# Game theory

a course for the

MSc in ICT for Internet and multimedia

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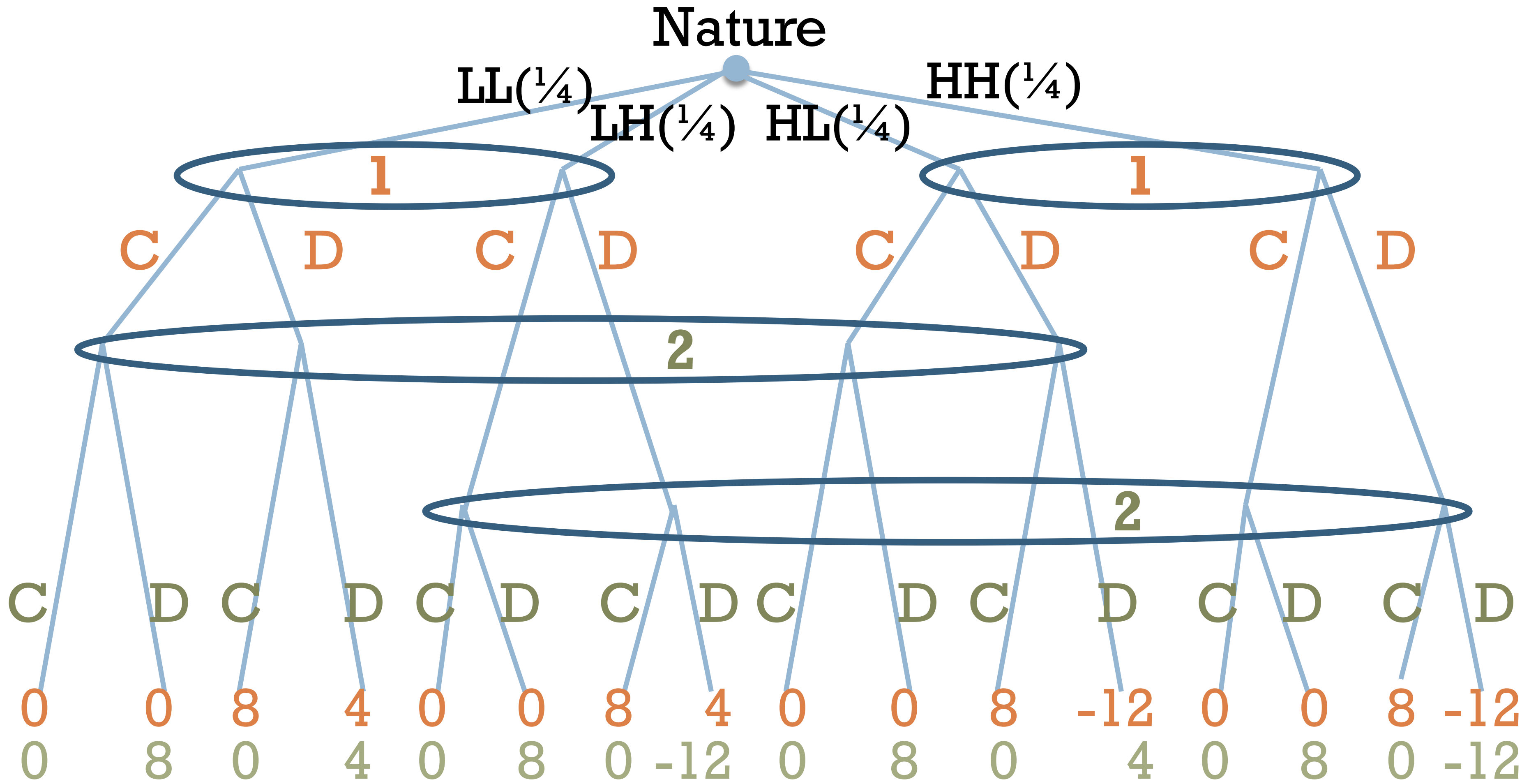
# Examples of Bayesian NE

## Useful applications

# Chicken game

- An anti-coordination game: youngster can (C)hicken (steer) or (D)rive toward the other
  - chickens always get nothing ($u=0$)
  - drivers gains Respect ($u=8$)
  - if both drive, they split Respect, plus an accident happens; they receive $u=4$ minus punishment P depending on their parents
  - parents can be of the (H)ard ($P=16$) type or the (L)enient ($P=0$) one, with 0.5 probability
  - kids know the type of their parents

# Chicken game



# Chicken game

- Matrix form

player 2

| player 1 | CC | CD | DC | DD |
|---|---|---|---|---|
| CC | 0, 0 | 0, 4 | 0, 4 | 0, 8 |
| CD | 4, 0 | -1, -1 | -1, 3 | -6, 2 |
| DC | 4, 0 | 3, -1 | 3, 3 | 2, 2 |
| DD | 8, 0 | 2, -6 | 2, 2 | -4, -4 |

- BNE is (DC, DC)
  - different punishments can lead to other BNEs

# Committee voting

- Many decisions are made by committees through majority voting
- Consider a jury with just two jurors deciding whether to (A)cquit or (C)onvict a defendant
  - Every juror casts a sealed vote
  - The defendant is convicted if both jurors vote C
- It is uncertain whether the defendant is (G)uilty or (I)nnocent: the prior probability of G is $q > ½$ and is common knowledge

# Committee voting (cont'd)

- Jurors desire to make the right decision, so their payoff is 1 if G→C and I→A, 0 otherwise
- If the only information is probability $q$, then

| | | juror 2: A | juror 2: C |
|---|---|---|---|
| juror 1 | A | $1-q$, $1-q$ | $1-q$, $1-q$ |
| | C | $1-q$, $1-q$ | $q$, $q$ |

- and since $q > ½$ then it is dominant to play C and the NE is (C,C)

# Committee voting: types

- Assume each player observes the evidence and independently gets a private **signal** (his/her idea about the case) $t_i \in \{ t_G , t_I \}$
  - It is more likely (but not certain) to receive signal "$t_x$" if the defendant status is x
  - $\text{Prob}[t_G \,|\, G] = \text{Prob}[t_I \,|\, I] = p > ½$ for both $i = 1,2$
  - clearly $\text{Prob}[t_G \,|\, I] = \text{Prob}[t_I \,|\, G] = 1\text{-}p < ½$
  - **Note.** These types are not about the player itself, but about the world; still, they affect payoffs

(btw, this is a binary symmetric channel = BSC )

# Committee voting: types

- Since each player has 2 types and 2 actions, → 4 possible strategies: AA, AC, CA, CC
  - strategy (xy) means that $t_G \rightarrow x$, $t_I \rightarrow y$
  - It is a coordination game, because both players have the same objective of a right judgment
- Consider for the moment a one-person problem where only one juror decides
  - Without the signal, he plays C of course
  - How would the signal affect this choice?

# Committee voting: types

- Check the posterior to see the signal effect!

$$P[G \mid t_G] = \frac{P[G \,\&\, t_G]}{P[t_G]} = \frac{qp}{qp + (1-q)(1-p)} > q$$

  - since $p > ½$, thus $qp + (1-q)(1-p) < qp + (1-q)p$

- and instead

$$P[G \mid t_I] = \frac{P[G \,\&\, t_I]}{P[t_I]} = \frac{q\ (1-p)}{q(1-p) + (1-q)\ p} < q$$

  - → if $t_G$: conviction is even surer; if $t_I$: is doubtful

# Committee voting: types

- Actually, if is $t_I$ received, it all depends on $p$:

$$P[G \mid t_I] = \frac{q\ (1-p)}{q(1-p) + (1-q)\ p}$$

may even be less than ½ in which case the juror prefers to acquit than to convict

- This happens if $p > q$
  - The reason is that the information content of the signal must be higher than the prior information
  - E.g. if $p = ½$ , the signal gives no information!

# 2-person decision

- Now, we check whether with $p > q$ we have a BNE given by (CA,CA) in the real problem
  - That would correspond to "following the signal"
- First, draw the probability of each type pair

| juror 1 \ juror 2 | type $t_G$ | type $t_I$ |
|---|---|---|
| type $t_G$ | $qp^2+(1-q)(1-p)^2$ | $p(1-p)$ |
| type $t_I$ | $p(1-p)$ | $q(1-p)^2+(1-q)p^2$ |

# 2-person decision

- Is strategy CA a best response to itself?
- With the rules of the jury, a player is decisive ("pivotal") only if the other juror chooses C
  - If 2 chooses A, that is the result regardless of $a_1$
  - →If 1 believes that 2 is playing CA, **any** strategy of 1 is always a best response if the type of 2 is $t_I$
  - This is to say, if 1 thinks that 2 received signal $t_I$ then everything 1 does is a BR and we are good
  - So we need only to check what happens if $t_2 = t_G$

# Committee voting: types

- Check the posterior to see the signal effect!

$$P[G \mid t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

  - since $p > ½$, thus $qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2$

$$P[G \mid t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

- $\rightarrow$ if also $t_1 = t_G$: conviction is even surer
- but if $t_1 = t_I$: useless signal (symmetry reason)

# Committee voting: conclusion

- Strategy CA is **not** a best response to itself!
  - → one may prove that (CC, CC) is a BNE
- Paradox: though signal is informative ($p > ½$) players go against it even if signal=$t_I$ for both
- The problem is in the bias of beliefs!
  - The fact that the action of a player is relevant only when the other player is inclined to convict tips the scale in favor of conviction

# Dynamic + Bayesian

dynamic games with incomplete information

# Refinements of NE concept

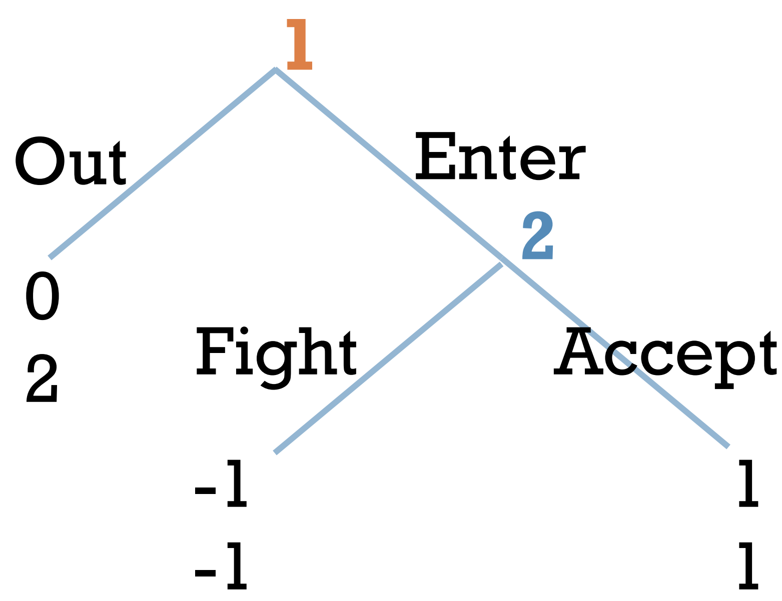
- Static games, complete information: plain NE
- Dynamic games, complete information: plain NE may be insufficient → SPE
  - in case information is perfect (sequential games) this is the result of backward induction
- Bayesian games: if "static" we can use the plain NE with the caveat that a strategy is now defining what different types do
- What about Bayesian + Dynamic:?

# Can we still use SPE?

- In dynamic games, we found SPE to be a peculiar "rational" outcome of the game
- Incomplete information translate a static game with types into a dynamic one
  - where Nature moves first, by choosing types
- However: trouble if two "dynamic" elements: Nature's choices + real gameplay evolution

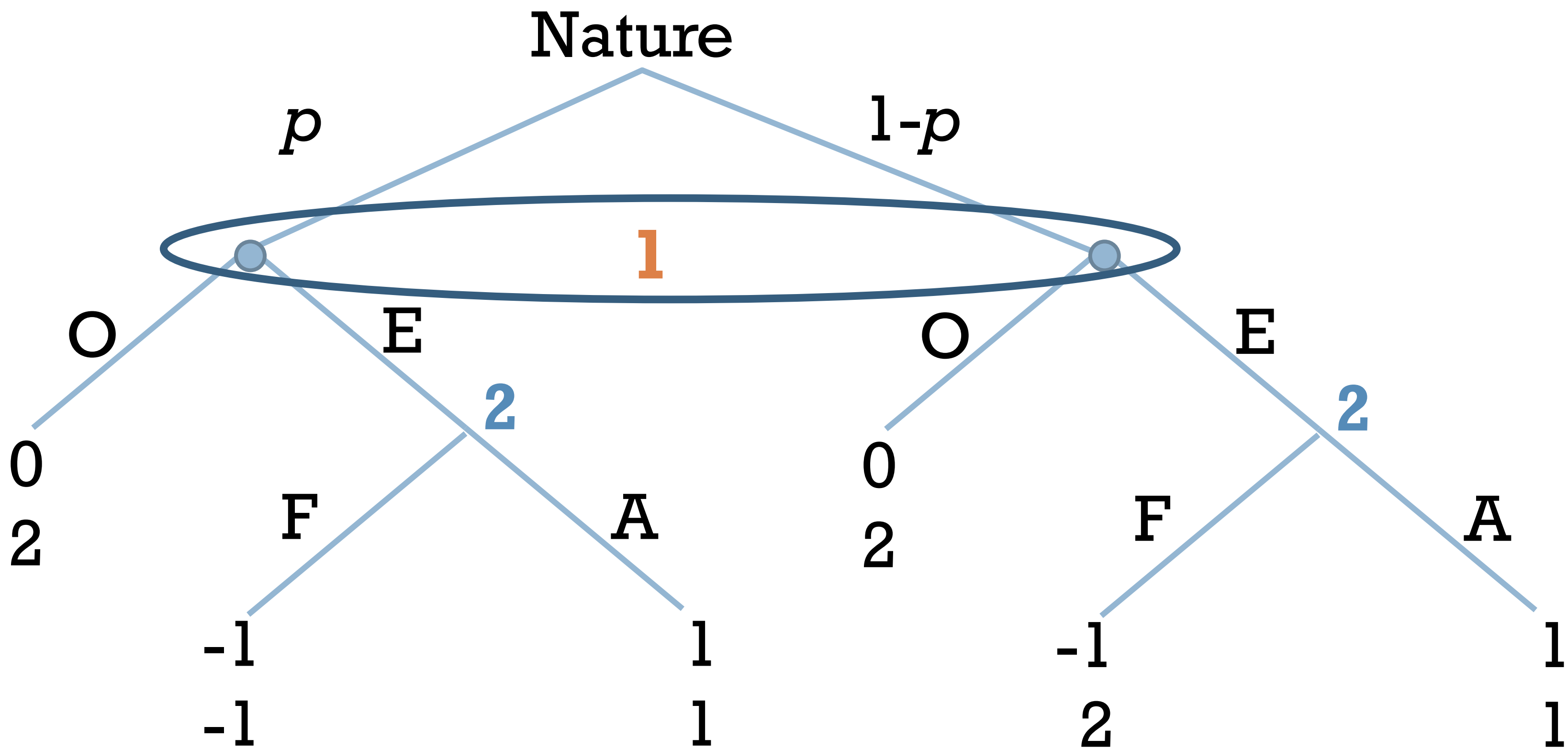
# Example: Entry Game

- Player 1 is a newcomer (e.g., in a market or network); he may (E)nter or stay (O)ut
- Player 2 is incumbent, if 1 enters, 2 may (A)ccept or (F)ight
- SPE outcome is (E,A)
  - (O,F) is a NE, but not SPE



# Entry game, with types (1)

- Player 2 can be Normal (left) or Crazy (right)



# Entry game, with types (1)

- Say that p = ⅔

|  | player 2: AA | AF | FA | FF |
|---|---|---|---|---|
| player 1: O | 0, 2 | 0, 2 | 0, 2 | 0, 2 |
| E | 1, 1 | ⅓, 4⁄3 | -⅓, -⅓ | -1, 0 |

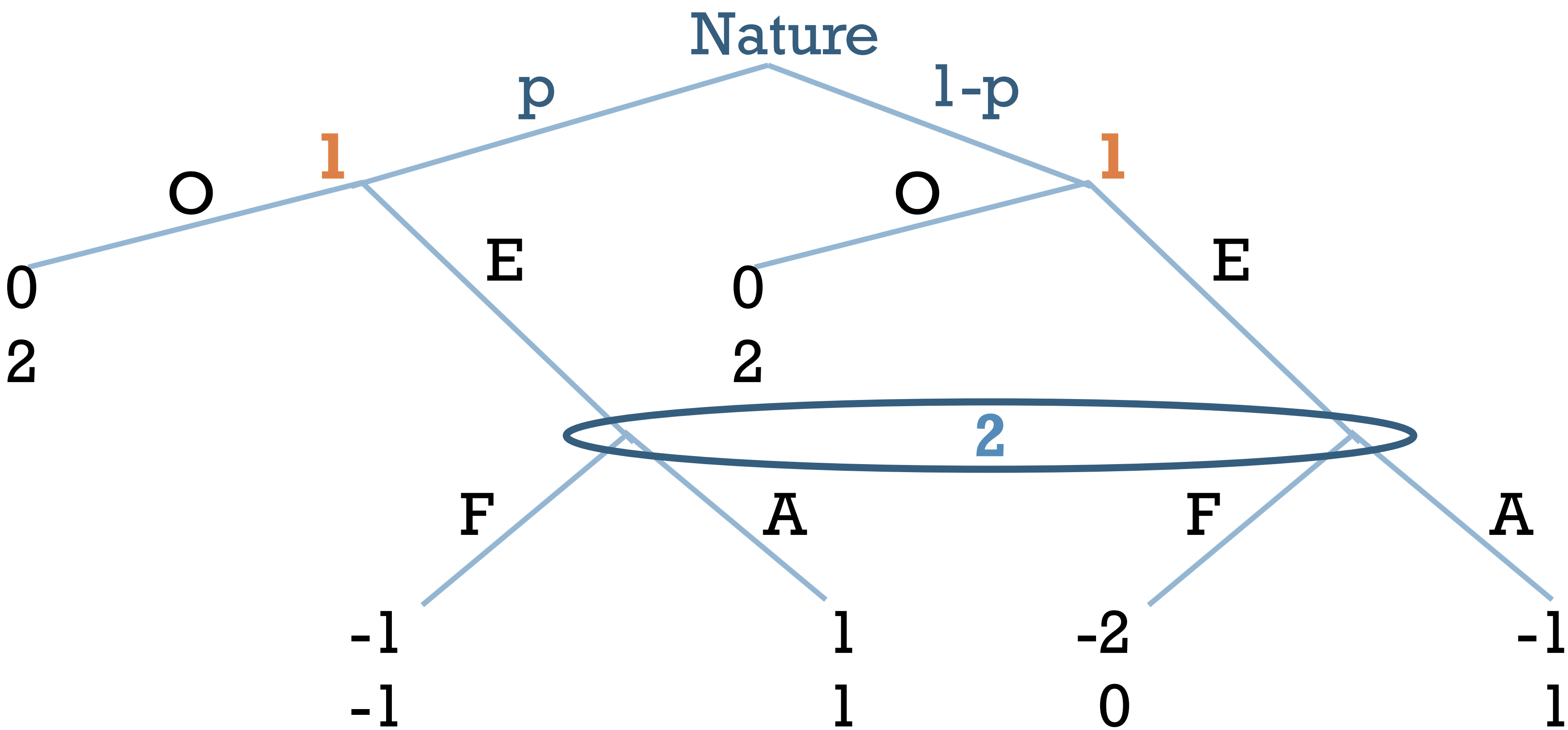
- 3 plain NEs: (O,FA), (O,FF), and (E,AF)
- However, only (E,AF) is an SPE (see why?)
  - If *p* is lower, this can change to (O,AF), meaning: 2 always plays AF, 1 acts based on the prior *p*

# Entry game, with types (2)

- What if the **entrant** (player 1) can have two types with probability $p$ and 1-$p$
  - First type describes the case where situation is as above, with a (**C**)ompetitive entrant
  - Or the entrant can be (**W**)eak, e.g., does not have technologies or plants to compete with the incumbent; in this case, the outsider **1** does not want to enter (always gets negative payoff)
  - In the following, to set numbers, let p=½

# Entry game, with types (2)

- Extensive form



# Strategies of the players

- This time, the situation is reversed
  - player 1 has types, not player 2
  - thus, dynamics must be taken into account, since player 1 is the first to move
- Player 1 has two types and thus 4 strategies: one per each of his types (OO, OE, EO, EE)
- Player 2 does not have types: his strategy is simply to (A)ccept or (F)ight

# Strategies of the players

- Note. We cannot apply backward induction as the last player (no. 2) does not know what to do (types of 1, unknown)
- We can reduce the extensive form to yet another normal (static) form
- This time, we need to computed expected payoffs of the player in every case
  - e.g, (OE,A) gives $\mathbb{E}[v_1]=p-1=-\frac{1}{2}$, while $\mathbb{E}[v_2]=2p+1-p=1+p=\frac{3}{2}$

# Strategies of the players

- We have 2 NEs
  - (OO,F): equilibrium where the incumbent threatens to fight
  - (EO,A): equilibrium where the incumbent accepts but only a competitive outsider enters (a weak one just stays out)

| player 1 \ player 2 | F | A |
|---|---|---|
| OO | 0, 2 | 0, 2 |
| OE | -1, 1 | -½, 3/2 |
| EO | -½, ½ | ½, 3/2 |
| EE | -½, -½ | 0, 1 |

# Sequential rationality

- (OO,F) does have some credibility problems
  - Player 2 always plays F even when it would be more logical to yield (i.e. play A)
  - This equilibrium therefore involves non credible behavior: it is not sequentially rational
- Now, is this a SPE? It surely is a NE
  - The problem is there is only one subgame! (the whole game itself)
  - Thus, this must also be a SPE by definition, although its "perfection" is questionable

# Perfect Bayesian NE

- The problem is that, due to the types of player 1, we are not able to extrapolate cases of player 2 within the information set
- For dynamic games, we had subgames being "on" or "off" the **equilibrium path**
  - Here, everything is "on" because uncertainty about player 1's type merges all the subtrees
  - We need to recover this distinction

# Perfect Bayesian equilibrium

A further extension of the NE concept

# Definitions for Bayesian games

- If we have a Bayesian NE $\boxed{s^*}$, we say that an information set is **on the equilibrium path** if, given the distribution of types, it is reached with probability >0
  - Note that this applies to a **Bayesian NE**
  - And also note that in the BNE given by (OO,F) the information set of node 2 is never reached!(so this ought to be **off** the equilibrium path)

# Definitions for Bayesian games

- In an extensive-form Bayesian game, a **system of belief** $\mu$ is a prob distribution over decision nodes for every information set
  - That is, the probability that when we are in an information set that spans over multiple nodes, we are really at a specific node of the tree
  - It is a conditional probability : prob(node | inf set) → as such, by Bayes' = prob(node)/prob(inf set)
  - In our entry game, the system of belief of player 1 is sure, while that of player 2 depends on the types of player 1 (i.e., its prior of being C or W)

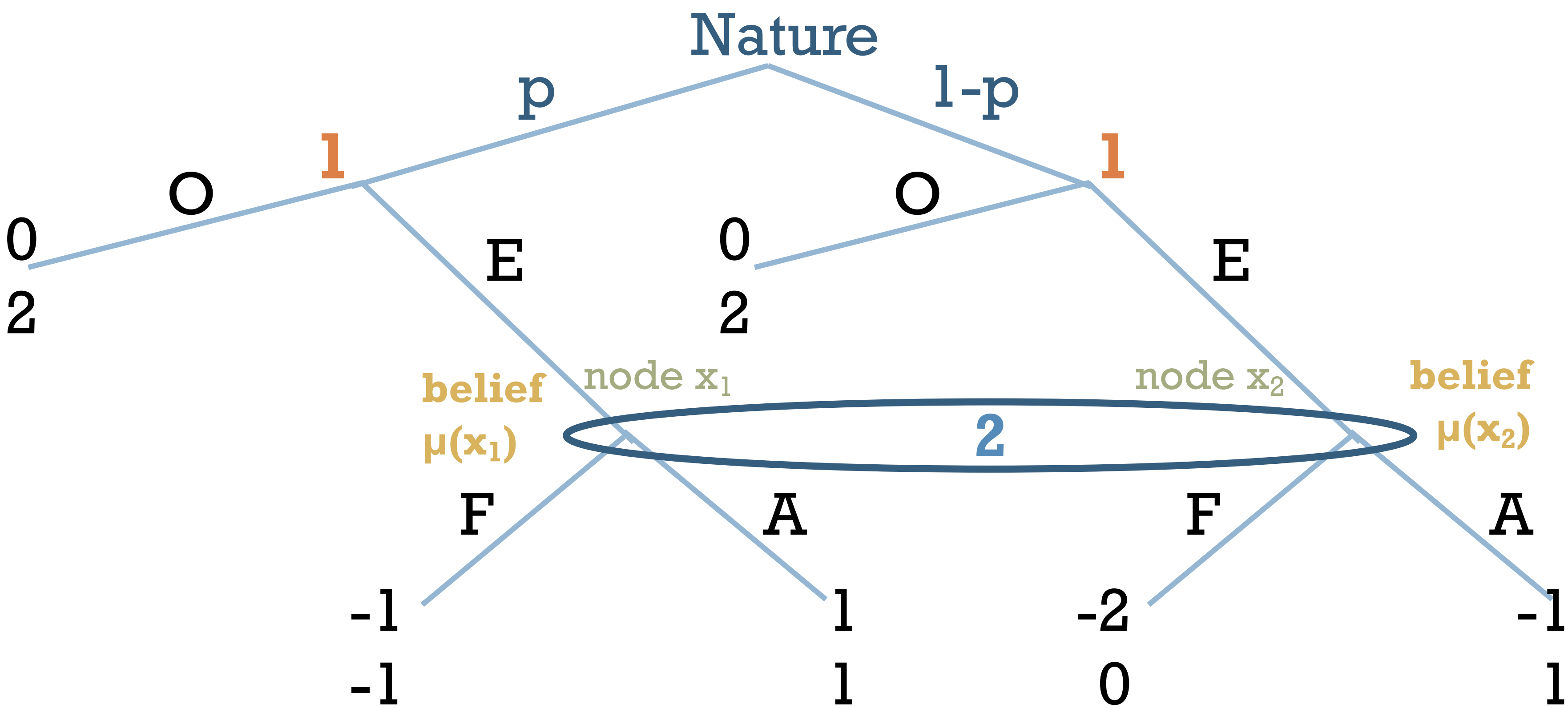
# Seq. rationality requirements

1. Players must have a system of beliefs
2. On the equilibrium path they must follow Bayes' rule on conditional probability
3. Off the equilibrium path: arbitrary
4. Given the beliefs, players are sequentially rational: that is, they play a best response

- A pair ($s^*$, $\mu$) of a BNE $s^*$ and its system of beliefs $\mu$, meeting requirements 1-4 is said to be a **perfect Bayesian equilibrium** (PBE)

# Why PBE works in Entry (2)

- First of all, a PBE is not just a pair of strategies: there must be a system of beliefs associated



# Why PBE works in Entry (2)

- A strategy pair must be **sustained** by a system of beliefs: $\mu(x_1)$ (and $\mu(x_2) = 1-\mu(x_1)$) for player 2
  - e.g.: if 2 believes 1 plays OE, then $\mu(x_1) = 0$
  - this can also work for mixed strategies
  - Bayes' rule must apply to any case where 1 plays a strategy that leads 1 to enter with probabilities $q_C$ and $q_W$ when 1's type is **C** or **W**, resp.

$$\boxed{\mu(x_1) = \frac{p\, q_C}{p\, q_C + (1-p)\, q_W}}$$

# Why PBE works in Entry (2)

- Rational Bayesian NE: $\boxed{(EO,A)}$
  - sustained by system of belief $\mu(x_1) = 1$
  - all players play in a sequentially rational way
- Illogical Bayesian NE: $\boxed{(OO,F)}$
  - Bayes' rule cannot be applied: $q_C = q_W = 0$
  - but whatever $\mu(x_1)$, either $\mu(x_1)$ or $\mu(x_2)$ are >0 thus making the choice of F by 2 to be irrational
- Compare with off-equilibrium choices in SPE!

# Further discussion (optional)

Even PBE can be insufficient, still

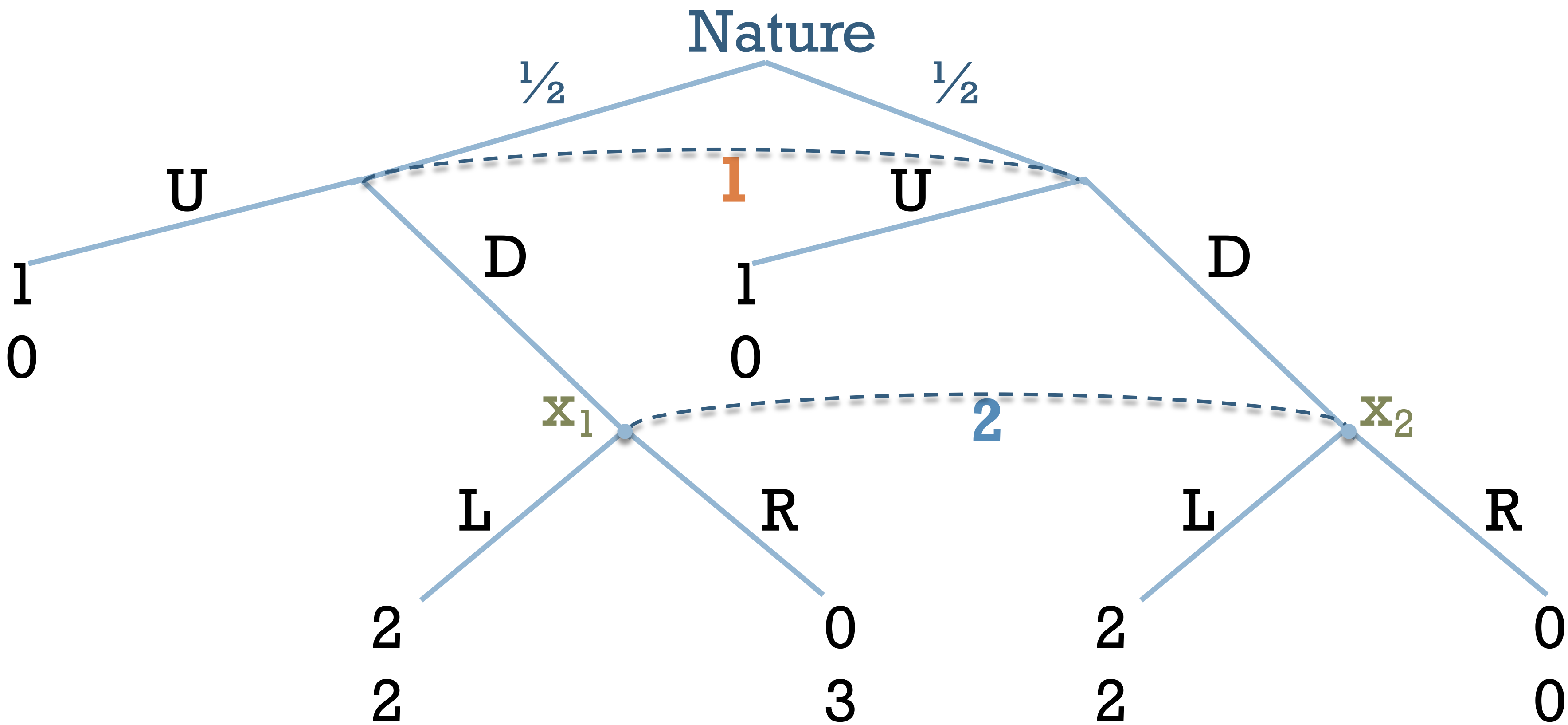
# Is PBE enough?

- Our definition is sometimes called a "weak" PBE because of requirement 3
  - more stringent requirements can be set for information sets off the equilibrium path
- **Theorem**. If $\boxed{s^*}$ is a profile of (possibly mixed) strategies $\boxed{s^* = (s_1^*, s_2^*, \ldots, s_n^*)}$ inducing system of beliefs $\mu$ where every information set is reached with probability>0 → $(s^*, \mu)$ is a PBE

# Weak spots of PBE

- "Refinements" of PBE may be required
- Possible reasons:
  - weak requirement 3
  - requirement 2 does not fully specify the system of beliefs to be consistent to what happens at nodes that are never reached
- This may lead to "odd" PBE where, even with the sequential rationality requirement, the players do not behave "rationally" at all!

# Another entry-like game



# PBE "solution" of this game

- If player 1 plays D with probability >0, then the belief of player 2 is that both $x_1$ and $x_2$ have equal probability
- Thus, best response is to play L → PBE=(D,L)
- However.. if player 1 never plays D, then the system of beliefs at these nodes is arbitrary
  - for example $\mu(x_1)>⅔$ is admissible!
  - then 2's best response is R, to which always playing U is 1's best response → PBE=(U,R)

# Sequential equilibrium

- A better requirement for "solving" the game may then be as follows
- A joint (possibly mixed) strategy $\mathbf{s}^*$ and its associated system of beliefs $\mu$ are said to be **consistent** if they are the limit of a sequence of non-degenerate strategies-beliefs pairs: $(\mathbf{s}^*, \mu) = \lim_{k \to \infty} (\mathbf{s}^*_k, \mu_k)$
- A sequential equilibrium is a consistent PBE
  - i.e., it can be reached through subsequent steps

# Further discussion

- Even sequential equilibrium may be insufficient sometimes!
- Also, it is much harder to check than PBE
- Thus, the "solution concept" to use can be adapted case-by-case
- Also note that these different concepts are useful to characterize human behavior (as well as to argue about human rationality)