Game theory

a course for the
MSc in ICT for Internet and multimedia

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Dynamic Nash equilibria

Direct extension of the definition

NE of a dynamic game

- Going back to the normal form seems to lose the dynamic character of the problem
- Yet, it is interesting for aspects, such as the Nash equilibrium, that are inherently static!

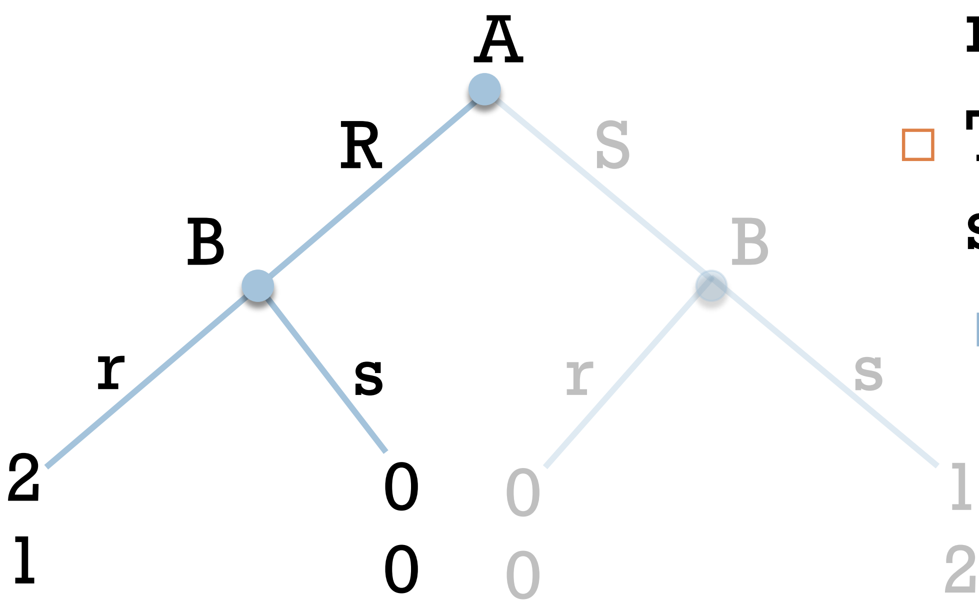
Ann \ Brian	rr	rs	sr	ss
R	2, 1	2, 1	0, 0	0, 0
S	0, 0	1, 2	0, 0	1, 2

NE of a dynamic game

- For the sequential-move Battle of the Sexes, we have three (pure) NE:
  - (R,rr): Ann plays R, Brian "always plays R"
  - (R,rs): Ann plays R, Brian "copies Ann's move"
  - (S,ss): Ann plays S, Brian "always plays S"
- Remember these strategies are chosen by Brian as though he is moving first!

NE of a dynamic game

- Compare two equilibria: (R, rr) and (R, rs)
  - Are they really different?



- Ann does not play S, no right-side tree
- They end up in the same outcome
  - Only part "r" of Brian strategy counts?

NE of a dynamic game

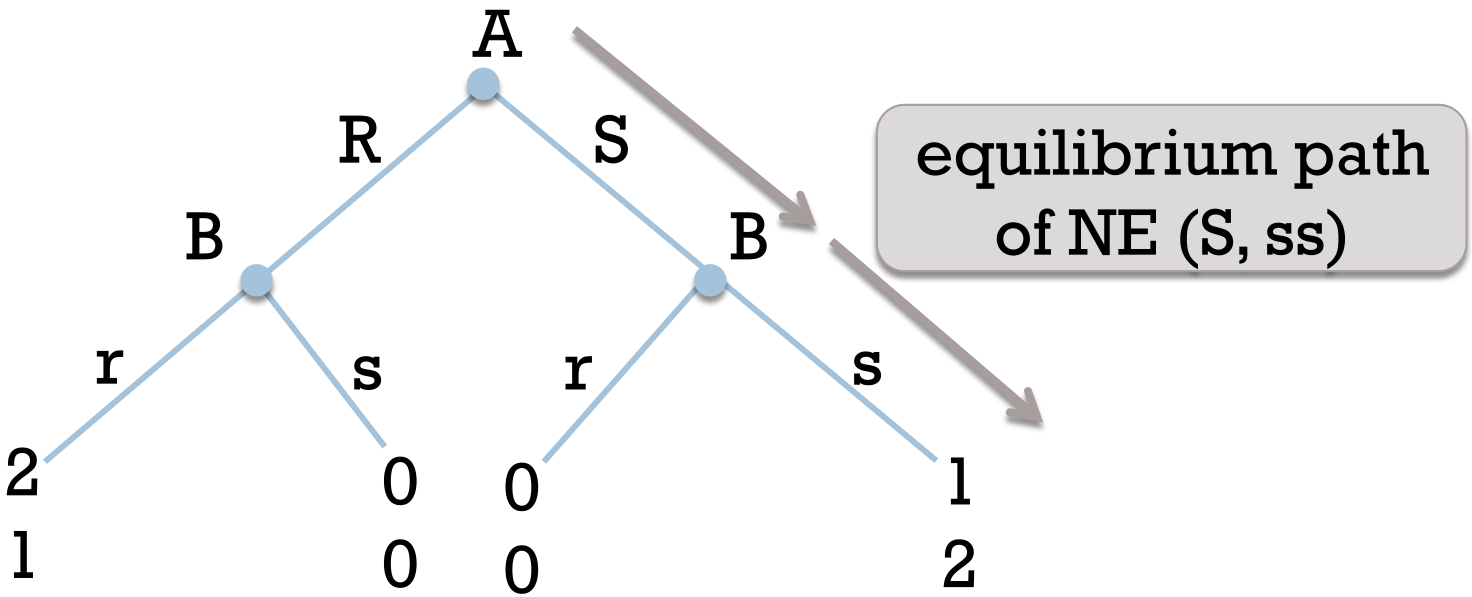
- There is indeed a difference between the two strategies: they are the same in equilibrium, but they are not identical off-equilibrium
  - Actually, when we are not in equilibrium, playing "rs" for Brian seems to be smarter (it is the strategy "do what Ann says")
  - "rr" has a non-rational answer (r to Ann's S): the thing is, it never comes into play!

NE of a dynamic game

- Representing situations that will not come into play is not really strange
- Remember that in 2-night Battle of the Sexes we included also strategies such as "Go to R the 1st night. If 1st outcome is Rr, then go to S the 2nd night, else go to R" = (r, s, r, r, r)
  - Do we need this part of the strategy? (reply to Ss)
  - The strategy demands the 1st move to be r, so Ss cannot happen. Yet, we need this specification, not for this player, but for the others!

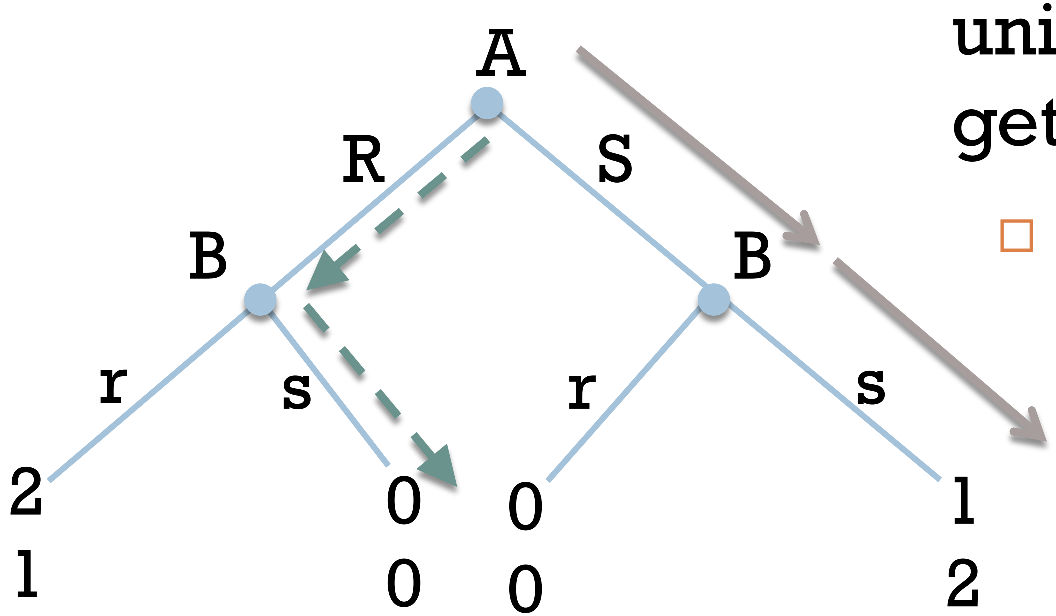
Equilibrium path

- Given a joint profile of behavioral strategies $m^* = (m_1^*, m_2^*, \ldots, m_n^*)$ that is a NE, its equilibrium path contains the decision tree nodes that are reached with probability > 0



Equilibrium path

- NEs are determined by belief about what other people are doing both **on and off** the equilibrium path!

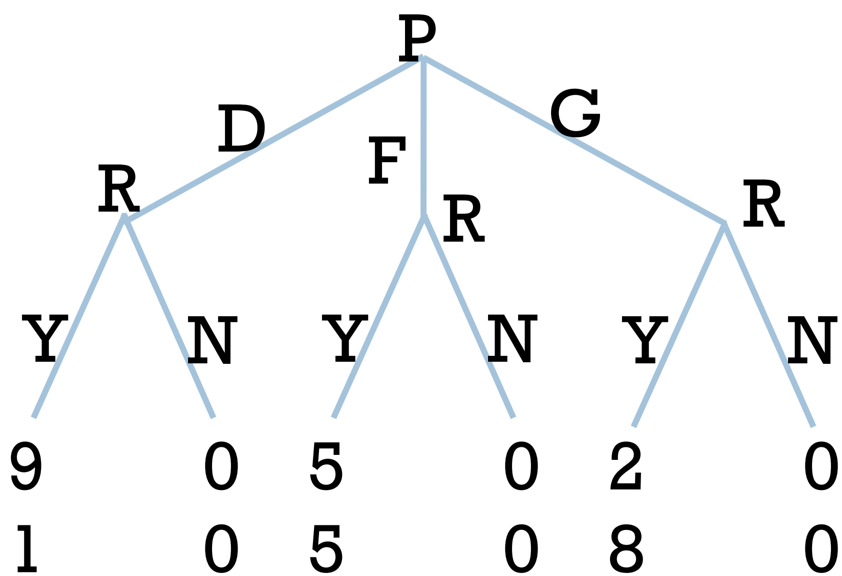


- (S, ss) is a NE since A's unilateral deviation gets "punished" by B
  - B "threatens" A
    - is this credible?

Example: Ultimatum game

- Two players share 10 candies as follows
  - Player 1(Proposer) presents a split
  - Player 2(Responder) decides whether to accept it
  - If Player 2 refuses, they both get nothing
- For simplicity, A_P={"D"(9-1), "F"(5-5), "G"(2-8)}
- Actions A_R={"Y"(accept), "N"(refuse) }
  - The **strategies** of R are more complex, e.g., "play Y if the offer is D or F but not G"
  - Thus, they are a triple (x_1, x_2, x_3) where x_j = Y or N

Example: Ultimatum game



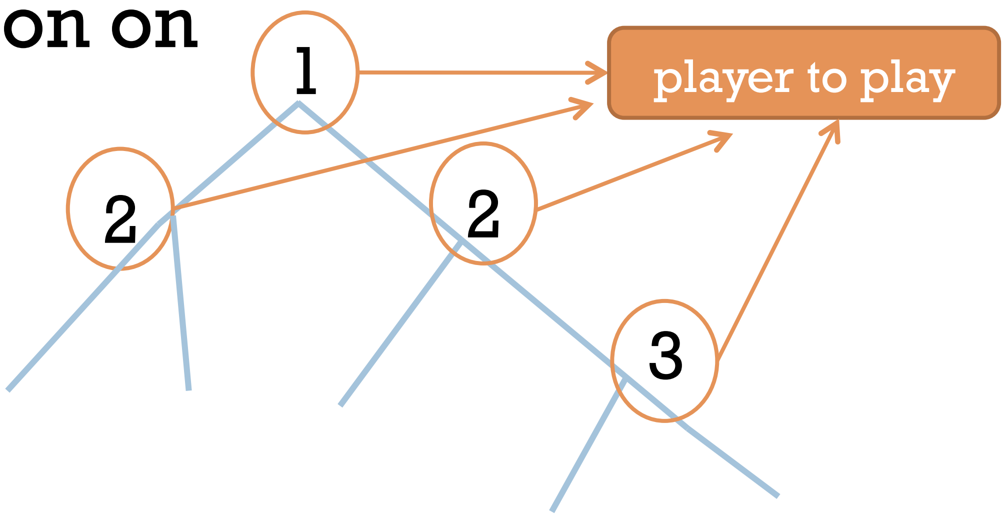
- Joint strategies "offer x/10-x" (proposer) and "refuse if P's share is more than x" (responder) are NEs: no player has incentive to deviate

Rationality and credibility

How to solve dynamic games

Perfect vs imperfect information

- A dynamic game with perfect information is a **sequential game** that can be represented with a regular decision tree (all the information sets are singletons)
- Players move one after another; later players have full information on previous players' choices and can exploit it



Dynamic game, perfect inf.

- This class of games can be solved by means of **backward induction**
- To see why, consider just a 2-players setup
  - Player **1** chooses action $\mathbf{a}_1$ from set A_1
  - Player **2** sees $\mathbf{a}_1$ and chooses action $\mathbf{a}_2 \in A_2$
  - A_2 depends on $\mathbf{a}_1$ (the game can even end after player 1's move, if $A_2 = \{\mathbf{a}_2^*\}$, so 2 has no choice)
  - Players receive payoffs $u_1(\mathbf{a}_1, \mathbf{a}_2)$ and $u_2(\mathbf{a}_1, \mathbf{a}_2)$

Dynamic game, perfect inf.

- We can assume that player 2 can always optimize his/her move
  - Because of **perfect** information, player 2 knows has the right information set (singleton)
  - Thanks to **complete** information, player 1 can anticipate the optimization and do the same
- Theorem (~Zermelo). Any dynamic game of perfect information has a backward induction solution that is sequentially rational; if terminal payoffs are all different, it is unique

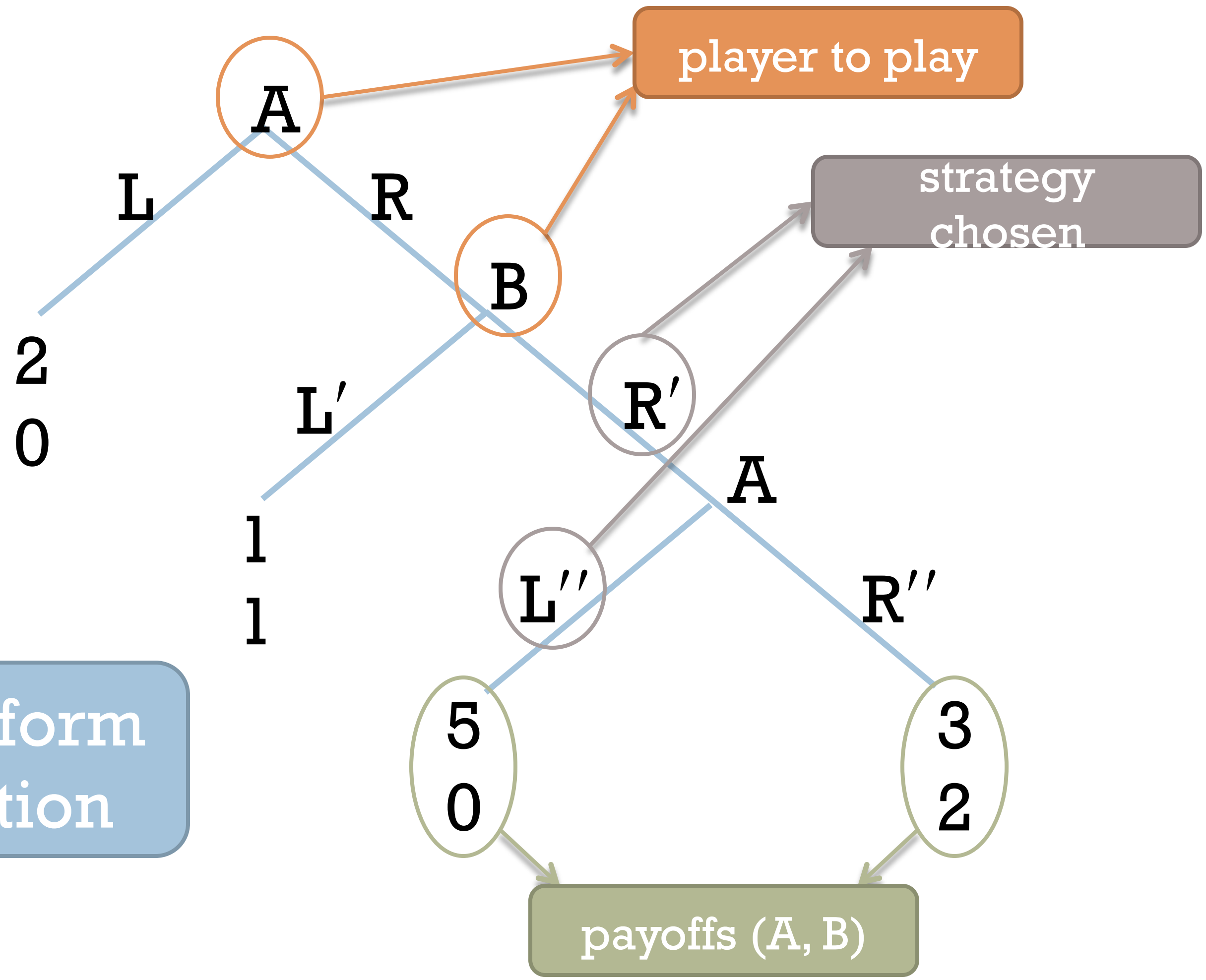
Backward induction

- When it is his/her turn, Player 2 sees Player 1's move a_1^h and solves the optimization problem
$$\max_{a_2 \in A_2} u_2(a_1^h, a_2)$$
- Call $R_2(a_1^h)$ the argmax solving the problem, i.e., a_2 yielding the max. Due to complete info, 1 anticipates 2's reaction and solves
$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$$
- 1's solution is a_1^*, the outcome is a_1^*, $R_2(a_1^*)$
It is a **Nash equilibrium in pure strategies**

Example: Trust game

- Consider the following game
  1. A chooses either L or R. L ends the game with payoffs 2 for A and 0 for B. R gives B the right to move (step 2)
  2. B chooses either L' or R'. L' ends the game with payoffs 1 for A and 1 for B. R' gives A the right to move (step 3)
  3. A chooses either L'' or R''. Both end the game, with respective payoffs 5 or 0 for L'' and 3 or 2 for R''
- We can represent this sequence with a tree

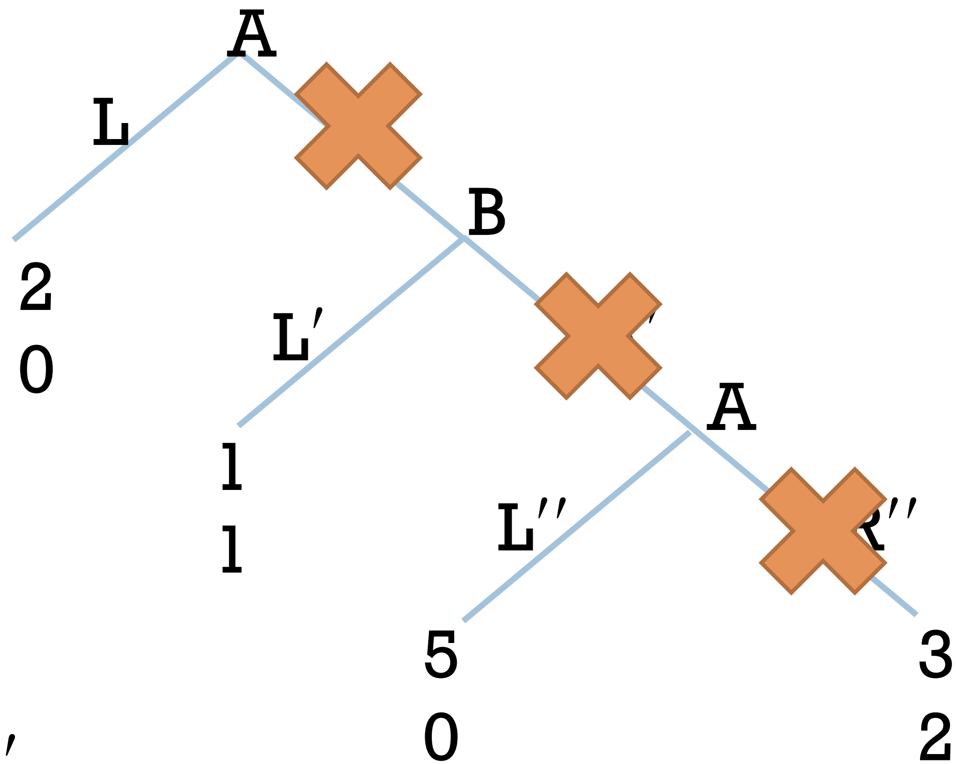
Example: Trust game



Extensive – form representation

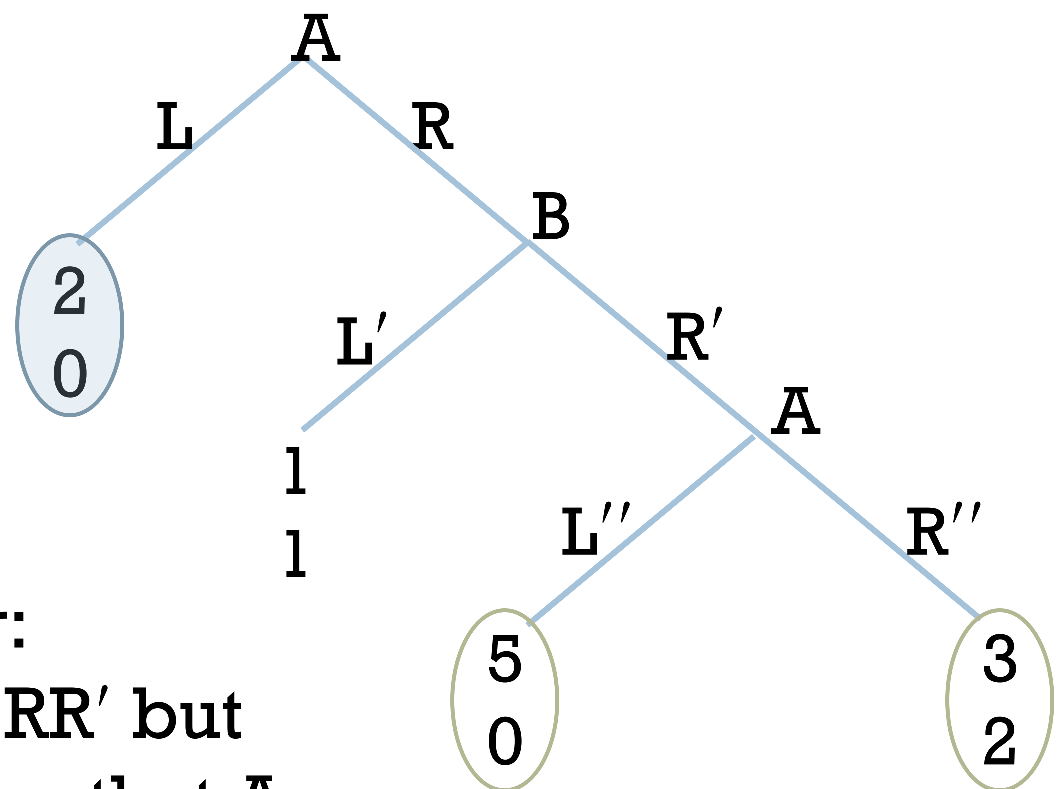
Example: Trust game

- Apply backward induction.
- A prefers L′′ over R′′
- Knowing R′ will end up in A playing L′′, B will choose to play L′
- Knowing this, A plays L



Example: Trust game

- Payoffs: 2 and 0
  - inefficient solution (in Pareto sense)
- Rational players do not trust each other: A can ask B to play RR′ but there is no guarantee that A will play R′′ (not L′′), nor that B plays L′ instead



Imperfect information

- Consider now a dynamic game with complete but imperfect information
- A basic model for this kind of games is
  - Players 1 and 2 choose actions a_1 and a_2 from sets A_1 and A_2, respectively
  - Players 3 and 4 observe the outcome of this and choose a_3 and a_4 from A_3 and A_4
- Payoffs are $u_j(a_1, a_2, a_3, a_4)$ for $j = 1,2,3,4$

Note: players are not necessarily distinct or all present

Imperfect information

- Use an approach akin to backward induction.
- For every choice ($\mathbf{a}_1$,$\mathbf{a}_2$) of the first two players, players 3 and 4 play a Nash equilibrium ($\mathbf{a}_3$*($\mathbf{a}_1$, $\mathbf{a}_2$), $\mathbf{a}_4$*($\mathbf{a}_1$, $\mathbf{a}_2$))
  - Players 1 and 2 know and anticipate it, as if they play a simultaneous-move game with payoffs u_j($\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$*($\mathbf{a}_1$, $\mathbf{a}_2$), $\mathbf{a}_4$*($\mathbf{a}_1$, $\mathbf{a}_2$)) for $j = 1,2$
  - They take $\mathbf{a}_1$* , $\mathbf{a}_2$* as NE of this game
- ($\mathbf{a}_1$*, $\mathbf{a}_2$*, $\mathbf{a}_3$*($\mathbf{a}_1$*, $\mathbf{a}_2$*), $\mathbf{a}_4$*($\mathbf{a}_1$*, $\mathbf{a}_2$*)) is the outcome resulting from backward induction

Subgame-perfect NE

Extending the Nash equilibrium concept

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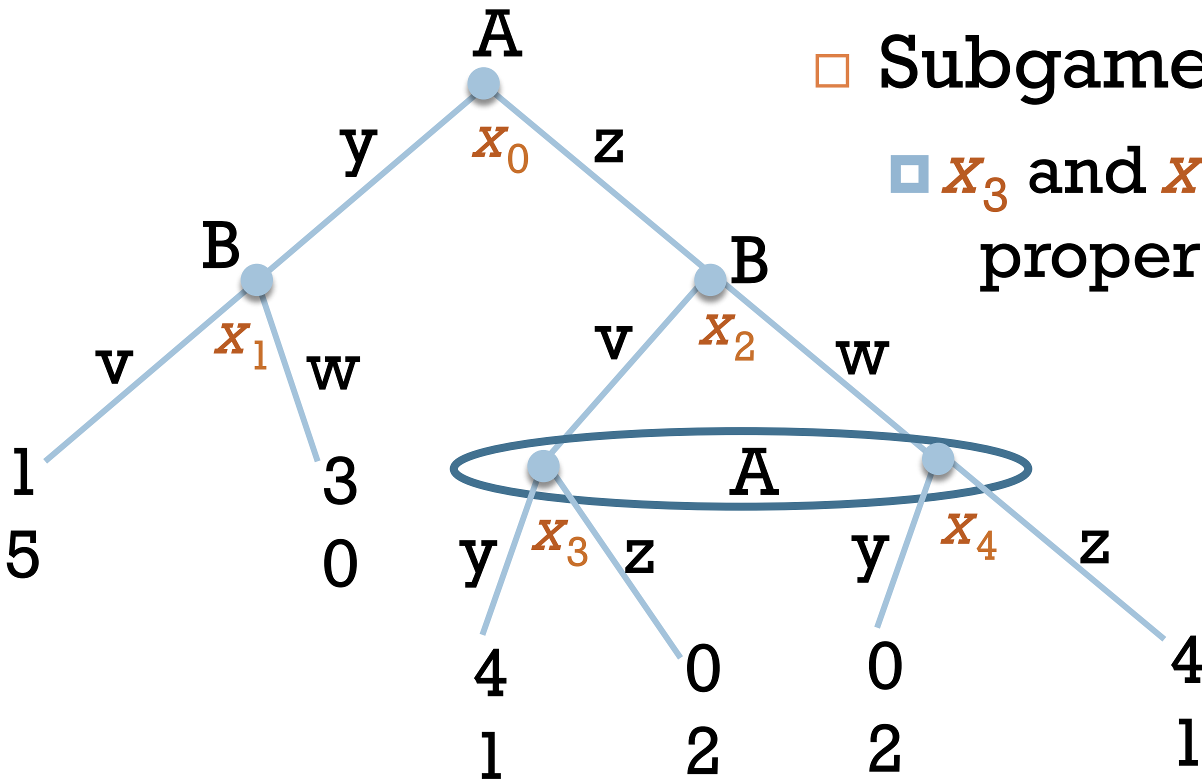
Subgames

- Game follows a tree: what about subtrees?
- A (proper) **subgame** **G** contains a single node of the tree and all of its successor nodes, with the requirement: $x_j \in \mathbf{G},\ x_k \in h_i(x_j) \Rightarrow\ x_k \in \mathbf{G}$
- All other requirements (players, payoffs, common knowledge) are left unchanged
- The whole game is a subgame of itself

Subgames



- Subgames: x_0, x_1, x_2.
  - x_3 and x_4, are not proper subgames

Subgame-perfect NE

- Definition (R. Selten). A Nash equilibrium is **subgame-perfect** if the strategies chosen by the players give a NE in **every** subgame
  - It is a refinement of NE. In a subgame-perfect Nash equilibrium (SPE) the players strategies must first be a NE and then must pass an additional test
- Every finite extensive form game has an SPE
  - This means that every game, from tic-tac-toe to chess or go, has an optimal way to be played

Subgame-perfect NE

- How to prove that SPE is unique? For perfect information game with finite horizon, SPE is the outcome of backward induction
- This can be somehow extended for other dynamic games, by taking into account the **credibility** of the threats
- Credibility: Player 1 knows $\mathbf{a}_1$ implies response $\boldsymbol{R}_2$ ($\mathbf{a}_1$), so strategies "if $\mathbf{a}_1$ then $\mathbf{a}_2$ $\neq \boldsymbol{R}_2$ ($\mathbf{a}_1$)" are classified as non-credible

Credibility of threats

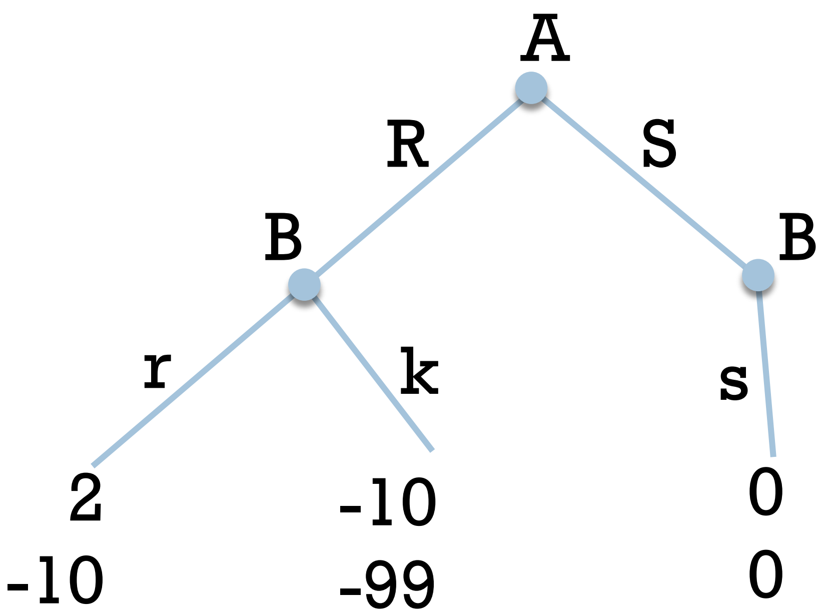
- Consider again the dynamic battle of the sexes with Ann moving first
  - Brian can play (ss) meaning that, even when Ann selects R, he goes to S
  - Ann does not believe him and decides what she prefers, knowing Brian's threat is empty
- In Hawk-and-Dove the threat to deviate from NE is non-credible (it hurts both)

Credibility of threats

- An extreme version of incredible threat
  - There is no (S)ci-fi movie at the theater, just one (R)omance movies that Brian hates
  - Now, option S means = "stay at home" that is probably the best option for Brian: if Ann chooses this, then the game ends
  - But if Ann decides to go (R), then Brian has just two options: to comply (r) or to kill himself (k)
  - Brian may consider strategy (s,k)

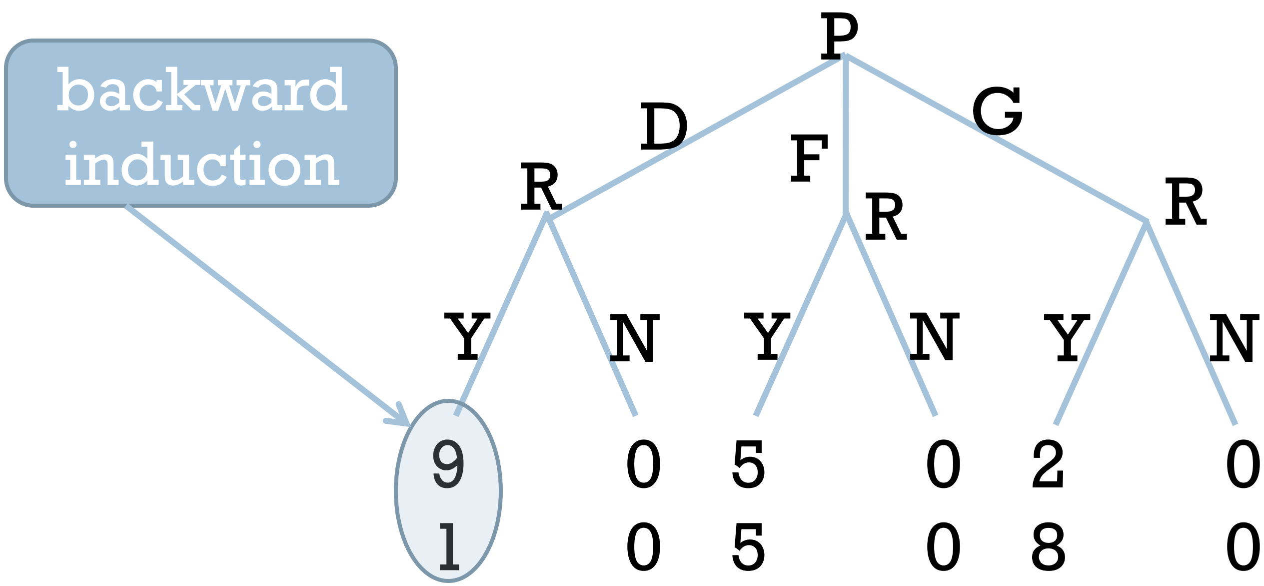
Credibility of threats

- Brian may consider strategy (k,s)
  - This means to threaten Ann to commit suicide if she insists in going to R



- Ann can be tempted to play S to avoid this
- However, B choosing k instead of r would be irrational
  - Non-credible threat!

back to Example 11



- Many NEs, **one** SPE: "P chooses D" "R accepts"
- P knows that R is better off if accepting any proposal, since it is "something" against "nothing"
- Not accepting is a non-credible threat