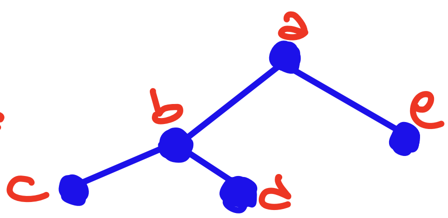
RECAP

- TSP is not $g(n)$-approximable for ANY $g(n)$ ptc
- T_TSP (NPH): c satisfies $\triangle$-inequality

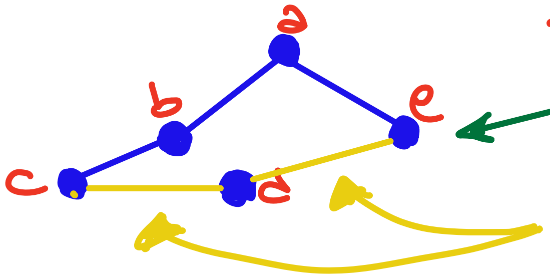
MST-BASED APPROXIMATION

APPROX_T_TSP(G, c)
$/*\ V = \{v_1, v_2, \dots, v_{|V|}\}\ */$
$T^* = (V, E_T) \leftarrow$ PRIM(G, c)
$r = v_1$
$\gamma_{T^*} \leftarrow \langle$PREORDER$(T^*, r), r\rangle$
return γ_{T^*}

$T^* =$  $\rightarrow$ PREORDER: $\langle a, b, c, d, e\rangle$

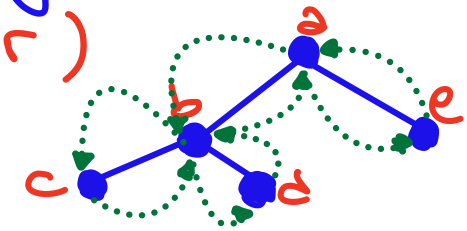
$\downarrow$

TOUR: $\langle a, b, c, d, e, a\rangle$

$\gamma_{T^*} =$ 

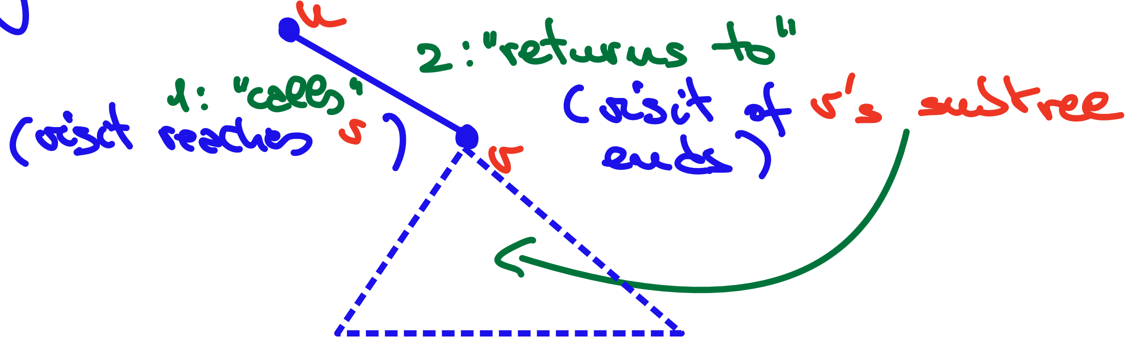
γ_{T^*} uses edges NOT in T^*

FRW: Nonsimple cycle associated to calls of PREORDER(T^*, r)

$\langle a, b, c, b, d, b, a, e, a\rangle$ 

PROPERTIES OF A FPW

1. The first occurrence of each node gives the preorder visit of T^*
2. Each edge of T^* is used $\boxed{2 \text{ times}}$ by FPW:

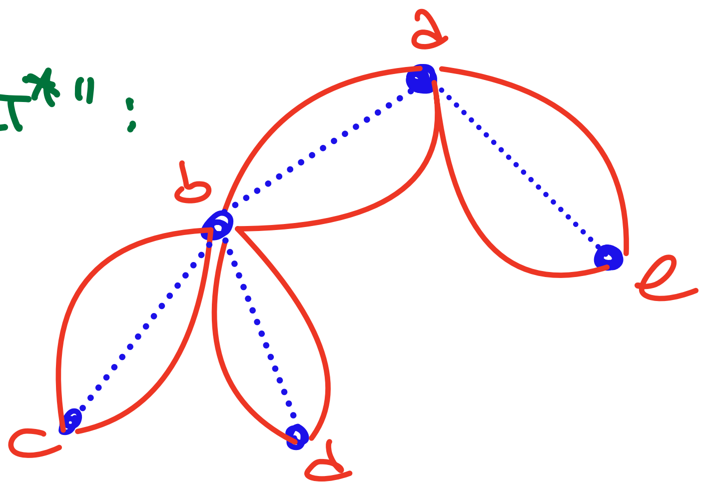


Therefore: $c(FPW) = 2c(T^*)$

OBSERVATION (useful for later)

The preorder visit on T^* follows a nonsimple cycle (circuit) which uses each edge of T^* twice!

"$2T^*$":



Multigraph obtained from T^* replicating each edge twice

FPW = $\langle a, b, c, b, d, b, a, e, a \rangle$
touches all edges of the multigraph once!

A nonsimple cycle touching ALL EDGES of a (multi-)-graph ONCE is called EULER TOUR

We are ready to prove:

LEMMA

$$c(\gamma_{T^*}) \leq c(FPW) = 2c(T^*)$$

PROOF

Consider the FPW associated to the call PREORDER(T^*, r) in A-T-T(G, c)

$$FPW = \langle v_1 = r, v_2, \ldots, v_{2|V|-1} = v_1 = r \rangle$$

(the FPW contains $2|V|-2$ edges)

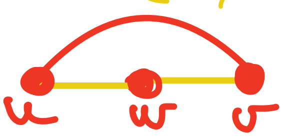
Thanks to the $\triangle$-inequality, we can use shortcutting to eliminate all duplicate nodes (but $v_{2|V|-1} = r$) from FPW without increasing the cost :

duplicate

$$FPW = \langle \ \ldots \ u, w, v \ \ldots \rangle \overset{\triangle}{\leadsto} \quad c(u,w) + c(w,v) \geq c(u,v)$$

$$FPW' = \langle \ldots \ \boxed{u, v} \ \ldots \rangle$$

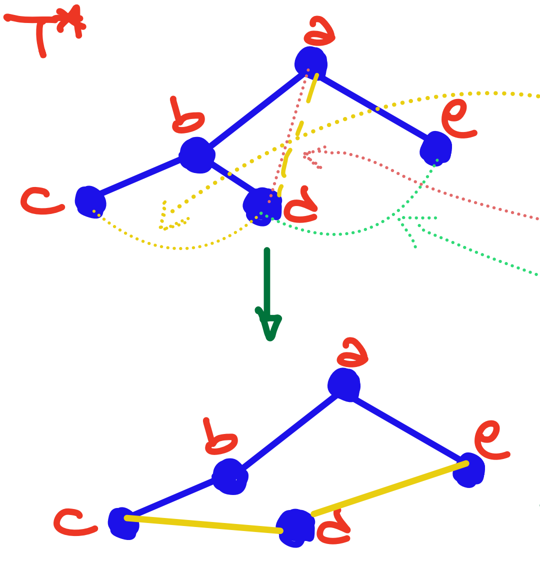
$$c(FPW) \geq c(FPW') \ !$$



By removing all these duplicates, we obtain $\langle \text{PREORDER}(T^*, r), r \rangle = \gamma_{T^*}$!

EXAMPLE

T^*



$FPW = \langle a, b, c, b, d, b, a, e, a \rangle$

$\leadsto \langle a, b, c, d, b, a, e, a \rangle$

$\leadsto \langle a, b, c, d, a, e, a \rangle$

$\leadsto \langle a, b, c, d, e, a \rangle$

$= \langle \text{PREORDER}(T^*), r \rangle$

$= \gamma_{T^*}$

Since each shortcut can only decrease $c(FPW)$, we have proved that

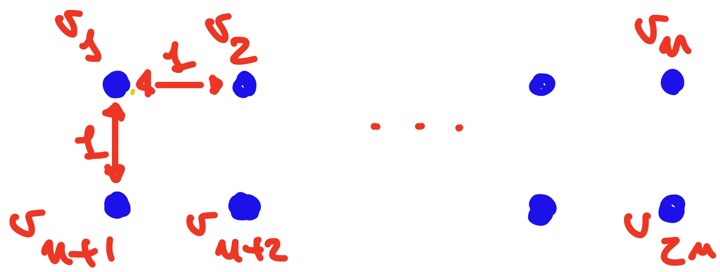
$$c(\gamma_{T^*}) \le c(FPW) = 2c(T^*)$$

Therefore

$$\rho = \frac{c(\gamma_{T^*})}{c(\gamma^*)} \le \frac{2c(T^*)}{c(T^*)} = 2$$

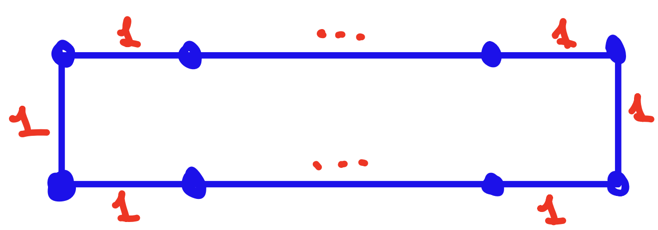
The bound on the approximation is asymptotically tight:

G: $2n$ points in $\mathbb{R}^2$ (unit grid)



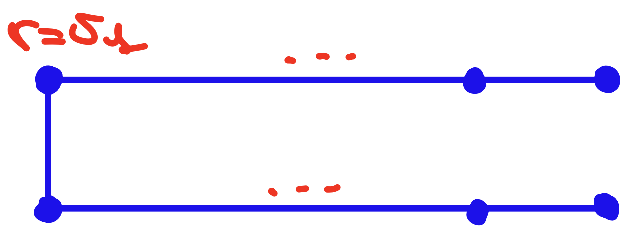
$c(u,v) = \|u-v\|_2$
Euclidean distance

Optimal tour: $c(\gamma^*) = 2n$ (perimeter)

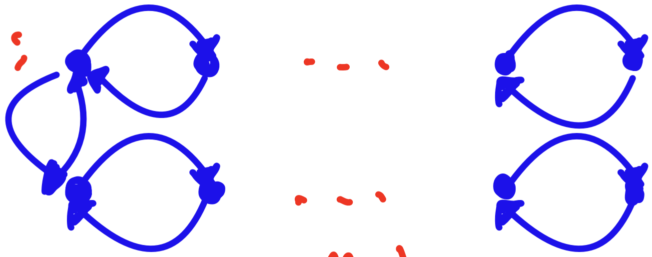


(edges have length ≥ 1)

MST:

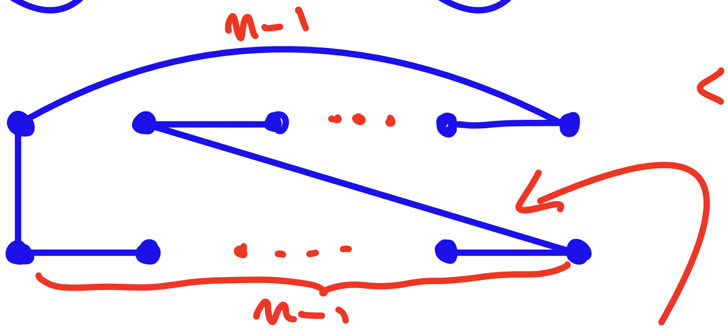


FPW :



$\langle s_1, s_{n+1}, \dots, s_{2n}, s_{2n-1}, \dots, \dots, s_{n+1}, s_1, \dots, s_n, s_{n-1}, s_1 \rangle$

γ_{T^*} :



$\langle s_1, s_{n+1}, \dots, s_{2n}, s_2, \dots, s_n, s_1 \rangle$

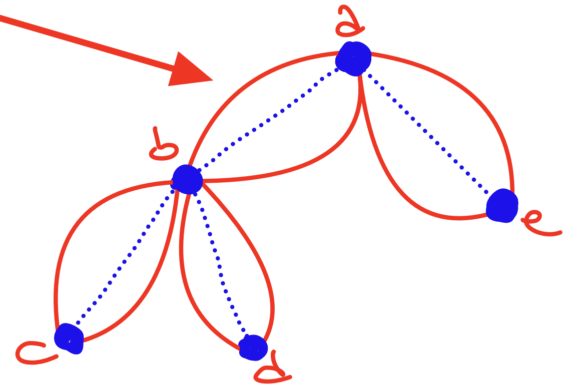
$$C(\gamma_{T^*}) = 3n-3+\sqrt{1+(n-2)^2} > 4n-5$$

Therefore $\rho(n) > \dfrac{4n-5}{2n} = 2 - \dfrac{5}{2n} \xrightarrow[n\to+\infty]{} 2$

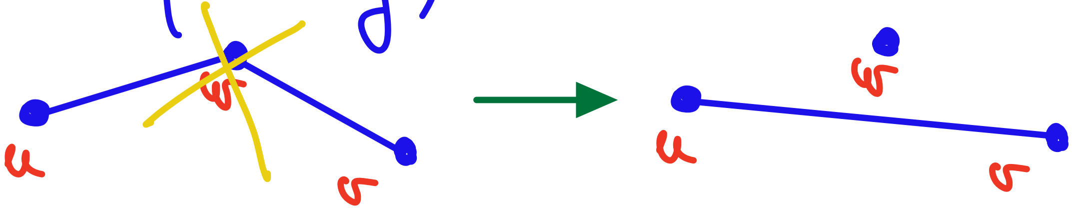
LESSON LEARNED from MST-based approximation:

- MST has been used to create the FULL-PREORDER-WALK (FPW), a nonsimple cycle of small cost ($2 \cdot c(T^*)$) touching all nodes (multiple times)

FPW: Duplicate all edges of T^* and traverse all replicas: $\boxed{\text{Euler tour}}$ of the (multi)-subgraph "$2T^*$"



- ANY nonsimple cycle can be made simple using shortcutting without increasing the cost (thanks to Δ-inequality)

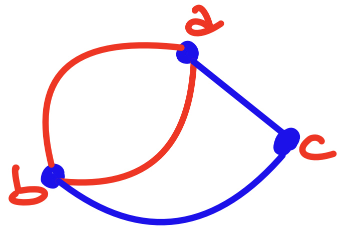


→ Any Euler tour yields a tour of lesser cost!

Christofides algorithm: determine a better subgraph than the double replica "$2T^*$" characterized by a Euler tour of smaller cost.

DEF A multigraph $G=(V,E)$ is such that E is a multiset (each edge is associated to a multiplicity

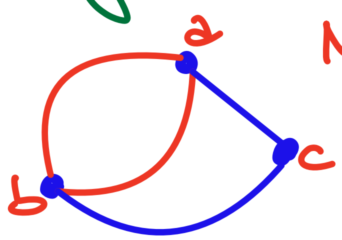
E.g.



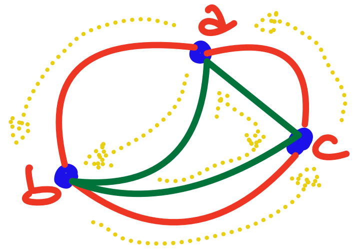
Edge $e=\{a,b\}$ has multiplicity $m(e)=2$

DEF An undirected multigraph $G=(V,E)$ is Eulerian if $\exists$ a Euler tour (nonsimple cycle) traversing each edge $m(e)$ times

E.g.



NOT EULERIAN



EULERIAN

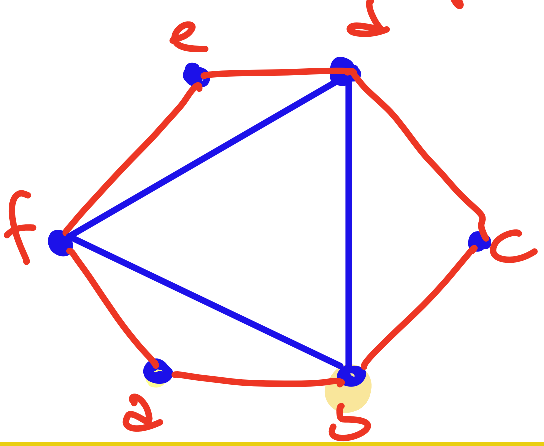
ET: $\langle a,b,c,a,b,c,a\rangle$

Observe: A Euler tour enters and exits each node the same number of times

THEOREM (Euler) A connected multi-graph $G=(V,E)$ is Eulerian if and only if all nodes have even degree

FACT A Euler tour of a Eulerian multi-graph can be found in $O(|E|)$ time:

- Find an initial cycle from an initial node v and remove the edges
- Starting from v, follow the cycle: if the degree of the current node >0:
  - Find a cycle going through node
  - Splice the cycles together, eliminating the edges



$\langle a, b, c, d, e, f, a \rangle$

$\langle a, b, d, f, b, c, d, e, f, a \rangle$

GENERALIZATION OF THE MST APPROACH

Given an instance $\langle G=(V,E) \rangle$ let us determine a (multi)-subgraph of G, $G'=(V,E')$ that is Eulerian and such that

$$\boxed{\sum_{e\in E'} m(e)\cdot w(e)}$$

is small

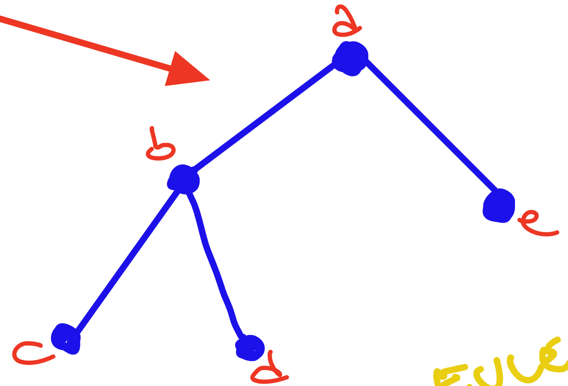
multiplicity

cost of the Euler tour

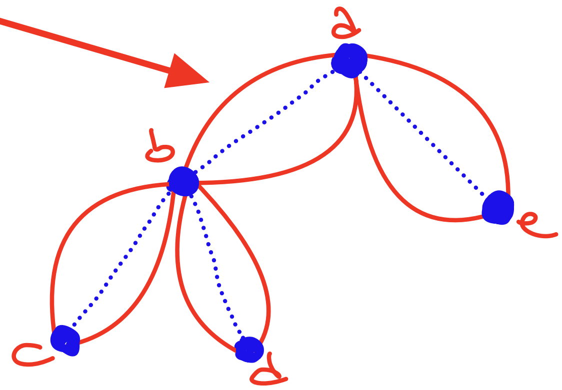
Once the Euler tour has been obtained, we can use shortcutting to obtain a simple tour whose cost is not larger.

OBSERVATION: This is exactly what we did in the MST approach:

T^*



"$2T^*$"



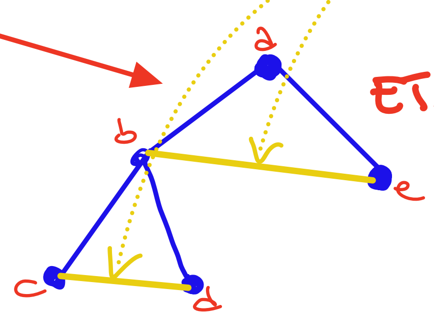
FPW: $\langle a,b,c,d,b,a,e,a \rangle$ = EULER($2T^*$) $\rightsquigarrow$ Tour: $\langle a,b,c,d,e,a \rangle$

CHRISTOFIDES IDEA:

"Doubling" T^* is too expensive. Let us add a subset of edges of small cost to T^* to make it an Eulerian (multi)-subgraph!

EXAMPLE:

T^*



only two edges to shown

$ET = \langle a,b,c,d,b,e,a \rangle$

$\gamma = \langle a,b,c,d,e,a \rangle$

If $c(c,d) + c(b,e) < \alpha \cdot c(\gamma^*)$, $\alpha < 1$ we get a better quality guarantee, since

$$c(\hat{\gamma}) \leq c(T^*) + \alpha\, c(\gamma^*) \leq (1+\alpha)\, c(\gamma^*)$$

$$\rightarrow \rho \leq (1+\alpha)$$

Let us see how to do it!

PROPERTY Let $G=(V,E)$ be an arbitrary undirected graph, and let V_{ODD} be the subset of nodes of odd degree.
Then $|V_{ODD}|$ is even

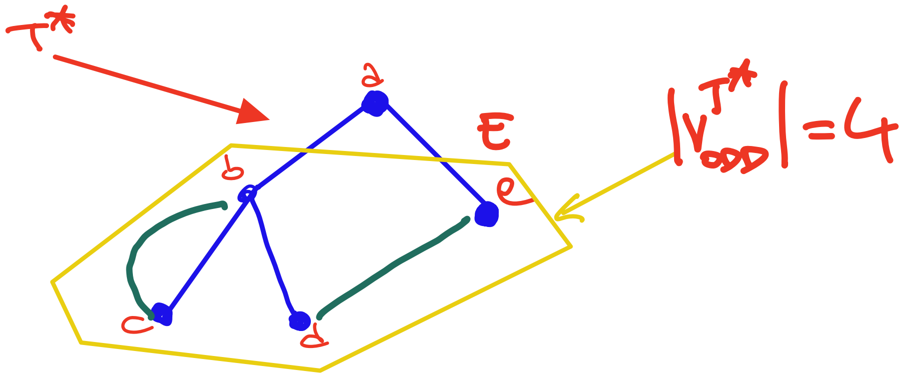
PROOF In any graph, the sum of the degrees $\sum_{v \in V} deg(v) = 2|E|$ (each e counted twice)

Then

$$\underbrace{2|E|}_{\text{EVEN}} = \sum_{v \in V} deg(v) = \underbrace{\sum_{v \in V_{EVEN}} deg(v)}_{\text{EVEN}} + \underbrace{\sum_{v \in V_{ODD}} deg(v)}_{\text{it must be even!}}$$

Since $\sum_{v \in V_{ODD}} deg(v)$ is even and the single terms are odd, the number of these terms $|V_{ODD}|$ must be even!

Let us apply the property to the MST T^*: we have that $|V_{ODD}^{T^*}|$ is even!



CRUCIAL IDEA: Let us complete T^* into a Eulerian multi-subgraph by adding $|V_{ODD}|/2$ edges between pairs of nodes in V_{ODD} and use the resulting Euler Tour instead of the FPW.

(with the new edges, all degrees become even)

DEF Given a weighted undirected complete graph $G' = (V', E')$ with $|V'|$ even, a perfect matching is a matching $M \subseteq E'$ of size $|V'|/2$ (a perfect matching "touches" all nodes in V'). Its cost is
$$c(M) = \sum_{\{u,v\} \in M} c(u,v)$$

FACT A perfect matching M^* of minimum cost can be found in $O(|V'|^3)$ time (blossom algorithm) (extremely complex)

We are ready to specify the algorithm:

CHRISTOFIDES$(G=(V,E), c)$

$T^* = (V, E_{T^*}) \longleftarrow \mathrm{MST}(G, c)$

* let $V_{ODD}^{T^*}$ be the set of nodes of odd degree of T^* and let $E_{ODD}^{T^*} = \{ e = \{u,v\} \in E : u,v \in V_{ODD}^{T^*} \}$ *

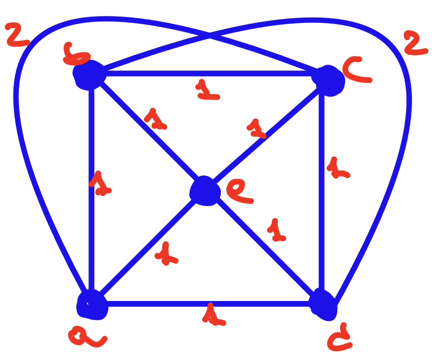
$M^* \leftarrow \mathrm{M\_C\_MATCHING}(G' = (V_{ODD}^{T^*}, E_{ODD}^{T^*}), c)$

$W \leftarrow \mathrm{EULER\_TOUR}(\bar{G} = (V, E_{T^*} \cup M^*))$

$\rho \leftarrow \mathrm{SHORTCUT}(W)$

return ρ

EXAMPLE

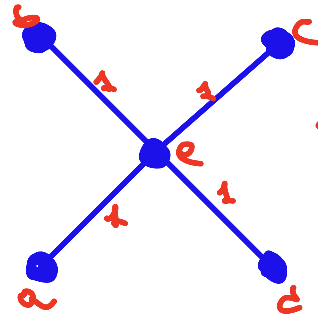


complete graph with 5 nodes

$c(u,v)$ satisfies $\triangle$

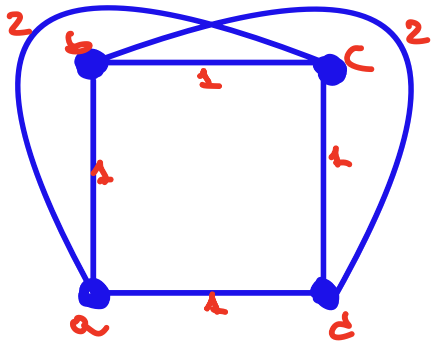
$c(\rho^*) = 5$

$\langle a,b,c,d,e,a \rangle$



T^*: $c(T^*) = 4$

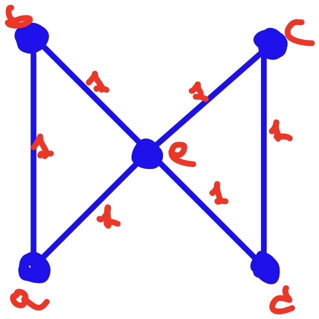
$V_{ODD}^{T^*} = \{a,b,c,d\}$



$G' = (V^{T^*}_{odd}, E^{T^*}_{odd})$

$M^* = \{\{a,b\}, \{c,d\}\}$

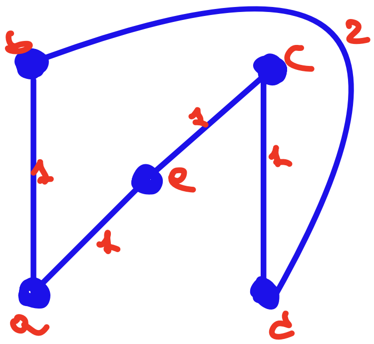
min-cost perfect matching



$\bar{G} = (V, E_{T^*} \cup M^*)$

$W = \langle d, c, e, a, b, e, d \rangle$

EULER TOUR



$\hat{\gamma} = \text{SHORTCUT}(W) =$

$= \langle d, c, e, a, b, d \rangle$

$c(\hat{\gamma}) = 6$

Correctness: the algorithm returns a tour $\hat{\gamma}$ (since it shortcuts a Euler Tour W of the Eulerian multigraph $\bar{G} = (V, E_{T^*} \cup M^*)$)

Running time: $O(|V|^3)$ (dominated by M_C_MATCHING)

Approximation ratio: It suffices to bound $c(W) = c(E_{T^*}) + c(M^*)$, since $c(\hat{\gamma}) \leq c(W)$ (shortcutting). We already know that $c(E_{T^*}) \leq c(\gamma^*)$.

LEMMA $c(M^*) \le c(\gamma^*)/2$

PROOF Consider an optimal tour

$$\gamma^* = \langle s_{i_1}, s_{i_2}, \dots, s_{i_{|N|}}, s_{i_1} \rangle$$

We can use shortcutting to eliminate all nodes in $V^{T^*}_{EVEN}$. We are left with a simple cycle of all nodes in $V^{T^*}_{ODD}$!

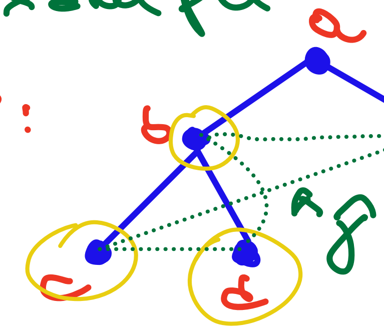
$$\hat{\gamma} = \langle \hat{s}_{i_1}, \hat{s}_{i_2}, ---, \hat{s}_{i_{|V^{T^*}_{ODD}|}}, \hat{s}_{i_1} \rangle$$

EVEN!

Clearly, $c(\hat{\gamma}) \le c(\gamma^*)$

Example

T^*:

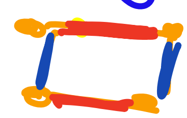


SHORTCUT

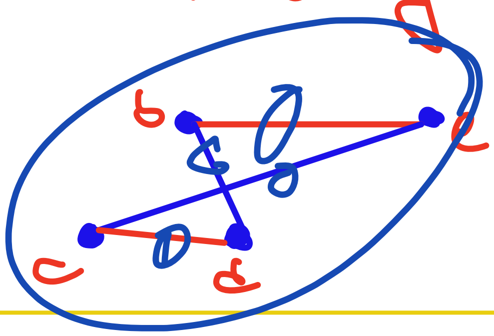
If $\gamma^* = \langle b, e, a, c, d, b \rangle$

$\hat{\gamma} = \langle b, e, c, d, b \rangle$

$\hat{\gamma}$ is a simple cycle of $|V^{T^*}_{ODD}|$ (even!) edges containing all nodes in $V^{T^*}_{ODD}$



Let us color the edges in the cycle alternating between RED and BLUE!



I obtain TWO PERFECT MATCHINGS of $V^{T^*}_{ODD}$! M_{RED} and M_{BLUE}

We have:

$$c(\hat{\gamma}) = c(M_{RED}) + c(M_{BLUE})$$

therefore one of the two matchings has cost $\leq c(\hat{\gamma})/2 \leq c(\gamma^*)/2$!

Therefore:

$$c(M^*) \leq \min\{c(M_{RED}), c(M_{BLUE})\} \leq c(\gamma^*)/2$$

Putting it all together:

Let $\gamma = \text{CHRISTOFIDES}(G=(V,E), c)$

$$c(\gamma) \leq c(E_{T^*}) + c(M^*) \leq c(\gamma^*) + c(\gamma^*)/2$$

$$= \frac{3}{2} c(\gamma^*)$$

$$\rightarrow \quad \rho = \frac{c(\gamma)}{c(\gamma^*)} \leq \frac{3}{2} \frac{\cancel{c(\gamma^*)}}{\cancel{c(\gamma^*)}} = \frac{3}{2}$$