Game theory

a course for the
MSc in ICT for Internet and multimedia

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Minimax

Optimization approach to game theory

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Maximin

- Consider a "two-"player game (*i* vs -*i*)
- We define $f_i : S_i \rightarrow \mathbb{R}$ as $f_i(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$
- $s_i^* = \arg\max_{s_i \in S_i} f_i(s_i)$ is a **security strategy** (maximinimizer) for *i* (may not be unique)
- We say that $w_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ is the **maximin** or the security payoff of *i*
- A security strategy is a conservative approach allowing *i* to achieve the highest payoff in case of the worst move by -*i*

Minimax

- Similarly, $F_i : S_{-i} \rightarrow \mathbb{R}$ as $F_i(s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i})$
- $z_i = \min_{s_{-i} \in S_{-i}} F_i(s_{-i}) = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$ is called the **minimax** for player *i*
- If *i* could move after -*i*, the minimax would be the minimum payoff which is guaranteed to player *i*
  - so, if player *i* can perfectly predict the move of the other players (it is not known yet, but player *i* just knows that it will be possible to predict it), minimax = what can be expected

Example

player B (-*i*)

player A (*i*)

	L	C	R	*f* (min)
T	5, -	3, -	4, -	3
D	2, -	6, -	1, -	1
F (max)	5	6	4	

- maximin$_A$ = 3: player A can secure this payoff by playing the security strategy T
- minimax$_A$ = 4: knowing with certainty what B will play guarantees at least this payoff to A

Minimax, maximin, NE

- We can prove:

(1) For every player i , maximin$_i$ ≤ minimax$_i$

(2) If joint strategy s is a Nash equilibrium, then for every player i, minimax$_i$ ≤ $u_i\,(s)$

- The first relationship is obvious
- The second follows from every player not desiring to deviate from the NE

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Example

player B

player A

	L	C	R
T	5, 6	3, 2	4, 1
D	2, 0	6, 8	1, 2

- As previously observed, $\text{maximin}_A < \text{minimax}_A$
- Moreover, there are two Nash equilibria:
  - (T,L) where $u_A = 5 > \text{minimax}_A$
  - (D,C) where $u_A = 6 > \text{minimax}_A$
- Check for B!

Another example

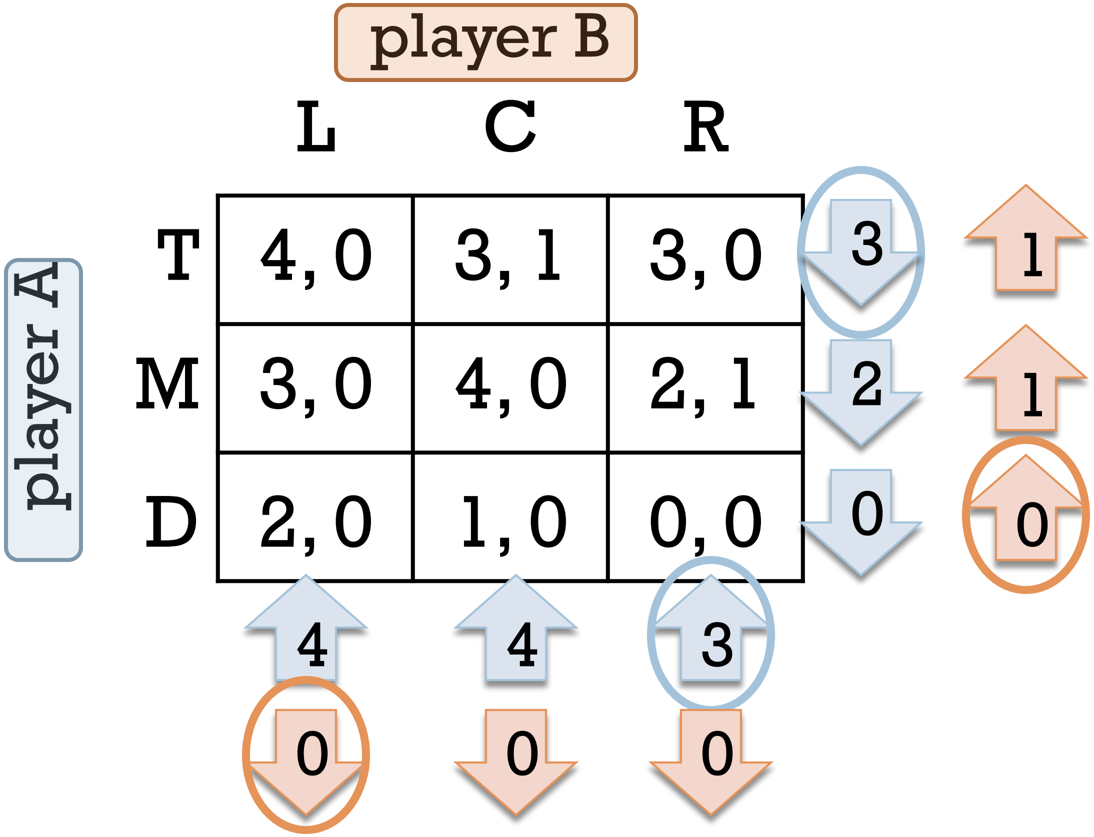
player B

		L	C	R
player A	T	3, 4	5, 0	3, 1
	D	5, 4	6, 2	7, 2

- Here, there is one NE (D, L). For both players, maximin = payoff at the NE, so it must be:

$$\text{maximin}_i = \text{minimax}_i = u_i\,(\text{NE})$$

And yet another example



	L	C	R
T	4, 0	3, 1	3, 0
M	3, 0	4, 0	2, 1
D	2, 0	1, 0	0, 0

- In general, the Lemma does not guarantee a NE
- Here, maximin$_i$ = minimax$_i$ for each player *i*

And yet another example

player B

player A

	L	C	R
T	4, 0	3, 1	3, 0
M	3, 0	4, 0	2, 1
D	2, 0	1, 0	0, 0

- However, there is no NE in pure strategies

Zero-sum games

A special class of games, easier to solve

Zero-sum

- We speak of **zero-sum game** if $u_i(s) = -u_{-i}(s)$

		player B		
		L	C	R
player A	T	-9,9	8,-8	-5,5
	M	-2,2	6,-6	2,-2
	D	-1,1	3,-3	4,-4

- Odds&Evens, rock/paper/scissors, chess… are all zero-sum games

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Competitive/adversarial games

- A more general class of games where two players *i* and -*i* (adversaries) have utilities s.t. $\boxed{u_i \nearrow \quad \Leftrightarrow \quad u_{-i} \searrow}$
  - zero-sum games are a special category of this
- If the utilities have just ordinal meaning and/or they can be rescaled by constant terms, easy connection with zero-sum games
  - Chess: +1 to winner,+0.5 if tie → like zero-sum
  - Serie A: +3 to winner, +1 if tie → not exactly

Minimax Theorem (1)

- G = zero-sum game with finitely many strategies

(1) G has a NE iff maximin$_i$ = minimax$_i$ for each i

(2) All NEs yield the same payoff (= maximin$_i$)

(3) NEs have form (s_i^*, s_{-i}^*), with s_i^* = security strategy

player B

player A

	L	C	R
T	-9, 9	8, -8	-5, 5
M	-2, 2	6, -6	2, -2
D	-1, 1	3, -3	4, -4

for player A:

- maximin = -1
- minimax = -1
- (L,D) is a NE, $u_A = -1$

Remarks

- Since the game is zero-sum, it is sufficient to check maximin = minimax for one player only
- It also holds

$$\boxed{\begin{aligned} \text{maximin}_i &= -\,\text{minimax}_{-i} \\ \text{minimax}_i &= -\,\text{maximin}_{-i} \end{aligned}} \Rightarrow \text{da definizione}$$

- The common value of $\text{maximin}_1 = \text{minimax}_1$ is called the **value** of the game
  - Some games with infinitely many strategies are "without value" (theorem does not hold)
- A joint security strategy (if any), i.e., a NE, is called a **saddle point** of the game

Remarks

- The bi-matrix for this special kind of games can be represented with a regular matrix (utility of player $-i$ is implicit)
- The proof of the theorem is due to von Neumann (1928) and makes use of linear programming (constrained optimization)
- The criterion of minimaximizing the utility has been widely employed in artificial intelligence applications: e.g., chess, which is a zero-sum (although sequential) game

Mixed maximin/minimax

the extensions to mixed strategies

Mixed security strategy

- Consider a "two-"player game (*i* vs -*i*), and take $f_i : \Delta S_i \rightarrow \mathbb{R}$ as $f_i(m_i) = \min_{m_{-i} \in \Delta S_{-i}} u_i(m_i, m_{-i})$
- Any mixed strategy m_i^* maximizing $f_i(m_i)$ is a **mixed security strategy** for *i*
- This max, i.e. $\max_{m_i \in \Delta S_i} \min_{m_{-i} \in \Delta S_{-i}} u_i(m_i, m_{-i})$ is the maximin_i^m or the **mixed security payoff** of *i*
- A mixed security strategy is the conservative mixed strategy guaranteeing the highest payoff for *i* in case of the worst mixed strategy by -*i*

Mixed minimax

- Also if $F_i : \Delta S_{-i} \rightarrow \mathbb{R}$ is $F_i(m_{-i}) = \max_{m_i \in \Delta S_i} u_i(m_i, m_{-i})$ $\min_{m_{-i} \in \Delta S_{-i}} F_i(m_{-i}) = \min_{m_{-i} \in \Delta S_{-i}} \max_{m_i \in \Delta S_i} u_i(m_i, m_{-i})$ is the **minimax** for i in mixed strategy, minimax_i^m
- If i could move after $-i$, there is a mixed strategy which guarantees i to achieve at least minimax_i^m
- **Note 1.** $f_i(m_i)$ can be found minimizing $u_i(m_i, s_{-i})$, i.e., using pure strategies only. $F_i(m_{-i})$ can be defined maximizing $u_i(s_i, m_{-i})$
- **Note 2.** maximin_i^m and minimax_i^m always exist and are equal, as payoff $u_i(m_i, m_{-i})$ is continuous

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maximin^m vs minimax^m

- From pure minimax:

  If joint mixed strategy m is a Nash equilibrium, then for every player i, $\boxed{\text{minimax}_i^m \leq u_i(m)}$

		Joe	
		S	C
Jim	T	3,-	0,-
	M	1,-	2,-

(only Jim's payoffs are shown)

- Jim: maximin = 1, minimax = 2
- Jim can increase his maximin if he plays ¼ T + ¾ M. maximin^m = 1.5
- For Jim, the worst strategy Joe can play is ⅓ S + ⅔ C, minimax^m = 1.5

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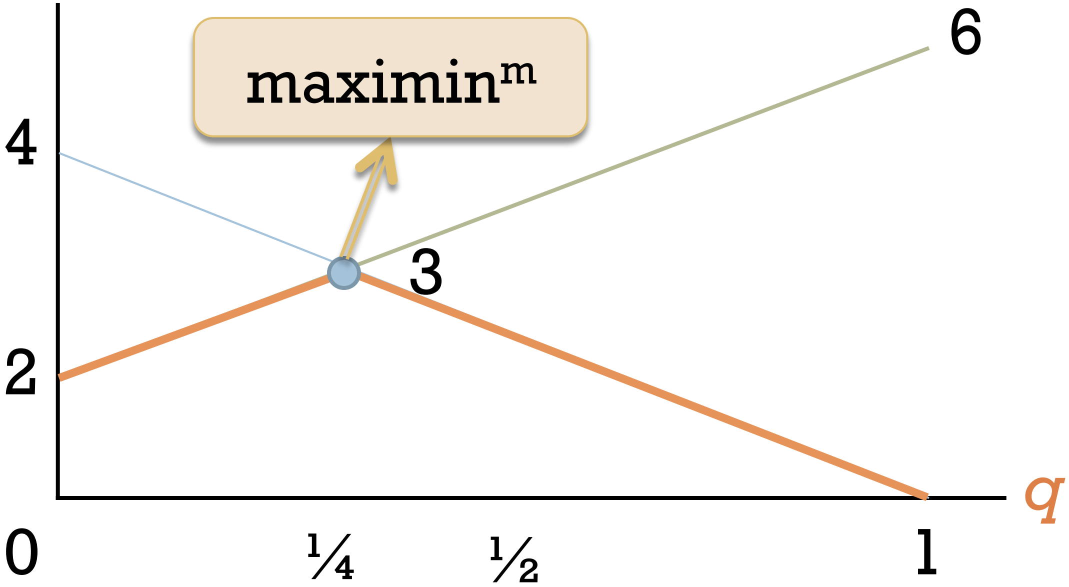
maximin^m vs minimax^m

		Bea	
		F	G
Art	U	6, 1	0, 4
	D	2, 5	4, 0



- ☐ Art's mixed strategies are uniquely described by q
- ☐ $f_A(q) = \min_{s_B \in \{F,G\}} u_A(q,s_B) = \min\{\, u_A(q,F), u_A(q,G) \,\} =$

  $= \min\{\, 6q+2(1-q), 4(1-q) \,\} = \min\{\, \underline{2+4q}, \underline{4-4q} \,\}$

maximin^{m} vs minimax^{m}

- Check yourself that minimax$_{A}^{m}$ is also 3
- So it is verified that
  maximin$_i$ ≤ maximin$_i^m$ = minimax$_i^m$ ≤ minimax$_i$

- Note that we found a Nash equilibrium at (⅝, ½), so Art's payoff at NE is also 3

- As an exercise, do the same check for Bea, her maximin$_i^m$ = minimax$_i^m$ = u_B(NE) = 2.5

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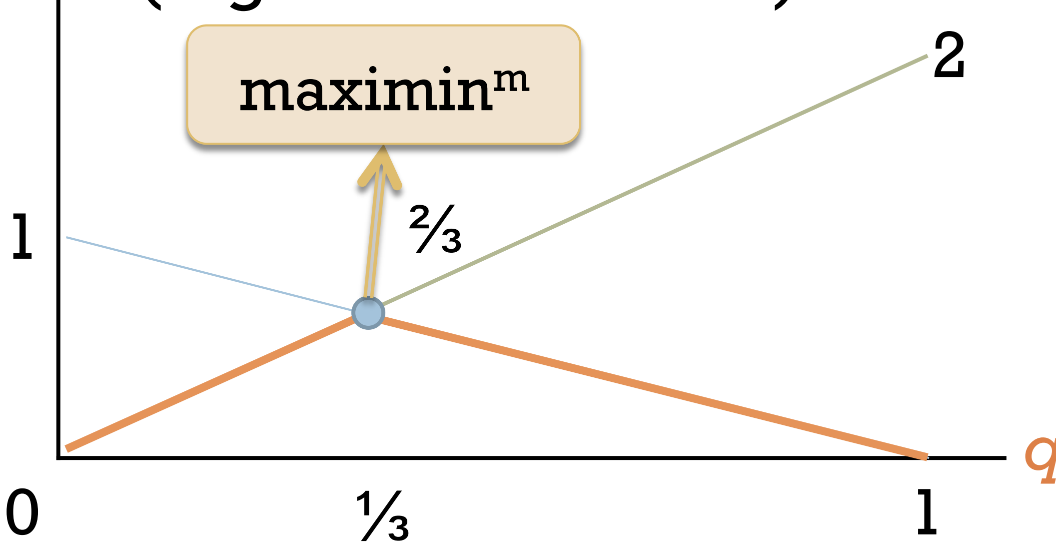
back to Example 5

- You can have more NEs (e.g. Battle of Sexes)

		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2



- You can check that the maximin = 0, minimax = 1 for both players. But maximin^m = minimax^m= ⅔
- You have three NEs whose payoffs are 1, 2, 1.67

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back to Example 3

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

- Also for this game (which is zero-sum)

$$\text{maximin} = -4 < \text{maximin}_i^{m} = \text{minimax}_i^{m} = 0 < \text{minimax} = 4$$

- 0 was the payoff at the (mixed) NE

Minimax Theorem (2)

- G = zero-sum game with finitely many strategies

(1) For every player *i*, $\text{maximin}_i^m = \text{minimax}_i^m$ → thus, G must have a Nash equilibrium! (this is actually how to find it)

(2) All Nash equilibria in mixed strategies are security strategies for player *i* and yield a payoff to player *i* equal to maximin_i^m

- **Note**. In zero-sum games $\text{maximin}_1^m = -\text{minimax}_2^m$
- All Nash equilibria are "equivalent" (same payoff)
- maxmin_1^m is called the **value** of the game.

Linear Programming

- The search of minimax solutions (i.e., NEs) of a zero-sum game is a nice application of LP
- Player 1 has pure strategies $\{A_1, A_2, \dots, A_L\}$
- A mixed strategy $\mathbf{a} = \{ a_j \}$ is a linear combination
  $a_1 A_1 + \dots + a_L A_L$
- Player 2 has pure strategies $\{B_1, B_2, \dots, B_M\}$
- A mixed strategy $\mathbf{b} = \{ b_j \}$ is a linear combination
  $b_1 B_1 + \dots + b_M B_M$

- **Note.** We only need $u = u_1$ as $u_2 = - u_1$

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Linear Programming

- $a_j \geq 0, \sum_j a_j = 1$
- $\sum_j a_j u(A_j B_k) \geq W \; (\forall k)$
- **maximize** W

The a_j s are a probability distribution

M constraints

strategie pure di B

The payoff of $(\mathbf{a}, B_k)$. We check **a** against M pure strategies only

W must be maximized. W cannot be increased, when some constraints become active. These constraint describe the support of player 2's **b**.

- In general, find a mixed minimax strategy for player 1

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Linear Programming

- Since $\text{maximin}_i^m = \text{minimax}_i^m$
  - $a_j \geq 0, \sum_j a_j = 1$
  - $\sum_j a_j u(A_j B_k) \geq W \quad (\forall k)$
  - **max**imize W

  minimax version

  - $b_j \geq 0, \sum_j b_j = 1$
  - $\sum_j b_j u(A_k B_j) \leq W \quad (\forall k)$
  - **min**imize W

  maximin version

- The two problems yield the same solution
- **Note.** This formulation can be made for every problem, but solution is not always guaranteed
- Zero-sum games are special in that $u_2 = - u_1$

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How to solve minimax

- LP problems can be solved via optimization
- Polynomial-time techniques exist
- Simplex method is widely used (CPLEX, lpsolve): (worst-case) exponential, often fast in practice
- Meta-heuristic techniques (Genetic Algorithms, Tabu search): sometimes even faster, but they do not guarantee to find the solution

Stackelberg games

From static to sequential games

Stackelberg games

- Proposed by von Stackelberg (1934) to model incumbent vs. outsider competition
- It is a sequential version of a static game (analogous to the sequential Battle of Sexes)
- Players move alternately
  - First player 1 (**leader**), then player 2 (**follower**)
- Can be represented again with a bi-matrix
- The backward induction outcome is called the **Stackelberg equilibrium**

Stackelberg games

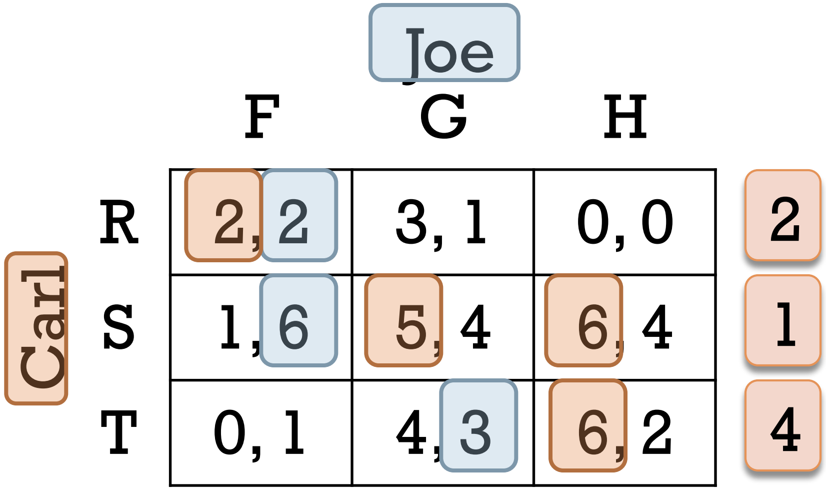
		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- ☐ If Ann is leader, Stackelberg equilibrium is (R,R)
- ☐ Brian achieves his minimax=1

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

- ☐ Stackelberg eq = same as NE, i.e., both players choose 50/50
- ☐ Both achieve minimax=0.

Example 12



		Joe			
		F	G	H	
Carl	R	2, 2	3, 1	0, 0	2
	S	1, 6	5, 4	6, 4	1
	T	0, 1	4, 3	6, 2	4

- (R,F) is a pure Nash equilibrium
- If Carl is leader, he knows Joe's best responses
- Stackelberg equilibrium with Carl leader = (T,G)
- Joe obtains payoff 3, his minimax was 2

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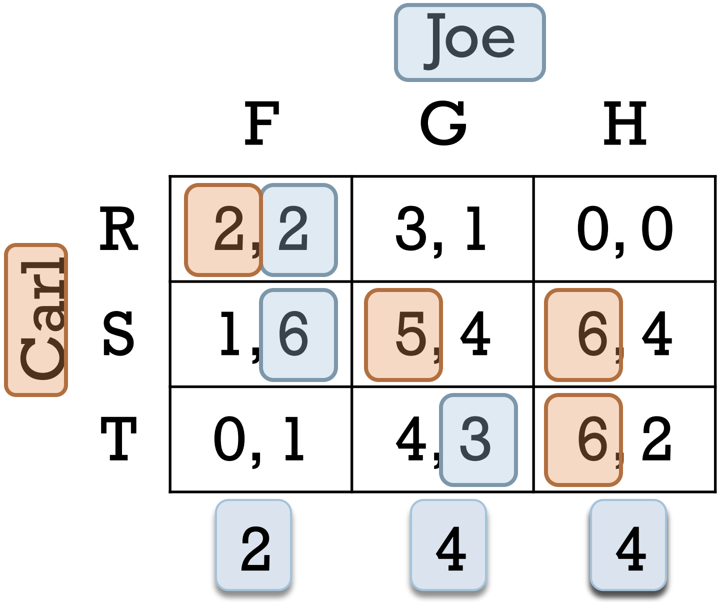
Example 12



	F	G	H
R	2, 2	3, 1	0, 0
S	1, 6	5, 4	6, 4
T	0, 1	4, 3	6, 2
	2	4	4

- If Joe is leader, we need other information to solve
  - We assume that Carl solves ties with the choice which is best for Joe (generous follower)
  - Also assume Joe solves ties with what is best for the follower (generous leader)
- Stackelberg equilibrium with Joe leader = (S,H)
- Carl obtains payoff 6, his minimax was 2

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Comments on Stackelberg

- The leader has "first-move advantage"
  - His/her payoff ≥ that in Nash equilibrium
  - See that if Ann leads, she has a guaranteed payoff greater than in any of the NEs
- The follower is not necessarily worse off in the Stackelberg setup
  - His/her payoff ≥ minimax

		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

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Comments on Stackelberg

- For adversarial/competitive setups, more specifically for zero-sum games, however:
  - the leader being better off means that the follower is worse off
- Strange: the follower has more information!
  - but more information → lower payoff
  - in classic optimization, knowledge is power
  - in game theory, ignorance is bliss
  - it is a consequence of rationality: player 2 has more information but player 1 can anticipate this