RECAP

- Congruent structures: $\mathbb{Z}_n$
  Operations on $\mathbb{Z}_n$: $+, \cdot$
- $(\mathbb{Z}_n, +)$ is an additive group
- $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n : \boxed{\gcd(a,n) = 1}\}$ COPRIME
  $(\mathbb{Z}_n^*, \cdot)$ is a multiplicative group
- Computation of a^{-1}, $a \in \mathbb{Z}_n^*$
  via $EE(a, n)$
- Euler's function $\varphi(n) = |\mathbb{Z}_n^*|$
- Euler's theorem $\forall a : \gcd(a,n) = 1$
  $a^{\varphi(n)} \bmod n = 1$
- Fermat's little theorem
  $\forall p$ prime $\forall a \in \mathbb{Z}_p^* \; (= \mathbb{Z}_p^+)$
  $a^{p-1} \bmod n = 1$

THE $\boxed{\text{CHINESE REMINDER THEOREM}}$

(Sun-Tzu, 300 AD)

Important result: establishes a bijection between congruent structures.

Can be seen as a conversion between different representations of certain congruent structures

APPLICATIONS: Proof of correctness of RSA, fast computations mod n, solution of systems of congruences ...

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THEOREM Let $\boxed{n = n_1 \cdot n_2 \cdot \ldots \cdot n_k}$, with $\boxed{k > 1}$ and $\boxed{\gcd(n_i, n_j) = 1 \ , 1 \leq i \neq j \leq k}$.

Let $\boxed{f : \mathbb{Z}_n \to \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}}$

defined as $\boxed{f(a) = (a \bmod n_1, a \bmod n_2, \ldots, a \bmod n_k)}$

Then:

1. f is bijective
2. $f((a+b) \bmod n) = ((a+b) \bmod n_1, \ldots, (a+b) \bmod n_k)$
3. $f((ab) \bmod n) ((ab) \bmod n_1, \ldots, (ab) \bmod n_k)$

Function f "converts" reminders mod n into k-ples of reminders (mod n_1, ..., mod n_k) so that arithmetic mod n can be executed equivalently by doing arithmetic w.r.t. smaller moduli!

(COMPUTATIONAL ADVANTAGE! computing mod n is quadratic in $|\langle n\rangle|$!)

Eg: $n = p \cdot q$, with $p, q \simeq \sqrt{n}$

$\Rightarrow |\langle p\rangle|, |\langle q\rangle| \simeq |\langle n\rangle|/2$

$|\langle p\rangle|^2, |\langle q\rangle|^2 \simeq |\langle n\rangle|^2/4$

$\Rightarrow$ Computing $\{x \bmod p, x \bmod q\}$ costs about a half as computing $x \bmod n$

PROOF

Let us first prove:

2. $f((a+b) \bmod n) = ((a+b) \bmod n_1, \dots, (a+b) \bmod n_k)$

3. $f((ab) \bmod n) ((ab) \bmod n_1, \dots, (ab) \bmod n_k)$

These immediately follow from property m4. For $m, n > 0$:

$$m \mid n \Rightarrow (x \bmod n) \bmod m = x \bmod m$$

since $n_i \mid n$, $1 \leq i \leq k$, thus

$$[(a+b) \bmod n] \bmod n_i = (a+b) \bmod n_i$$
$$[(ab) \bmod n] \bmod n_i = (ab) \bmod n_i$$

Therefore

$$f((a+b) \bmod n) = ([(a+b) \cancel{\bmod n}] \bmod n_1, \dots, [(a+b) \cancel{\bmod n}] \bmod n_k)$$
$$= ((a+b) \bmod n_1, \dots, (a+b) \bmod n_k)$$

Let us prove 1. Since

$$\boxed{|\mathbb{Z}_n| = n = n_1 \cdot n_2 \cdot \ldots \cdot n_k = |\mathbb{Z}_{n_1}| \times |\mathbb{Z}_{n_2}| \times \ldots \times |\mathbb{Z}_{n_k}|}$$

to prove bijectivity it is sufficient to prove surjectivity (or injectivity).

$|\mathbb{Z}_n| = |\mathbb{Z}_{n_1}| \cdots |\mathbb{Z}_{n_k}| \Rightarrow$ una implica l'altra

Given $\boxed{(a_1, \ldots, a_k) \in \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}}$ we will determine $\boxed{x \in \mathbb{Z}_n}$ such that

$$\boxed{f(x) = (x \bmod n_1, \ldots, x \bmod n_k) = (a_1, \ldots, a_k)}$$

REMARK We need to show that the following system of congruences:

$$\begin{cases} x \equiv a_1 \bmod n_1 \\ x \equiv a_2 \bmod n_2 \\ \vdots \\ x \equiv a_k \bmod n_k \end{cases}$$

always admits a (unique) solution $x \in \mathbb{Z}_n$

The proof is constructive: it gives a procedure to compute x!

Define $\boxed{m_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} n_j \qquad 1 \leq i \leq k}$

In other words, $\boxed{m_i = \frac{n}{n_i} = n_1 \cdot \ldots \cdot n_{i-1} \cdot n_{i+1} \cdot \ldots \cdot n_k}$

Clearly $\boxed{\gcd(m_i, n_i) = 1}$ (since $\gcd(a, n) = 1$ and $\gcd(b, n) = 1 \Rightarrow \gcd(ab, n) = 1$)

(use Bezout's id to prove this)

Therefore $\boxed{[m_i]_{n_i} \in \mathbb{Z}^*_{n_i}} \Rightarrow \boxed{\exists [m_i]^{-1}_{n_i} \in \mathbb{Z}^*_{n_i}}$: $\boxed{(m_i \cdot m_i^{-1}) \bmod n_i = 1}$

Set $\boxed{c_i = m_i \cdot (m_i^{-1} \bmod n_i)}$ principal representative of $\left([m_i]_{n_i}\right)^{-1}$

We set
$$\boxed{x = \left(\sum_{i=1}^{k} a_i \cdot c_i\right) \bmod n}$$

and prove that
$$\boxed{f(x) = (a_1, a_2, \dots, a_k)}$$

Observe that for $1 \leq i \neq j \leq k$:
$$\boxed{c_i \bmod n_j = 0}$$
since $\boxed{m_i (m_i^{-1} \bmod n_i) = k\, n_j}$ (m_i is a multiple of n_j)

Therefore
$$\boxed{\begin{aligned}
&\left[\left(\sum_{i=1}^{k} a_i \cdot c_i\right) \bmod n\right] \bmod n_j = \\
&\overset{m4}{=} \left(\sum_{i=1}^{k} a_i \cdot c_i\right) \bmod n_j \\
&\overset{m_1, m_2}{=} \left(\sum_{i=1}^{k} (a_i \bmod n_j)\underbrace{(c_i \bmod n_j)}_{=0 \text{ for } i \neq j}\right) \bmod n_j \\
&= \left((a_j \bmod n_j)(c_j \bmod n_j)\right) \bmod n_j \\
&\overset{m_2}{=} (a_j\, c_j) \bmod n_j \\
&= \left(a_j\, m_j (m_j^{-1} \bmod n_j)\right) \bmod n_j
\end{aligned}}$$

$\overset{m_2}{=} \left(a_j \left(\underbrace{(m_j m_j^{-1}) \bmod m_j}_{=1}\right)\right) \bmod m_j$

$= a_j \bmod m_j = a_j .$

Q.E.D.

COROLLARY 1 Let $\boxed{m_1, \dots, m_k : \gcd(m_i, m_j) = 1}$ for $\underline{1 \le i \ne j \le k}$. Then the system of congruences

$$\boxed{\begin{cases} x \equiv a_1 \bmod m_1, & a_i \in \mathbb{Z}_{m_1} \\ \vdots & \\ x \equiv a_k \bmod m_k, & a_k \in \mathbb{Z}_{m_k} \end{cases}}$$

has a $\underline{\text{unique solution in } \mathbb{Z}_m}$:

$$\boxed{x = f^{-1}(a_1, a_2, \dots, a_k)}$$

COROLLARY 2 Let $\boxed{m_1, \dots, m_k : \gcd(m_i, m_j) = 1}$ for $\underline{1 \le i \ne j \le k}$, $\boxed{m = \prod m_i}$. Then

$$\boxed{(x \equiv y) \bmod m \iff (x \equiv y) \bmod m_i \,, \ 1 \le i \le k}$$

$\Rightarrow$ If $\boxed{x \bmod m = y \bmod m}$

then $\boxed{(x \bmod m) \bmod m_i = (y \bmod m) \bmod m_i}$

that is, from m4: $\boxed{x \bmod m_i = y \bmod m_i}$

for all $1 \le i \le k$

$\Leftarrow$ If $\boxed{x \equiv y \bmod m_i, 1 \le i \le k}$, then $\boxed{f(x) = f(y)}$

(since $f(a) = (a \bmod m_1, \dots, a \bmod m_k)$)

Therefore $\boxed{x \equiv y \bmod m}$ because $\underline{f \text{ is}}$ $\underline{\text{injective over } \mathbb{Z}_m}$

EXAMPLE

Consider the system

$$\begin{cases} x \equiv 2 \mod 5 \\ x \equiv 3 \mod 13 \end{cases}$$

we determine a solution in $\mathbb{Z}_n$, with $n = 5 \cdot 13 = 65$

$n_1 = 5 \quad n_2 = 13 \quad \gcd(5, 13) = 1$ ok

$m_1 = \frac{n}{n_1} = 13 \qquad m_2 = \frac{n}{n_2} = 5$

$m_1^{-1} \mod 5 = 13^{-1} \mod 5 = 2$, since

$13 \cdot 2 \mod 5 = 26 \mod 5 = 1$

$m_2^{-1} \mod 13 = 5^{-1} \mod 13 = 8$, since

$5 \cdot 8 \mod 13 = 40 \mod 13 = 1$

Therefore $c_1 = 13 \cdot 2 = 26$, $c_2 = 5 \cdot 8 = 40$

and $x = f^{-1}(2, 3) = (2 \cdot 26 + 3 \cdot 40) \mod 65$

$= (52 + 120) \mod 65 =$

$= 172 \mod 65 = 172 - 130 = 42$

THE RSA PUBLIC-KEY CRYPTOSYSTEM

↳ [Rivest, Shamir, Adelman, 1978]

A $\boxed{\text{cryptosystem}}$ provides a family of coding functions. A function from the family can be used by a sender to send a message to a recipient along a nonsecure communication channel.

An essential property of a cryptosystem is that the encoded message can be decoded efficiently only in the presence of extra information (key)

FORMALLY:

$\boxed{\mathcal{D} = \text{message domain}}$ (usually, binary strings of given length representing plain text: e.g. concatenation of ASCII codes) & (TREAT AS MEMBERS OF $\mathbb{Z}_n$)

$\boxed{\text{Cryptosystem } \mathcal{E} = \{e_K : \mathcal{D} \to \mathcal{D} : K \in \mathcal{K}\}}$

family of encodings parametric in $K \in \mathcal{K}$ (key)

PROPERTIES:

- e_K is invertible and efficiently computable, given K.

- $\boxed{\forall M \in \mathcal{D}}$: given $\boxed{e_K(M) = y}$, it is computationally hard to compute $M = e_K^{-1}(y)$ in the essence of any other information (not necessarily K)

Public-Key Cryptosystem

Each participant X has two keys:

- P_X : public key
- S_X : secret key

EXAMPLE

Alice : P_A, S_A

Bob : P_B, S_B

Each participant creates its own keys

- The public key P_X is distributed to all participants
- The secret key S_X is known only to X

Each key corresponds to an encoding function:

EXAMPLE: $M \in \mathcal{D}$

$$e_{P_A}(M), e_{S_A}(M), e_{P_B}(M), e_{S_B}(M)$$

$$|||$$

$$P_A(M), S_A(M), P_B(M), S_B(M)$$

PROPERTIES

1. $P_X(M)$, $S_X(M)$ "efficiently computable", given key P_X or S_X

2. $\boxed{S_X(M) = P_X^{-1}(M)}$, that is

$$\boxed{S_X(P_X(M)) = P_X(S_X(M)) = M}$$

3. Even if $P_X(M)$ is known, computing $S_X(M) = P_X^{-1}(M)$ is computationally unfeasible if S_X is not known

Point 3. is critical. For many functions knowing the analytical description of $P_X(M)$ yields $S_X(M) = P_X^{-1}(M)$ easily

EXAMPLE $\mathcal{D} = \mathbb{Z}_n$

$P_A(M) = M + 7 \mod n$

$\Rightarrow \quad S_A(Y) = Y - 7 \mod n$

$P_A(M) = aM \mod n, \quad a \in \mathbb{Z}_n^*$

$\Rightarrow \quad S_A(Y) = a^{-1} Y \mod n$

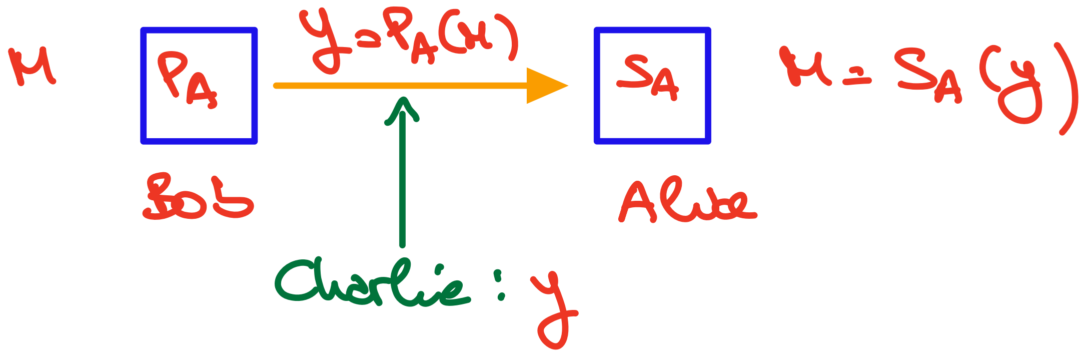
NOTE For long messages we use repeated application on k-tuples, $k \geq 1$:

$P_A(\langle M_1, \ldots, M_k \rangle) = \langle P_A(M_1), \ldots, P(M_k) \rangle$

COMMUNICATION PROTOCOLS

SECRET MESSAGE PASSING

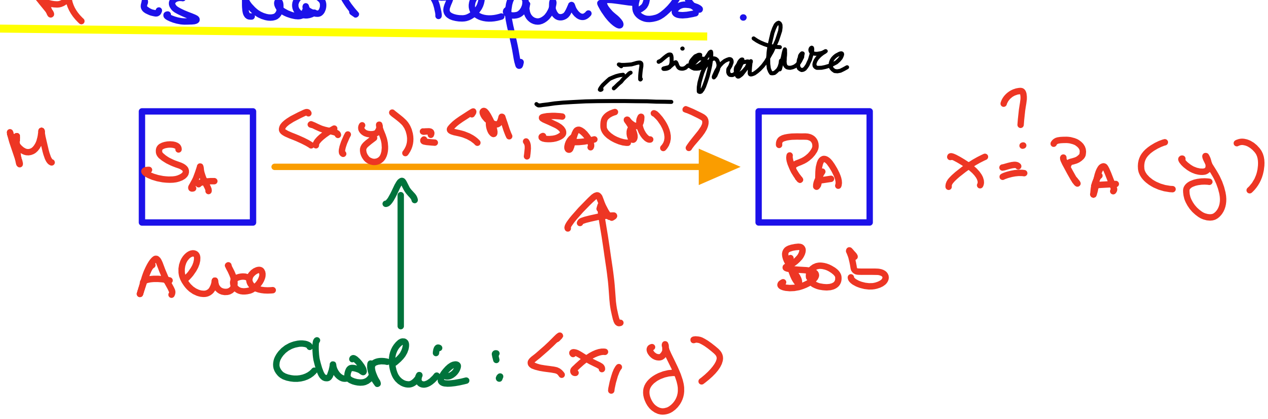
Bob wants to send M to Alice secretly:



Charlie (catching $y = P_A(M)$ over channel) is unable to obtain M

AUTHENTICATION

Alice wants to send message M to Bob so that Bob can be sure that Alice is the true sender. Assume that secrecy of M is not required.



If $x = P_A(y)$, Bob can be sure that the message was generated by Alice, since $S_A(M)$ cannot be computed by Charlie

APPLICATIONS: Electronic checks: bank "signs" credit messages. Certification authorities that distribute public keys

The two protocols can be combined: if Alice does not want M to be visible to everyone:

Alice sends $z = P_B(\langle M, S_A(M)\rangle)$

Bob computes $\langle x, y\rangle = S_B(z)$ and then checks $x = P_A(y)$

THE RSA (PUBLIC-KEY) CRYPTOSYSTEM

The asymmetry needed in constructing $P_A(M)$ and $S_A(M)$ in the RSA cryptosystem relies on the different complexity of these two decision problems!

PRIMALITY

$$\begin{cases} I : \langle n \rangle : n \in \mathbb{N}^+ \\ Q : \text{is } n \text{ prime?} \end{cases}$$

FACTORING

$$\begin{cases} I : \langle n, k \rangle \\ Q : \exists p,q \in \mathbb{N}-\{1,0\} \; n = p \cdot q, \; 1 < p \leq k \end{cases}$$

In FACTORING the second parameter is needed to actually compute the factorization.

Without k, we get

COMPOSITE

$$\begin{cases} I : \langle n \rangle \\ Q : \text{is } n \text{ composite?} \end{cases}$$

which is just PRIMES^c

These decision problems are related to these non decision ones (used by RSA)

LARGE-PRIMES : $\mathcal{I} = \mathbb{N}^+$, $\mathcal{S} = \mathbb{N}^+$

$n \Pi p \iff p$ prime, $p \geq n$

(ability to compute arbitrarily large primes)

FACTORS : $\mathcal{I} = \mathbb{N}^+$, $\mathcal{S} = \mathbb{N}^+ \times \mathbb{N}^+ \cup \{\perp\}$

$n \Pi \perp$ if n prime

$n \Pi (p,q)$ if $p, q > 1$ and $n = p \cdot q$

(ability to factor composite numbers)

$\boxed{\text{PRIMES} \in P}$

- 2004 : first (inefficient) deterministic polynomial algorithm
- 1976 : Miller-Rabin (efficient) randomized test (can be incorrect)

$\boxed{\text{FACTORING} \in NP}$

- No efficient algorithms are known for FACTORING

Clearly COMPOSITE = $(\text{PRIMES})^C \Rightarrow$ $\boxed{\text{COMPOSITE} \in P\,!}$

- It is not known whether FACTORING ∈ NPC
  (probably not, since FACTORING ∈ NP ∩ co-NP) (PROVE IT)
- Difficult instances: $n = p \cdot q$, with p, q large primes

These two problems are somehow "inverse problems": I can easily generate two large primes p, q and multiply them together into $n = p \cdot q$. However given ONLY $n = p \cdot q$ I am not able to obtain p, q efficiently.
This asymmetry is at the base of the RSA cryptosystem.