Game theory

A course for the
MSc in ICT for Internet and multimedia

Leonardo Badia

leonardo.badia@gmail.com

Rationalizing solutions

Best responses and beliefs

meglio : non giocare strategie sempre dominate

Leonardo Badia – leonardo.badia@gmail.com

Single- vs. multi-player games

- For single-agent problems, once the setup is known, the solution can be found directly
- Not so for multi-player games
  - Here the solution depends on other players
  - Sometimes rationality can help (eg. we identify a dominated strategy → we do not play it)
  - We can extend this reasoning by assuming rationality of other players, which leads to IESDS
  - But still most of the times no solution is found

Best response

- Strategy $s_i \in S_i$ is i's best response to ~~his/her~~ opponent's moves $(s_1,..,s_{i-1}, s_{i+1},..,s_n)$ if: $n \leftarrow$ giocatori

$$u_i(s_1,..,s_{i-1}, s_i, s_{i+1},..,s_n) \geq u_i(s_1,..,s_{i-1}, s_i', s_{i+1},..,s_n)$$
$$\text{for every } s_i' \in S_i$$

- Notation: $(s_1,..,s_{i-1}, s_{i+1},..,s_n) \in S_1 \times .. \times S_{i-1} \times S_{i+1} \times .. \times S_i$
- This is often shortened to "$s_{-i} \in S_{-i}$"
- Thus: $s_i \in S_i$ is a best response to $s_{-i} \in S_{-i}$ if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i$

Best response

- There may be more than one best response!
  - All with identical value of $u_i(s_i, s_{-i})$ of course

$s_i = U/D$

$s_{-i} = R$

		player B		
		L	C	R
player A	U	3, 3	5, 1	6, 2
	M	4, 1	8, 4	3, 6
	D	4, 0	9, 6	6, 8

  - So, U and D are both best responses to player B's strategy to play R. While for strategy L?

$\Downarrow$

M, D

Best response

- Claim: a rational player who believes that the opponents are playing some $s_{-i} \in S_{-i}$, will always choose a best response to s_{-i}
- Theorem: if $s_i \in S_i$ is a strictly dominated strategy, it is no best response to any $s_{-i} \in S_{-i}$
  - Proof: there must be $s_i' \in S_i$ dominating it
  - It is immediate to see that the definition of best response applied to s_i is violated by s_i'

Beliefs

- A **belief** of player i is a possible profile of opponents' strategies, ie., an element of set S_{-i}
  - Beliefs are connected to best responses!
- We define a best-response-correspondence $\boxed{\text{BR}: S_{-i} \rightarrow \wp(S_i)}$ that associates to $s_{-i} \in S_{-i}$ a subset of S_i such that each $s_i \in \text{BR}(s_{-i})$ is a best response to s_{-i}
  - This is not a function: but $\text{BR}(s_{-i})$ can be a singleton (if the best response is unique)

Nash equilibrium

the key tool of game theory

Leonardo Badia – leonardo.badia@gmail.com

Nash equilibrium

- We want to strengthen the dominated strategy concept with this idea in mind:
  - game theory should make predictions about the outcome of games played by a rational players
  - a prediction is correct if the players are **willing** to play their predicted strategy
- That is, players choose their **best response** to the predicted strategy of the others
  ⇒ qualunque sia
- If this happens, the prediction is said to be **self-enforcing** (or also strategically stable)

Formal definition

- In a n-player game G = $\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, strategies $(s_1^*,\ldots,s_n^*)$ are a **Nash equilibrium** if, for any i , s_i^* is the best response of player i to $(s_1^*,\ldots,s_{i-1}^*,s_{i+1}^*,\ldots,s_n^*)$
- That is, $\forall s_i \in S_i$:

$$u_i(s_1^*,\ldots,s_{i-1}^*, s_i^*, s_{i+1}^*,\ldots s_n^*)$$
$$\geq u_i(s_1^*,\ldots,s_{i-1}^*, s_i, s_{i+1}^*,\ldots s_n^*)$$

$$s_i^* = \operatorname{argmax}_s\ u_i(s_1^*,\ldots,s_{i-1}^*, s, s_{i+1}^*,\ldots s_n^*)$$

Motivation

- Take a possible combination $(s_1',\ldots,s_n')$
- If this is not a Nash equilibrium then there exist some player i , such that s_i' is **not** the best response to $(s_1',\ldots,s_{i-1}',s_{i+1}',\ldots,s_n')$.
- That is, $\exists\ s_i'' \in S_i$ such that

$$\boxed{\begin{aligned} & u_i(s_1',\ldots,s_{i-1}', s_i',s_{i+1}',\ldots s_n') \\ & < u_i(s_1',\ldots,s_{i-1}', s_i'',s_{i+1}',\ldots s_n') \end{aligned}}$$

- Thus, there exists an incentive for player i to deviate from $(s_1',\ldots,s_n')$

Comment

- Remember this is a static (one-shot) game
- A NE can also be seen as the case where nobody has **regrets** on his/her choice
  - it is intended as a forecast of the outcome, not as the final result of several moves
  - repeated games will disprove this wrong (but diffuse) misconception
- We will also discuss how useful it is to know that there is such a "natural" outcome

back to Example 1

- Combination (M,R) is a Nash equilibrium

player B

player A		L	R
	U	6, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

- (M,R) satisfies the NE condition
- A first way to find Nash equilibria is brute force search: here, (M,R) is the only one

back to Example 2

- Another way is to focus on "best responses"

player B

player A		L	C	R
	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

- (D,R) is the only Nash equilibrium, found by checking the cell with both entries highlighted

back to Example 3

- Here there is no Nash equilibrium
- We will see that there is actually one, but we need to “extend” the game somehow

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

back to Example 5

- (R,R) and (S,S) are both Nash equilibria
- This reflects our previous intuition

Ann \ Brian	R	S
R	2, 1	0, 0
S	0, 0	1, 2

- However, here the NE concept is less useful as it cannot be used to make predictions

back to Example 6

- Combination (F,F) is a Nash equilibrium

Al \ Bob	M	F
M	-1, -1	-21, 0
F	0, -21	-20, -20

- It seems that Nash equilibrium extends iterated elimination of strictly dominated strategies (i.e., if any exists, it is a NE)

Theorem

- In a finite game, if $\boxed{(s_1^*, \ldots, s_n^*)}$ is:
  - the only survivor of IESDS
  - or the only rationalizable profile ⇒ inferenza dopo eliminazione strategie stupide

  then $\boxed{(s_1^*, \ldots, s_n^*) \text{ is a NE}}$

- Lemma: a NE $(s_1^*, \ldots, s_n^*)$ survives iterated elimination of strictly dominated strategies
- Another result: IESDS is order irrelevant

To sum up

- Two requirements must be satisfied by a NE
  - Everyone plays a best response to their beliefs
  - Everyone's beliefs are **correct**
- Actually the first requirement is quite logical and consequent from rationality, while the second requirement is quite demanding
  - It may be inferred only from some external reasoning (for example, one player being particularly "influential" in the game)

Dominance, efficiency

further comparisons

Strict/weak dominance

- For brevity, we write thereafter

$$s_{-i} = (s_j)_{j \neq i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

- Recall that s_i' **strictly dominates** s_i if

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for every } s_{-i}$$

- We say that s_i' **weakly dominates** s_i if

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \qquad \text{for } \textbf{every } s_{-i}$$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for } \textbf{some } s_{-i} \quad \textbf{(*)}$$

- Without **(*)**, we say that s_i' **dominates** s_i

Dominance/Nash equilibrium

- A strategy that (strictly, weakly) dominates every other strategy of a user is said to be **(strictly, weakly) dominant**
- **Lemma**

  If every user i has a dominant strategy s_i* then (s_1*,…,s_i*,…,s_n*) is a Nash equilibrium.
- It directly follows from the definition of NE
- The reverse statement is false (only sufficient condition, not necessary)

Do not eliminate weakly dom.

- Enlarge the Odd/Even game with a third strategy "Punch the opponent" (P)
- P is weakly dominated, yet it is a NE
- If we eliminate it, we lost the only NE

(a strange NE: later in the course we will see a similar situation)

Odd \ Even	0	1	P
0	-5, 5	5, -5	-5, -5
1	5, -5	-5, 5	-5, -5
P	-5, -5	-5, -5	-5, -5

NE vs. Pareto efficiency

- Pareto efficiency is different from NE:
  - Pareto efficiency: no way (in the **whole game**) a user can improve without somebody else being worse
  - Nash equilibrium: no way a user can improve **with a unilateral change**
- The outcome of the Prisoner's Dilemma is not "efficient!"

		Bob M	Bob F
Al	M	-1, -1	-21, 0
Al	F	0, -21	-20, -20

These strategies are Pareto efficient

(F,F) is the only Nash equilibrium

NE vs. Pareto efficiency

non formulato bene

- Pareto inefficient Nash equilibria arise as we assume players are only driven by egoism
- To estimate the inefficiency of being selfish (or distributed) one can compare Nash equilibria with Pareto efficient strategies
- To this end, assume that a joint strategy $\boxed{s}$ has a social cost $\boxed{K(s)}$
  - E.g., $\boxed{K(s) = \sum_j -u(s_j)}$, or $\boxed{K(s) = \max_j -u(s_j)}$

(this means overall welfare) (this is *minmax fairness*)

Price of anarchy

- The **price of anarchy** is the ratio between the social costs in the worst NE s^* and in the best Pareto efficient strategy (i.e., social optimum)

$$\boxed{A = K(s^*) \,/\, (\min K(s))}$$

- If the best NE is considered, it is sometimes spoken of **price of stability**

- For certain classes of problems, there are theoretical results on the price of anarchy

Fun game

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game
- You are paired with an unknown classmate
- You secretly choose an integer from 18 to 30, and so does the classmate
- Then the numbers are checked
  - If they match, this is the score you both get.
  - If they don't, let L be the lower number. Who proposed L gets L+R, the other gets L-R (score <18 means rejection, >30 means honors)
- Play the game with R=2… Now with R=10

Solution of the game

- If R > 1, there is a unique Nash equilibrium, which is to play 18 for both students
- However, cooperative behaviors may arise, even though they are not NE
  - Criticism against rationality of players
- Usually, a high R (for example 10) dampens the cooperation