# Game theory

a course for the
MSc in ICT for Internet and multimedia

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# Repeated games

Same game – played many times

# Repeated games

- A $\boxed{\text{repeated game } G(T, \delta)}$ is a dynamic game where the same static game G is played as a stage game $T$ times and payoffs are discounted by δ and cumulated
- Finitely repeated games (finite horizon, e.g., same game played $T$ ($T$=2,3,..) times
  - infinitely repeated games (infinite horizon) – remember this also models random exit; also, if horizon is infinite, it δ must be $\leq 1$

# Two-stage Repeated games

- Look again at the Prisoner's Dilemma
- No discounting ($\delta=1$)
- Does the second stage have a NE for every outcome of the first stage?

| Al \ Bob | M | F |
|---|---|---|
| M | 4, 4 | 0, 5 |
| F | 5, 0 | 1, 1 |

# Two-stage Repeated games

- Regardless of the first stage, (F, F) is the only NE
- So the final payoffs are simply the payoffs of the stage game plus +1
- And the final outcome is to play (F, F) twice

| Al \ Bob | M | F |
|---|---|---|
| M | 4, 4 | 0, 5 |
| F | 5, 0 | 1, 1 |

⇩

| Al \ Bob | M | F |
|---|---|---|
| M | 5, 5 | 1, 6 |
| F | 6, 1 | 2, 2 |

# Finitely repeated games

- As a consequence of multi-stage games:
- **Theorem.** The outcome of last stage is a NE
- **Theorem.** If stage game G only has NE $\boxed{\mathbf{s^*}}$, then $\boxed{G(T, \delta)}$ has a unique subgame-perfect outcome, i.e., play **s*** in every stage
- Hence, repetitions of stage games with a single NE are not very interesting

# Finitely repeated games

- (F, F) and (H,H) are NEs
- Players know the last stage will end at one of those
- They may anticipate to agree on (H,H) at the last stage only if they played (M,M) in the first stage

| | | Bob M | Bob F | Bob H |
|---|---|---|---|---|
| Al | M | 4, 4 | 0, 5 | 0, 0 |
| Al | F | 5, 0 | 1, 1 | 0, 0 |
| Al | H | 0, 0 | 0, 0 | 3, 3 |

- no real information exchange between Al and Bob, just speculation!

# Finitely repeated games

payoffs of (M,M) += 3
every other payoff += 1

- We can build an SPE where (M,M) is played (though not a NE of G)
  - subgame ≠ stage game!
- In case of multiple NEs, a SPE may not play a NE of G at stage $t < T$

| | | Bob | | |
|---|---|---|---|---|
| | | M | F | H |
| Al | M | 4, 4 | 0, 5 | 0, 0 |
| | F | 5, 0 | 1, 1 | 0, 0 |
| | H | 0, 0 | 0, 0 | 3, 3 |

| | | Bob | | |
|---|---|---|---|---|
| | | M | F | H |
| Al | M | 7, 7 | 1, 6 | 1, 1 |
| | F | 6, 1 | 2, 2 | 1, 1 |
| | H | 1, 1 | 1, 1 | 4, 4 |

# A remark on cooperation

- Repeated games tend to introduce cooperation (though to a limited extent)
- In fact, for finitely repeated games
  - the last stage is always egoistically played
  - collaborative Nash equilibria exist only in the presence of multiple egoistic NEs
- The main influence to the game is the credibility of threats or promises about future
  - No guarantee that the previous anticipation is kept and nobody prefers to "renegotiate"

# A better example

| | | Bob | | | | |
|---|---|---|---|---|---|---|
| | | M | F | H | P | Q |
| Al | M | 4, 4 | 0, 5 | 0, 0 | 0, 0 | 0, 0 |
| | F | 5, 0 | 1, 1 | 0, 0 | 0, 0 | 0, 0 |
| | H | 0, 0 | 0, 0 | 3, 3 | 0, 0 | 0, 0 |
| | P | 0, 0 | 0, 0 | 0, 0 | 4, ½ | 0, 0 |
| | Q | 0, 0 | 0, 0 | 0, 0 | 0, 0 | ½ , 4 |

- Four NEs. (F,F) (H,H) (P,P) (Q,Q).
- (H,H) Pareto dominates (F,F)

# A better example

| | | Bob M | F | H | P | Q |
|---|---|---|---|---|---|---|
| Al | M | 4, 4 | 0, 5 | 0, 0 | 0, 0 | 0, 0 |
| | F | 5, 0 | 1, 1 | 0, 0 | 0, 0 | 0, 0 |
| | H | 0, 0 | 0, 0 | 3, 3 | 0, 0 | 0, 0 |
| | P | 0, 0 | 0, 0 | 0, 0 | 4, ½ | 0, 0 |
| | Q | 0, 0 | 0, 0 | 0, 0 | 0, 0 | ½ , 4 |

**anticipated strategy**

| first stage | second stage |
|---|---|
| (M, ¬M) | (P,P) |
| (¬M, M) | (Q,Q) |

| first stage | second stage |
|---|---|
| (M, M) | (H,H) |
| (¬M, ¬M) | (F, F) |

# SPE outcomes

- The subgame-perfect outcome is (M,m) followed by (H,h): no better deviation.
  - Playing (F, f ) is punished as before
  - But (H,h) punishes the punisher as well!
  - Better if we also have strategies P and Q where the punisher benefits: cooperation appealing
- **To sum up**. Cooperation is possible when punishment strategies are available; multiple punishment options are better

# Infinitely repeated games

Extending cooperation to infinite horizons

# Infinitely repeated games

- Infinitely repeated game, with stage game G and discount factor $\delta$: denoted as $G(\infty, \delta)$
  - remember we need to have discount $\delta < 1$ this time, if we want the game to be meaningful
- In infinitely repeated games we cannot apply backward induction (no "last" stage)
- Surprisingly, this leads to a conclusion even more powerful than the finite horizon
  - We do not need "external" punishments!

# SP outcomes of the game

- There may be SPE of $G(\infty, \delta)$ in which no stage's outcome is a NE of G
- The argument can be shown again with the Prisoner's Dilemma
- Define a **grim trigger** strategy (GrT) as:
  - Start playing M at stage 1
  - At stage $t > 1$, play M only if outcome of all $t - 1$ previous stages was (M,M), otherwise play F

# Is "All play GrT" a SPE?

- **Proposition.** For $\delta$ "close enough" to 1, the joint strategy where both users play GrT is a SPE
- 1) We show GrT is a NE = best response to itself
- If Bob assumes that Al plays GrT, he knows that, whenever outcome ≠ (M,M), Al plays F forever
- Thus, also for Bob it is optimal to play F forever if outcome ≠ (M,M)
- We just need to find Bob's best first move

| Al \ Bob | M | F |
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# Is "All play GrT" a SPE?

- Choosing F in the first stage yields payoff 5 but triggers non-cooperation by Al forever after; also Bob will play F forever after
- Present value of this sequence is

$$V = 5 + \delta \cdot 1 + \delta^2 \cdot 1 + \ldots = 5 + \delta / (1-\delta)$$

- Choosing M yields 4 and iterates the same (sub)game. Call the present value of this sequence as $V'$

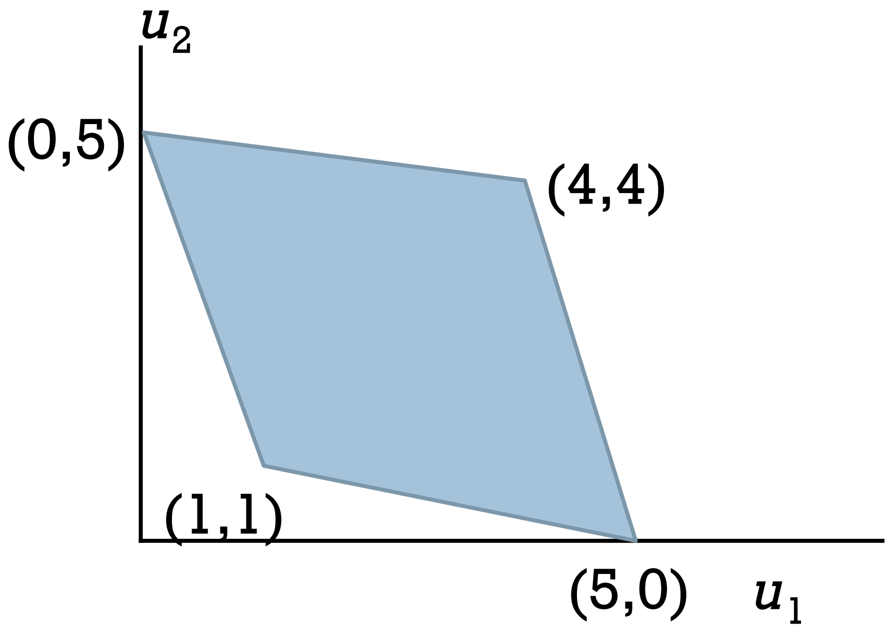
$$V' = 4 + \delta \cdot 4 + \delta^2 \cdot 4 + \ldots = 4 / (1-\delta)$$

# Is "All play GrT" a SPE?

- $V = 5 + \delta/(1-\delta)$, $V' = 4/(1-\delta)$. Solve $V \leq V'$
  - Thus, M is optimal if $\delta \geq 1/4$
- BUT we still need to prove NE→SPE, i.e. to show that GrT is a NE in every subgame.
- We have two classes of subgames: (i) all previous stages are (M, M) as outcome (ii) at least one stage deviated
  - (i) is the same as the whole G, GrT is a NE of G.
  - (ii) GrT becomes "always play (F, F)", a NE

# The Friedman Theorem

- The key result for infinitely repeated games. Also called Folk Theorem
- Preliminaries
  - A **feasible payoff** is any convex combination (i.e., weighted average with sum(weights)=1 of pure-strategy payoffs

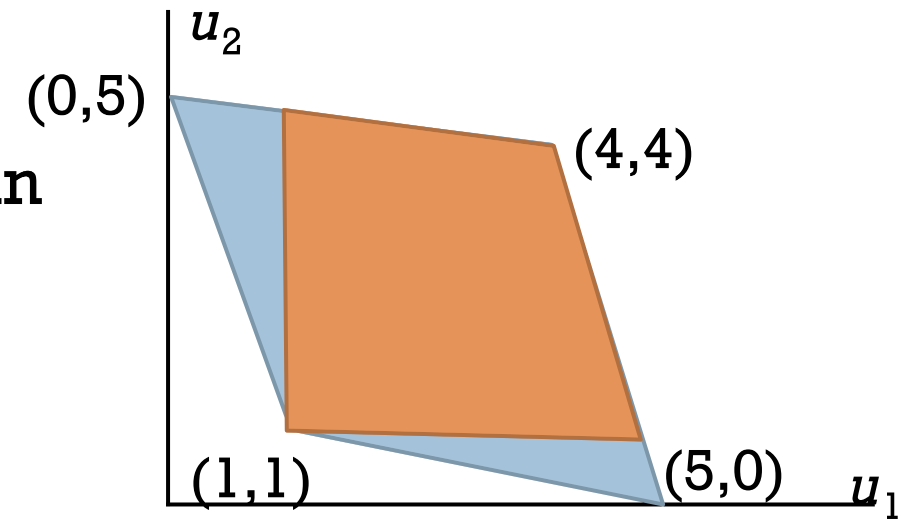


# The Friedman Theorem

- Let $\boxed{G}$ be a finite static game of complete info
- Let $\boxed{(e_1, e_2, \ldots, e_n)}$ be the payoffs from a NE of G
- Let $\boxed{(x_1, x_2, \ldots, x_n)}$ be feasible payoffs s.t. $\forall j, x_j > e_j$

→ If $\delta$ close to 1, $G(\infty, \delta)$ has a SPE with payoffs $(x_j)$

- **Proof:** as per repeated Prisoner's Dilemma
  - Any point in the area can be achieved with a GrT



# Developments: punishment

- In the Prisoner's Dilemma, both players have security payoff = 1 (also the payoff at NE)
- But stage game G has maximin≤payoff(NE)
  - Security payoffs $(r_1,\ldots,r_n)$ can replace $(e_1,\ldots,e_n)$
- What if $\delta$ is not close to 1?
  - Smaller $\delta$ makes the punishment less effective
  - In some games (not Prisoner's Dilemma) there may be better (credible) punishments than using a GrT, i.e. worse than deviating from cooperation

# Developments: Tit-for-Tat

- It may be unnecessary to keep punishment forever (holding a grudge)
  - Assume the stage game has two actions (Cooperate & Defect) → GrT can be replaced by "Tit-for-Tat"
- **Tit-for-Tat (TFT)**: At stage $t$, $i$ chooses the move (cooperate, defect) played by $-i$ at stage $t$-1
- Tit-for-Tat punishes immediately deviation from cooperation but is also forgiving (1-step history)
- Behavioral analogous: Eye-for-an-eye, Live-and-let-live, Biological reciprocal altruism

# Developments: Tit-for-Tat

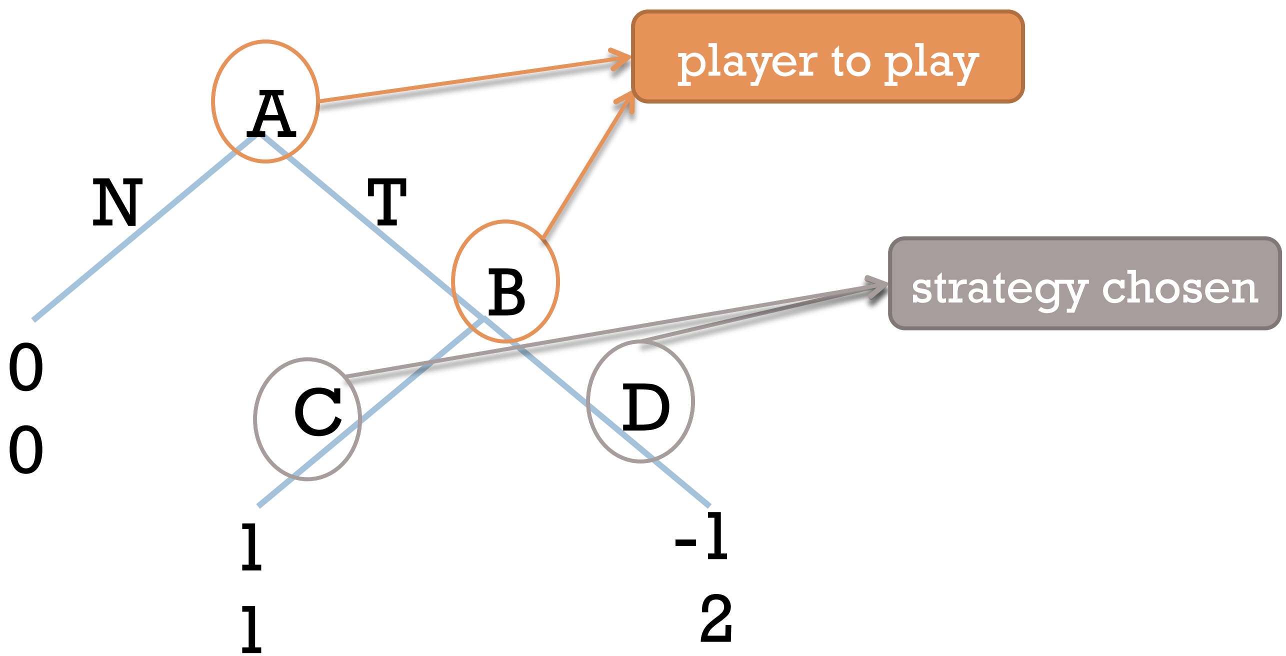
- Even though TFT is often effective, it may be unstable in certain conditions
- Two "unsynchronized" TFT players trigger "**death spiral**"
- Hence, the NE achieved by TFT is **not** subgame-perfect (it must be NE in every subgame, in the death spiral case players do not take a NE)
- Analogous "Tit for Two Tats." First defection forgiven, second is punished with defection
  - Highly forgiving strategy, avoiding death spiral, it is often worse off against aggressive strategies

# Reputation

Building trust over sequential iterations

# Trust game

- Consider this simple Trust game
  - A can either (T)rust B or not; if trusted, B can either (C)ollaborate or (D)efect



# Trust game

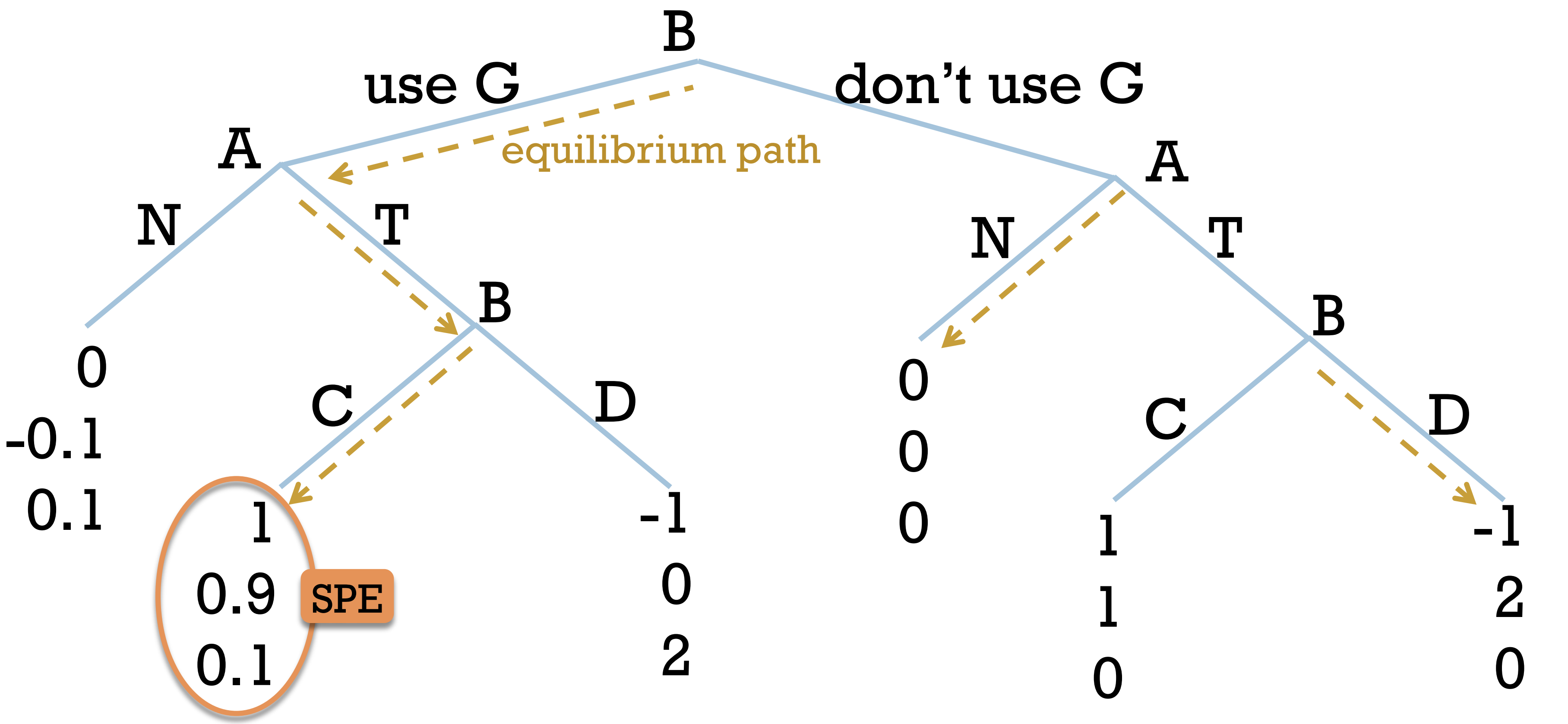
- From backward induction, we know that A does not trust B and decides not to cooperate
- What if this game is repeated?
- We can build a GrT as follows
  - In period 1, A chooses T
  - A chooses T as long as the previous outcomes are (T,C); at the first deviation, A plays N
  - B chooses "always play C" as best response as long as myopic deviation is worse, i.e. for $\delta \geq ½$

# Certified reputation

- Can we give further incentives to B to extend cooperation beyond this long-term reward?
- One way is to introduce a guarantor G
  - G gives an aura of good reputation to B in exchange for some reward
- For example, B gives G an insurance of 2
  - If A trusts B and B defects, G keeps it
  - Otherwise, G returns this insurance to B, keeping a small fraction (0.1) for the service

# Certified reputation

- With G in the game, it becomes



# Certified reputation

- Why should such a guarantor exist in the first place? Why does not G keep the insurance?
  - If seen as a one-shot game, then G has a dominant strategy: to keep the insurance!
  - But if the game is repeated, also G want to establish a reputation of a certified guarantor
- Myopic: keep insurance (2) + 0 afterwards
- Cooperate: $0.1 + \delta\, 0.1 + \delta^2\, 0.1 \ldots = 0.1 / (1-\delta)$
- Cooperate is better if $\delta \geq 0.95$