EXERCISE (Complementary languages)

Let $L^c = \{0,1\}^* - L$. Prove that

$(L^c)^c = L$

$\forall L \subseteq \{0,1\}^*$:

$$\underbrace{(L \in NPC)}_{HP1} \wedge \underbrace{(L^c \in NP)}_{HP2} \Rightarrow \underbrace{L^c \in NPC}_{TH}$$

HP1 $\forall \hat{L} \in NP : \hat{L} <_p L$ $\quad (L \in NP)$

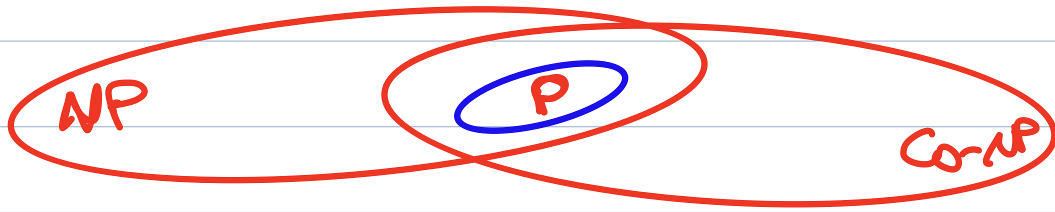
HP2 $L^c \in NP$

EXERCISE Define

$$co\text{-}NP = \{L \subseteq \{0,1\}^* : L^c \in NP\}$$

$$\equiv L' = L^c \longrightarrow \left(= \{L'^c \subseteq \{0,1\}^* : L' \in NP\}\right)$$

1) Prove that $P \subseteq NP \cap co\text{-}NP$



2) Prove that

if $NPC \cap co\text{-}NP \neq \emptyset \Rightarrow NP = co\text{-}NP$

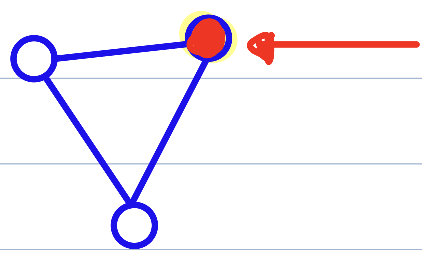
EXERCISE

DEF Given an undirected graph $G=(V,E)$ a dominating set $V' \subseteq V$ is such that:

$$\forall v \in V: (v \in V') \vee (\exists \{u,v\} \in E : u \in V')$$

In other words, a node is either in the dominating set or adjacent to a node in the dominating set

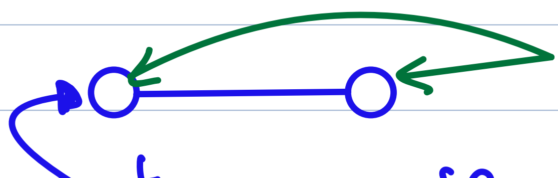
EXAMPLE:



dominating set for G

Observe that a dominating set is not a vertex-cover

VICEVERSA: In a graph with no isolated nodes a vertex cover is also a dominating set:



one or the other in the VC

(each vertex is the endpoint of an edge: either it is in the vertex cover or the other endpoint must be for the edge to be covered)

DOMINATING SET (DS)

$\begin{cases} I : \langle G=(V,E), k \rangle \\ Q : \text{Does } G \text{ have a DS of size } k\text{?} \end{cases}$

Prove that DS $\in$ NPH

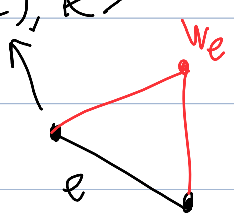
$VC \leq_p DS$

- rimuoviamo nodi isolati

$\langle G=(V,E),k \rangle \rightsquigarrow \langle G=(V',E),k \rangle$, $V' = V \setminus \{\text{nodi isolati}\}$

$(\langle G,k \rangle \in VC \iff \langle G',k \rangle \in VC)$

$\langle G'=(V',E),k \rangle$



$\forall e \in E$

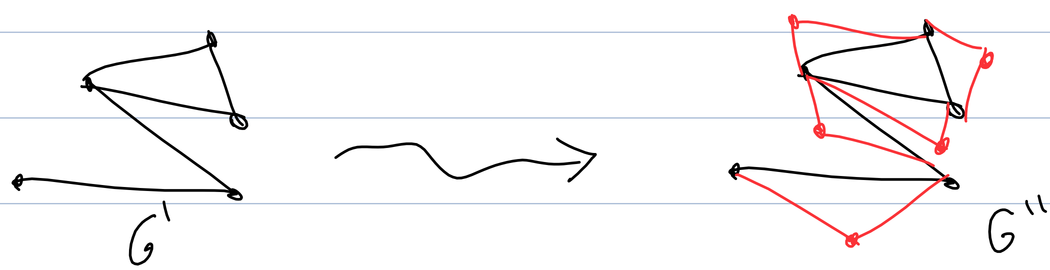
almeno 1 nodo di triangolo necessario per dominare triangolo

$\Downarrow$

se prendo w_e, posso sostituirlo con u/v

$\Downarrow$

così, DS fatto solo da vertici di V' e ogni arco ha nodo in DS ($\Rightarrow$ VC)



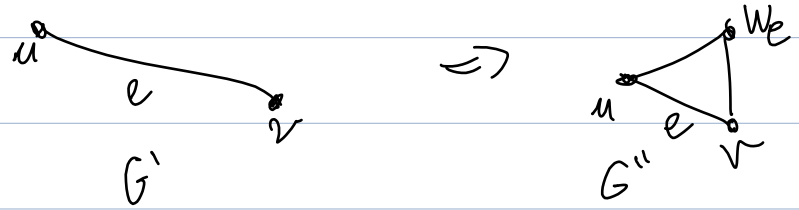
$<G'=(V',E),K> \overset{f}{\rightsquigarrow} <G''=(V'',E''),K>$,

$V''=V'\cup\{w_e\ \forall e\in E\}$, $E''=E\cup\{\{u,w_e\},\{v,w_e\}:\{u,v\}\in E\}$

Nodi aggiunti: $|V''|=|V'|+|E|\le|<G'>|$, $|E''|=|E|+2|E|\le 3|<G'>| \Rightarrow$
$\Rightarrow f$ é ptc

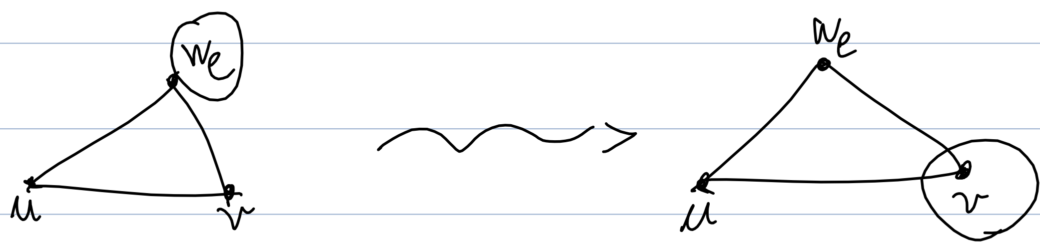
$<G'=(V',E),K>\in VC \iff <G''=(V'',E''),K>\in DS$

$\Rightarrow$: G' has VC $\bar V$ di dim. $K \Rightarrow \bar V$ tocca tutti lati di E



Supponiamo $v\in\bar V \Rightarrow v$ può dominare sia u che $w_e \Rightarrow$
$\Rightarrow \bar V$ domina $V\setminus\bar V \Rightarrow \bar V$ é DS di dim. K in G''

$\Leftarrow$: $<G''=(V'',E''),K>\in DS \Rightarrow \bar V$: DS di G'' di dim. K



Supponiamo $w_e\in\bar V \Rightarrow$ può dominare solo $w_e, u, v \Rightarrow$ non ha senso $\Rightarrow$ prendo u o $v \Rightarrow$ ottengo $\bar{\bar V}\subseteq V' \Rightarrow$ ogni lato ha nodo in $\bar{\bar V} \Rightarrow \bar{\bar V}$ é VC per G' dim. $\le K$

EXERCISE Consider the following problem:

L-PATH

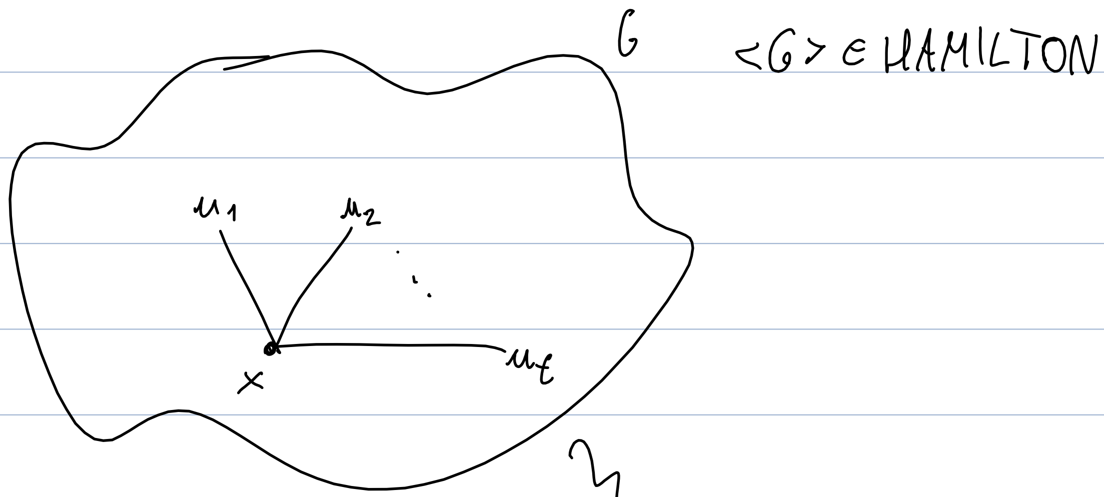
I: $\langle G=(V,E), u, v, k \rangle$,
$G=(V,E)$ undirected graph, $u, v \in V$, $1 \le k \le |V|$

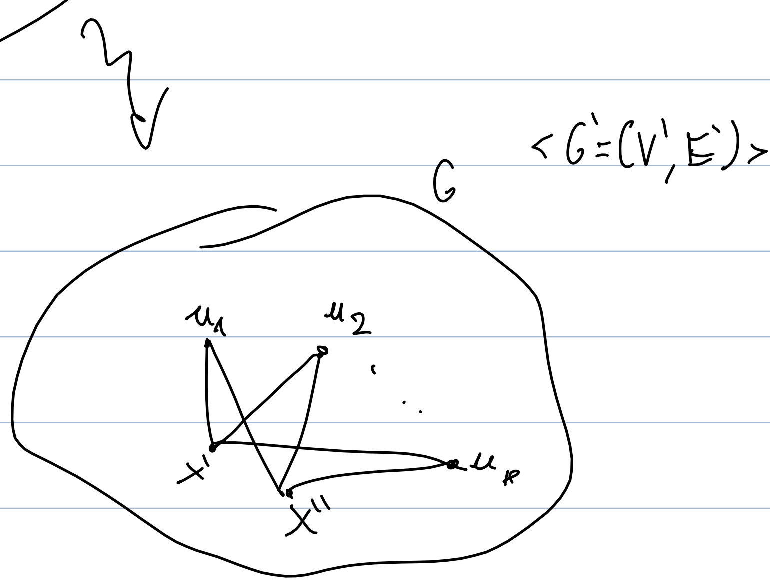
Q: Is there a simple path from u to v of length $\geq k$?

(decision problem for LONGEST-PATH)

Prove that HAMILTON $\le_p$ L-PATH

(thus L-PATH $\in$ NPH)





$f(<G=(V,E)>) = <G'=(V',E'), x', x'', k_{G'}=|V|>$
$V' = V \setminus \{x\} \cup \{x', x''\}$, $E' = E \setminus \{\{x,u\} \in E\} \cup \{\{x',u\}, \{x'',u\} : \{x,u\} \in E\}$
f é ptc

$\Rightarrow$: $<G> \in$ HAMILTON $\Rightarrow$ prendiamo ciclo C

$C = <\ldots u' \, x \, u'' \ldots>$

percorso: $x' \to u'' \overset{C}{\rightsquigarrow} u' \to x'' \Rightarrow$ lunghezza: $|V|$

$\Leftarrow$: facile

EXERCISE Assume that there exists a polynomial algorithm $A_{SS}(\langle S,t\rangle)$ for SS:

$$\begin{cases} I: \langle S,t\rangle : S \subseteq \mathbb{N} \text{ finite}, t \in \mathbb{N} \\ Q: \exists S' \subseteq S : \sum_{s \in S'} s = t ? \end{cases}$$

Design a polynomial algorithm SUBSET($\langle S,t\rangle$) which uses A_{SS} as a subroutine and, in case $\langle S,t\rangle \in SS$, returns S' : $\sum_{s \in S'} s = t$.

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SUBSET(<S,t>):
  if A_SS(<S,t>) = 0 then return "∄S'";
  n ← |S|    // S = {s_1,...,s_n}
  S' ← S;
  // invariante: istanza positiva ⇒ ∃ S̃ ⊆ S' : Σ_{s∈S̃} s = t
  for i ← 1 to n do:
     if A_SS(<S' \ {s_i}, t>) = 1 then S' ← S' \ {s_i};
  return S';
```

Dimostriamo correttezza $\Rightarrow$ dimostrare S' alla fine è tale che $\sum_{s \in S'} s = t$

Dim: contraddizione: $\exists s_i \in S' \setminus \tilde{S} \Rightarrow$ sia S_i' insieme S' all'inizio di it. i

$(S_1' = S;$ return $S_{n+1}')$

$S_{i+1}' \subseteq S_i' \; \forall i \Rightarrow S_i' \supseteq S_{n+1}' \supseteq \tilde{S} \not\ni s_i \Rightarrow S_i' \setminus \{s_i\} \supseteq \tilde{S} \Rightarrow$

$\Rightarrow A_{SS}(<S_i' \setminus \{s_i\}, t>) = 1 \rightarrow$ tolgo $s_i \Rightarrow$ contraddizione

n chiamate a $A_{SS} \Rightarrow n = |S| \leq |<S,t>|$

Se $T_{ASS}(|<S,t>|) = O(|<S,t>|^{k_1})$, $T_{SUBSET}(|<S,t>|) = O(|<S,t>|^{k_1+1})$

EXERCISE Consider the following problem:

PARTITION

$$\begin{cases} I: \langle S \rangle,\ S \subset \mathbb{N},\ \text{finite} \\ Q: \exists S_1, S_2 \subset S, (S_1 \cup S_2 = S) \wedge (S_1 \cap S_2 = \phi): \\ \qquad \sum_{s \in S_1} s = \sum_{s \in S_2} s \ ? \end{cases}$$

Show that PARTITION $\in$ NPH

$SS \leq_p$ PARTITION

$\langle S, t \rangle \overset{f}{\rightsquigarrow} \langle \tilde{S} \rangle$

Se $\exists S' \subseteq S : \sum_{s \in S'} s = t$, allora $\exists S_1, S_2 \subseteq S : (S_1 \cup S_2 = S) \wedge (S_1 \cap S_2 = \emptyset) : \sum_{s \in S_1} s = \sum_{s \in S_2} s$

$\langle S, t \rangle$, $M := \sum_{s \in S} s$

1. $t > M \Rightarrow$ istanza negativa

   $\langle S, t \rangle \rightsquigarrow \langle \{1\} \rangle$ (sicuro $\notin$ PARTITION)
2. $t = M/2 \Rightarrow$ istanza positiva

   $\langle S, t \rangle \rightsquigarrow \langle S \rangle$
3. $t \in [0, M/2]$

   aggiungo a S elementi t, $M-t$

$<S,t> \rightsquigarrow <\tilde{S} = S \cup \{t, M-t\}>$

$\exists S' \subseteq S : \sum_{s \in S'} s = t \iff S_1 = S' \cup \{M-t\},\ S_2 = (S \setminus S') \cup \{t\}$

$\sum_{s \in S_1} s = t + M - t = M,\ \sum_{s \in S_2} s = M - t + t = M$

Trasforma sempre in istanze positive

Cambio riduzione

1) scelgo S_1, S_2 con stessa somma se $<S,t> \in SS$

2) essere sicuri che $s', s'' \notin S$ (s', s'': nuovi numeri)

3) // // // /// non possono essere in stessa partizione

$s' = X + t,\ s'' = X + M - t$ (aggiungo offset)

$<S,t> \in SS,\ S' \subseteq S : \sum_{s \in S'} s = t \Rightarrow S' \cup \{X + M - t\} = S_1 \Rightarrow \sum_{s \in S_1} s = \cancel{t} + X + M - \cancel{t}$

$S_2 = S \setminus S' \cup \{X + t\} \Rightarrow \sum_{s \in S_2} s = M - \cancel{t} + X + \cancel{t}$

Posso scegliere $X = M + 1$ (o valori maggiori) $\Rightarrow s' = M + t + 1,\ s'' = 2M - t + 1$

$s', s'' > M \Rightarrow s', s'' \notin S,\ s' \neq s'' \Rightarrow s' + s'' = 3M + 2,\ \sum_{s \in S_1 \cup S_2} s = 2(M + 1 + M) = 4M + 2 \Rightarrow$

$\Rightarrow$ così, riduzione garantita giusta