Game theory

A course for the

MSc in ICT for Internet and multimedia

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General applications of NE

- Prediction instrument
  - In many cases, there is a NE (as we will see, we can extend the problem so there is always one)
  - But it is not unique: we may need other points to decide which of the NEs is the most likely one
  - Prediction is more accurate than IESDS
- Criticism of competition
  - NE does not guarantee a "good" (Pareto efficient) solution, because players are driven by egoism

Duopolies

An interesting application of NE

Cournot duopoly

- Cournot (1838) anticipated Nash's results in a particular context: a special duopoly
- In the Cournot model, we have two firms (called 1 and 2) producing a good in quantities q_1 and q_2. Let $Q = q_1 + q_2$
- The cost to produce q is the same for both firms and equals $C(q) = c\,q$ (with constant c)
- When the good is sold on the market, its price is $P(Q) = a - Q$. (with constant $a > c$)
- More precisely, $P(Q) = (a - Q)\, h\,[a - Q]$

Cournot duopoly

- If the firms choose q_1 and q_2 simultaneously, can we predict their optimal production?
- I.e., is there a Nash equilibrium of the game?
- Both firms $i = 1,2$ have a single-move strategy represented by $\boxed{q_i}$ and $\boxed{S_i = [0, \infty)}$; actually, any $q_i > a$ is pointless, we can put $\boxed{S_i = [0, a)}$
- The payoff of a firm is simply its profit (revenue minus cost):

$$\boxed{u_i(q_i, q_j) = q_i [P(q_i + q_j) - c] = q_i (a - q_i - q_j - c)}$$

NE of a Cournot duopoly

- Is there any NE (q_1^*, q_2^*)?
- For each player i, q_i^* must satisfy:

$$q_i^* = \text{argmax}_{q_i}\, u_i\,(q_i, q_j^*)$$

- $q_i \in [0,\infty)$: $\text{argmax}_{q_i}\, q_i\,(a - q_i - q_j^* - c)$ for $i = 1,2$

$$(a - 2\,q_i^* - q_j^* - c\,) = 0$$

  - Solution (for both) $q_1^* = q_2^* = (a - c\,)/3$
- Profit (for both) $u_1^* = u_2^* = (a - c\,)^2/9$

Monopoly solution

- In case of a single firm (monopoly) the optimum production would be (set $q_2^* = 0$) :

$$q_\mathrm{m} = \mathrm{argmax}_{q_1}\, q_1 (a - q_1 - c)$$

$$q_\mathrm{m} = (a - c)/2$$

- In which case the profit is $u_\mathrm{m} = (a - c)^2/4$
- We call it q_m not q_1^* because it is different
  - The monopolist produces less than the two firms together (at monopoly, $Q = q_\mathrm{m} < q_1^* + q_2^*$)
  - Lower production → higher price → profit!

Trust case

- The two firms could compare their NE, which achieves profit $u^* = (a - c)^2/9$, with the following alternate solution
- They could cooperate as it were a monopoly
- The produce half of q_{m}, so they could share $u_{\mathrm{m}} = (a - c)^2/4$. Hence, profit is higher
- In other words, they produce less than the equilibrium so the price is higher and therefore the revenue is increased

Why is it not a NE?

- Each firm has an incentive to deviate from such a strategy ($q_1 = q_m/2$ is not best response to $q_2 = q_m/2$ and vice versa)
- As the price is high, unilaterally increasing the production level will raise the revenue (while decreasing that of the other firm)
- At the same time, this decreases the price, so this deviation goes on as long as there is no longer incentive in betraying the trust

Bertrand duopoly

- Bertrand (1883) argued against Cournot model that firms choose prices, not q_j s
- Now, we have an **entirely different** game. Strategies are prices p_i and $p_i \in S_i = [0, \infty)$
- E.g., assume people buy $q_i = a - p_i$ from the firm with cheaper price and 0 from the other (if the p_i s are equal, share q_i between them)
- Cost is $C(q) = c\, q$ (as in Cournot case, $a > c$)
- Competition leads to lowering the price
- NE of this game is $p_1^* = p_2^* = c$

Bertrand duopoly

- Similarly to Cournot's, Bertrand equilibrium is clearly not the best outcome for the firms
- In fact, they could agree on a higher price and share the market. The price can be pushed up to $(a + c)/2 > c$
- However, this is not a NE as each of the firm has a (selfish) incentive to deviate, i.e., decrease price, so as to conquer the market
- This process is indefinitely repeated as long as the price is c

Bertrand duopoly

- Interestingly, both firms set price=cost
  - which means they have zero profit
- The reason of this strange outcome is in the best response to the beliefs of the players
  - if firm 1 has the belief that firm 2 sets $p_2=c$, profit for firm 1 will be 0 anyways
- Be careful! Profit is 0 even if firm 1 sets $p_1 > c$
  - but (c + ε , c) is NOT a Nash equilibrium, because not all players choose a best response (firm 1 is, but firm 2 is not)

Bertrand duopoly

- Economic-wise, Bertrand equilibrium is nice for the customers. But, is it realistic?
- Explanation? Imperfect substitutes
- Let $q_i = a - p_i + b\,p_j$ (with constant $b < 2$)
- Note: this is **yet another game**!
- b is a sort of exchange rate between goods.

- It can be shown that there is a Nash eq.

$$p_1^* = p_2^* = (a + c)/(2 - b)$$

Bertrand duopoly

- Or, consider a case with different costs
  - For example, $c_1 = 1$, $c_2 = 2$ (cost advantage for 1)
  - For simplicity, prices are set in steps of $\varepsilon = 0.01$
- Now, there is no way firm 2 can "win"
  - Firm 1 can set $p_1 = 1.99$ and becomes monopolist
  - One possible Nash equilibrium is (1.99, 2.00)
  - However, if $\varepsilon \to 0$ we have a problem
    (2,2) is not a NE as payoffs are discontinuous
  - Discretizing the state space is a trick often used to avoid this kind of problems

Hotelling model

- Hotelling (1929) proposed a model of competition, readjusted here as follows
- Two street vendors of ice-cream serve a seaside boulevard, assumed 1 km long
  - Ice-cream cones sold by the vendors are perfect substitutes for each other (→ same price)
  - People buy ice-cream by the nearer vendor
  - People distribution on the street is uniform
  - For modeling ease, assume 101 possible locations (one each 10 meters): 0, 1, .., 99, 100

Hotelling model

- If vendor A chooses 22 and vendor B chooses 35, vendor A gets all the people from 0 to 28
  - but A has an incentive to move right (actually, he can do better by moving to 36)
- Easy to see the only NE: they both choose 50
  - Such a result has often been used as a political paradigm (median voter theorem)
  - Political convergence "to the middle point"

The problem of commons

Why is common resource often wasted?

The tragedy of commons

- Many political philosophers and economists, since at least Hume (1739) have understood that, if moved only by private incentives, citizens tend to misuse public resources
- Environmental pollution is an example
- This problem is commonly referred to as the "tragedy of commons"
- There are several ways to see it

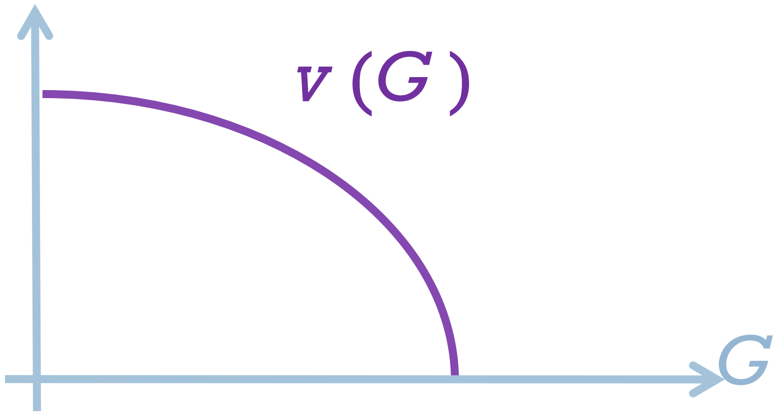
The tragedy of commons

- Classic version (Hardin, 1968):
  - We have n farmers in a village, which forage their goats in a common green
- Each farmer owns g_i goats
- $G = g_1 + g_2 + \ldots + g_n$
- Each goat costs c in caring expenses
- The use of the common green shared by G goats has a value of $v(G)$ per goat
- The value decreases with G

The tragedy of commons

- An information-theory version.
  - We have n users of a WiFi hotspot, accessing a shared spectrum. Each activates g_i processes
- The overall network throughput has a value of $v(G)$ per process (decreasing with G)

$$v(G_{\max}) = 0$$
$$v'(G) < 0$$
$$v''(G) < 0$$



The tragedy of commons

- The payoff to each user is $g_i\,(v\,(G\,) - c\,)$
- We write $g_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n\,)$
- Let find the Nash equilibrium g_i *

$$\max_{g_i} \; g_i\,(v\,(g_i + g_{-i}{}^*) - c\,)$$

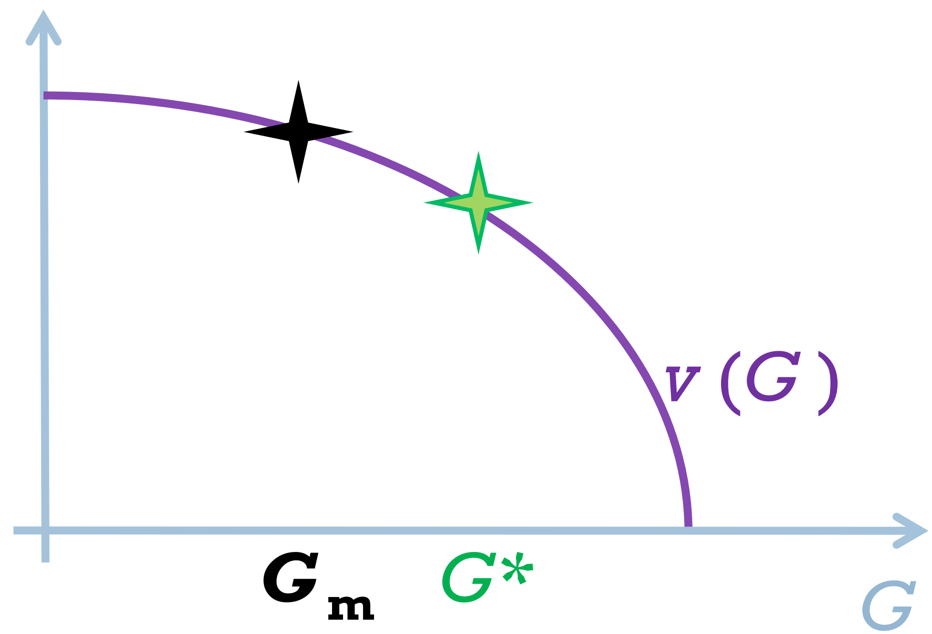
- Now replace g_i * with G^*/n

$$v\,(G^*) + G^*\,v'(G^*)\,/n \; - c = 0$$

The tragedy of commons

- For the NE $v(G^*) + G^* v'(G^*)/n - c = 0$
- The global welfare is $G(v(G) - c)$, so we have an optimum at $\mathbf{G_m}$ for which $v(\mathbf{G_m}) + \mathbf{G_m} v'(\mathbf{G_m}) - c = 0$

$$v(\mathbf{G_m}) + \mathbf{G_m} v'(\mathbf{G_m}) = v(G^*) + G^* v'(G^*)/n$$



The tragedy of commons

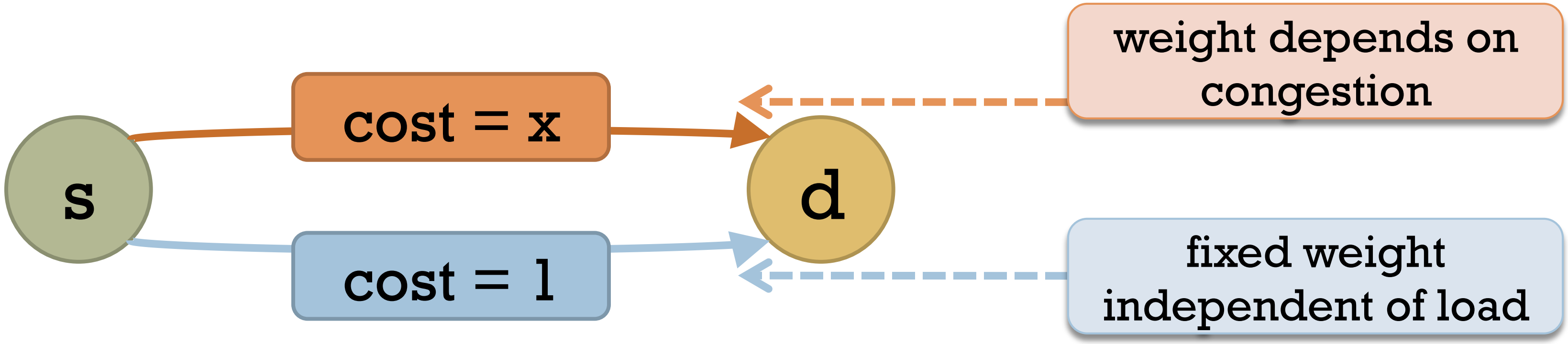
- At NE, it holds $v(G^*) + v'(G^*)\ G^*/n - c = 0$ which reflects the following fact.
- A user with g_i possessions (goat or processes) may consider adding an "increment" h :
  - the cost of the possessions increases by $c\ h\ /\ h = c$
  - its possessions lose value by $(v(G+h) - v(G))/h$ that is, $v'(G)$, summing to a total of $v'(G)\ g_i$
  - At the NE (symmetry) all users have $g_i = G^*/n$
- The global viewpoint considers the loss of all users, which is $v'(\boldsymbol{G}_\mathbf{m})\ \boldsymbol{G}_\mathbf{m}$ (no $1/n$ term)

Selfish routing

A scenario with high Price of Anarchy

Selfish routing

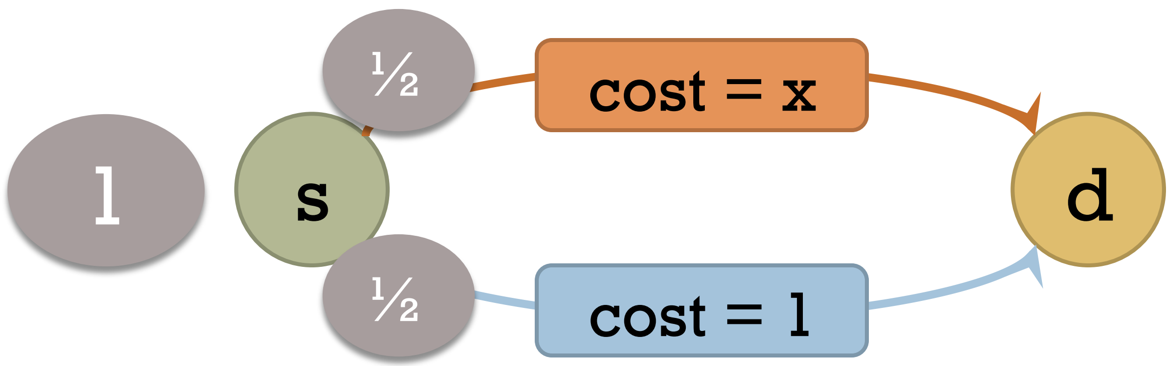
- Pigou (1920): 2 paths from s to d
  - Cost ∝ congestion for one path



- Say 1 unit of traffic goes from s to d
- Top edge is a dominant strategy
- All traffic incurs a cost of 1

Selfish routing

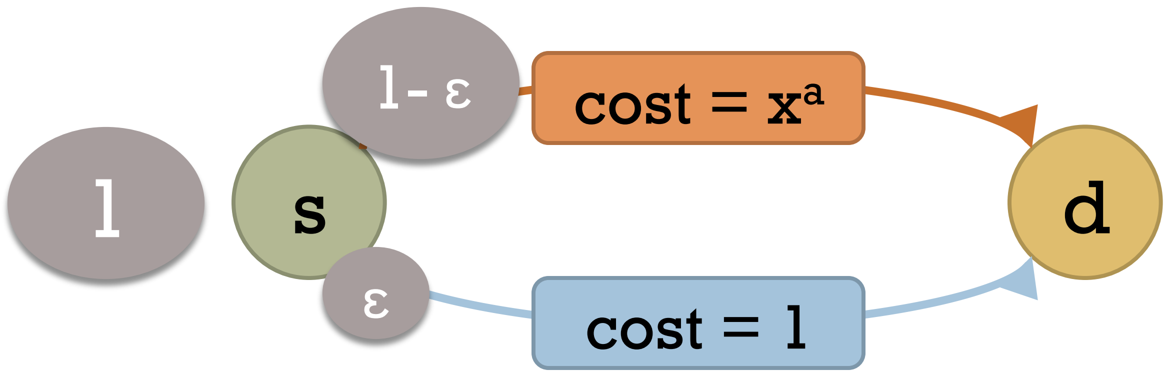
- Can we do better? Split traffic ½ and ½



- Unit cost is ½ on upper edge, 1 on lower
- Average cost is ¾. Overall optimum, but players have incentive to deviate
- Price of Anarchy = 4/3.

Selfish routing

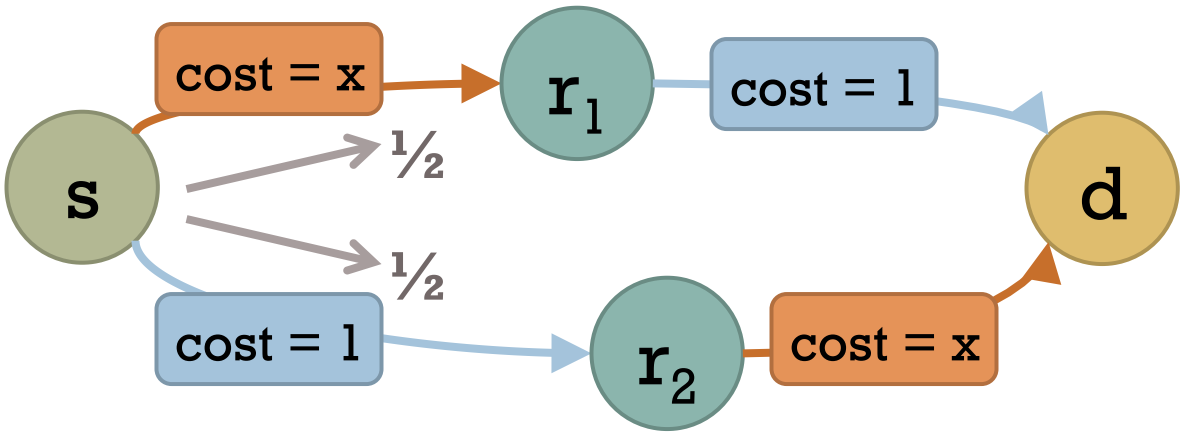
- Even worse with non-linear cost (a>1)
- Assume a is large



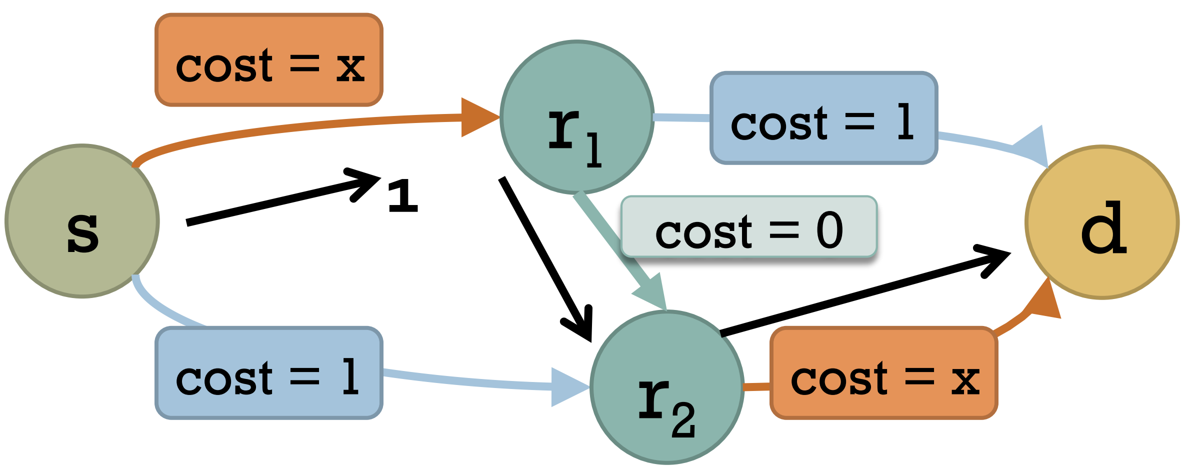
- Top path is again dominant, total cost = 1
- If a fraction ε goes below, cost = $(1-\varepsilon)^a + \varepsilon$
  - For $a \to \infty$, cost $\to 0$: Unbounded PoA

The Braess paradox

- A “better” network can have a “worse” NE



- Cost = 3/2



- Cost = 2
(despite adding a costless cut)

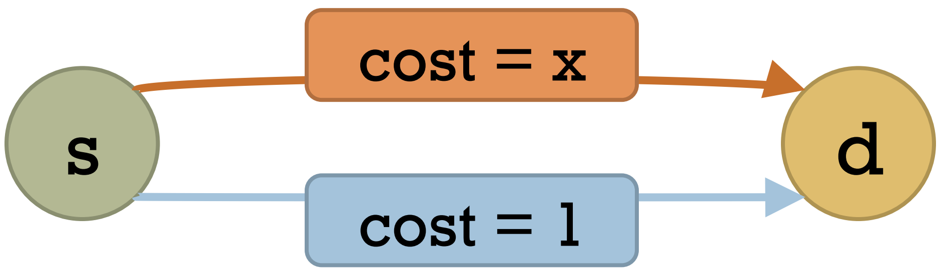
Linear costs

- Theorem

(for linear latencies)

Cost of Nash flow = $^{4}/_{3}$ Optimal flow cost

PoA



Two-path Pigou's graph is a worst case

- **Intuition**: When confronted with two choices, all selfish users take the better one → overload
- Intrinsic PoA, **does not depend on topology**