UNIVERSITÀ
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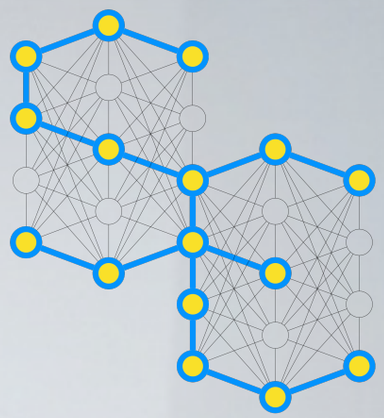
DI PADOVA



Statistical Hypothesis Testing

Search Engines

Master Degree in Computer Engineering
Master Degree in Data Science
Academic Year 2023/2024

Nicola Ferro

Intelligent Interactive Information Access (IIIA) Hub
Department of Information Engineering
University of Padua

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

Outline

- Hypothesis Testing
- Student's t test
- Analysis of Variance (ANOVA)

Hypothesis Testing

The Problem: Are They Different?

$$x = \begin{bmatrix} 3.1929 \\ 0.6575 \\ 1.4155 \\ -0.2043 \\ -1.6054 \\ 1.1446 \\ 1.1033 \\ 0.7418 \\ 0.5035 \\ 1.5477 \end{bmatrix} \qquad y = \begin{bmatrix} 0.7471 \\ 1.6357 \\ 1.0390 \\ 1.1421 \\ 1.5731 \\ 3.5389 \\ 1.9563 \\ 0.6754 \\ 2.6305 \\ 1.9505 \end{bmatrix} \qquad z = \begin{bmatrix} 0.6408 \\ 2.6859 \\ 3.0818 \\ 3.1258 \\ 2.9028 \\ 2.4371 \\ 1.7648 \\ 3.2616 \\ 2.0997 \\ 1.0346 \end{bmatrix}$$

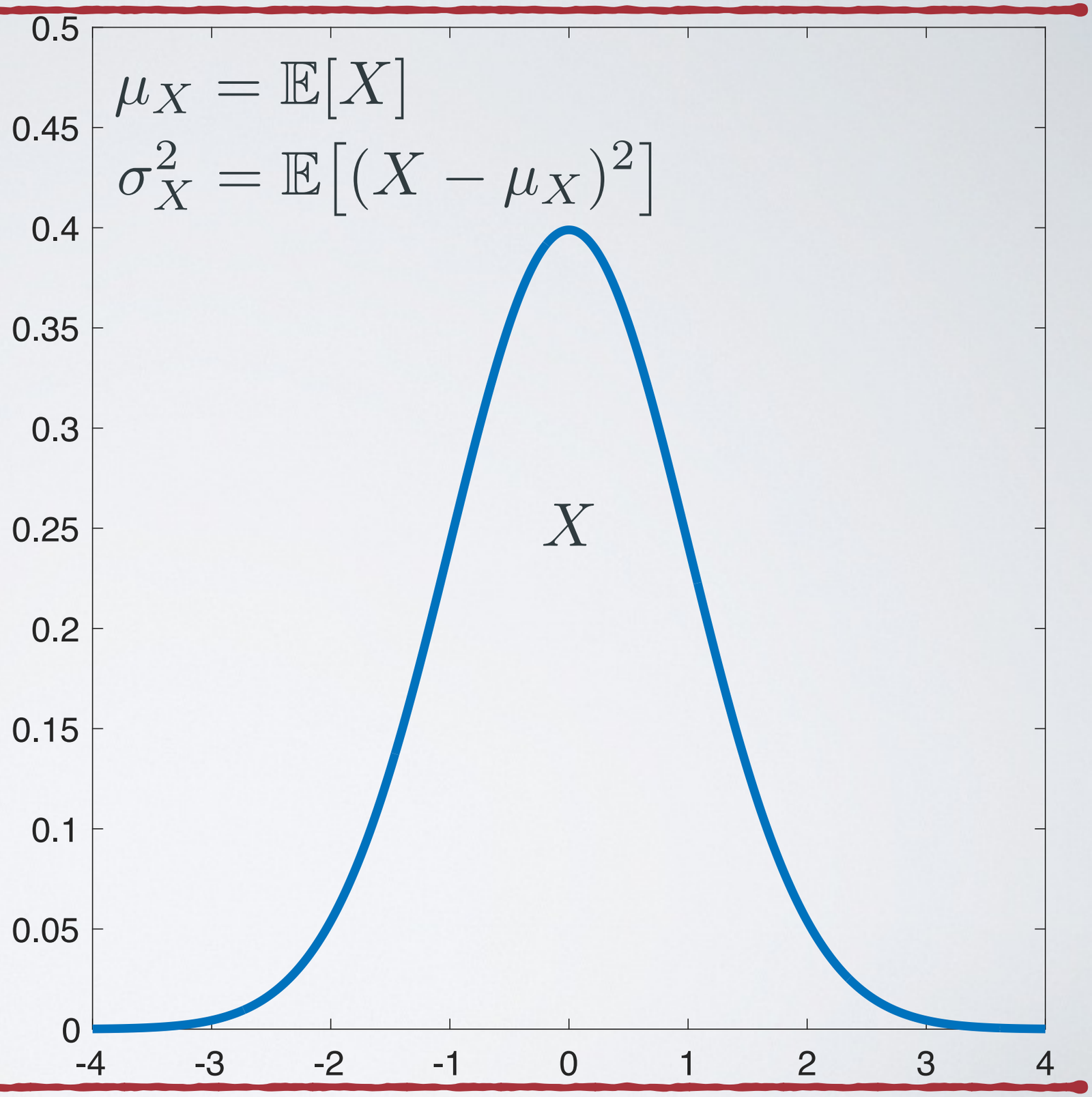
X → $\hat{\mu}_X = 0.8497$, $\hat{\sigma}^2_X = 1.5284$

Y → $\hat{\mu}_Y = 1.6889$, $\hat{\sigma}^2_Y = 0.7890$

Z → $\hat{\mu}_Z = 2.3035$, $\hat{\sigma}^2_Z = 0.8255$

What Did We Do?

$$x = \begin{bmatrix} 3.1929 \\ 0.6575 \\ 1.4155 \\ -0.2043 \\ -1.6054 \\ 1.1446 \\ 1.1033 \\ 0.7418 \\ 0.5035 \\ 1.5477 \end{bmatrix}$$



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$$\hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2_X = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu}_X)^2$$

- $\{X_1, X_2, \ldots, X_n\}$ is a **random sample** of size n, i.e. a sequence of **independent and identically distributed (i.i.d)** random variables drawn from a distribution
  - x_i is an **observation** (realisation) of the random variable X_i, i.e. the actual value assumed by that random variable in a given trial
- The **sample mean** $\hat{\mu}_X$ and the **sample variance** $\hat{\sigma}^2_X$ are **unbiased estimators** of the **population mean** μ_X and **population variance** σ^2_X

What Did We Do?

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Degrees of Freedom (DF)

- $\{X_1, X_2, \ldots, X_n\}$ is a **random sample** of size n, i.e. a sequence of **independent and identically distributed (i.i.d)** random variables drawn from a distribution
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Law of Large Numbers & Central Limit Theorem

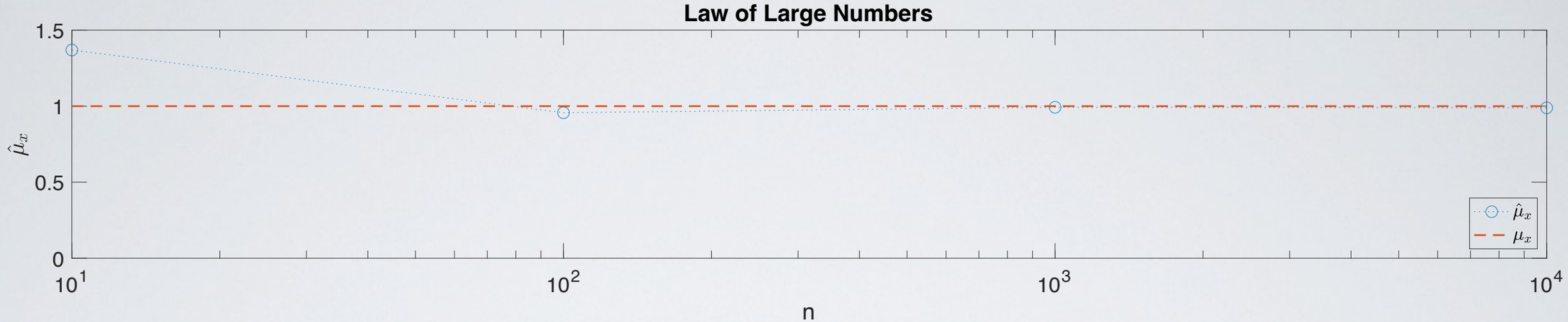
- The (strong) **Law of Large Numbers** states that the sample mean converges almost surely to the population mean

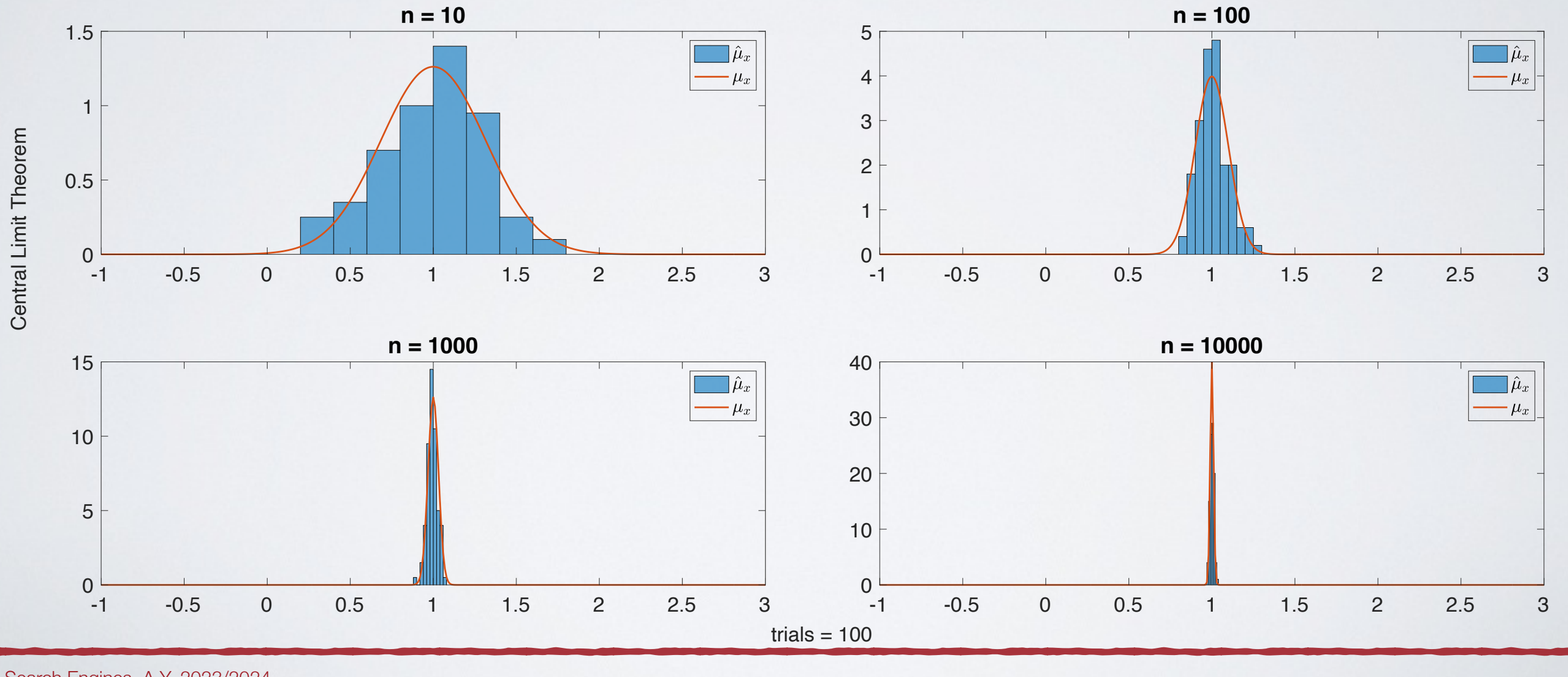
$$\hat{\mu}_X \xrightarrow{\text{a.s.}} \mu_X \quad \text{for } n \to \infty$$

- The **Central Limit Theorem** states that the sample mean follows a normal distribution

$$\hat{\mu}_X \sim \mathcal{N}\left(\mu_X, \frac{\sigma_X^2}{n}\right) \qquad \frac{\hat{\mu}_X - \mu_X}{\sigma_X / \sqrt{n}} \sim \mathcal{N}(0, 1)$$

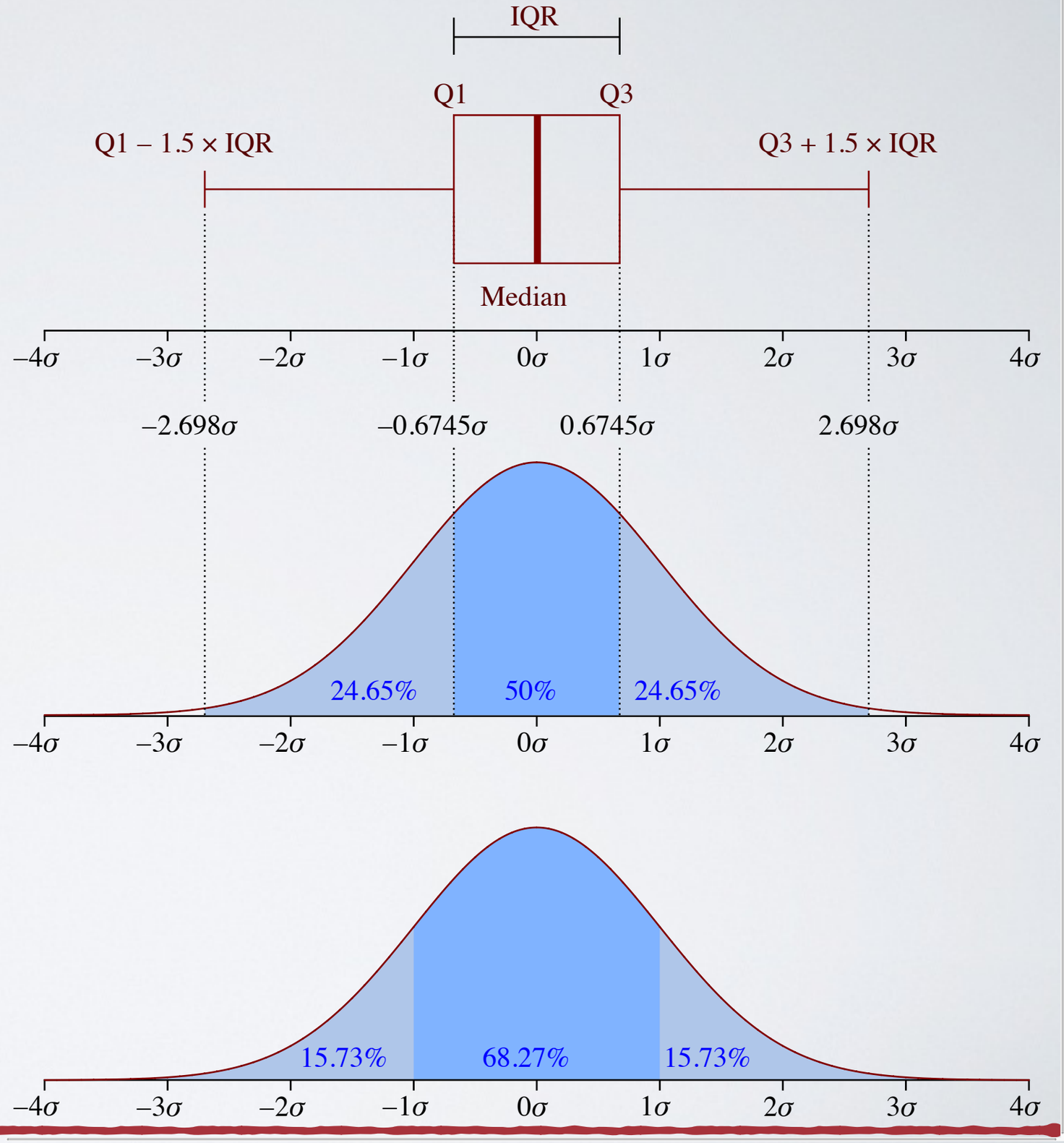
Law of Large Numbers & Central Limit Theorem





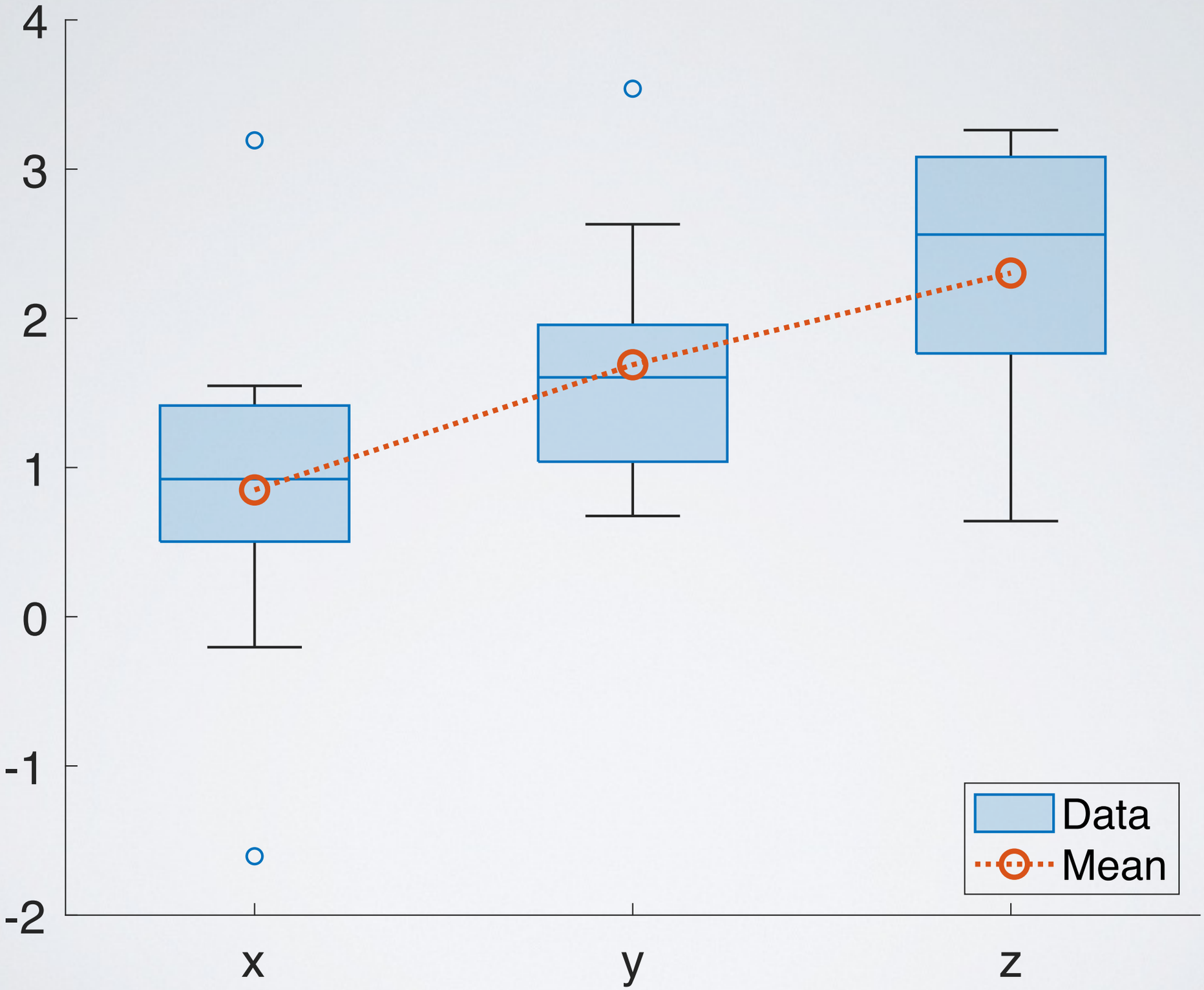
Box Plot

- A **boxplot** is a graphical tool to summarise a distribution of data
- The box shows the **first quartile** (Q1), the **second quartile** (Q2, the **median**) with a line inside the box, and the **third quartile** (Q3)
- The box represents the **Inter-Quartile Range (IQR)**, i.e. the difference Q3-Q1
- The extension of the **whiskers** represents 1.5·IQR
  - they roughly cover ~99% of the data, assuming a normal distribution
- Any data outside the whiskers is considered an **outlier**



McGill, R., Tukey, J. W., and Larsen, W. A. (1978). Variations of Box Plots. *The American Statistician*, 32(1):12–16.

The Problem: Are They Different?



Compute it Yourself!

Basic Statiscs, Law of Large Numbers, and Central Limit Theorem

This example shows the computation of some basics statistics on simulated data as well as the application of the Law of Large Numbers and the Central Limit Theorem.

Basic Statistics and Box Plot

Define the parameters to be used.

```
% set the random number generator for reproducibility
rng(6, 'twister');

% total number of samples to be used
n = 10;
```

Generate the random (simulated) data from normal distributions with same variance as follows:

- $x \sim \mathcal{N}(1, 1)$
- $y \sim \mathcal{N}(1, 1)$
- $z \sim \mathcal{N}(2.3, 1)$

```
x = 1 + randn(n, 1);
y = 1 + randn(n, 1);
z = 2.3 + randn(n, 1);
```

Compute basics statistics:

- sample mean

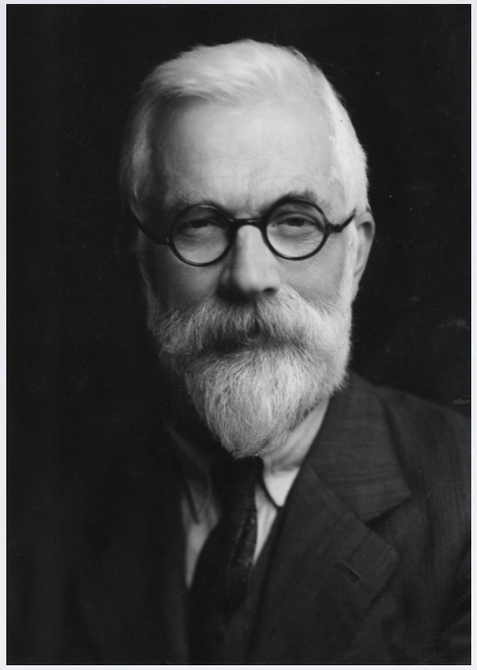
$$\widehat{\mu}_x = \frac{1}{n}\sum_{i=1}^{n} x_i = \texttt{mean}(x)$$

- sample variance

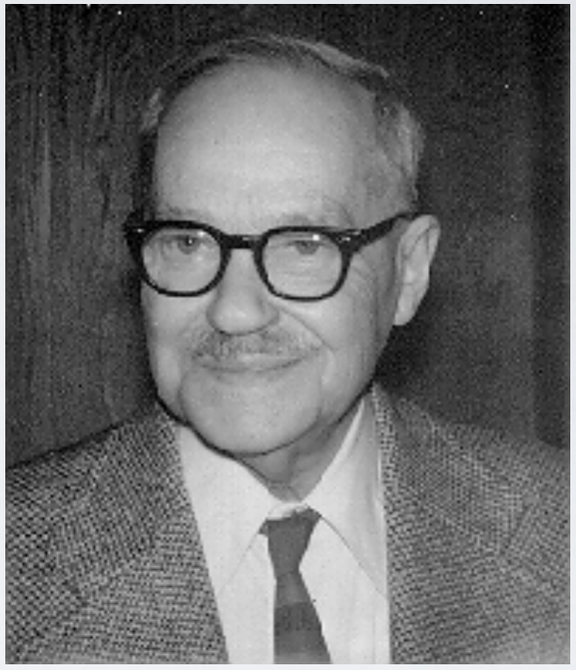
$$\widehat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \widehat{\mu}_x)^2 = \texttt{var}(x)$$

Statistical Hypothesis Testing

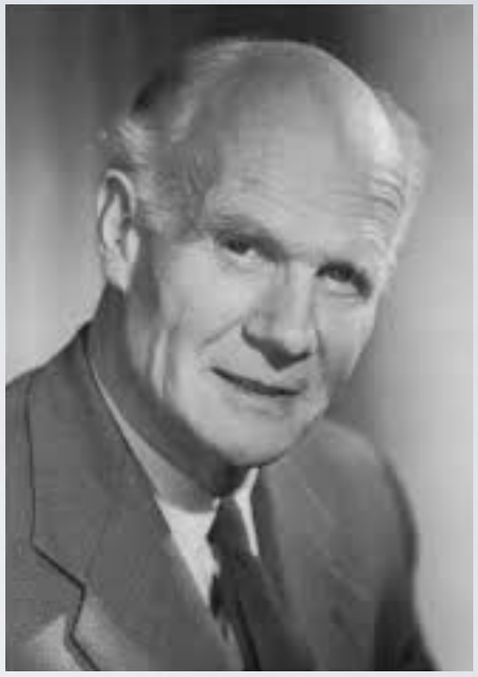
William Sealy Gosset "Student"



Ronald Aylmer Fisher



Jerzy Neyman



Egon Sharpe Pearson

- **Statistical hypothesis testing** provides us with a mathematical framework to conduct **statistical inference** from the data
- It compares the so-called **null hypothesis** H_0 against an **alternative hypothesis** H_1 or H_A
- The comparison is **statistically significant** if the data are unlikely to be a realisation of the null hypothesis with respect to a chosen threshold, called **significance level** α. In this case we **reject** the null hypothesis; in the opposite case, we **fail to reject** the null hypothesis

Student (1908). The Probable Error of a Mean. *Biometrika*, 6(1):1–25.
Fisher, R. A. (1925). Statistical Methods for Research Workers. Oliver & Boyd, Edinburgh, UK.
Neyman, J. and Pearson, E. S. (1928). On the Use and Interpretation of Certain Test Criteria for Purposes of Statistical Inference. *Biometrika*, 20A(1/2):175–240.
Lehmann, E. L. (1993). The Fisher, Neyman-Pearson Theories of Testing Hypotheses: One Theory or Two? *Journal of the American Statistical Association*, 88(424):1242–1249.

Formulating the Problem

$$x = \begin{bmatrix} 3.1929 \\ 0.6575 \\ 1.4155 \\ -0.2043 \\ -1.6054 \\ 1.1446 \\ 1.1033 \\ 0.7418 \\ 0.5035 \\ 1.5477 \end{bmatrix}$$

$$\begin{cases} H_0: & \mu_X = \mu_Y \\ H_1: & \mu_X \neq \mu_Y \end{cases}$$

$$y = \begin{bmatrix} 0.7471 \\ 1.6357 \\ 1.0390 \\ 1.1421 \\ 1.5731 \\ 3.5389 \\ 1.9563 \\ 0.6754 \\ 2.6305 \\ 1.9505 \end{bmatrix}$$

Random Sampling

Random Sampling

$\mu_X - \mu_Y$

$\hat{\mu}_X = 0.8497$

Sample Mean

$\hat{\mu}_Y = 1.6889$

Sample Mean

μ_X Population Mean

μ_Y Population Mean

Sakai, T. (2018). *Laboratory Experiments in Information Retrieval*, volume 40 of *The Information Retrieval Series*. Springer Singapore.

Test Statistic

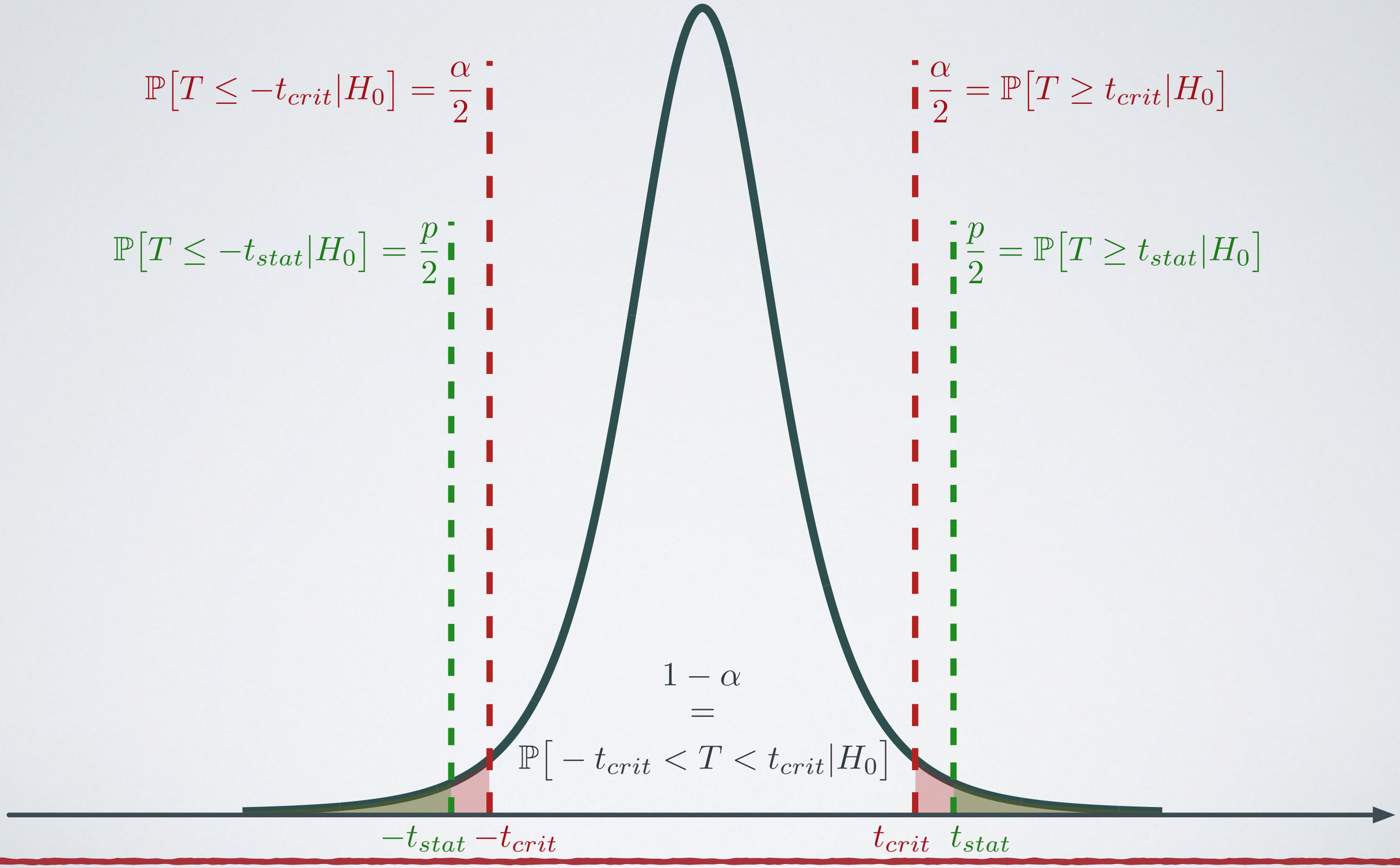
Test statistic distribution T under the null hypothesis H_0

$$\mathbb{P}\big[T \leq -t_{crit} | H_0\big] = \frac{\alpha}{2}$$

$$\mathbb{P}\big[T \leq -t_{stat} | H_0\big] = \frac{p}{2}$$

$$\frac{\alpha}{2} = \mathbb{P}\big[T \geq t_{crit} | H_0\big]$$

$$\frac{p}{2} = \mathbb{P}\big[T \geq t_{stat} | H_0\big]$$



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$$\mathbb{P}\big[T \leq -t_{stat} | H_0\big] = \frac{p}{2}$$

$$\frac{\alpha}{2} = \mathbb{P}\big[T \geq t_{crit} | H_0\big]$$

$$\frac{p}{2} = \mathbb{P}\big[T \geq t_{stat} | H_0\big]$$

Reject the null hypothesis H_0 when

$$t_{stat} \geq t_{crit}$$

or, equivalently, when

$$p \leq \alpha$$

$$1 - \alpha = \mathbb{P}\big[-t_{crit} < T < t_{crit} | H_0\big]$$

$-t_{stat}$ $-t_{crit}$ t_{crit} t_{stat}

Types of Error

	We fail to reject H_0 [not statistically significant]	We reject H_0 [statistically significant]
		
H_0 **is true** [e.g. systems are equivalent]	Correct conclusion [true negative] Probability $1 - \alpha$	**Type I error** [false positive] Probability α
H_0 **is false** [e.g. systems are not equivalent]	**Type II Error** [false negative] Probability β	**Power** (Correct conclusion) [true positive] Probability $1 - \beta$

Multiple Comparisons

- Type I errors concern the comparison of 2 alternatives
  - What happens when you need to compare c alternative?
  - They originate $m = \binom{c}{2}$ possible pairs to be compared, i.e. m hypotheses to be tested simultaneously
- Performing multiple comparisons increases the Type I error probability, i.e. it is easier the reject the null hypothesis when you should not, since the m pairwise comparisons are **independent**

$$\mathbb{P}(\text{No Type I Error}) = 1 - \alpha$$

$$\mathbb{P}(\text{No Type I Errors}) = \prod_{i=1}^{m}(1-\alpha) = (1-\alpha)^m$$

$$\mathbb{P}(\text{At Least One Type I Error}) = 1 - (1-\alpha)^m$$

Family-wise Error Rate (FWER)

Student’s t Test

sistemi eseguiti su stessi topic

(Paired) Student's t test

- Let us assume to have a random sample of size n (**independence**) from two **normally-distributed** random variables with **same variance** (homoskedasticity)

$$X \sim \mathcal{N}(\mu_X, \sigma^2) \qquad Y \sim \mathcal{N}(\mu_Y, \sigma^2)$$

- Let us define the difference among these two random variables

$$D_i = X_i - Y_i \sim \mathcal{N}(\mu_X - \mu_Y, 2\sigma^2)$$

$$\hat{\mu}_D = \frac{1}{n}\sum_{i=1}^{n} d_i$$

$$\hat{\sigma}_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \hat{\mu}_D)^2$$

(Paired) Student's t test

- The test statistic

$$T = \frac{\hat{\mu}_D - \mu_D}{\sqrt{\hat{\sigma}_D^2 / n}} \sim t(n-1)$$

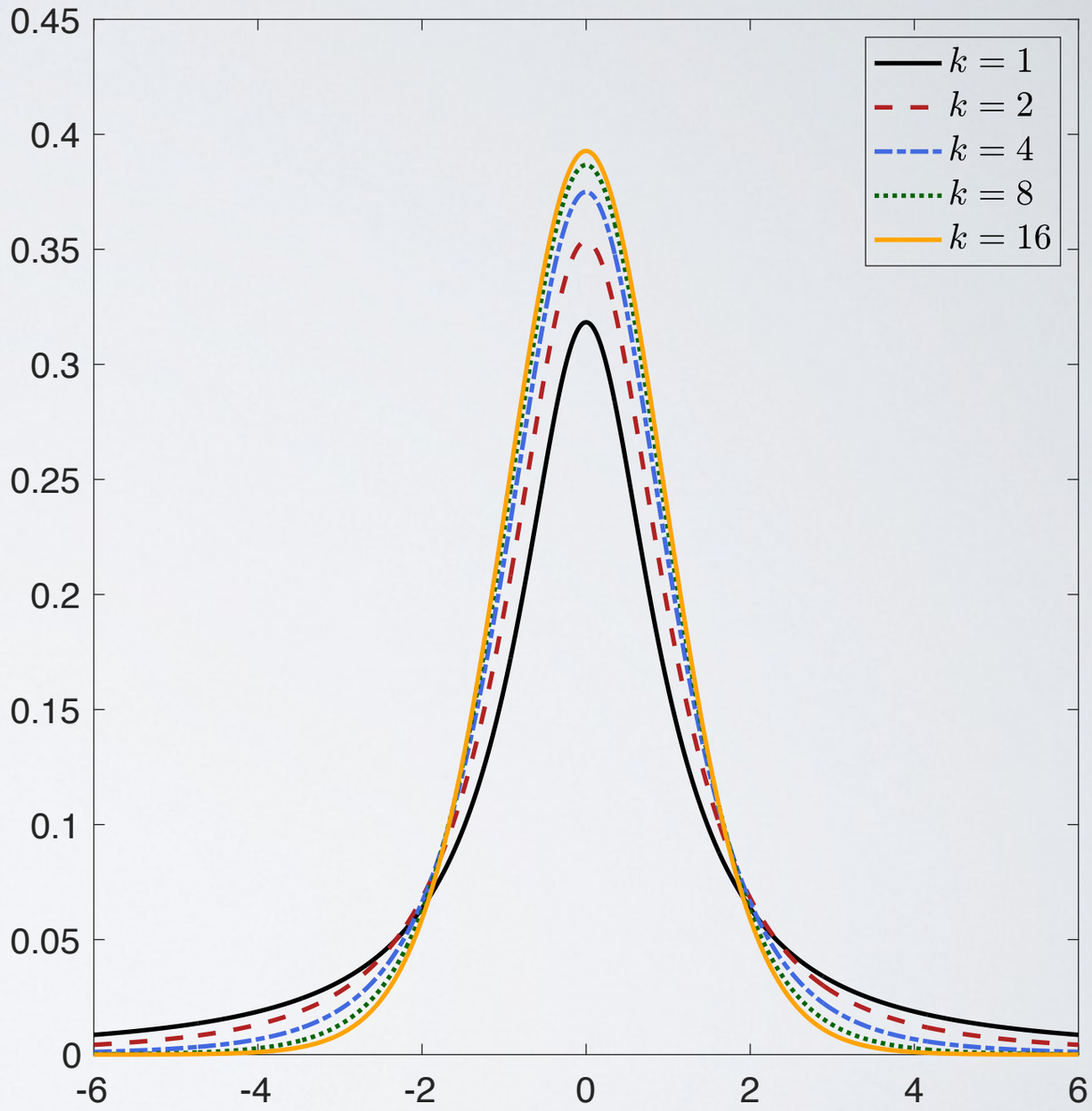
follows a **Student's t distribution** with $k = n - 1$ **degrees of freedom**

- Under the null hypothesis

$$H_0 : \mu_X = \mu_Y \Leftrightarrow \mu_D = 0$$

we obtain the test statistic value

$$t_{stat} = \frac{\hat{\mu}_D}{\sqrt{\hat{\sigma}_D^2 / n}}$$



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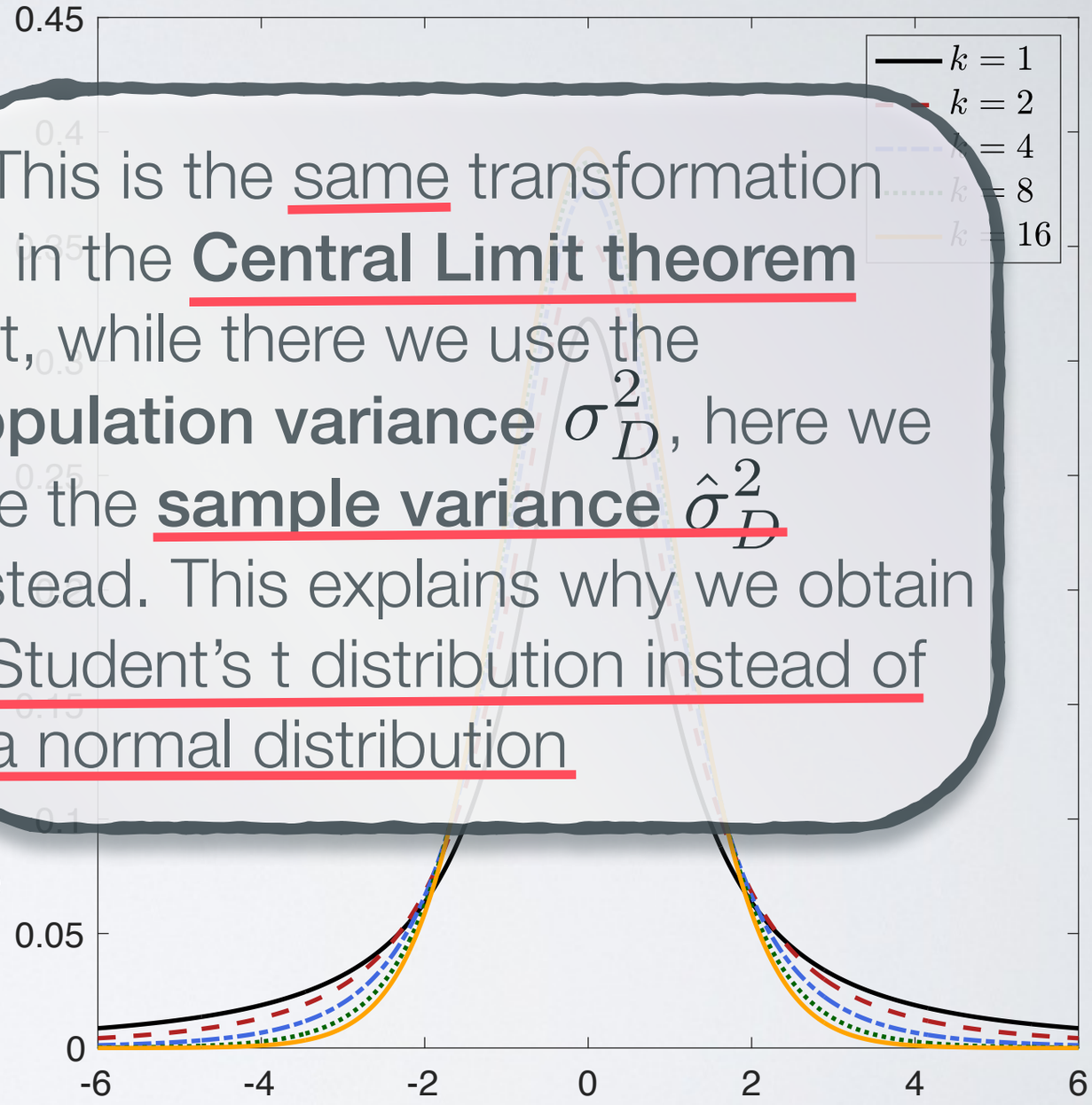
we obtain the test statistic value

$$t_{stat} = \frac{\hat{\mu}_D}{\sqrt{\hat{\sigma}_D^2 / n}}$$

This is the same transformation as in the **Central Limit theorem** but, while there we use the **population variance** σ_D^2, here we use the **sample variance** $\hat{\sigma}_D^2$ instead. This explains why we obtain a Student's t distribution instead of a normal distribution



Compute it Yourself!

Computation of (Paired) Student's t test on simulated data

This example shows step-by-step computation of the Student's t test on simulated data.

Setup

Define the parameters to be used.

```
% set the random number generator for reproducibility
rng(6, 'twister');

% total number of samples to be used
n = 10;

% the significance level
alpha = 0.05;
```

Generate the random (simulated) data from normal distributions with same variance as follows:

- $x \sim \mathcal{N}(1, 1)$
- $y \sim \mathcal{N}(1, 1)$
- $z \sim \mathcal{N}(2.3, 1)$

```
x = 1 + randn(n, 1);
y = 1 + randn(n, 1);
z = 2.3 + randn(n, 1);
```

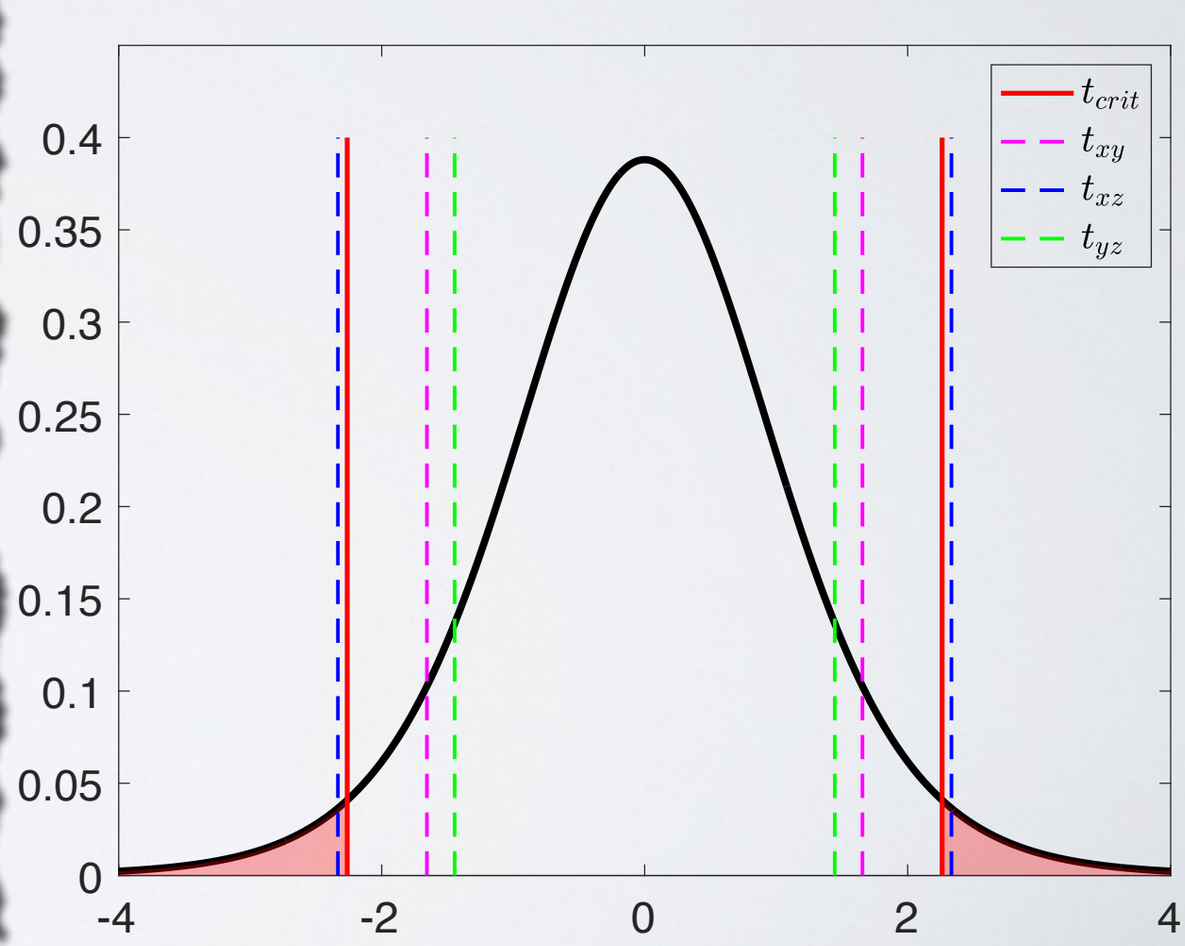
Compute the degrees of freedom of the Student's t test.

Note that they depend on the number of samples n but not on the actual simulated data.

```
df = n - 1;
```

Compute the critical value of the Student's t test under the null hypothesis H_0.

Notat that it depends on the significane level α and the degrees of freedom df but not on the actual simulated data.



Compute it Yourself!

Computation of (Paired) Student's t test on TREC 08, 1999, AdHoc

This example shows how to compare runs from TREC 08, 1999, AdHoc using the Student's t test.

Setup

Load the performance score about the runs. The file contains:

- `ap` (or another measure name): a matrix where rows are topics and columns are runs; each cell contains the AP score for a run on a given topic;
- `runs`: an array of strings containing the identifiers of the runs; elements in the array are in the same order as columns on `ap`;
- `topics`: an array of strings containing the

```
% Load Average Precision data
% Other measures available are: P@10
load ../../../data/T08/ap.mat

% rename ap into measure to make fo
% measure we load
measure = ap;

% remove the now useless ap matrix
clear ap;
```

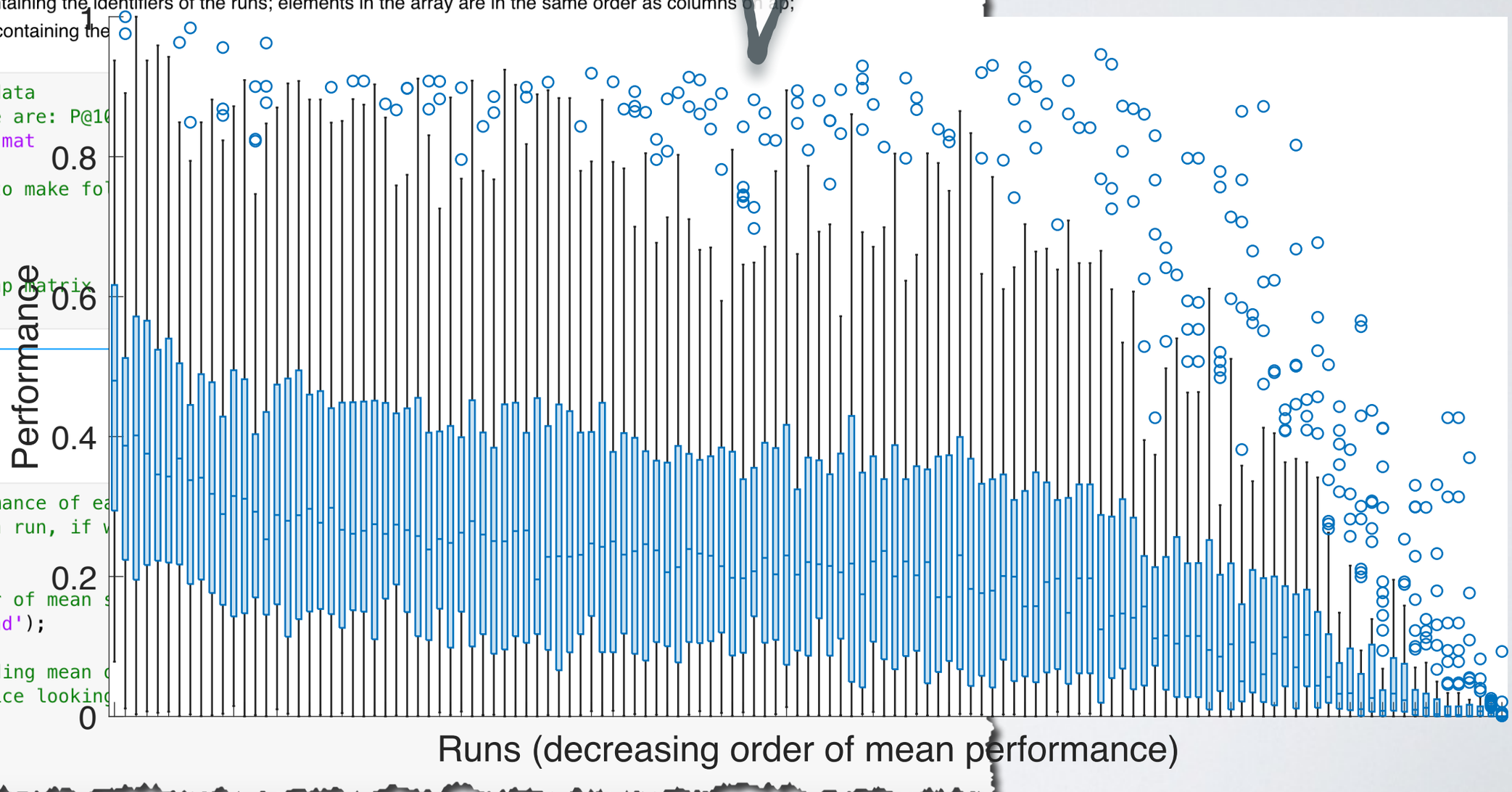
Box Plot

Show the box plot of the runs.

```
% Compute the mean performance of ea
% This is the MAP for each run, if w
m = mean(measure);

% sort in descending order of mean s
[~, idx] = sort(m, 'descend');

% re-order runs by descending mean
% needed to have a more nice looking
measure = measure(:, idx);
runs = runs(idx);
```

```
[~, p] = ttest(x, y)
```

computes the Student's t test among two vectors `x` and `y`



Type I and Type II Errors

Simulation of Type I and Type II Errors

This example simulates Type I and Type II errors in the case of Student's t test.

Setup

Define the parameters to be used.

```
% set the random number generator for reproducibility
rng(6, 'twister');

% total number of samples to be used
n = 10;

% the significance levels
alpha = [0.0001 0.001 0.01 0.05 0.10];

% the number of trials
trials = 1000;
```

Type I Errors

Generate the random (simulated) data from the same normal distributions as follows:

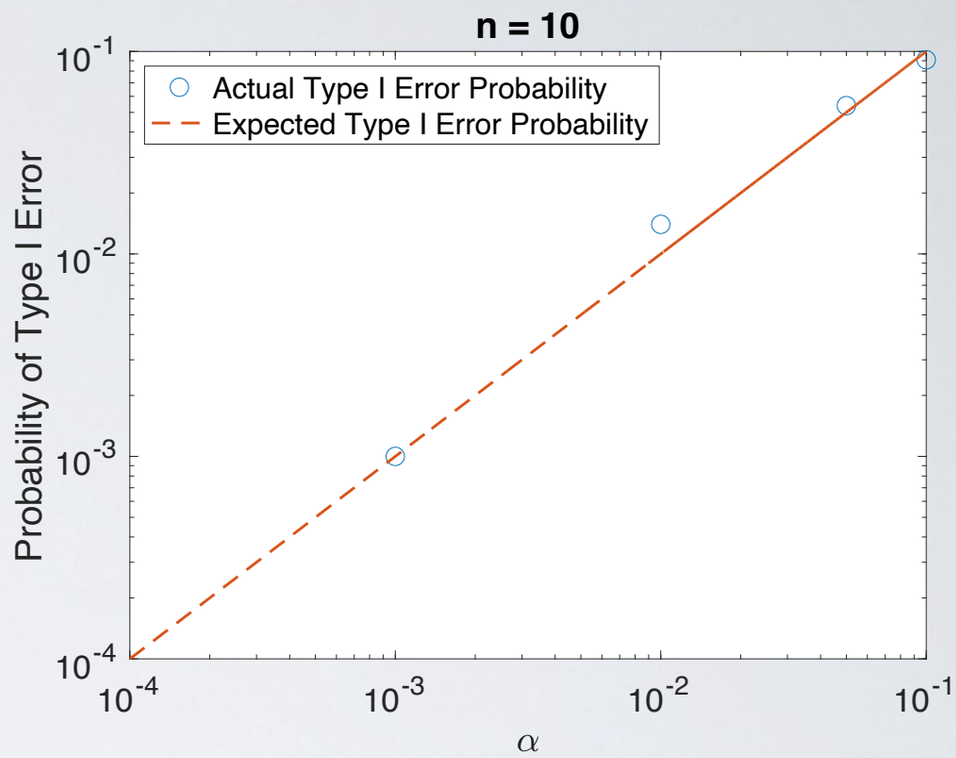
- $x \sim \mathcal{N}(1,1)$
- $y \sim \mathcal{N}(1,1)$

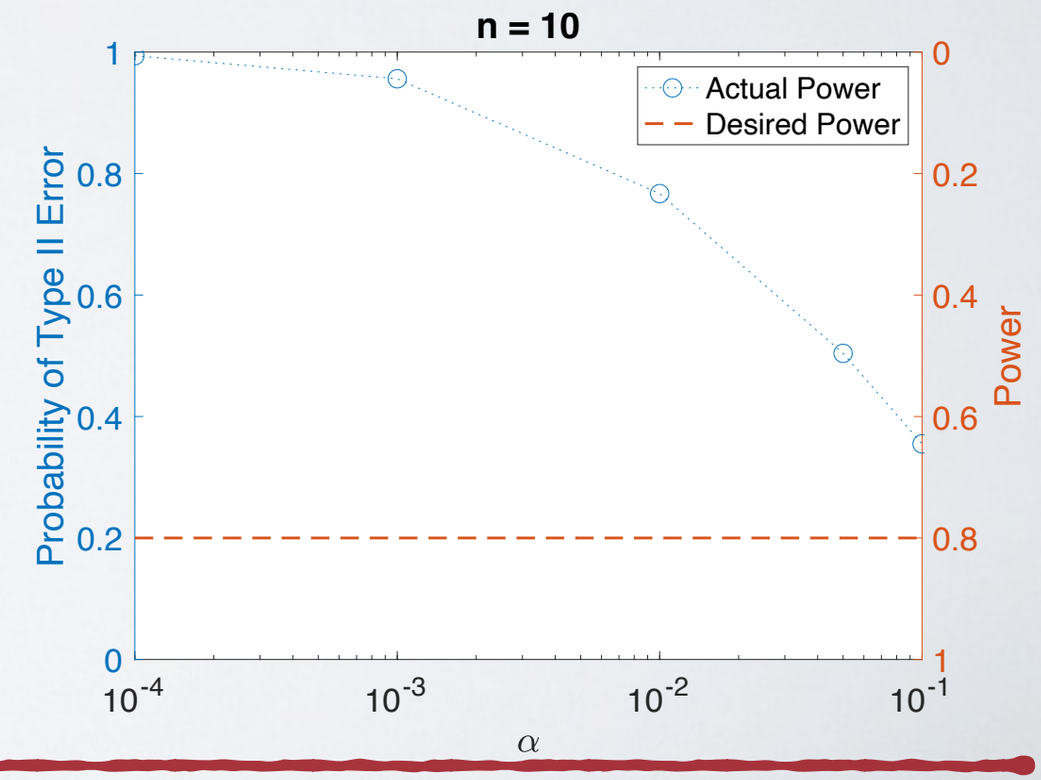
By construction, x and y are samples drawn from the same distribution, so the null hypothesis H_0 holds in their case.

```
% the first random sample of size n, repeated trials times (columns)
x = 1 + randn(n, trials);

% the second random sample of size n, repeated trials times (columns)
y = 1 + randn(n, trials);
```

Compute `trials` Student's t test between each pair of x and y samples

```
% the p-values computed in the t tests
p = NaN(1, trials);
```





Bonferroni Correction

come adattarsi a type I error

- Test each individual hypothesis at a significance level

$$\alpha' = \frac{\alpha}{m}$$

numero confronti

- Criticisms
  - too conservative
  - increases the likelihood of Type II errors

Carlo Emilio Bonferroni

Bonferroni, C. E. (1936). *Teoria statistica delle classi e calcolo delle probabilità*. Number 8 in Pubblicazioni del R. Istituto Superiore di Scienze Economiche e Commerciali di Firenze. Libreria internazionale Seeber, Firenze, Italia.
Perneger, T. V. (1998). What's wrong with Bonferroni adjustments. *BMJ*, 316(7139):1236–1238.

Student's t Test Assumptions?

- **Independence**: topics and systems can be considered (reasonably) independent
- **Normality**: typical IR measures are bounded in [0, 1] so they cannot be normal
  - the normal distribution is unbounded
- **Homoskedasticity**: variance changes across systems

- The Student's t test is considered robust to violations of normality and also homoskedasticy when the sample sizes are equal (our typical case)

Markowski, C. A. and Markowski, E. P. (1990). Conditions for the Effectiveness of a Preliminary Test of Variance. *The American Statistician*, 44(4):322–326.

Analysis of Variance (ANOVA)

General Linear Models (GLM)

$$\text{Data} = \text{Model} + \text{Error}$$

- A GLM explains the variation of a **dependent variable** (**Data**) in terms of a controlled variation of **independent variables** (**Model**) in addition to a **residual** uncontrolled variation (**Error**)
- Regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

- ANalysis Of VAriance (ANOVA)

unica media
e.g. sistemi: uno per ogni sistema

$$y_{ij} = \mu_{..} + \alpha_j + \varepsilon_{ij}$$

- the above regression model corresponds to the ANOVA one once you add as many x_{ij} predictors as many levels there are in the experimental condition α_j, e.g., by using dummy coding

Rutherford, A. (2011). *ANOVA and ANCOVA. A GLM Approach*. John Wiley & Sons, New York, USA, 2nd edition.

Modelling System Effects (one-way ANOVA)

ANOVA

$$y_{ij} = \underbrace{\mu_{..} + \alpha_j}_{\text{Model}} + \underbrace{\varepsilon_{ij}}_{\text{Error}}$$

Systems

Topics	α_1	α_2	$\cdots$	α_q	
τ_1	y_{11}	y_{12}	$\cdots$	y_{1q}	$\mu_{1.}$
τ_2	y_{21}	y_{22}	$\cdots$	y_{2q}	$\mu_{2.}$
$\vdots$	$\vdots$	$\vdots$	y_{ij}	$\vdots$	$\mu_{i.}$
τ_p	y_{p1}	y_{p2}	$\cdots$	y_{pq}	$\mu_{p.}$
	$\mu_{.1}$	$\mu_{.2}$	$\mu_{.j}$	$\mu_{.q}$	$\mu_{..}$

Marginal mean

e.g. valori AP

Grand mean

Marginal mean

Modelling System Effects (one-way ANOVA)

ANOVA

$$y_{ij} = \underbrace{\mu_{..} + \alpha_j}_{\text{Model}} + \underbrace{\varepsilon_{ij}}_{\text{Error}}$$

Regression

$$y_{ij} = \underbrace{\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_q x_{iq}}_{\text{Model}} + \underbrace{\varepsilon_{ij}}_{\text{Error}}$$

1 variable per column

Systems

Topics	α_1	α_2	$\cdots$	α_q	
τ_1	y_{11}	y_{12}	$\cdots$	y_{1q}	$\mu_{1.}$
τ_2	y_{21}	y_{22}	$\cdots$	y_{2q}	$\mu_{2.}$
$\vdots$	$\vdots$	$\vdots$	y_{ij}	$\vdots$	$\mu_{i.}$
τ_p	y_{p1}	y_{p2}	$\cdots$	y_{pq}	$\mu_{p.}$
	$\mu_{.1}$	$\mu_{.2}$	$\mu_{.j}$	$\mu_{.q}$	$\mu_{..}$

Estimators

- Grand mean

$$\hat{\mu}_{..} = \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}$$

- Marginal mean of the j-th system

$$\hat{\mu}_{.j} = \frac{1}{p} \sum_{i=1}^{p} y_{ij} \qquad \hat{\alpha}_{.j} = \hat{\mu}_{.j} - \hat{\mu}_{..}$$

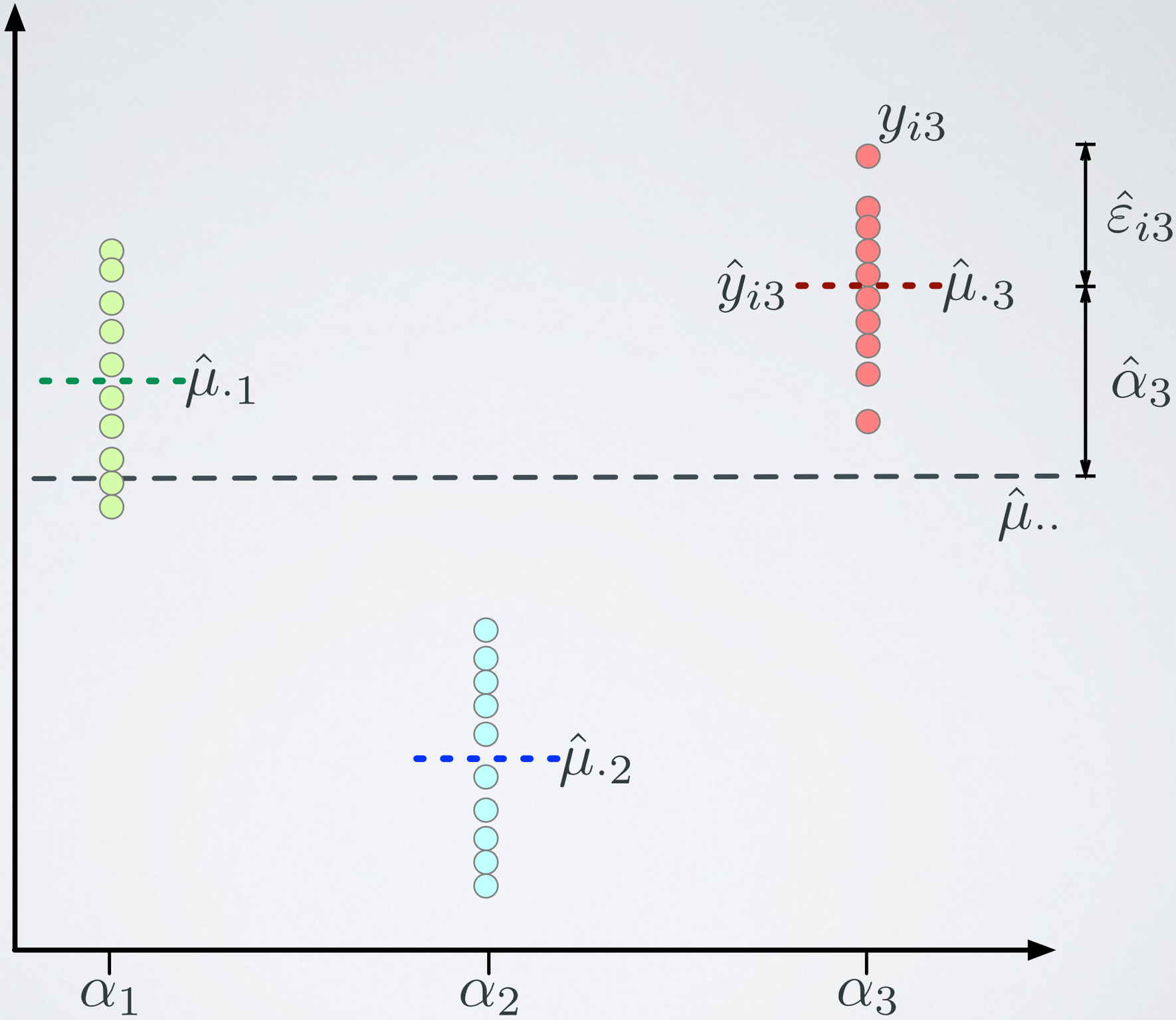
- Predicted score

$$\hat{y}_{ij} = \hat{\mu}_{..} + \hat{\alpha}_{.j} = \hat{\mu}_{.j}$$

- Prediction error

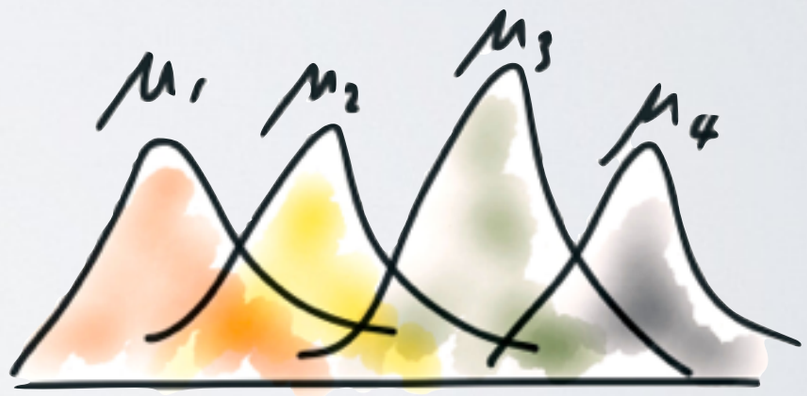
$$\hat{\varepsilon}_{ij} = y_{ij} - \hat{y}_{ij} = y_{ij} - \hat{\mu}_{.j}$$

Estimators



Assessment: ANOVA

- **Analysis of Variance (ANOVA)** was developed by statistician and evolutionary biologist Ronald Fisher (1890-1962)
- It provides a statistical test of whether or not the means of several groups are equal
  - H_0 is the **null hypothesis** that all the means are equal ⇒ non confronti singoli
- It partitions the observed variance in a particular variable into components attributable to different sources of variation



ANOVA

$\mu_1 = \mu_2 = \mu_3 = \mu_4$?

Assessment: Sum of Squares

non proprio varianza

$$\underbrace{y_{ij} - \hat{\mu}_{..}}_{\text{Total Effect}} = \underbrace{\hat{\mu}_{.j} - \hat{\mu}_{..}}_{\text{System Effect } \hat{\alpha}_j} + \underbrace{y_{ij} - \hat{\mu}_{.j}}_{\text{Error Effect } \hat{\varepsilon}_{ij}}$$

- Sum of squares (SS)

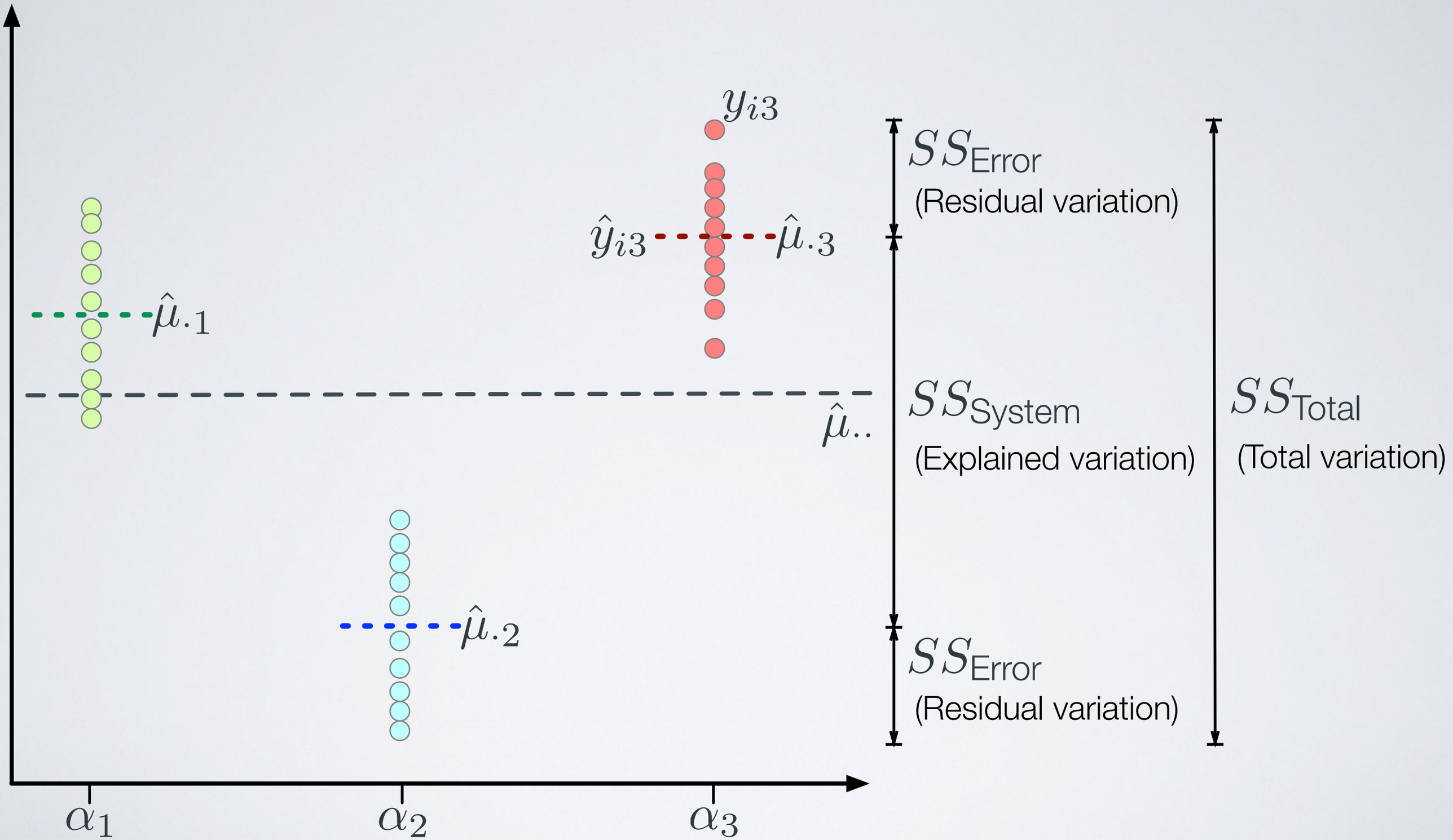
$$SS_{fact} = \sum_{i=1}^{p} \sum_{j=1}^{q} \left([\text{total} \mid \text{system} \mid \text{error}] \text{ effect}\right)^2$$

- Variance break-down

quanta variazione spiegabile da sistema

$$SS_{\text{Total}} = SS_{\text{System}} + SS_{\text{Error}}$$

Assessment: Sum of Squares



Assessment: Degrees of Freedom and Mean Squares

- Degrees of Freedom (DF)

$$DF_{\text{Total}} = pq - 1$$

$$DF_{\text{System}} = q - 1$$

$$DF_{\text{Error}} = (pq - 1) - (q - 1) = q(p - 1)$$

- Mean Squares (MS)

$$MS_{fact} = \frac{SS_{fact}}{DF_{fact}}$$

Walker, H. M. (1940). Degrees of freedom. *Journal of Educational Psychology*, 31(4):253–269.

Assessment: F-test

- Let us assume to have a random sample of size $n = pq$ (**independence**) from q **normally-distributed** random variables with **same variance** (homoskedasticity)

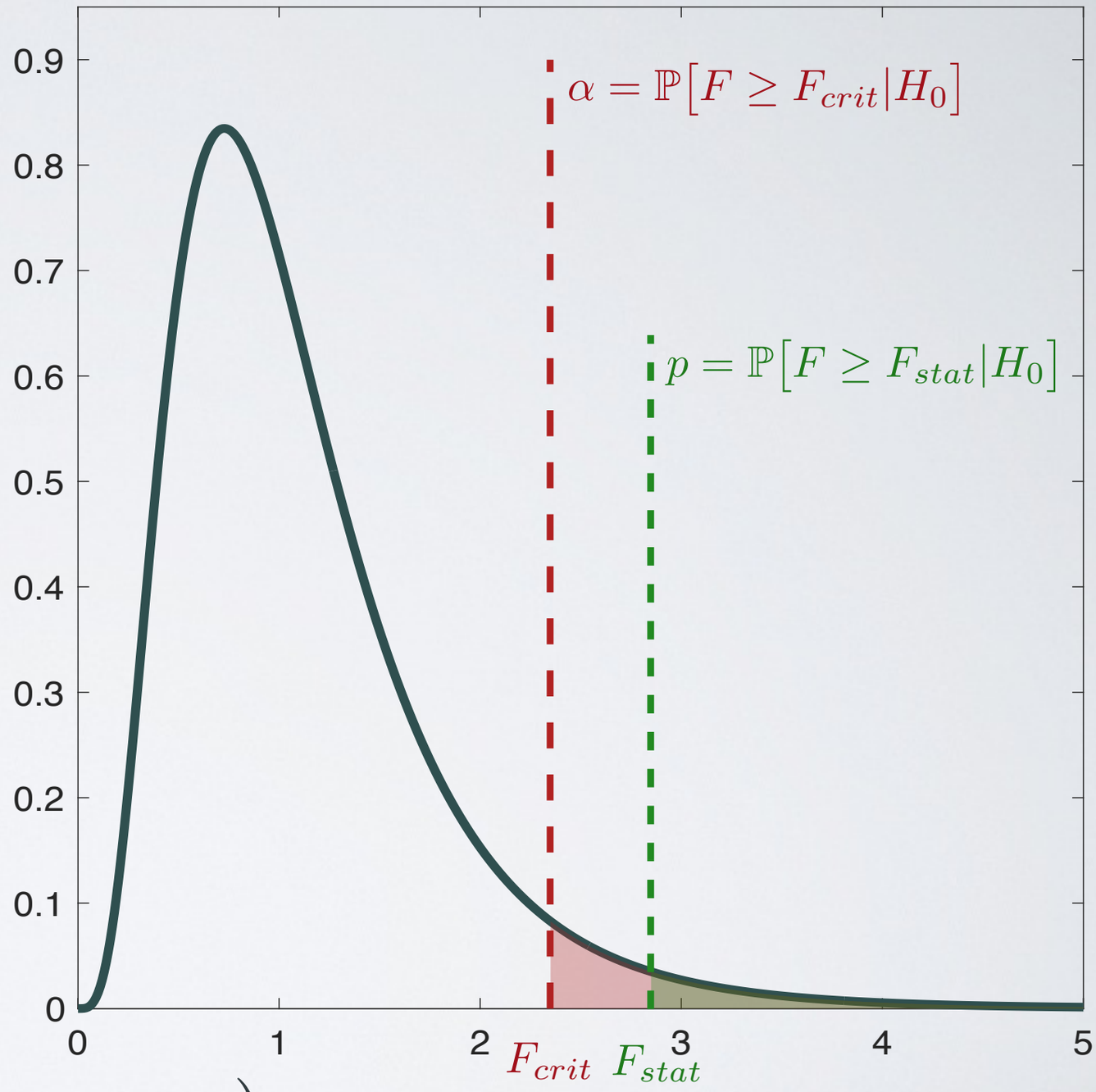
$$Y_{ij} \sim \mathcal{N}(\mu_j, \sigma^2)$$

- Under the null hypothesis

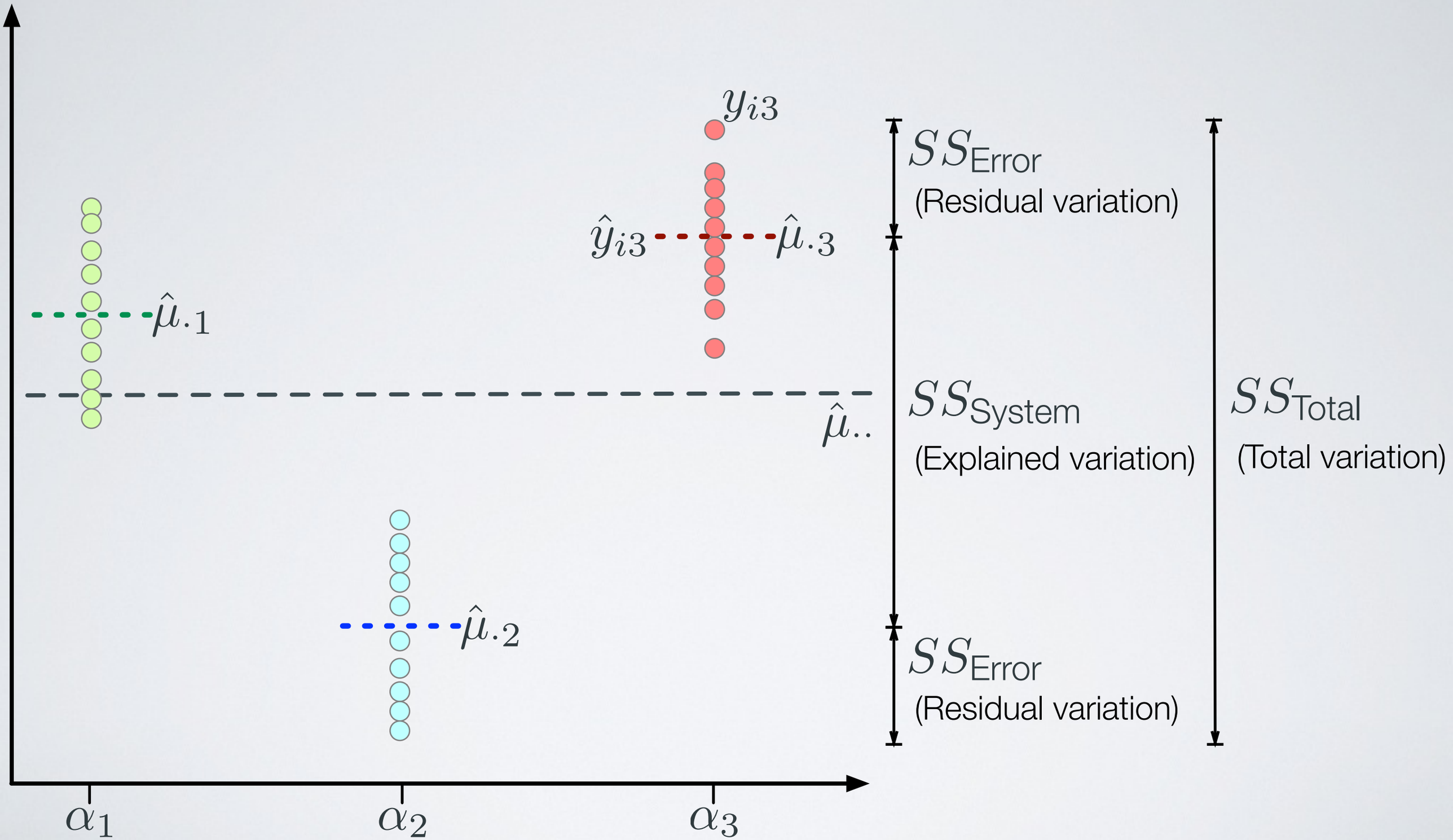
$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_q$$

the tests statistic is

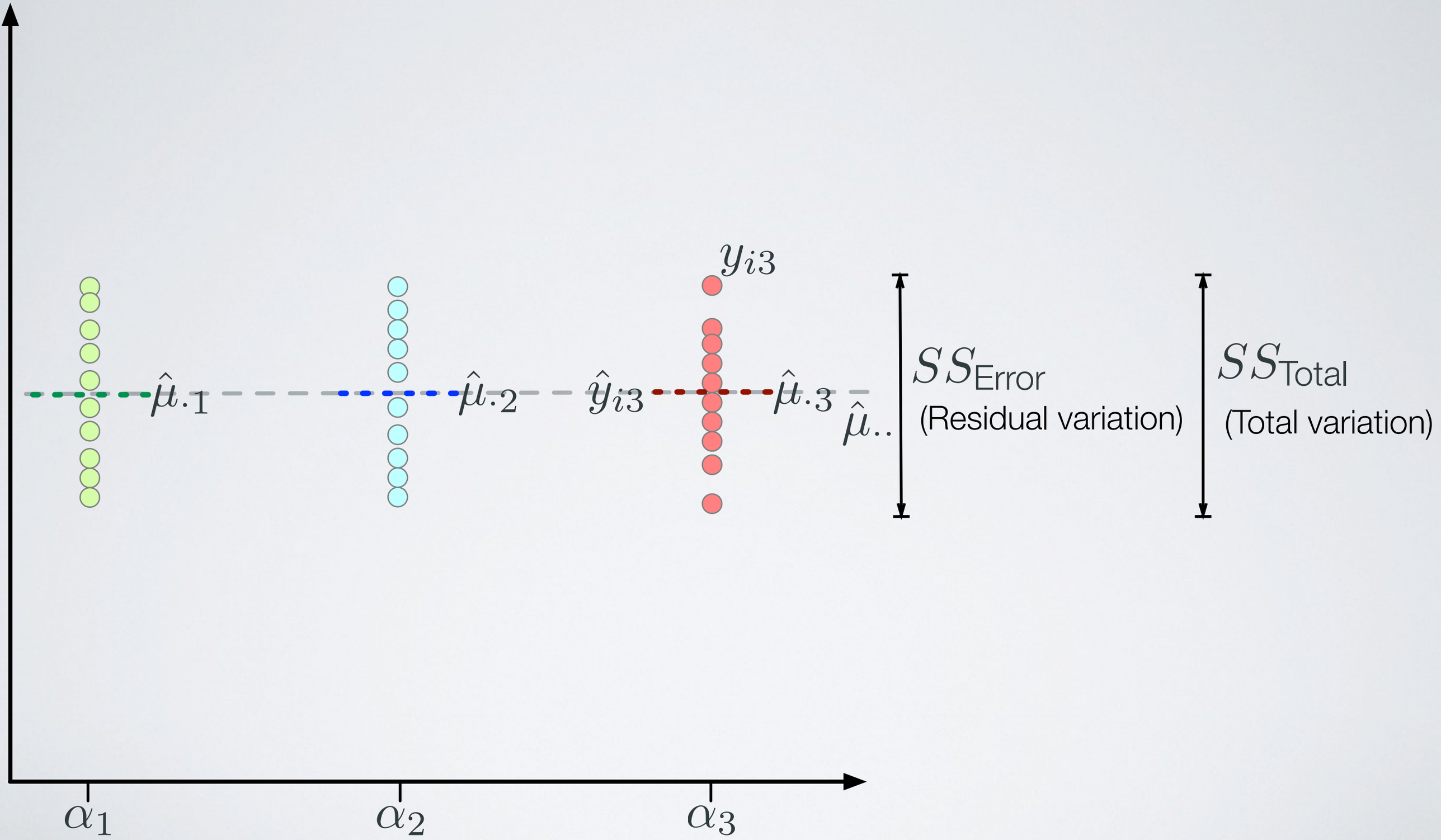
$$F_{stat} = \frac{MS_{\text{System}}}{MS_{\text{Error}}} \sim F_{\left(DF_{\text{System}}, DF_{\text{Error}}\right)}$$



Assessment: F-test under H_0



Assessment: F-test under H_0



ANOVA Assumptions?

- **Independence**: topics and systems can be considered (reasonably) independent
- **Normality**: typical IR measures are bounded in [0, 1] so they cannot be normal
  - the normal distribution is unbounded
- **Homoskedasticity**: variance changes across systems

- ANOVA is considered robust to violations of normality and also homoskedasticy when the sample sizes are equal and large (our typical case)

Eisenhart, C. (1947). The Assumptions Underlying the Analysis of Variance. *Biometrika*, 3(1):1–21.
Ito, P. K. (1980). Robustness of ANOVA and MANOVA test procedures. In Krishnaiah, P. R., editor, *Handbook of Statistics – Analysis of Variance*, volume 1, pages 199–236. Elsevier, The Netherlands.
Tague-Sutcliffe, J. M. and Blustein, J. (1994). A Statistical Analysis of the TREC-3 Data. In Harman, D. K., editor, *The Third Text REtrieval Conference (TREC-3)*, pages 385–398. National Institute of Standards and Technology (NIST), Special Publication 500-225, Washington, USA.
Carterette, B. A. (2012). Multiple Testing in Statistical Analysis of Systems-Based Information Retrieval Experiments. *ACM Transactions on Information Systems (TOIS)*, 30(1):4:1–4:34.

Tukey HSD Test

- The **Tukey Honestly Significant Difference (HSD)** test creates confidence intervals $|t|$ for all pairwise differences between factor levels, while controlling the family error rate

$$|t| = \frac{|\hat{\mu}_{.u} - \hat{\mu}_{.v}|}{\sqrt{\frac{2MS_{\text{Error}}}{p}}} > \frac{1}{\sqrt{2}} \mathrm{q}_{\alpha,q,q(p-1)}$$

  - $\hat{\mu}_{.u}$ and $\hat{\mu}_{.v}$ are the marginal means of the two factor levels, i.e. the two systems to be compared
  - $\mathrm{q}_{\alpha,q,q(p-1)}$ is the upper $100 * (1 - \alpha)$-th percentile of the **studentized range distribution**, i.e. the distribution of the range of samples drawn from a normal distribution, considering q systems to compare using p topics

John Wilder Tukey

Tukey, J. W. (1949). Comparing Individual Means in the Analysis of Vari- ance. Biometrics, 5(2):99–114.
Newman, D. (1939). The Distribution of Range in Samples from a Normal Population, Expressed in Terms of an Independent Estimate of Standard Deviation. *Biometrika*, 31(2):20–30.
Hochberg, Y. and Tamhane, A. C. (1987). *Multiple Comparison Procedures*. John Wiley & Sons, USA.

Compute it Yourself!

Computation of one-way ANOVA on simulated data

This example shows step-by-step computation of one-way ANOVA on simulated data.

Setup

Define the parameters to be used.

```
% set the random number generator for reproducibility
rng(6, 'twister');

% total number of samples to be used
n = 10;

% the significance level
alpha = 0.05;
```

Generate the random (simulated) data from normal distributions with same variance as follows:

- $x \sim \mathcal{N}(1,1)$
- $y \sim \mathcal{N}(1,1)$
- $z \sim \mathcal{N}(2.3,1)$

```
x = 1 + randn(n, 1);
y = 1 + randn(n, 1);
z = 2.3 + randn(n, 1);
```

Put everything together in a single matrix

```
data = [x y z];

% the number of rows and columns
[p, q] = size(data);
```

Compute the one-way ANOVA

We consider the following one-way ANOVA model

$y_{ij} = \mu_{..} + \alpha_j + \varepsilon_{ij}$

where y_{ij} is an element of the `data` matrix.

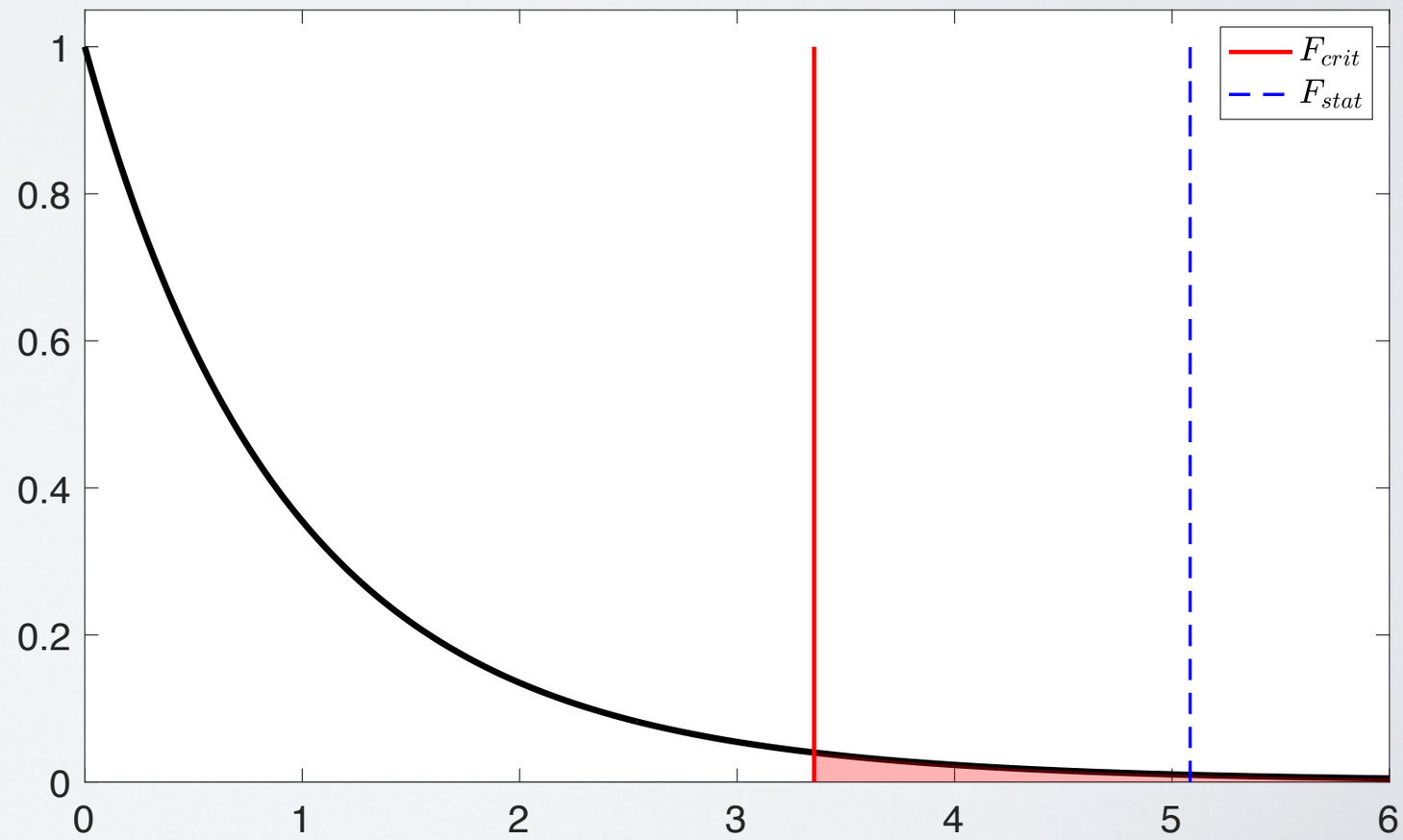
Fit the ANOVA Model

- Grand mean:

$\hat{\mu}_{..} = \frac{1}{pq} \sum_{i}^{p} \sum_{j}^{q} y_{ij}$

ANOVA Table

	1	2	3	4	5	6
1	'Source'	'SS'	'DF'	'MS'	'F'	'Prob>F'
2	'Columns'	10.6516	2	5.3258	5.0836	0.0134
3	'Error'	28.2863	27	1.0476	[ ]	[ ]
4	'Total'	38.9379	29	[ ]	[ ]	[ ]



Compute it Yourself!

Computation of one-way ANOVA on TREC 08, 1999, AdHoc

This example shows how to compare runs from TREC 08, 1999, AdHoc using one-way ANOVA.

Setup

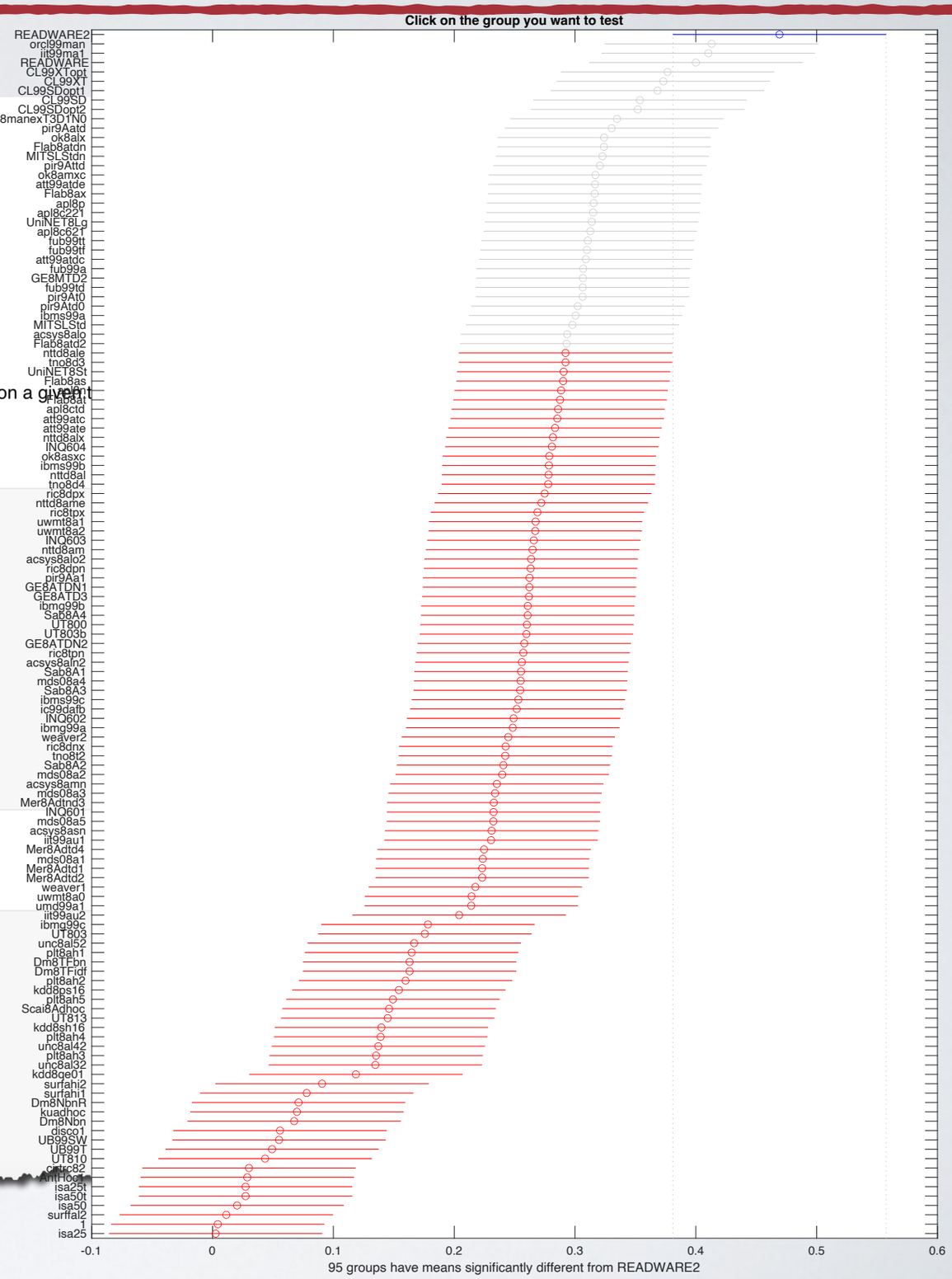
Load the performance score about the runs. The file contains:

- `ap` (or another measure name): a matrix where rows are topics and columns are runs; each cell contains the AP score for a run on a given topic
- `runs`: an array of strings containing the identifiers of the runs; elements in the array are in the same order as columns on ap;
- `topics`: an array of strings containing the identifiers of the topics; elements in the array are in the same order as rows on ap;

```
% Load Average Precision data
% Other measures available are: P@10, nDCG, RBP
load ../../../data/T08/ap.mat

% rename ap into measure to make follow-up processing the same for whatever
% measure we load
measure = ap;

% remove the now useless ap matrix
clear ap;

% the significance level
alpha = 0.05;
```

Reorder the measure by mean performance

```
% the mean for each run across the topics
% Note that if the measure is AP (Average Precision),
% this is exactly MAP (Mean Average Precision) for each run
m = mean(measure);

% sort in descending order of mean score
[~, idx] = sort(m, 'descend');

% re-order runs by descending mean of the measure
measure = measure(:, idx);
runs = runs(idx);
```



What About Two-Way ANOVA?

$$y_{ij} = \mu.. + \tau_i + \alpha_j + \varepsilon_{ij}$$

Topic Effect

System Effect

- Check-out **`anova2`** in Matlab
- When runs on two systems it is basically equivalent to a (paired) Student's t test

Tague-Sutcliffe, J. M. and Blustein, J. (1995). A Statistical Analysis of the TREC-3 Data. In Harman, D. K., editor, *The Third Text REtrieval Conference (TREC-3)*, pages 385–398. National Institute of Standards and Technology (NIST), Special Publication 500-225, Washington, USA.

Questions?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