Game theory

a course for the
MSc in ICT for Internet and multimedia

Leonardo Badia

leonardo.badia @gmail.com

No Nash Equilibrium?

- In general, finding Nash Equilibria is tricky
- Sometimes, we do not seem to have one
  - For example, in the Odds&Evens game…

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

which feels somehow incomplete (especially if we want to use Nash Equilibrium as solution/prediction)

Mixed strategies

Uncertainty makes the games interesting

Leonardo Badia – leonardo.badia@gmail.com

Missing outcome

- Expand Odds&Evens to find the outcome

forziamo NE

		Even		
		0	½	1
Odd	0	-4, 4	0, 0	4, -4
	½	0, 0	0, 0	0, 0
	1	4, -4	0, 0	-4, 4

scelta intermedia ⇙

- It seems that (½ ,½) is a NE. Let formalize this.

Mixed strategies

- If A is a non-empty discrete set, a **probability distribution** over A is a function $p : A \rightarrow [0,1]$, such that $\sum_{x \in A} p(x) = 1$
- The set of possible probability distributions over A is called the **simplex** and denoted as ΔA
- For a normal form game $(S_1,\ldots,S_n ; u_1,\ldots,u_n)$, a **mixed strategy** for player *i* is a probability distribution m_i over set S_i
- That is, *i* chooses strategies in $S_i = (s_{i,1},\ldots,s_{i,n})$ with probabilities $(m_i(s_{i,1}),\ldots, m_i(s_{i,n}))$

Leonardo Badia – leonardo.badia@gmail.com

Expected payoff

- Utility u_i can be extended to the expected utility, which is a real function over $\Delta S_1 \times \Delta S_2 \times \ldots \times \Delta S_n$
- If players choose mixed strategies $(\overset{s}{\cancel{m}}_1, \ldots, \overset{s}{\cancel{m}}_n)$, $=: s$ compute player i 's payoff by weighing on m_i's

$$u_i(m_1, \ldots, m_n) = \sum_{s \in S} m_1(s_1) \cdot m_2(s_2) \cdot \ldots \cdot m_n(s_n) \cdot u_i(s)$$

- In other words:
  - fix a global strategy s
  - compute its probability
  - weigh the utility of s on this probability and sum

Intuition

- Consider Odds&Evens game and assume Odd decides to play 0 with probability *q*, while Even plays 0 with probability *r*
  - Consequently 1 is played by Odd and Even with probability 1-*q* and 1-*r*, respectively

		Even	
		0 (prob r)	1 (prob 1-r)
Odd	0 (prob q)	$-4qr, 4qr$	$4q(1-r), -4q(1-r)$
	1 (prob 1-q)	$4(1-q)r, -4(1-q)r$	$-4(1-q)(1-r), 4(1-q)(1-r)$

this is a **single** global strategy $\boxed{m = (m_1, m_2) = (q, r)}$

Leonardo Badia – leonardo.badia@gmail.com

Intuition

- In other words, we revise the game so that each player can choose not only either 0 or 1, but also a value between them: q for Odd, r for Even
- Odd’s payoff is $-16qr + 8q + 8r - 4 = -4(2q-1)(2r-1)$

		Even	
		0 (prob r)	1 (prob $1-r$)
Odd	0 (prob q)	$-4qr$, $4qr$	$4q(1-r)$, $-4q(1-r)$
	1 (prob $1-q$)	$4(1-q)r$, $-4(1-q)r$	$-4(1-q)(1-r)$, $4(1-q)(1-r)$

Intuition

- In other words, we revise the game so that each player can choose not only either 0 or 1, but also a value between them: q for Odd, r for Even
- Odd's payoff is $-16qr + 8q + 8r - 4 = -4(2q-1)(2r-1)$

			Even	
		0	r	1
Odd	0			
	q		$-16qr + 8q + 8r - 4$, $16qr - 8q - 8r + 4$	
	1			

Pure strategies

- Given a mixed strategy $m_i \in \Delta S_i$ we define the **support** of m_i as $\{s_i \in S_i : m_i\,(s_i) > 0\}$
- Each strategy $s_i \in S_i$ (an element of S_i) can be identified with the mixed strategy p (which is an element of ΔS_i) such that $p(s_i) = 1$
  - Hence, $p(s'_i) = 0$ if $s'_i \neq s_i$ and also support(p)=$\{s_i\}$
- Thereafter, we identify p with s_i : Pure strategy s_i is seen as a degenerate probability distribution
  - Previous definitions of dominance and NE only refer to the pure strategy case

Strict/weak dominance

- Consider game $G = \{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$.
- If $m_i', m_i \in \Delta S_i$, m_i' **strictly dominates** m_i if

  $u_i(m_i', m_{-i}) > u_i(m_i, m_{-i})$ for every m_{-i}
- We say that m_i' **weakly dominates** m_i if

  $u_i(m_i', m_{-i}) \geq u_i(m_i, m_{-i})$ for **every** m_{-i}

  $u_i(m_i', m_{-i}) > u_i(m_i, m_{-i})$ for **some** m_{-i}
- Note: there are infinitely (and continuously) many m_{-i} in the set: $\Delta S_1 \times \ldots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \ldots \times \Delta S_n$

Strict/weak dominance

visto che considero solo i, posso togliere $m_i(s_i)$ da sommatoria di u_i

- However, it is possible to prove that:

  If m_i', $m_i \in \Delta S_i$, m_i' **strictly dominates** m_i if

  $u_i(m_i', s_{-i}) > u_i(m_i, s_{-i})$ for every $s_{-i} \in S_{-i}$

- Similarly, m_i' **weakly dominates** m_i if

  $u_i(m_i', s_{-i}) \geq u_i(m_i, s_{-i})$ for **every** $s_{-i} \in S_{-i}$

  $u_i(m_i', s_{-i}) > u_i(m_i, s_{-i})$ for **some** $s_{-i} \in S_{-i}$

- That is, we can limit our search to pure strategies of the opponents.

Nash equilibrium

- Consider game G = $\{S_1,\ldots,S_n\,;\,u_1,\ldots,u_n\}$.
- A joint mixed strategy $m \in \Delta S_1 \times \ldots \times \Delta S_n$ is said to be a Nash equilibrium if for all i :

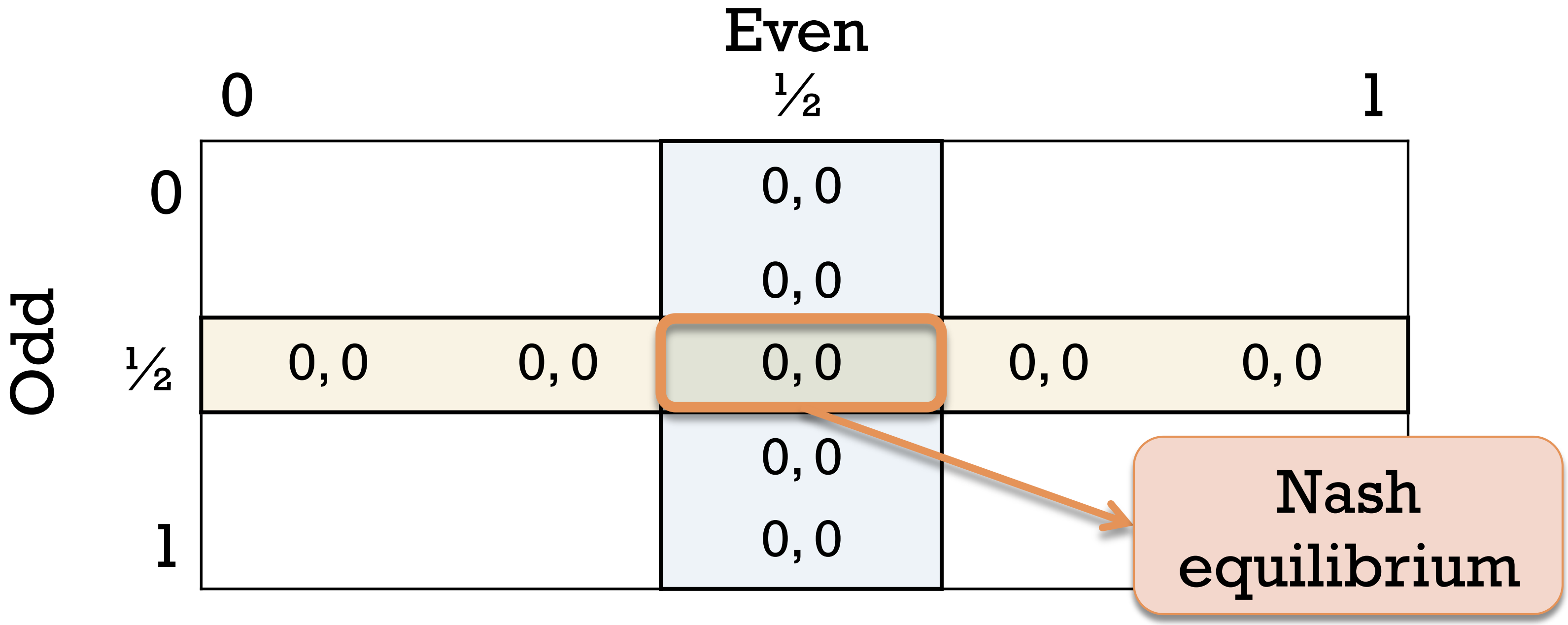
$$u_i(m) \geq u_i(m_i',m_{-i}) \text{ for every } m_i' \in \Delta S_i$$

- This reprise the same concept of NE in pure strategies: no player has an incentive to change his/her move (which is a mixed strategy now)

Leonardo Badia – leonardo.badia@gmail.com

back to Example 3

- In the Odds&Evens game, the payoff for Odd is $-4(2q - 1)(2r - 1)$, the opposite for Even.
- If $q = ½$, or $r = ½$, **both** players have payoff 0.
- If $q = r = ½$ no player has incentive to change.



Even

Odd	0	½	1
0		0, 0 0, 0	
½	0, 0 0, 0	0, 0	0, 0 0, 0
1		0, 0 0, 0	

Nash equilibrium

Leonardo Badia – leonardo.badia@gmail.com

back to Example 3

- As an exercise, prove that (½ , ½) is the **only** Nash Equilibrium of the Odds&Evens game
- How to proceed
  - Consider three cases, where the payoff of player Odd is <0, >0, =0 but joint strategy is not (½ , ½)
  - Show that in each case there is a player (who?) having an incentive in changing strategy
  - None of this is a NE. (½ , ½) is the only one

Using mixed strategies

and introducing the Nash theorem

Leonardo Badia – leonardo.badia@gmail.com

IESDS vs mixed strategies

player B

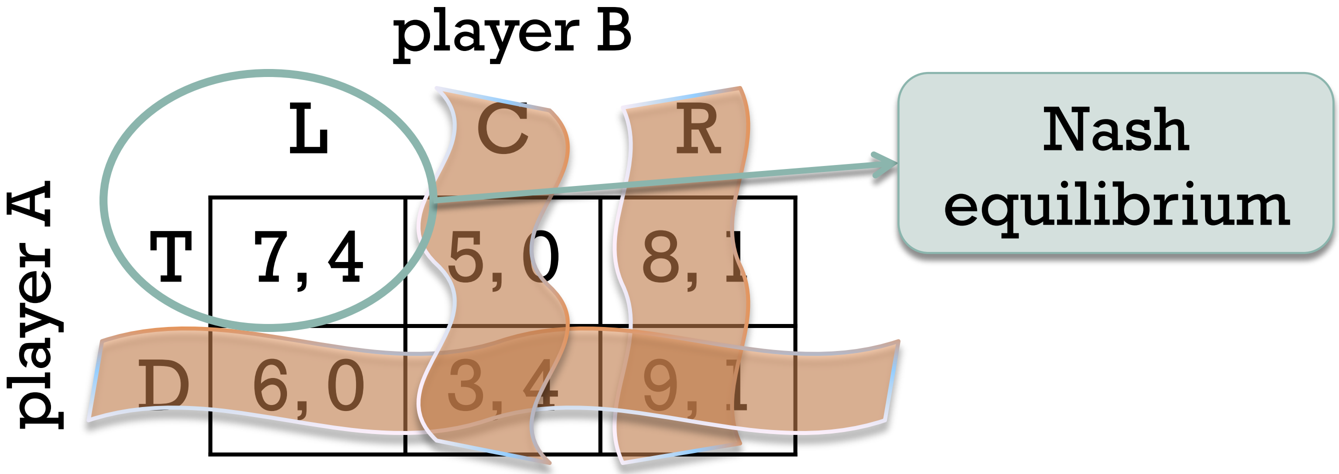
player A

	L	C	R
T	7, 4	5, 0	8, 1
D	6, 0	3, 4	9, 1

- R is not dominated by L or C. But mixed strategy $\boxed{m = ½\, L + ½\, C}$ gets $u_B = 2$ regardless of A’s move
- Pure strategy R is strictly dominated by *m*
  - R can be eliminated
  - Further eliminations are possible

Leonardo Badia – leonardo.badia@gmail.com

IESDS vs mixed strategies



	L	C	R
T	7, 4	5, 0	8, 1
D	6, 0	3, 4	9, 1

- R is not dominated by L or C. But mixed strategy $m = \tfrac{1}{2}\mathrm{L} + \tfrac{1}{2}\mathrm{C}$ gets $u_{\mathrm{B}} = 2$ regardless of A's move
- Pure strategy R is strictly dominated by *m*
  - R can be eliminated
  - Further eliminations are possible

Leonardo Badia – leonardo.badia@gmail.com

IESDS vs mixed strategies

- Similar theorems to the pure strategy case hold for IESDS in mixed strategies (IESDSm).
- **Theorem.** Nash equilibria survive IESDSm.
- **Theorem.** The order of IESDSm is irrelevant.

- **Note:** Use strict (not weak) dominance!
  A weakly dominated strategy can be a NE, or belong to the support of a NE

Leonardo Badia – leonardo.badia@gmail.com

Characterization

- **Theorem**. Take a game $G = \{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$ and a joint mixed strategy m for game G
The following statements are equivalent:

(1) Joint mixed strategy m is a Nash equilibrium

(2) For each i:

$u_i(m) = u_i(s_i, m_{-i})$ for every $s_i \in \text{support}(m_i)$

$u_i(m) \geq u_i(s_i, m_{-i})$ for every $s_i \notin \text{support}(m_i)$

- **Corollary**. If a pure strategy is a NE, it is such also as a mixed strategy

Leonardo Badia – leonardo.badia@gmail.com

back to Example 5

		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- This game had two pure NEs: (R,R) and (S,S)
- We show now that there is also a mixed NE
- Ann (or Brian) plays R with probabilities q (or r)
- A mixed strategy is uniquely identified by (q,r)
  - Ann's payoff is $\boxed{u_A(q,r) = 2qr + (1-q)(1-r)}$
  - Brian's is $\boxed{u_B(q,r) = qr + 2(1-q)(1-r)}$

Leonardo Badia – leonardo.badia@gmail.com

back to Example 5

- Assume (a,b) is a mixed NE.
  - Note: support (a) = support (b) = {R,S}. Pure strategies R/S correspond with q (or r) being 0/1
- Due to the Theorem, $\boxed{u_A(a,b) = u_A(0,b) = u_A(1,b)}$
- Now, use: $u_A(q,r) = 2qr + (1-q)(1-r)$
- $\boxed{2ab + (1-a)(1-b) = 1-b = 2b}$
- Solution: $\boxed{b = 1/3}$
- Similarly, $\boxed{u_B(a,0) = u_B(a,1)}$
- Solution: $\boxed{a = 2/3}$

Leonardo Badia – leonardo.badia@gmail.com

Nash theorem (intro)

- The reasoning we used to find the third (mixed) NE of the Battle of Sexes is more general
- Every two-player games with two strategies has a NE in mixed strategies
- This is easy to prove and is part of the more general Nash theorem
- **Theorem** (Nash 1950). Every game with finite S_i's has at least one Nash equilibrium (possibly involving mixed strategies)

teorema di esistenza ⇒ non dice come trovare NE

Understanding mixed strategy

- Mixed strategies are key for Nash Theorem
  - What does "mixed strategies as probabilities" mean?
  - In the end, players take pure strategies.
- Possible interpretations
  - Large numbers: If the game is played M times, mixed strategy q = to choose a pure strategy qM times (note: each of the M times is one-shot memoryless)
  - Fuzzy values: Unsure actions: players do not know
  - **Beliefs**: The probability q reflects the uncertainty that my opponent has about my choice (which is pure)

Beliefs

- A **belief** of player *i* is a possible profile of opponents' strategies: an element of set ΔS_{-i}
    - Same definition of pure strategies (but here ↑)
- As before, a best-response-correspondence BR: $\Delta S_{-i} \rightarrow \wp(\Delta S_i)$ associates to $m_{-i} \in \Delta S_{-i}$ a subset of ΔS_i such that each $m_i \in \mathrm{BR}(m_{-i})$ is a best response to m_{-i}
    - Also, best responses are still not unique

NE as best responses

- Using beliefs, we can speak of **best response** to an opponent's (mixed) strategy
  - Intuition

		Bea	
		F	G
Art	U	6, 1	0, 4
	D	2, 5	4, 0

  - Bea ignores what Art will play
  - So she assumes he will play U with probability q
  - And, Art thinks Bea will play F with probability r

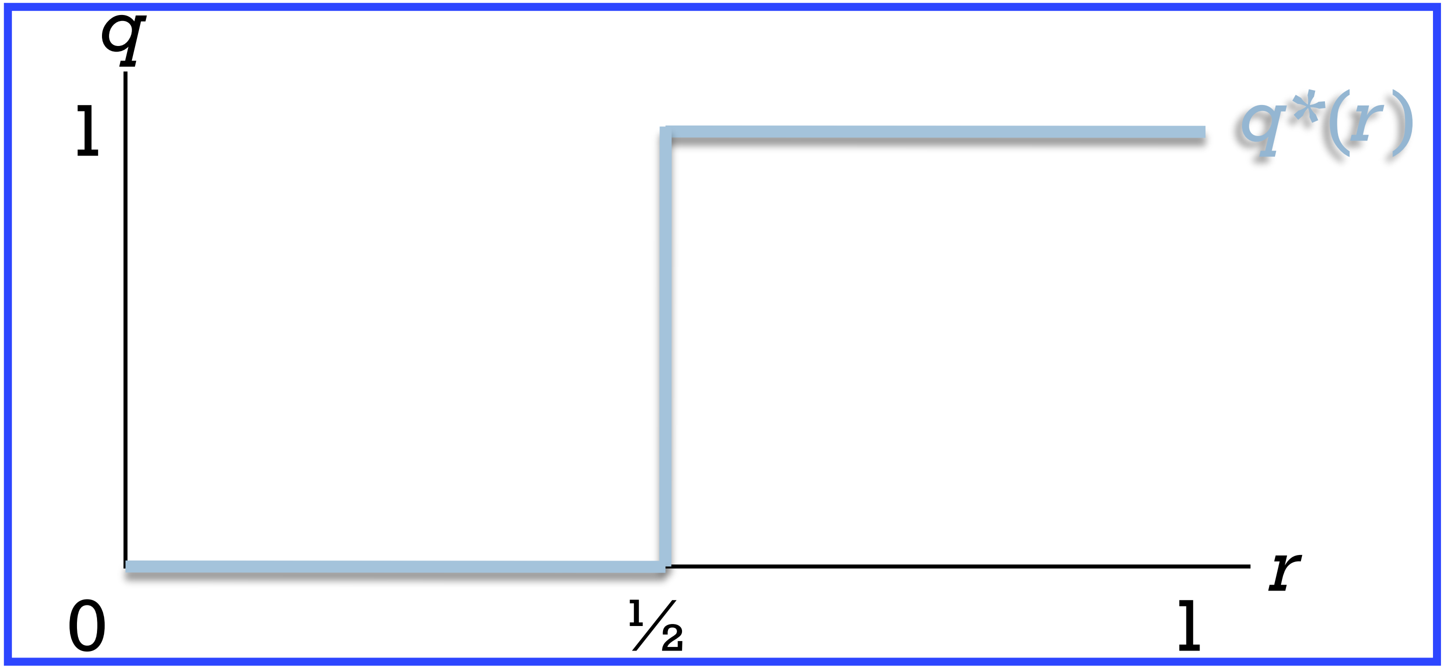
NE as best responses

		Bea	
		F	G
Art	U	6, 1	0, 4
	D	2, 5	4, 0

- E.g., if Bea is known for always playing F ($r = 1$), Art's best response is to play U ($q = 1$). In general?
- It holds: $\boxed{u_A(\mathrm{D}, r) = 2r + 4(1-r),\ u_A(\mathrm{U}, r) = 6r}$
- U is actually Art's best response as long as $r > ½$, else it is D. If $r = ½$ they are equivalent
- Denote Art's best response with $\boxed{q^*(r)}$

Leonardo Badia – leonardo.badia@gmail.com

NE as best responses

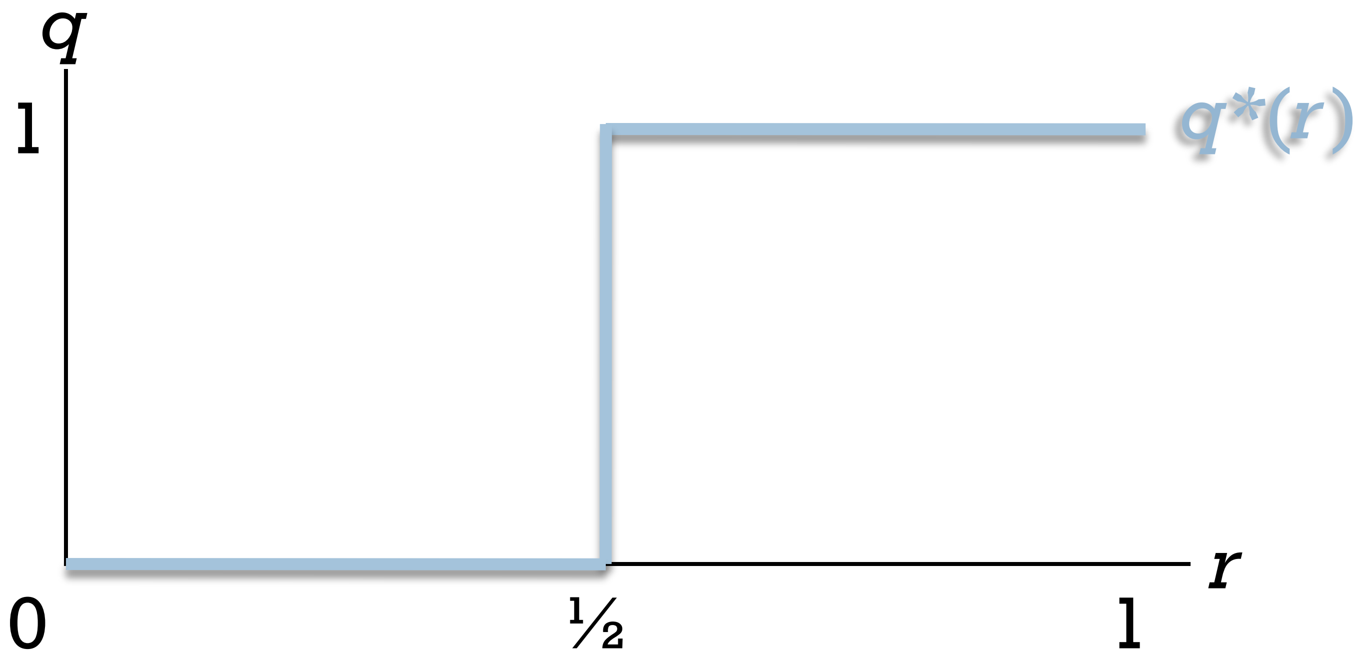


- Art’s best response is either U or D means that $q^*(r) = 1, 0$, respectively; then, $q^*(r)$ is step-wise

$$q^*(r) = 0 \text{ if } r < ½ \ , \qquad q^*(r) = 1 \text{ if } r > ½$$

Leonardo Badia – leonardo.badia@gmail.com

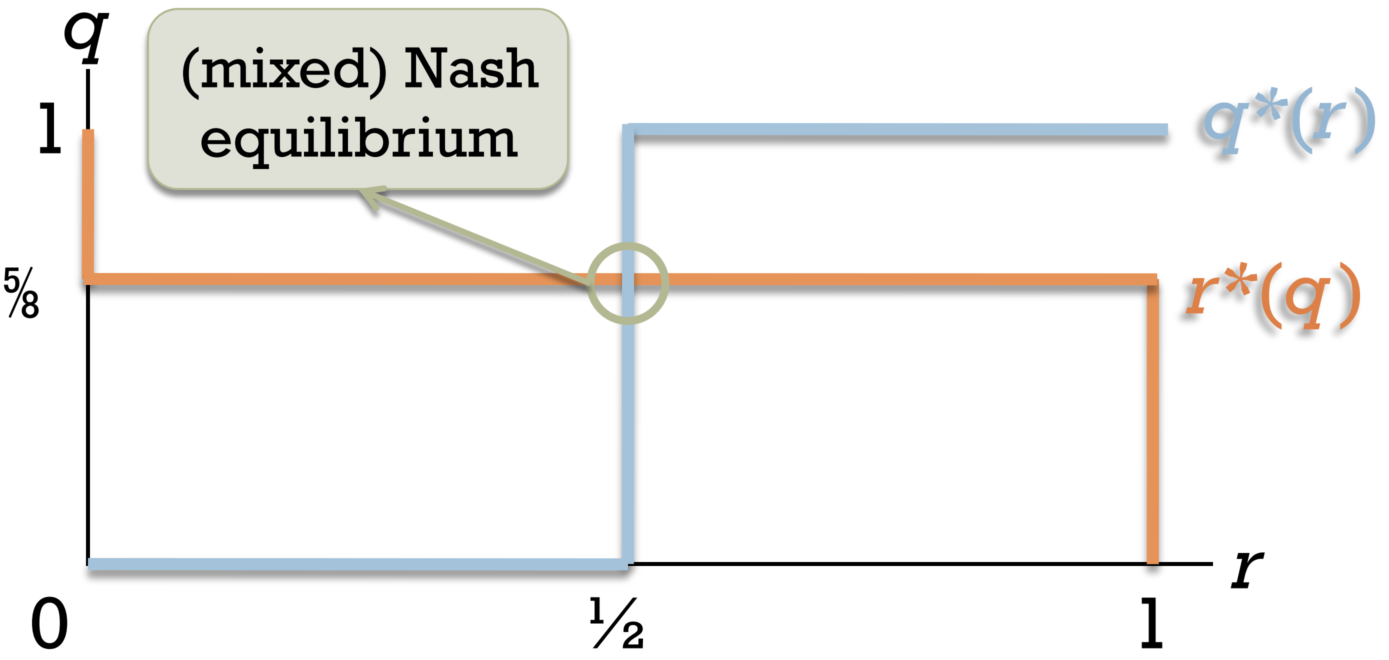
NE as best responses



- For Bea: $u_B(q,F) = q + 5(1-q)$, $u_B(q,G) = 4q$
- Thus, Bea's best response $r^*(q)$ is step-wise

  $r^*(q) = 1$ if $q < 5/8$, $r^*(q) = 0$ if $q > 5/8$

NE as best responses



(mixed) Nash equilibrium

- Joint strategy m = (q =½, r =⅝) is a NE.
- NE are points were the choice of each player is the best response to the other player’s choice.

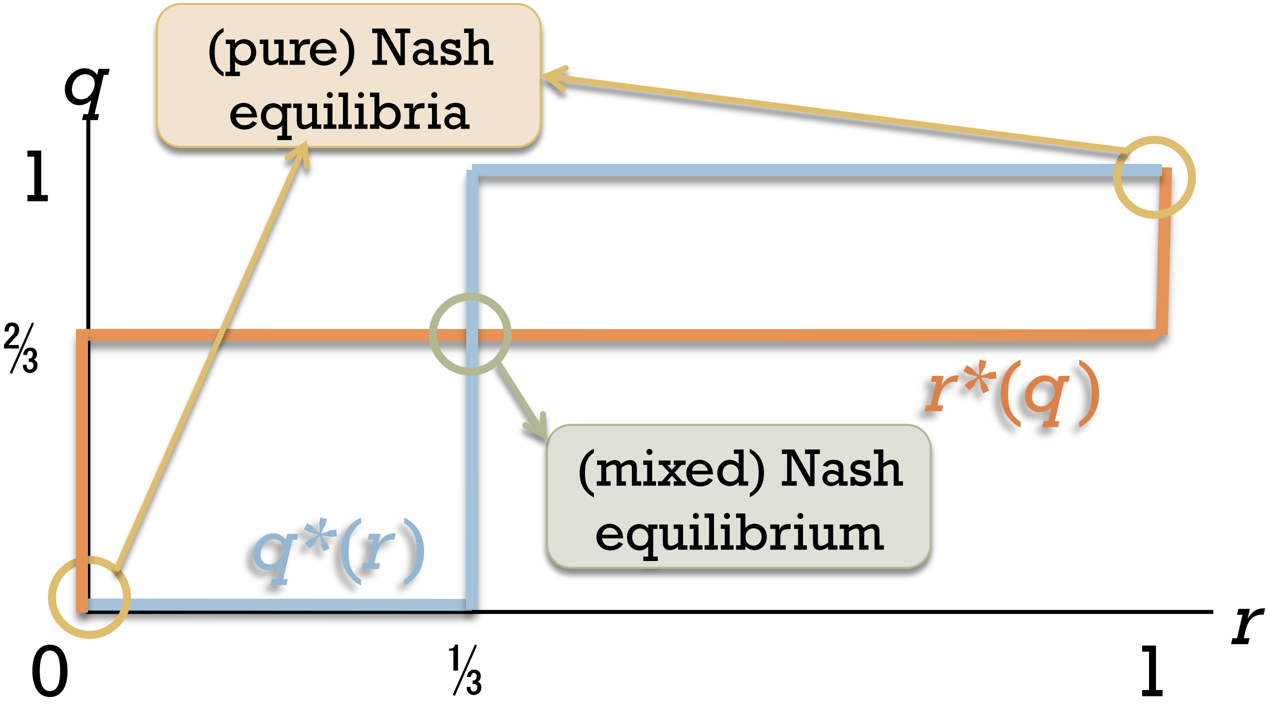
Existence of NE

- Clearly, the existence of at least one NE is guaranteed by topological reasons.
- There may be more NEs (e.g. Battle of Sexes).

		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- $u_A(\mathrm{R},r) = 2r$, $u_A(\mathrm{S},r) = 1 - r$, $q^*(r) = 1 - \mathrm{h}(r - \frac{1}{3})$
- $u_B(q,\mathrm{R}) = q$, $u_B(q,\mathrm{S}) = 2(1 - q)$, $r^*(q) = 1 - \mathrm{h}(q - \frac{2}{3})$

Existence of NE



- Anyway, $q^*(r)$ must intersect $r^*(q)$ at least once.
- The Nash theorem generalizes this reasoning.

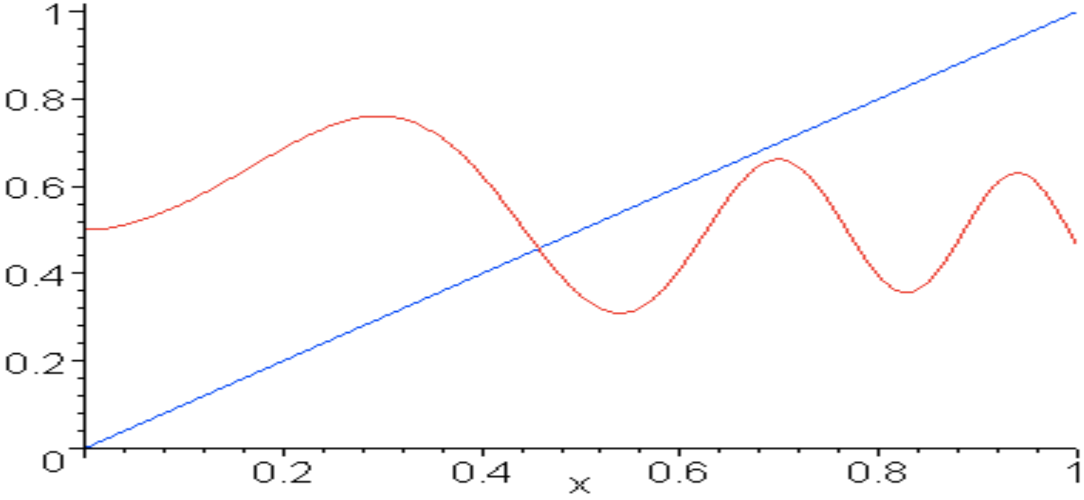
The Nash theorem

- For game G = $\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, define:
  $BR_i : \Delta S_1 \times \ldots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \ldots \times \Delta S_n \rightarrow \wp\, \Delta \mathrm{S}_i$
  $BR_i\,(m_{-i}) = \{m_i \in \Delta S_i : u_i\,(m_i, m_{-i})$ is maximal $\}$
- Then define **BR**: $\Delta S \rightarrow \wp\, \Delta \mathrm{S}$ as
  $\mathbf{BR}(m) = BR_1\,(m_{-1}) \times \ldots \times BR_n\,(m_{-n})$
- $BR_i\,(m_{-i})$ is the set of best responses of *i* to what others may do (m_{-i}); **BR** is their aggregate.
  - m is a NE if $m \in \mathbf{BR}(m)$
  - Properties of $BR_i\,(m_{-i})$: (1) is always non-empty (2) always contains at least a pure strategy

Leonardo Badia – leonardo.badia@gmail.com

The Nash theorem

- **Brouwer's Fixed Point Theorem**
- If $f(x)$ is a continuous function from a closed real interval $\mathcal{J}$ to itself, $\exists\, x^* \in \mathcal{J}$ such that $f(x^*) = x^*$
- **Proof:** consider $\mathcal{J} = [0,1]$. If f(0) > 0 and f(1) < 1, apply Bolzano-Weierstrass theorem to $f(x) - x$



Leonardo Badia – leonardo.badia@gmail.com

The Nash theorem

- **Kakutani's Fixed Point Theorem**
- If A is a non-empty, compact, convex subset of $\mathbb{R}^n$
- If correspondence $F : A \rightrightarrows A$ is such that
  - For all $x \in A, F(x)$ is non-empty and convex
  - If $\{x_i\}, \{y_i\}$ are sequences in $\mathbb{R}^n$ converging to x and y, respectively: $y_i \in F(x_i) \Rightarrow y \in F(x)$ (F has **closed graph**)
- Then there exists $x^* \in A$ such that $x^* \in F(x^*)$.

- **Nash theorem.** Nothing but Kakutani theorem applied to the global best-response **BR**

Adding a time dimension

Still “static” games?

Fictitious Play

- In fictitious play (G.W. Brown, 1951), regrets become actual changes of moves
  - Each player *i* assumes the (possibly mixed) strategies played by –*i* as fixed
  - If *i* gets a chance to play again, it best responds to what see the other players just did
  - Somehow, "full rationality" is denied (we acknowledge predictions may be incorrect)
- How does fictitious game evolve?
  - Nash equilibrium points are **absorbing** states. So, are they always convergence points?

Fictitious Play

- Not always! Players can also keep "cycling" (we will see examples of this)
  - In Rock/Paper/Scissors, FP does not converge.
- FP converges to a NE in some relevant cases:
  - The game can be solved by IESDS
  - **Potential games**
  - (also other cases such as 2xN games with generic payoffs – which means every outcome has a different payoff for all the players)

Leonardo Badia – leonardo.badia@gmail.com

Potential games

- Take G = $\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$. $S = S_1\times\ldots\times S_n$
- Function $\Omega: S\rightarrow\mathbb{R}$ is an (**exact**) **potential** for G if:

$$\Omega(s_i',s_{-i}) - \Omega(s_i,s_{-i}) = u_i(s_i',s_{-i}) - u_i(s_i,s_{-i}) = \Delta u_i$$

- $\Omega: S\rightarrow\mathbb{R}$ is a **weighted potential** with weight vector **w** = $\{w_i>0\}$ if: $\Omega(s_i', s_{-i}) - \Omega(s_i,s_{-i}) = w_i\Delta u_i$
- $\Omega: S\rightarrow\mathbb{R}$ is an **ordinal potential** for G if:

$$\Omega(s_i',s_{-i}) > \Omega(s_i,s_{-i}) \Leftrightarrow u_i(s_i',s_{-i}) > u_i(s_i,s_{-i})$$

- If G admits a potential (ordinal potential), it is called a **potential** (**ordinal potential**) **game**.

Leonardo Badia – leonardo.badia@gmail.com

Potential games

- Potential games have nice properties
- If G = $\{S_1, S_2, .., S_n ; u_1, u_2, .., u_n\}$ has an ordinal potential Ω, it is immediate that its set of NEs is the same of G' = $\{S_1, S_2, .., S_n ; \Omega, \Omega, .., \Omega\}$
- I.e., all the players want to max the potential
  - Multi-person reduces to single-goal optimization
  - To some extent, enables distributed optimization
  - The physical meaning of the potential may not be always immediate

Examples of potential

- The Prisoner's Dilemma is a potential game.

		Bob	
		M	F
Al	M	-1, -1	-9, 0
	F	0, -9	-6, -6

		Bob	
		M	F
Al	M	0	1
	F	1	4

potential Ω

- This potential is exact
- However, the players are not very smart (they do not maximize their global welfare!)
- So, there must be some dummy somewhere

Examples of potential

- The game of Cournot oligopoly is an ordinal potential game.
  - Recall that firms choose q_1 and q_2;
  - the market clearing price is $a - q_1 - q_2$;
  - unit cost is c (so cost to produce $q_i = c\, q_i$)
- Thus $u_i(q_i, q_j) = q_i(a - q_i - q_j - c)$

  and an ordinal potential function is:

  $$\Omega(q_i, q_j) = q_i q_j (a - q_i - q_j - c)$$

Leonardo Badia – leonardo.badia@gmail.com

Potential games

- **Theorem**. Every finite ordinal potential game has (at least) a pure strategy Nash eq.
  - This NE can be found deterministically
- **Proof**: a consequence of fictitious play
  - All players move, one at a time, to maximize their utility → they also maximize the potential
  - Repeat this until a local maximum of Ω is found

Congestion games

- Congestion games are a special case of potential game. They involve the choice of “least congested resources”
  - Especially found in network problems (finding the least congested route on a graph)
  - Or in resource allocation (minority games)
- It can actually be found that:
  - congestion games are potential games
  - for every potential game, there exists a congestion game with the same potential

Coordination game

- A **coordination game** models situations where players are required to act together
  - They give higher payoffs to the players when they make the same choice
  - An example is the Battle of sexes
  - In the historical "Stag Hunt" (proposed by Rousseau) 2 hunters may decide to hunt a deer (value 20), but they succeed only together; or, each one can hunt a hare (worth 7), even alone

Coordination game

- A coordination game has multiple pure strategy NEs
- It can be seen as a potential game, with coordination points as potential maxima
  - For the Stag Hunt:

		Grunt	
		S	H
Brunt	S	10,10	0,7
	H	7,0	7,7

payoffs

		Grunt	
		S	H
Brunt	S	-4	-7
	H	-7	0

potential Ω

Leonardo Badia – leonardo.badia@gmail.com

Coordination game

- Another case is the **anti-coordination game**
  - For example the Hawk-and-Dove, Chicken
  - Players try not to select the same thing

	Hawk	Dove
Hawk	-99,-99	10,-10
Dove	-10,10	0,0

- Hawk = buy nuclear weapons
  Dove = be peaceful
- Hawk = hold the wheel; if you win, the other is a chicken
  Dove = steer the wheel

- Note: Odd/Even and similar ones (a player is for =, the other ≠) are called **discoordination games**

Leonardo Badia – leonardo.badia@gmail.com

Potential=coordination+dummy

- Finally, a **dummy** (or pure externality) game is such that for all s_{-i}, $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$, i.e., payoff of player *i* only depends on s_{-i}
- Every potential game is a sum of a pure coordination and a dummy game

Bob

		M	F
Al	M	-1,-1	-9,0
	F	0,-9	-6,-6

=

coordination (greedy)

	M	F
M	-1,-1	0,0
F	0,0	3,3

+

dummy (externality)

	M	F
M	0,0	-9,0
F	0,-9	-9,-9

Leonardo Badia – leonardo.badia@gmail.com

Computational complexity

Is a NE easy to find?

How easy is to find a NE?

- Since Nash equilibria are regarded as the "natural" evolution of the system, one may wonder how much it takes to find them
- We already have the Nash theorem, which is an existence theorem
- Plus, there are notable results for certain specific games

A negative result

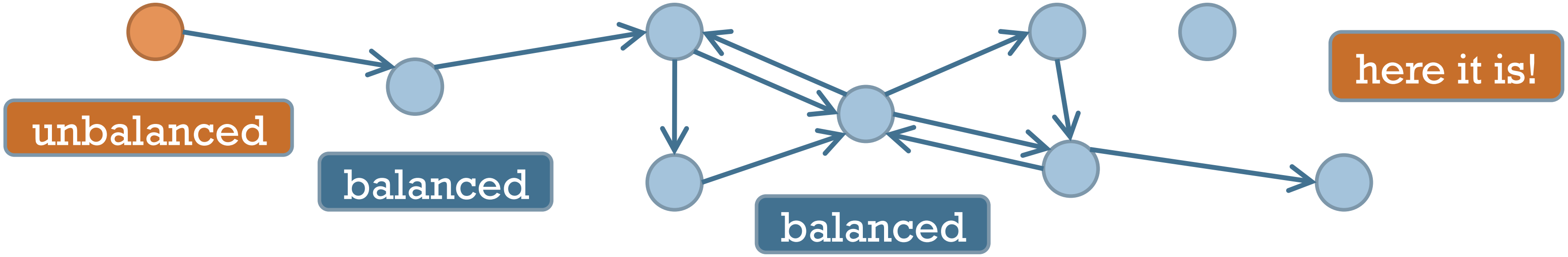
- Unfortunately, in the general case, finding a NE is **computationally hard**
- This has been proven in some recent papers by Papadimitriou et al.
- However, computationally hard does not mean NP-complete
- The search for a NE cannot be NP-complete as a solution *must* exist (there may even be multiple solutions, which complicates things)

The PPAD class

- The NASH problem is PPAD-complete
  - PPAD = Polynomial Parity Arguments on Directed graphs (Papadimitriou, 1994)
  - The PPAD class is somehow intermediate between P and NP
  - More or less, P<PPAD<NP. This means it is computationally hard, unless P=NP
  - This class includes the problem equivalent to the end-of-line problem

The PPAD class

- Consider the end-of-line problem:
  - "Take a directed graph with an unbalanced node. There must be another (at least). Find it."



- This problem is bound to have a solution
- However, finding it without exploring the whole graph is far from trivial (and in certain cases cannot be avoided)

How is NASH a PPAD problem?

- The NASH problem corresponds to find a fixed point of the **BR** function
- Finding a fixed point over a compact set can be shown to be equivalent to finding the end of a proper path on a directed graph
- There are elegant (not difficult but very long) proofs of it, involving graph coloring and compact partitioning

Consequences on NE?

- This may imply bad consequences on the practical usefulness of Nash Equilibrium
- To be optimistic:
  - Certain simple problems can be shown to have a NE which can be found through constructive steps (good for engineers)
  - one may be "close" to a NE (maybe it is enough) → relaxation: ε-**Nash Equilibrium**, i.e., instead of checking for "no unilateral improvements," ignore all improvements less than a given $\varepsilon > 0$