Game theory

A course for the
MSc in ICT for Internet and multimedia

Leonardo Badia

leonardo.badia@gmail.com

Dynamic games

Game development over time

Dynamic games

- A dynamic game involves some players moving first, others moving later
- **Complete** information: everyone knows the payoff, and knows everybody knows…
- However, a further distinction appears:
  - **perfect information** means that every player can make a decision with full awareness
  - **imperfect information** means that some decisions are "simultaneous" or Nature plays

Battle of the sexes, revisited

- Ann and Brian agreed to meet at either the romance (R) or the sci-fi (S) movie
  - (lower-case for Brian's actions for better clarity)

Ann \ Brian	r	s
R	2, 1	0, 0
S	0, 0	1, 2

- To frame this as a normal form game, they must act unbeknownst of each other
  - which is not very realistic…

Battle of the sexes, revisited

- Let's add a more sensible time sequence
- Assume Ann decides (before Brian does) which movie to see, and calls Brian to tell him
  - What should she decide? R or S?
  - Ann knows (being completely informed) that whatever she chooses, Brian's best response is to play along and choose the same thing
  - Since Ann prefers R over S, Ann chooses R (no uncertainty on this outcome, we will see why)

Extensive form

- To unfold the time dimension, we may want more than just the bi-matrix of payoffs
- We need to connect possible choices to the knowledge of what happened before
  - (or not: Brian may not receive Ann's call!)
  - such a knowledge conditions the development of the game, as per the previous example
- Normal form replaced by the **extensive form**
  - Graphically, we use a decision tree

Extensive form: ingredients

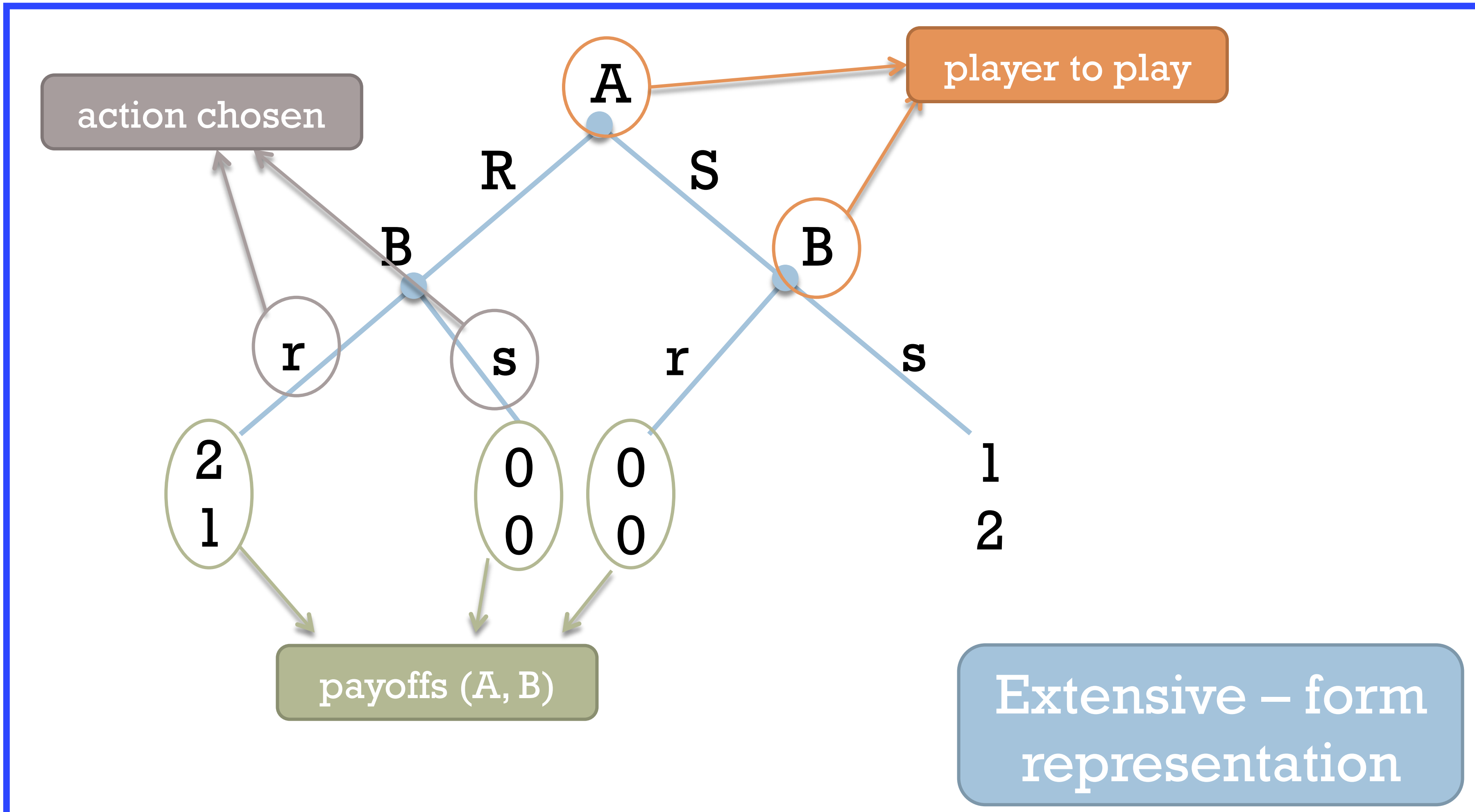
1. Set of players
2. Their payoff functions
3. Order of their move turns
4. Actions allowed to players when they can move
5. Information they have when they can move
6. Probability of external events
7. All of this: common knowledge

also true for normal form

added time dimension

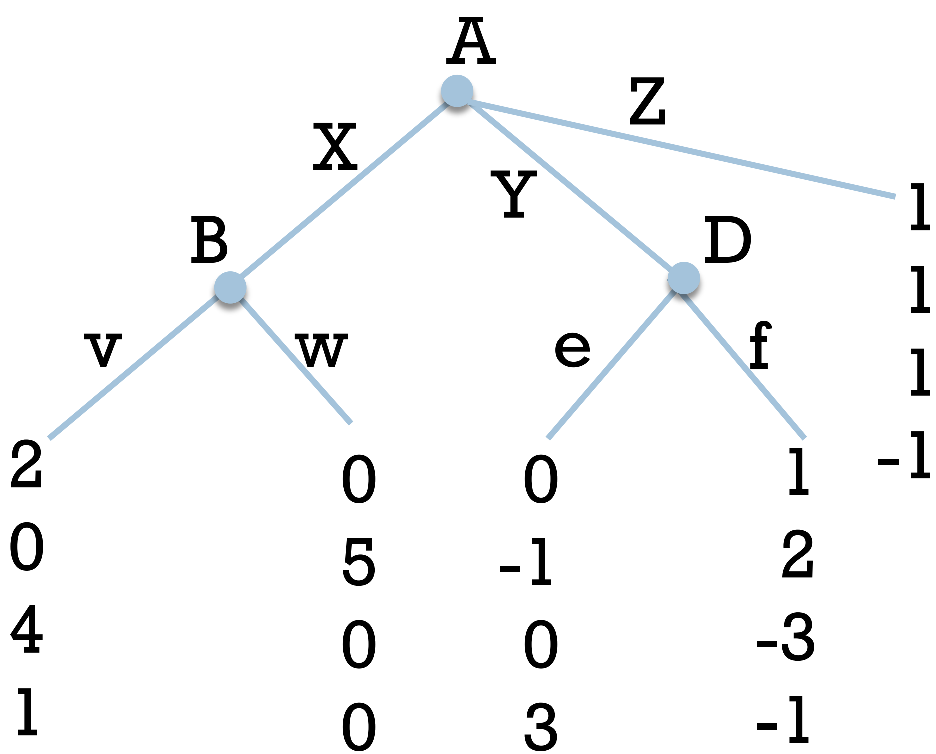
complete information

Sequential Battle of Sexes



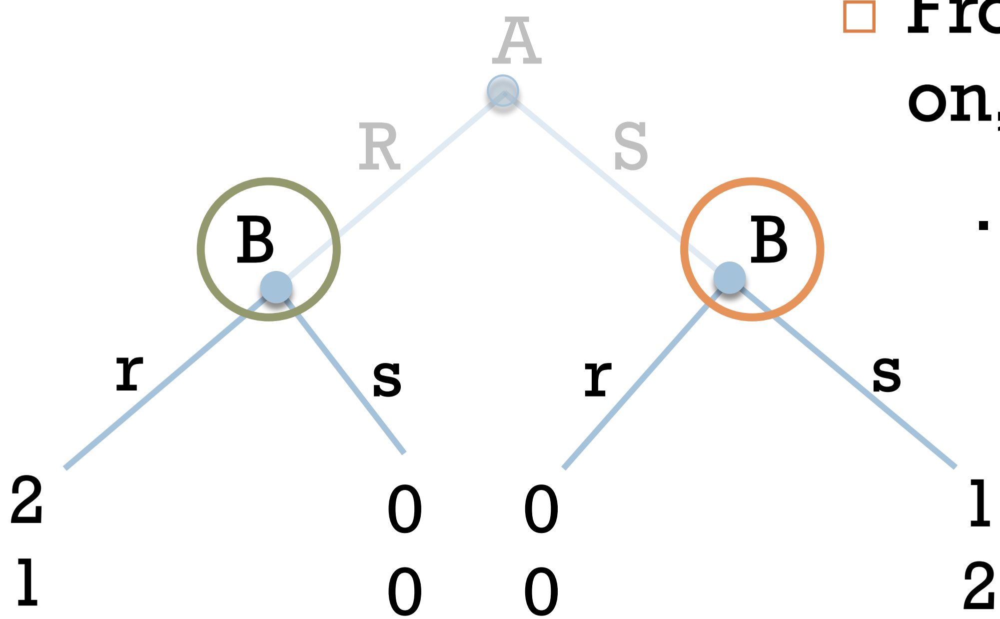
Dummy players

- An extensive form game may omit some of the players, e.g. if they have a single action



- Players = A,B,C,D
  - (see the outcomes)
- C never moves
- Also B and D may not have choices

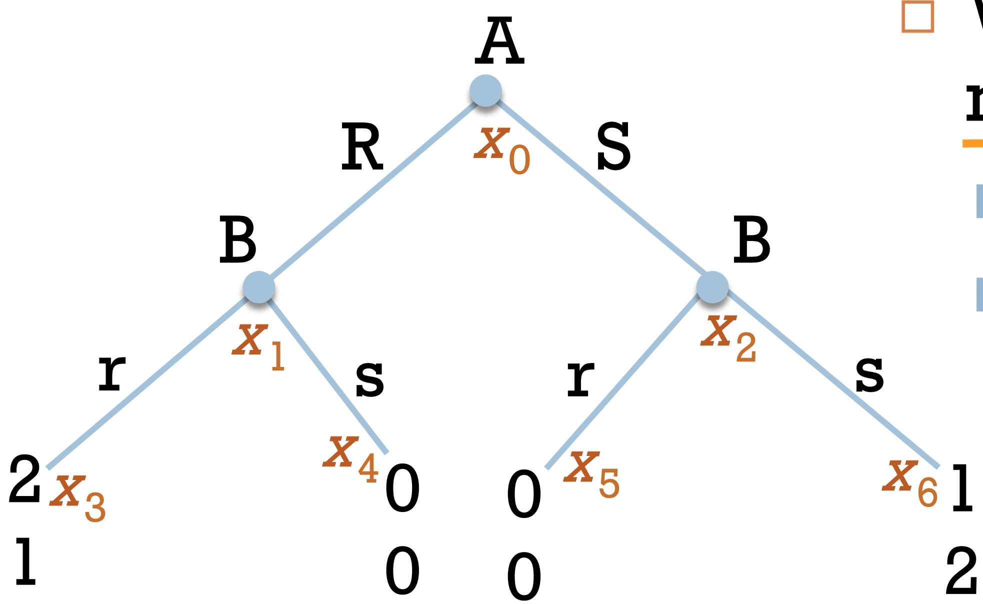
Information sets



- From these two points on, same actions for B
  ...but different payoffs
  why the difference?

- At these points, B is aware of A's choice
- Information is captured by different nodes

Information sets



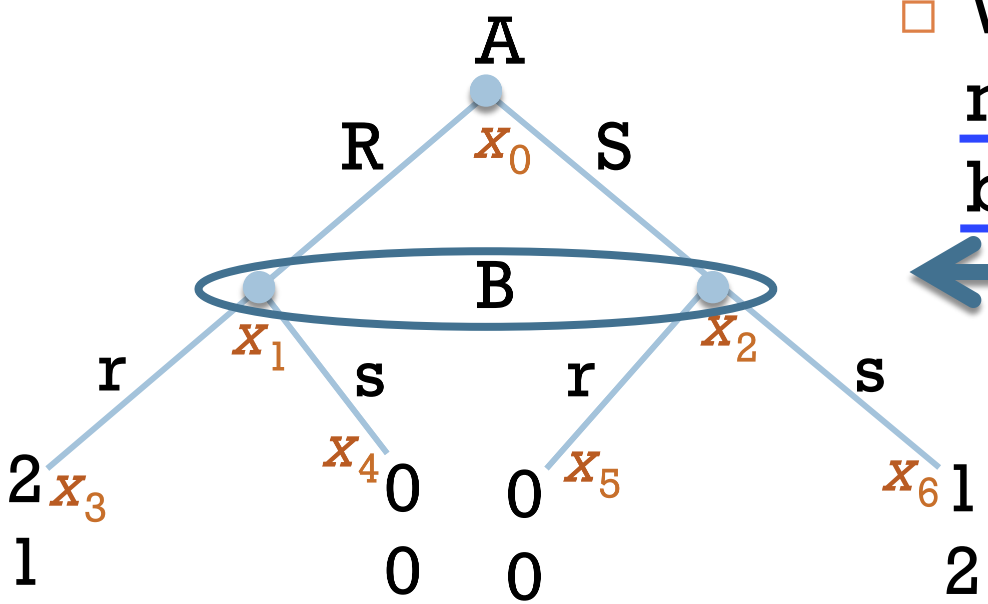
- We can label the nodes of the tree
  - x0 is the **root**
  - x3, x4, x5, x6 are the **terminal nodes**

- We can use a precedence relation (parent)
- Either terminal (→ payoffs) or parent node
  - (use this heavy notation only when needed)

Information sets

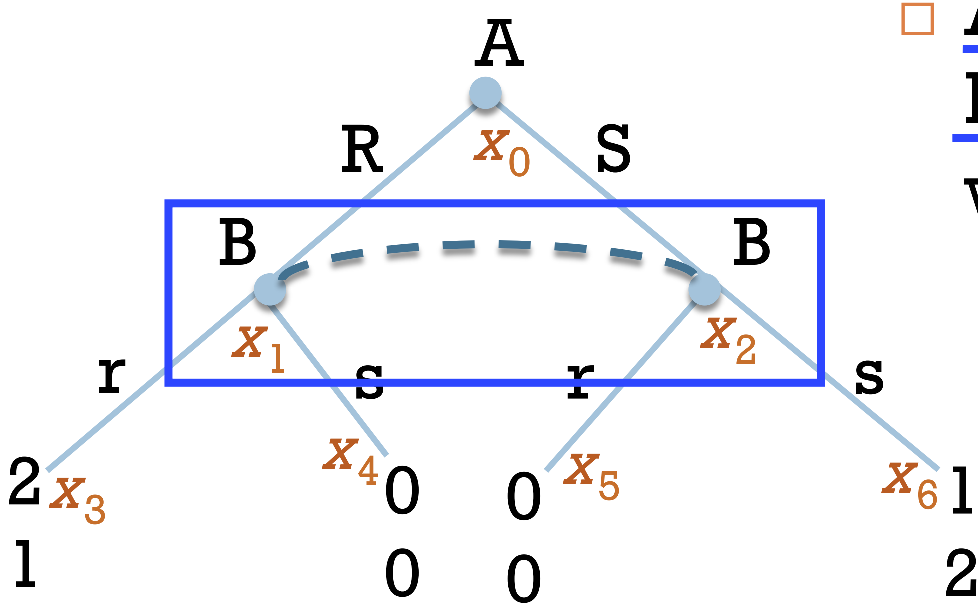
- Nodes go beyond denoting the game stage
- They also describe the **information set** h_i available to the player i that is to move
- If the information set is a singleton $\{x_j\}$ then the node is fully aware of the previous moves
- What if a node does not know?
  - In the original Battle of the Sexes, Brian does not know whether Ann chose R or S
  - Brian does not know whether he is at x_1 or x_2

Information sets



- We represent B not distinguishing between x_1 or x_2 as here
- In this case, we say that the information set of player B is $\boxed{\{x_1, x_2\}}$ (not a singleton)

Information sets



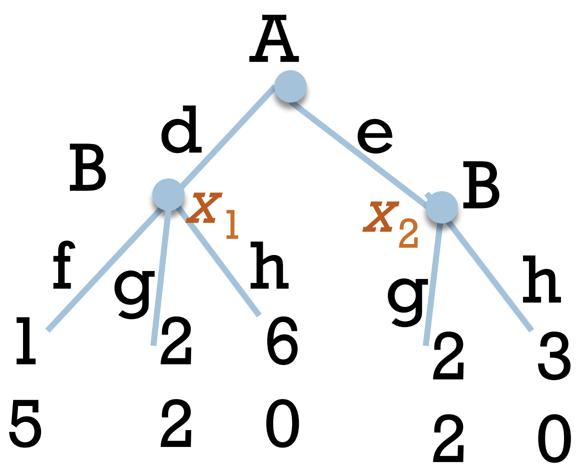
- Alternate notation for lack of information on whether B is @ x_1 or x_2, i.e. $h_B = \{x_1, x_2\}$

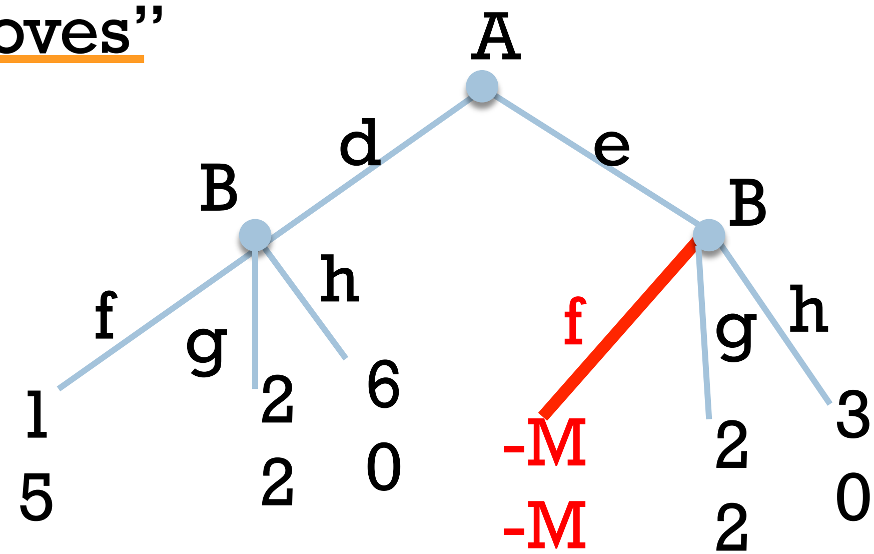
Information sets

- Every player i has a collection of information sets H_i, such that $h_i(x_j) \in H_i$ is associated to a node x_j where it is i's turn to move
  - It makes sense to assume that $x_j \in h_i(x_j)$
  - If $h_i(x_j)$ is a singleton $\{x_j\}$, then player i can move knowing for sure he/she is at node x_j
  - If h_i is a not singleton, i.e., $x_k \in h_i(x_j)$ and $x_j \neq x_k$ then i does not know if he/she is at x_j or x_k
  - In the latter case, it must be $A(x_j)=A(x_k)$, where $A(x)$ contains the actions available at node x

Set of available actions

- It can actually be made the same
  - exploit "forbidden moves"





- where –M = -∞ , or -10^{100} , or min(payoff)-1
- **not** the same game but every strategy choosing f at x_2 becomes strictly dominated, so…

Perfect/imperfect info. (repr'd)

- In dynamic games with **perfect information**, (1) all information sets are singletons and (2) there is no choice of Nature
- Instead, we have **imperfect information** if information sets contains multiple nodes or there is a choice of Nature

exogenous uncertainty (single-person decision problems with lotteries)

endogenous uncertainty (simultaneous moves)

  - In these two cases, players form beliefs

Re-defining strategies

- **Action** vs. **strategy**: In static games of complete information, pure strategy = action
- Then we saw mixed strategies = Prob(action)
- In a dynamic game, additional role by the **history of play** (through information sets)
- A player's **pure strategy** specifies an action according to what happened in the game
- Think of it as a program: a countermove for any possible case happened up to now

Re-defining strategies

- For the Battle of the Sexes with Ann moving first, both players choose a move within set A = {R,S}
- Brian has 2 actions, but more strategies
- A **strategy** is now a pair of elements of A

(a_R, a_S)

what to do if A played R

what to do if A played S

  - (s,s) = "I go to S no matter what"
  - (r,s) = "I do what Ann says"
  - (s,r) = "I avoid Ann"

Re-defining strategies

- If Ann and Brian repeat the original (static) battle for two consecutive nights
- A **strategy** is now a quintuple of moves:

first move

$(a_1, a_{2Rr}, a_{2Rs}, a_{2Sr}, a_{2Ss})$

answer to Ss

answer to Rr

answer to Rs

answer to Sr

  - Always go to R for both nights = (r, r, r, r, r)
  - Go to R the 1st night. If 1st night outcome is Rr, then go to S the 2nd night, else go to R = (r, s, r, r, r)

Re-defining strategies

- In principle, we may describe an "algorithm" for of all possible strategies

troppe strategie

- Yet, even a simple game with 2 sequential moves and $|A_1|=|A_2|=3$ has 27 possible strategies for player 2, since strategy $\in (A_2)^3$
- Therefore, we will often rely to some implicit description apart from very simple cases
  - Implementation: "left as a simple exercise"

What about mixed strategies?

- Previous definition: mixed strategies are probability distributions over the strategy set
- Now $\boxed{S = \text{all possible plans of actions}}$
  → $\boxed{m \in \Delta S = \{\text{prob(plan 1), prob(plan 2)}\ldots\}}$
- This does not look very "dynamic"
  - Probabilities are drawn **at the beginning** only, so that the player chooses a plan and sticks to it
  - Can we draw probabilities as the game unfolds?

Behavioral strategies

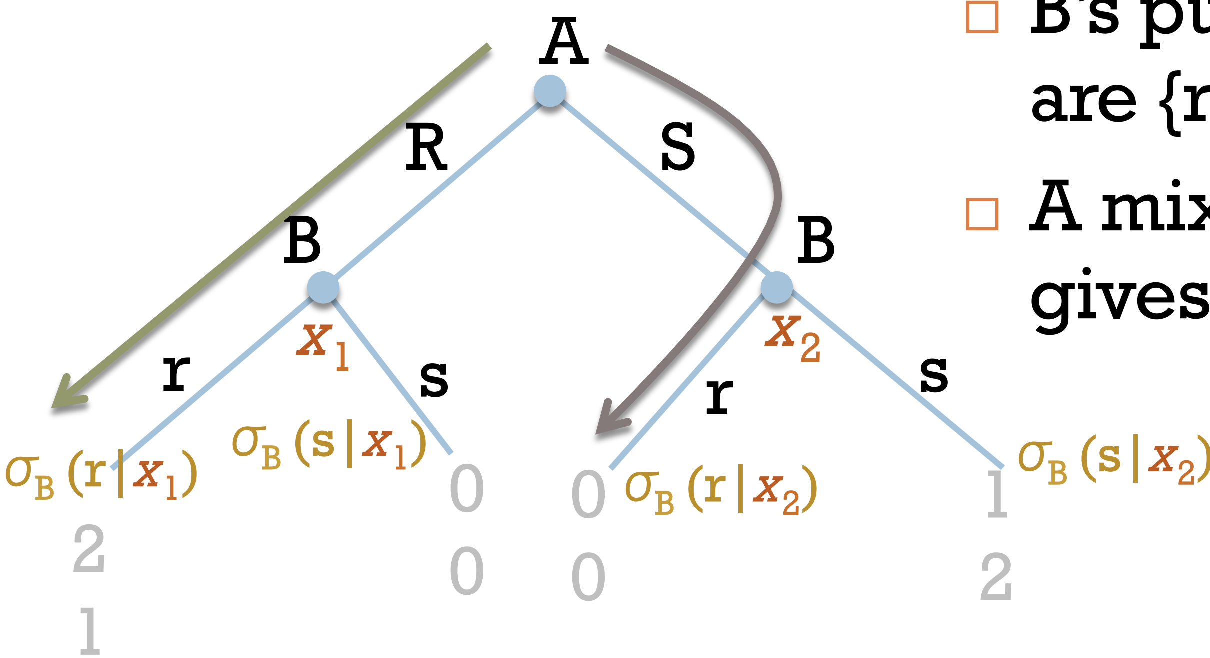
- A **behavioral strategy** specifies for any information set $h_i(x_j) \in \mathrm{H}_i$ an independent probability distribution over $A_i(h_i)$
- Denote this as $\sigma_i : \mathrm{H}_i \rightarrow \Delta A_i(h_i)$
  - Then $\sigma_i(\mathrm{a}_i \mid h_i)$ is the probability that *i* plays action $\mathrm{a}_i \in A_i(h_i)$ given information set $h_i = h_i(x_j)$, i.e., when he/she is at any x_j belonging to it
  - Note: the destination set cannot depend on h_i but we can use ΔA_i and set $\sigma_i(\mathrm{a}) = 0$ if $\mathrm{a} \notin A_i(h_i)$

Mixed ↔ behavioral

- By the analogy strategy = plan of action, think of it as a handbook (set of instructions)
  - a mixed strategy = take N handbooks, and select one of them at random
  - a behavioral strategy = a single handbook, which gives random instructions at any page

- Are these two descriptions equivalent?
- Luckily, yes (under some mild conditions)

Mixed ↔ behavioral



- B's pure strategies are {rr, rs, sr, ss}
- A mixed strategy gives p_{rr}, p_{rs}, p_{sr}, p_{ss}

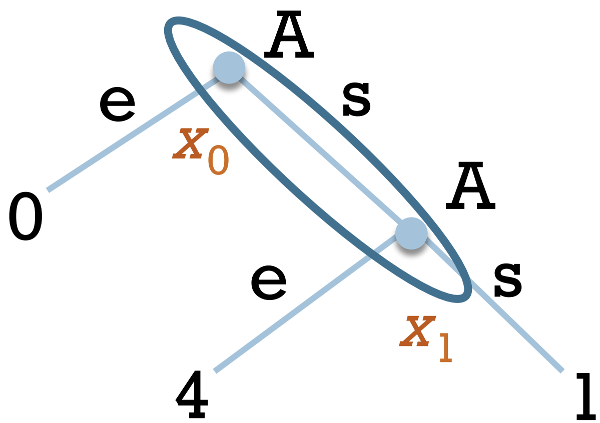
- A behavioral strategy is defined by $\sigma_i(a_i \mid h_i)$
  - Equal if: $\boxed{\sigma_B(r|x_1) = p_{rr} + p_{rs} \,;\ \sigma_B(r|x_2) = p_{rr} + p_{sr}}$
  - 4 equations in 4 unknowns (also $\boxed{\sigma_B(r) + \sigma_B(s) = 1}$)

Perfect recall

- The previous reasoning can be generalized
  - This can be extended also for the case with non-singleton information sets
- behavioral=mixed if this property holds
- **Perfect recall**: no player forgets information that he/she previously knew
  - This seems quite legitimate, and it is true for almost every game studied in the literature

Counter-example

- (Absent-minded driver) Andrew is driving on the highway and is now close to home
  - first exit: to a bad neighborhood → payoff 0
  - second exit: direct way home → payoff 4
  - third exit /road end: long route home → payoff 1

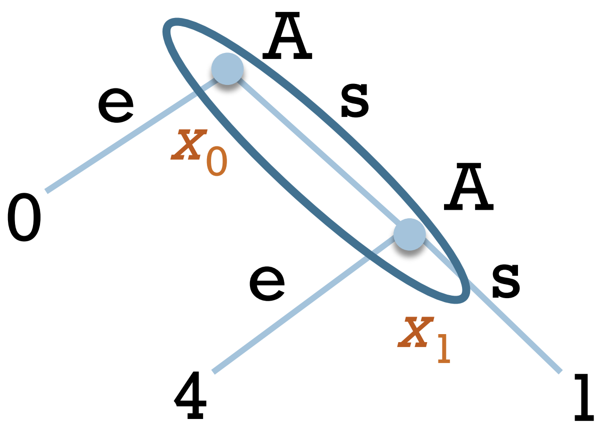


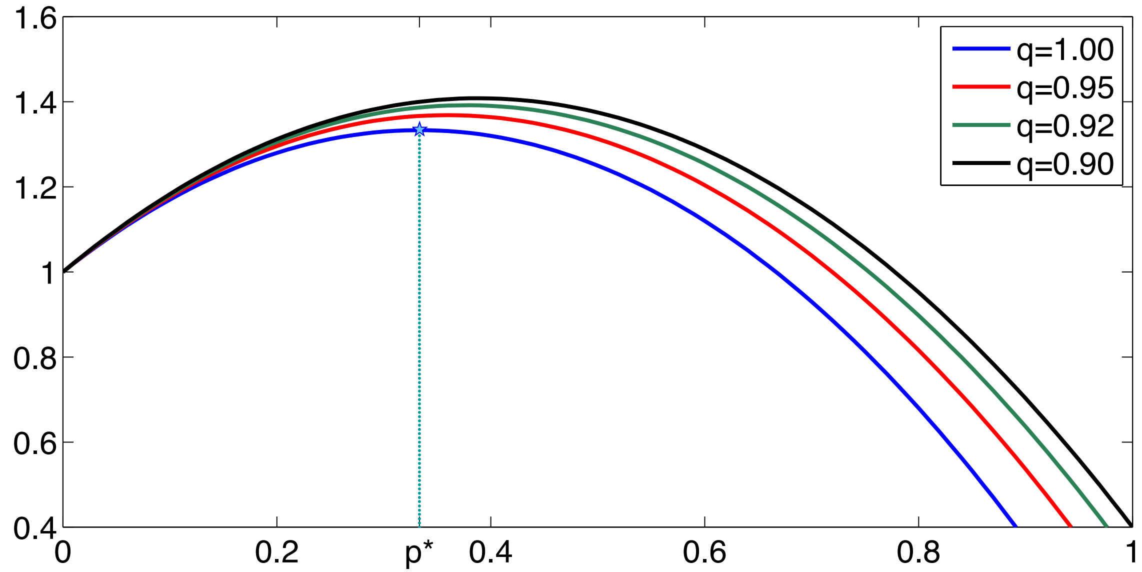
But Andrew is tired and when he passes an exit he is unsure of which is it

Information set = $\{x_0, x_1\}$

Counter-example

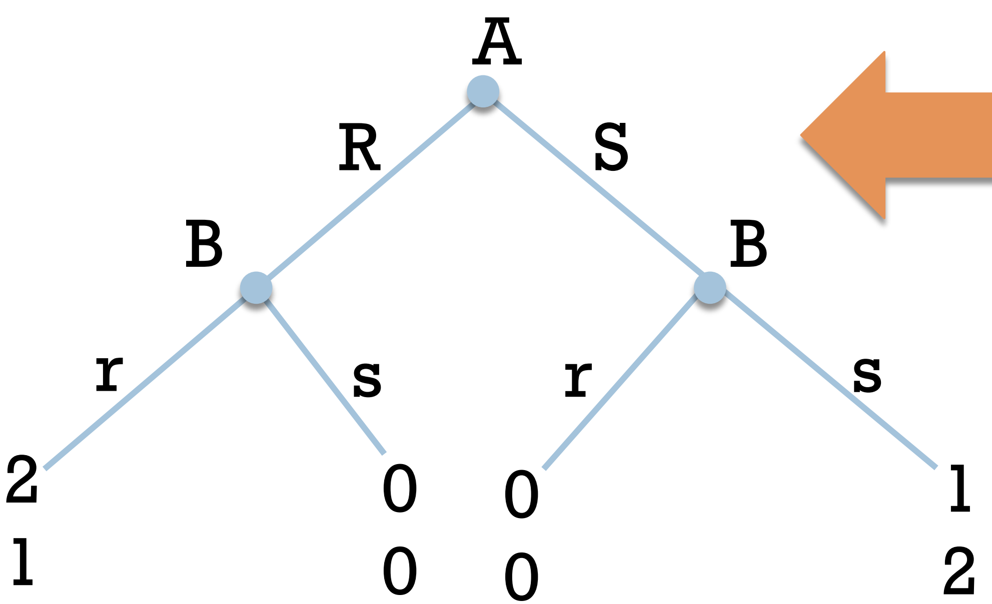
- Mixed strategy. Set p = Prob[e]
  $\mathbb{E}[u_A] = -3p^2+2p+1 \rightarrow$ Optimal $p^* = ⅓$
- Behavioral strategy. Set q = Prob[x_0]
  $\mathbb{E}[u_A] = -3qp^2 - qp + 3p + 1 \rightarrow$ same p^* only if $q = 1$





Extensive ↔ normal form

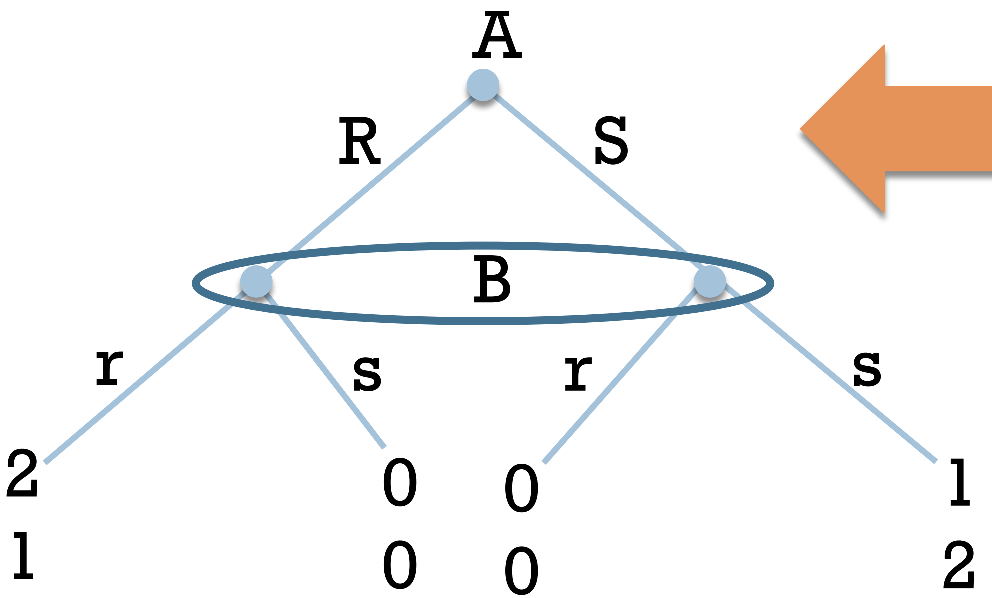
- The enumeration of strategies allows us to switch between extensive and normal form



		Brian			
		rr	rs	sr	ss
Ann	R	2,1	2,1	0,0	0,0
	S	0,0	1,2	0,0	1,2

Extensive ↔ normal form

- Similar equivalence for the simultaneous-play original version of the Battle of the Sexes



		Brian	
		r	s
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

Extensive ↔ normal form

- Multiple equivalences are possible
  - e.g. when terminal nodes have identical payoffs

		Beth	
		f	g
Alex	K	1, 5	3, 0
	E	2, 2	2, 2