Game theory

a course for the
MSc in ICT for Internet and multimedia

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Constitutions

Trying to unify preferences

e.g. elezioni -> NE non hanno senso

Constitution

- Let $R(A)$ be a set of rational preferences on A
- A **constitution** (or **social welfare function**) is

$$f : R(A)^n \longrightarrow R(A)$$

prima, abbiamo aggregato utilità

- A constitution makes profile $\succcurlyeq_{(i)} = (\succcurlyeq_1, \succcurlyeq_2, \ldots \succcurlyeq_n)$ into a unique social preference $f(\succcurlyeq_{(i)})$.

in realtà ristretta a $(Y \times Y)$

- Restricting preference $\succcurlyeq$ over A to $Y \subseteq A$:

$$\succcurlyeq | Y = \succcurlyeq \cap (Y \times Y)$$

Properties of constitutions

- A constitution f satisfies the Independence of Irrelevant Alternatives (**IIA**) if $\forall$ pairs of profiles $(\succcurlyeq_{(i)})$, $(\succcurlyeq'_{(i)})$ and $\forall$ a, b $\in A$

$$\forall i, \succcurlyeq_i |\{a,b\} = \succcurlyeq'_i |\{a,b\}$$

$$\text{implies } f(\succcurlyeq_{(i)}) |\{a,b\} = f(\succcurlyeq'_{(i)}) |\{a,b\}$$

- that is, adding or removing elements to the alternative set does not change the priority of a and b

Properties of constitutions

- Constitution f is **Pareto efficient** if $\forall$ profiles $(\succcurlyeq_{(i)})$, $\forall$ a, b $\in A$

$$\boxed{\forall i, a \succcurlyeq_i b \text{ implies } a \succcurlyeq b \text{, where } \succcurlyeq \, = f(\succcurlyeq_{(i)})}$$

  - that is, if everybody prefers a over b, so does the society as a whole as dictated by the social rule

- Pareto efficiency relates to the concept of "being better for everybody"

Properties of constitutions

- f is a **dictatorship** if there exists i such that

  $$\boxed{a \succcurlyeq_i b \quad \text{implies} \quad a \succcurlyeq b \text{ , where } \succcurlyeq\, = f\,(\succcurlyeq_{(i)})}$$

  - i.e., the constitution simply mimics i's preference

- f is **monotonic** if, when a single individual modifies his/her preference ranking something better, f does not rank it worse
- f satisfies **non-imposition** if all rational preferences can be outputs, i.e., is surjective

Arrow's Theorem

- Theorem (Arrow, 1951).
- Impossible to design a constitution which is:
  - non-dictatorship
  - monotonic
  - satisfies IIA and non-imposition
- A more synthetic version (1963) says that if *f*
  - is Pareto efficient
  - satisfies IIA

  …then it is a dictatorship!

Elections and Paradoxes

which do not hold only for elections

Elections and democracy

- What is **democracy**?
- Usually we immediately connect democracy with elections, as well as with "majority rule"
- What does majority means?
- Things get complicated in the case of multiple choices

Elections and democracy

- Say we have 3 voters and 2 candidates
- The preference are as follows

voter	1	2	3
best	A	A	B
worst	B	B	A

- A beats B by majority rule since 2 people prefer A over B and only 1 does the opposite
- A democratic society should choose A

Elections and democracy

- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	1	2	3
best	A	A	B
	B	C	C
worst	C	B	A

- A>B, B>C, A>C. A beats all other candidates
- A democratic society should choose A

Elections and democracy

- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	1	2	3
best	A	C	B
	B	A	C
worst	C	B	A

- A>B, B>C, C>A. There is no "best" candidate.
- What should a democratic society choose?

Cycle → Paradox!

Terminology

- A candidate that beats majority-wise all the others is called the **Condorcet winner**
- If there is no winner, then there must be a cycle, formally called a **Condorcet cycle**
- Also mixed cases are possible for >3 candidates (e.g., a winner, and a cycle among the remaining 3)

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A		
	B	A	B
		B	
worst			A

- It all depends where we put C between A and B

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A	C	C
	B	A	B
	C	B	
worst			A

- In this case, C is the Condorcet winner

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A		
	B	A	B
	C	B	C
worst		C	A

- C is the worst of all ("Condorcet loser")

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A	C	
	B	A	B
	C	B	C
worst			A

- Condorcet cycle!

Remark 2

- Condorcet cycles cannot occur when only two alternatives are present
- With ≥3 alternatives there may be cycles
- The probability of Condorcet cycles grows with the number of candidates
- If preferences are sufficiently randomized, for large (→∞) number of candidates, Condorcet cycles are sure to occur

Remark 2

- Probability of at least one cycle (random preferences)

voters→ choices↓	3	5	7	9	∞
3	5.6%	6.9%	7.5%	7.8%	8.8%
5	16.0%	20.0%	21.5%	23.0%	25.1%
7	23.9%	29.9%	30.5%	34.2%	36.9%
∞	100.0%	100.0%	100.0%	100.0%	100.0%

Remark 3

- Even though we speak of candidate and elections, the same thing could apply to:
- Scheduling: think of candidate A, B, C, as users/ packets/ objects to allocate and voters 1, 2, 3, as criteria to choose among them
- Optimization: think of candidate A, B, C, as possible solutions to an optimization problem and voters 1, 2, 3, as possible goal functions

Some "real world" examples

- Fiscal politics of governments

	liberals	anti-deficit	conservatives
best	Taxes ↑ Spending ↑	Taxes ↑ Spending ↓	Taxes ↓ Spending ↓
	Taxes ↓ Spending ↓	Taxes ↑ Spending ↑	Taxes ↑ Spending ↓
worst	Taxes ↑ Spending ↓	Taxes ↓ Spending ↓	Taxes ↑ Spending ↑

Some "real world" examples

- Quality of Service

	"well behaved"	high delay	high losses
best	Voice over IP	Video Streaming	Best Effort Data
	Video Streaming	Best Effort Data	Voice over IP
worst	Best Effort Data	Voice over IP	Video Streaming

Search for a perfect system

does it exist, actually?

Setting the agenda

- Assume 3 competitors A, B, and C: we choose between A and B in a first round, then the winner goes up against C
- Seems fair? It is not in a Condorcet cycle!
- Assume the cycle is A<B<C<A: C wins, while he would lose in a different setup
- For example: choose between C and B first, then the winner goes up against A: A wins

Other methods

- There are actually many electoral systems (which work also as selection rules in allocation problems), such as
- Plurality voting
- Two-phase Run-off
- Borda counting
- Approval voting
- Instant run-off

Plurality voting

- Let each voter sort the candidates in order of personal preference
  - Some candidates will get “first place” by some voters
- In the “plurality voting” criterion, the winner is who has most first places among the voters
- Is this mechanism immune to paradoxes?

Plurality voting

- Assume we have 9 voters

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	B	C	B
worst	C	A	A

- A wins (4 votes vs. 3 votes of B and 2 of C)
- However a majority prefers B>A
- A majority also prefers C>A
- There even is a Condorcet winner (B), as B>C

Two-phase Run-off

- We make a two-round voting
- First we select the two best candidates
- In a second round, we choose between them in a ballot

Two-phase Run-off

- Again, assume we have 9 voters

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	C	C	B
worst	B	A	A

- A and B go to the ballot, B wins 5-4
- However a majority prefers C>A and C>B
- C is the Condorcet winner, but C does not even make it to the ballot

Borda count

- Plurality and Run-off favor "polarized" solutions over "compromise" solutions
- A strong candidate in a (large) minority wins over a weak one even if appreciated by many
- Borda count tries to solve this:
  - If we have M candidates, the voter gives a score
  - M-1 points go to the best one, M-2 to the next one and so on; the last one gets 0 points
- Is this method better?

Borda count

- We have again 9 voters (assigning 27 points)

	1-5 (5 voters)	6-8 (3 voters)	9
best	A	B	C
	B	C	B
worst	C	A	A

- A achieves 10 points, B 12, C 5. B wins
- However, A is the Condorcet winner, since A>B and A>C
- Similar paradoxes hold for different scores

Borda count with dropout

- Borda-like counts are used, e.g., for sports

	1-5 (5 voters)	6-7 (2 voters)	8-9 (2 voters)
best	D	A	A
	C	D	B
	B	B	D
worst	A	C	C

- Total points: A 12, B 11, C 10, D 21
- Thus: D gold, A silver, B bronze

Borda count with dropout

- But D retires (e.g. anti-doping or naked photo)

	1-5 (5 voters)	6-7 (2 voters)	8-9 (2 voters)
best		A	A
	C		B
	B	B	
worst	A	C	C

- Total points: A 8, B 9, C 10
- Thus: C gold, B silver, A bronze
- The retirement entirely reverse the order

Approval voting

- Each voter can give more than one preference
- Each preference assigns one point
- The number N of preferences must be between 1 and M (no. of candidates)
- For N=1 we fall back into plurality case

Approval voting

- Again, an example with the 9 voters

	1-3 (3 voters)	4-6 (3 voters)	7-8 (2 voters)	9
best	A	D	B	A
	C	B	D	B
	D	C	C	C
worst	B	A	A	D

- Top 2 approvals: A 4, B 6, C 3, D 5. B wins
- Top 3 approvals: A 4, B 6, C 9, D 8. C wins
- The result depends on N

Approval voting

- Every system has a different outcome.

	1-3 (3 voters)	4-6 (3 voters)	7-8 (2 voters)	9
best	A	D	B	A
	C	B	D	B
	D	C	C	C
worst	B	A	A	D

- Plurality -Top 1 approvals- prefers A (4 votes)
- Borda winner is D with 16 (A 12, B 14, C 12)

Instant Run-off

- Again, we ask each voter for its “order of preference”
- Only top preferences count to reach a majority
- We make (“instantaneously”) subsequent rounds, each time removing the candidate with least top preferences

Instant Run-off

- Let see an example with 17 voters

	6 voters	5 voters	4 voters	2 voters
best	A	C	B	B
	B	A	C	A
worst	C	B	A	C

- No majority, so candidate C is eliminated
- A gains 5 votes, and wins with 11 votes

Instant Run-off

- What if the last 2 voters chose A first instead of B

	6 voters	5 voters	4 voters	2 voters
best	A	C	B	A
	B	A	C	B
worst	C	B	A	C

- This causes A to lose! B is now eliminated at the first round. 4 votes go to C, who wins with 9 votes
- A loses due to an increasing consensus

Setting the agenda

- The selection of a particular method may advantage some competitors in an almost invisible way
- This is a very subtle factor in many fields: politics, sports, sciences, everyday life

- Fortunately, this power is not almighty

Setting the agenda

- A>B>C>A are in Condorcet cycle. D is worst.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	C	B
	B	A	C
	C	B	A
worst	D	D	D

- There is no way for D to win (A>D, B>D, C>D)
- However, if we make semifinals and final, it always win who goes against D first

Setting the agenda

- A>B>C>A are in Condorcet cycle. D is best.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	D	D	D
	A	C	B
	B	A	C
worst	C	B	A

- Here, D always wins and the order of A, B, C depends on the agenda setting

Cheating: Condorcet cycles

another consequence of this paradox

Cheating

- A>C>B>A are in a Condorcet cycle.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	D	A	B
	C	D	D
worst	B	C	A

- However, A is the winner in many systems (plurality, Borda count, Top 2 approval…)
- Assume we choose plurality: A wins

Cheating

- 8 and 9 are disappointed. For them A is worst

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- They decide to cheat and indicate B as preferred choice, instead of C.
- Now B wins. For them it is an improvement.

Cheating

- For the first 4 voters this is bad.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- They may protest and ask for help from 5-7, but these are happy, since B is best for them

Cheating

- But if they can act first, they can cheat too

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	~~A~~ C	B	C
	~~D~~ A	A	B
	~~C~~ D	D	D
worst	B	C	A

- It counteracts cheating by 8-9, who vote C
- Bad for 5-7 but they can't do anything
- C wins with only 2 "natural" votes

Cheating

- There is also a chance for B's supporters.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	~~B~~ A	C
	D	~~A~~ B	B
	C	D	D
worst	B	C	A

- They can change and support A (whom they prefer better than C): now A wins again…

…in the end it depends on who *cheats first*

Extensions to Arrow's Theorem

- Social function f is **strategy-proof** (non-manipulable) if for any profile $(\succcurlyeq_{(i)})$ and a certain preference $\succcurlyeq'_j$

$$f(\succcurlyeq_{(i)}) \succcurlyeq_j f(\succcurlyeq'_j, \succcurlyeq_{-j})$$

  - that is, no one has incentive to cheating

- **Gibberard-Satterthwaite theorem**. Any strategy-proof constitution that does not forbid anyone to win… must be a dictatorship!

Problems of electoral systems

- It seems that no good system exists
- Recall Arrow's Theorem if a constitution:
  - is Pareto efficient
  - satisfies IIA

  …then it is a dictatorship!
- "Ways out"
  - some conditions are weakened
  - use free approval voting (vote "for" or "against")
  - we restrict the profile

Majority rule

- This last solution has been proposed in various ways by many economists and is in short a way to apply **majority rule**
- Formally, majority rule $\succcurlyeq$ can be defined as:

$$\boxed{\mathrm{a} \succcurlyeq \mathrm{b} \quad \Leftrightarrow \quad |\{i : \mathrm{a} \succcurlyeq_i \mathrm{b}\}| \geq |\{i : \mathrm{b} \succcurlyeq_i \mathrm{a}\}|}$$

  - is Pareto efficient
  - satisfies IIA
  - is not a dictatorship

…but is not a constitution!

Majority rule

no preferenze razionali

- Majority rule is complete but **non-transitive**
- The reason is the existence of Condorcet cycles
- If we are able to eliminate Condorcet cycles, majority rule becomes a constitution and possesses "nice" properties (Sen)
- Alternative: focus only on cases with a *linear* (i.e., total) order relationship on set *A*
  - This also guarantees to avoid Arrow's theorem by using majority rule (Black)