m/2$ as the discount factor δ goes from 0 to 9/17

Cournot duopoly + collusion

- Contrarily to the Prisoner's Dilemma, the Cournot duopoly can include worse punishments than simply play the NE
- A "Carrot-and-Stick" strategy successfully builds cooperation at $q_m/2$ even for $\delta < 9/17$
- Such a strategy has two possible actions.
  - (R)eward: produce $q_m/2$
  - (P)unishment: produce x, with properly chosen x ($>q_C$ - but not too high)

Cournot duopoly + collusion

- The strategy is defined as follows
- At stage 1, start with R
- At stage *t*:
  - Choose R if both firms played R at stage *t*-1
  - Choose R if both firms played P at stage *t*-1
  - Else play P
- Verify this works for $\delta = ½$, $x = 2(a - c)/5$

Cournot collusion: implication

- Governments often punish firms for cartels: if a meeting is held where two duopolists agree on acting like that, they are fined
- Problem: no need for holding meetings!
- The agreement between firms 1 and 2 is just reached as a GrT (no communication or cheap talk, just in the CEO's head!)