Learning from Networks

Graphlets and Motifs

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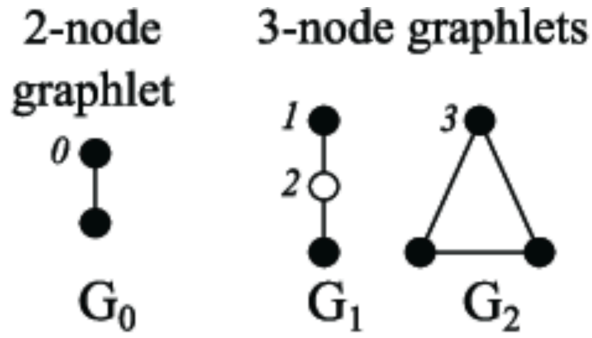
Graphlets

Other features that can be computed for a graph or for a node: the counts (or other scores) of *graphlets*.

Definition

A *graphlet* is a small connected subgraph.

importante:
- dimensione (# nodi)



We are interested in counting the number of times a graphlet appears *exactly* as a subgraph.

appare 5 volte, ma con questa condizione appare 2 volte

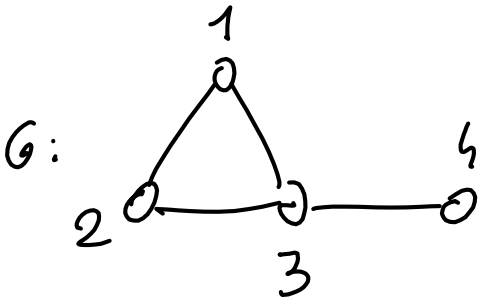
Graphlets (continue)

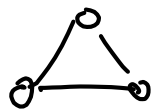
Definition

Given a graph $G = (V, E)$, let $S \subset V$. The *induced subgraph* $G[S] = (S, E')$ is the graph with:

- vertex set S;
- edge set E' with $E' = \{(u, v) \in E : u \in S, v \in S\}$

What does it mean that a graphlet *appears* as an induced subgraph?



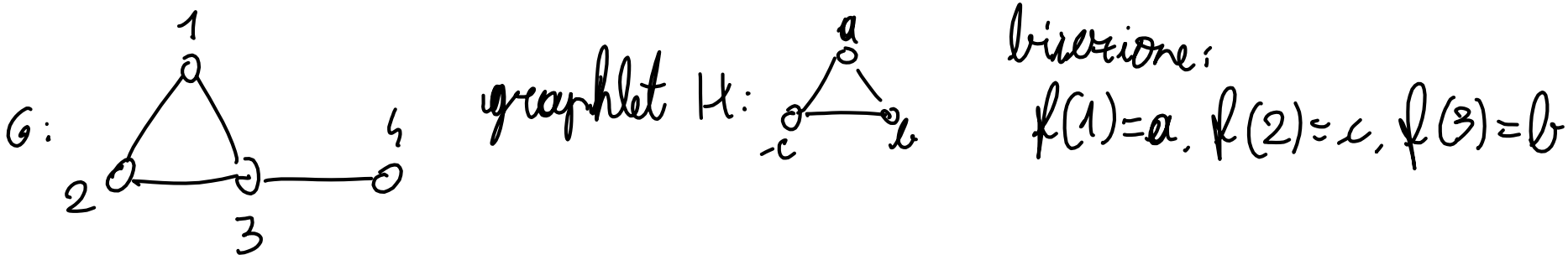
graphlet H:  ⇒ quante volte appare?

in teoria 6 (possibili mapping fra nodi di G e di H), ma possibile anche 1

Graphlets (continue)

Definition

Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are *isomorphic*, denotes as $G \simeq H$ if there exists a *bijection* $f : V_G \to V_H$ such that $(u, v) \in E_G$ if and only if $(f(u), f(v)) \in E_H$.



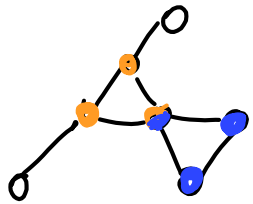
Informally: the count of a graphlet $H = (V_H, E_H)$ in a graph G is the number of subgraphs of G to which H is isomorphic to.

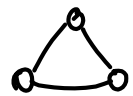
The count/score of a graphlet can be a feature at the *graph level* or at the *node level*.

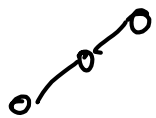
Graphlets: Graph Level

Definition

Given a graphlet $H = (V_H, E_H)$ and a graph $G = (V, E)$, the count $c(H, G)$ of H in G is $c(H, G) = |\{S \subset V : H \simeq G[S]\}|$

G: 

H:  $c(H, G) = 2$

H:  $c(H, G) = 8$

Let's define the corresponding computational problem.

Definition

Given a graphlet $H = (V_H, E_H)$ and a graph $G = (V, E)$, the graphlet counting problem asks to compute $c(H, G)$.

Graphlet Counting: Computational Complexity

Let's define the Subgraph Isomorphism Problem (decision)

Definition

Given a graph H and a graph G, the subgraph isomorphism problem asks to determine whether G contains a subgraph isomorphic to H.

Is it an easy problem? Or a difficult one?

Proposition

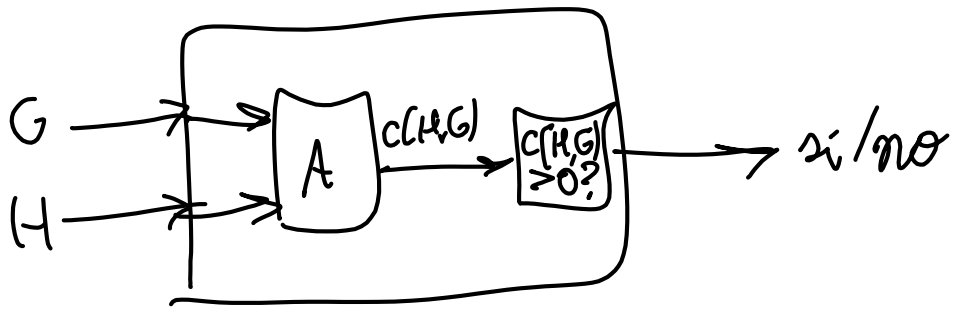
The subgraph isomorphism problem is NP-complete.

Graphlet Counting: Computational Complexity (continue)

Proposition

A polynomial time algorithm for the graphlet counting problem implies a polynomial time algorithm for the subgraph isomorphism problem.

Poniamo alg. A poly-time per graphlet counting =>
=> posso costruire alg. B per subgraph isomorphism



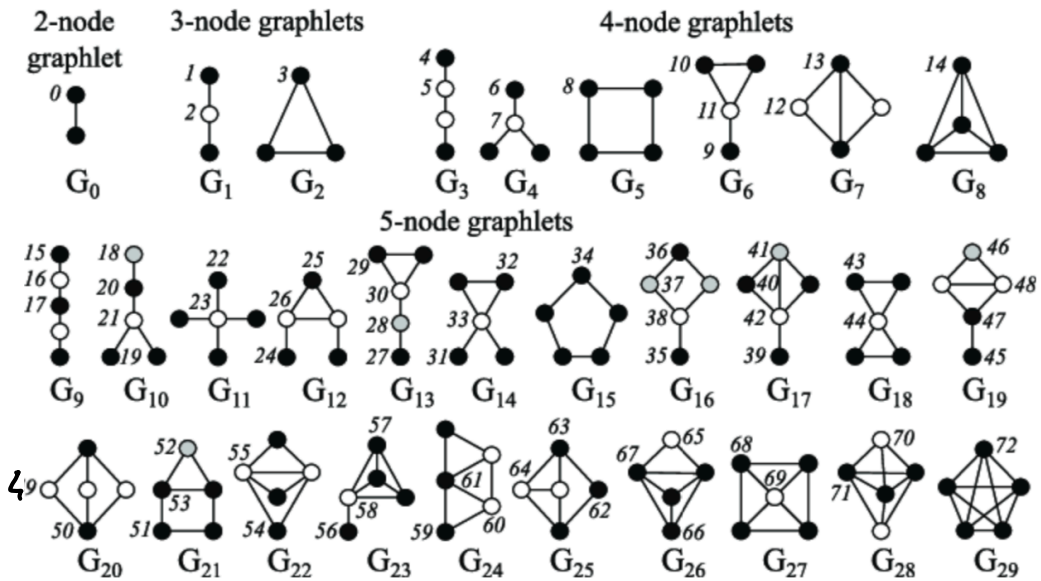
Graphlet Counting: Algorithms

Several algorithms have been developed for the graphlet counting problem.

In practice: often want to solve the following problem

Input: graph G, integer $k \in \mathbb{N}^+$
Output: the counts in G of all graphlets with k nodes

Note: number of graphlets with k nodes increases rapidly with k



numeri sui nodi per capire quante volte contare

Graphlet Counting: Naïve Algorithm

In: graph $G=(V,E)$, $k \in \mathbb{N}^+$

Out: $c(H,G)$ $\forall$ graphlet H con k nodi

1 - $\forall H : c(H,G) = 0$

2 - $\forall S \subset V$, $|S| = k$ do:

$\forall H$ di dim k do:

if $H \simeq G[S]$ then $c(H,G) += 1$

3 - return $c(H,G)$ $\forall H_i$

subgraph isomorphism ⇓ può essere risolto altrimenti efficientemente per k piccoli

Complessità: $\Omega\left(\binom{n}{k}\right) \in \Omega(n^k)$, $n = |V|$

ESU Algorithm

ESU (Exact Subgraph Enumeration) algorithm presented in:
Wernicke, S. (2006) *Efficient detection of network motifs.* IEEE/ACM Transactions on Computational Biology and Bioinformatics.

Input: graph G, integer $k \in \mathbb{N}^+$
Output: the counts in G of all graphlets with k vertices

Works in two phases:

- first phase: enumerate all connected subgraphs with k nodes in G
- second phase: compute the count $c(H, G)$ of each graphlet H with k vertices

An Approximation Algorithm

We are now going to describe an approach to approximate the count of graphlets in a graph

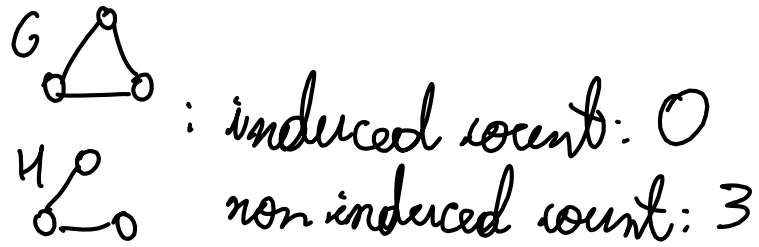
Based on **color-coding** technique: to count subgraphs of size k, randomly color the vertices with k colors, and count only *colorful* subgraphs

Colorful subgraph: every vertex has a different color

Color-coding technique: introduced in "Alon, Noga, Raphael Yuster, and Uri Zwick. Color-coding. Journal of the ACM (JACM) 42.4 (1995): 844-856."

First used for graphlet counting in "Alon, N., Dao, P., Hajirasouliha, I., Hormozdiari, F., Sahinalp, S. C. (2008). Biomolecular network motif counting and discovery by color coding. Bioinformatics, 24(13), i241-i249."

Color-Coding for Subgraph Counting



For simplicity:

- count the number of *non induced* occurrences of H in G
- we present the version of the algorithm that approximate the count of *given* graphlet H
- we look at the details only when the graphlet is a path of length k

Color-Coding for Subgraph Counting

Algorithm `ColorCodingCount`(G, H, t)
Input: graph $G = (V, E)$ with $|V| = n$ and $|E| = m$; graphlet H with k nodes; $t \in \mathbb{N}^+$
Output: estimated count of H in G

non induced

percorso

$count \leftarrow 0$;
forall $i \leftarrow 1$ *to* t **do**
    $G' \leftarrow$ `RandomColor`(G, k);
    $tmp \leftarrow$ `CountColorful`(G', H);
    $count \leftarrow count + tmp$;
return $\frac{1}{t} count \frac{k^k}{k!}$;

Color-Coding for Subgraph Counting

Algorithm `RandomColor`(G, k)
Input: graph $G = (V, E)$ with $|V| = n$ and $|E| = m$; $k \in \mathbb{N}^+$
Output: graph G' with each vertex having a color

$G' \leftarrow (V, E)$;
forall $v \in V$ **do**
    color v in G' with a value (color) chosen uniformly at random in $\{1, \ldots, k\}$;
return G';

Algorithm `CountColorful`(G', H)

Input: graph $G = (V, E)$ with $|V| = n$ and $|E| = m$; graphlet H with k nodes

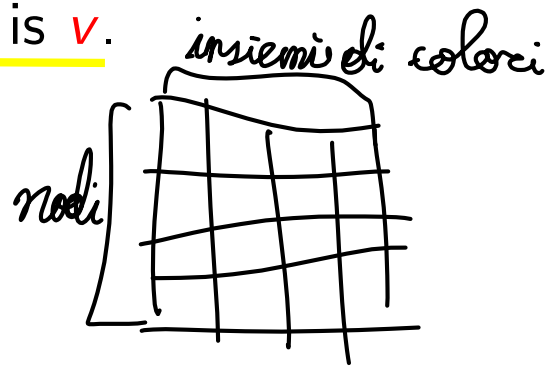
Output: number of colorful occurrences of H in G

The algorithm is based on a *dynamic programming* approach.

We consider the approach when H is a path (with k vertices)

For each vertex v:

- $col(v) =$ the color of v
- for each subset $S \subseteq \{1, ..., k\}$: $c(v, S) =$ number of colorful paths (with colors S) for which one endpoint is v.



Let $\ell \in \{1, \ldots, k\}$:

$$c(v, \ell) = \begin{cases} 1 \text{ if } col(v) = \ell \\ 0 \text{ otherwise} \end{cases}$$

For each vertex v and color set $S \subseteq \{1, \ldots, k\}$ with $|S| > 1$:

$$c(v, S) = \begin{cases} \sum_{u:(u,v)\in E} c(u, S \setminus \{col(v)\}) \text{ if } col(v) \in S \\ 0 \text{ otherwise} \end{cases}$$

Then the number of colorful paths of length k is

$$\frac{1}{2} \sum_{v \in V} c\left(v, \{1, \ldots, , k\}\right)$$

Proposition

Let $\tilde{c}(H, G)$ be the output of `ColorCodingCount`(G, H, t), and $\varepsilon, \delta \in (0, 1)$.

If $t \geq \frac{1}{\varepsilon^2} \frac{1}{\delta} \frac{k^k}{k!}$ then with probability $\geq 1 - \delta$ we have

$$\tilde{c}(H, G) \in [(1 - \varepsilon)c(H, G), (1 + \varepsilon)c(H, G)].$$

Dim: basata su Chebyshev $\Rightarrow$ bound per $\text{var}[\text{temp}] \Rightarrow$
$\Rightarrow \text{Var}[\sum Z_j] \Rightarrow Z_j$ non ind. $\Rightarrow$ bound $\text{Cov}[Z_i, Z_k]$

miglior bound: $t \geq \frac{1}{\epsilon^2} \frac{k^k}{k!} \log \frac{1}{\delta}$

Analysis: Time Complexity

$\forall$ it:

- randomGraph(G,k): $\Theta(n)$, $n=|V|$
- countColorful(G,H): $\forall$ set $S \subseteq \{1,\dots,k\}$, calcolo $\underbrace{\sum_{u \in V_i} c(u, S\setminus\{col(v)\})}_{\text{già calcolato}}$, $\forall v \Rightarrow$

$\Rightarrow \Theta(\sum_{v\in V} deg(v)) \in \Theta(m)$ se G è connesso

complex. totale: $\Theta(2^k m)$

totale per it: $\Theta(2^k m) \Rightarrow$ totale: $\Theta(t 2^k m)$

lower bound dato in Prop: $(=)$

$\Theta\left(2^k m \cdot \frac{1}{\varepsilon^2}\frac{1}{\delta}\frac{k^k}{k!}\right) \Rightarrow$ dato che $\boxed{k! \geq \left(\frac{k}{e}\right)^k}$ (Stirling)

$\Theta\left(\frac{1}{\varepsilon^2}\frac{1}{\delta} \cdot e^k 2^k m\right) \Rightarrow$ sempre $\exp(k)$, ma non è n

$\mathbb{E}\left[\frac{1}{t} \text{count} \frac{k^k}{k!}\right] = c(H,G)$

Dim: X_i = val. di temp in it. i

$Y = \frac{1}{t}\,\text{count}\,\frac{k^k}{k!}$ } v.c.

$Y = \frac{1}{t}\left(\sum_{i=1}^{t} X_i\right)\frac{k^k}{k!} \Rightarrow \mathbb{E}[Y] = \frac{1}{t}\frac{k^k}{k!}\, t\,\mathbb{E}[X_i]$

ordiniamo non-induced occ. di H in G:

$j = 1,2,\dots < c(H,G)$

$Z_j = \begin{cases} 1 & \text{se occ } j \text{ di } H \text{ è colorata in it. } i \\ 0 & \text{else} \end{cases}$

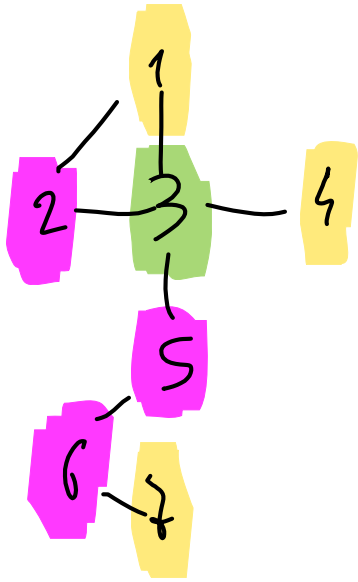
$X_i = \sum_{j=1}^{c(H,G)} Z_j \Rightarrow \mathbb{E}[X_i] = \sum_{j=1}^{c(H,G)} \mathbb{E}[Z_j]$

$\mathbb{E}[Z_j] = \mathbb{P}[Z_j = 1] = \frac{\#\text{colorful}}{\#\text{colorings}} = \frac{k!}{k^k}$

$\mathbb{E}[X_i] = c(H,G)\frac{k!}{k^k} \Rightarrow \mathbb{E}[Y] = \frac{1}{t}\frac{k^k}{k!}\, t\, c(H,G)\frac{k!}{k^k} =$

$= c(H,G)$ ∎

G



H: o—o—o (k=3)

6 occorrenze colorate

v \ S							
1	1			1		1	3
2		1		1	1		3
3			1		2	2	2
4	1					1	2
5		1			1		2
6		1		1			
7	1			1			

totale: 6

$c(1, \square) = c(2, \square) + c(3, \square) = 1$

$l \in \{1, \dots, k\}$

$$c(v,l) = \begin{cases} 1 \text{ se } col(v) = l \\ 0 \text{ else} \end{cases}$$

$$c(v,S) = \begin{cases} \sum_{u \in N} c(u, S \setminus \{col(v)\}) & \text{if } col(v) \in S \\ 0 & \text{else} \end{cases}$$

Color-Coding: Improved Algorithms for Subgraph Counting

All *non-induced* graphlets with small treewidth

- Alon, N., Dao, P., Hajirasouliha, I., Hormozdiari, F., Sahinalp, S. C. Bioinformatics, 2008
- Chakaravarthy, V. T., Kapralov, M., Murali, P., Petrini, F., Que, X., Sabharwal, Y., Schieber, B. IEEE International Parallel and Distributed Processing Symposium, 2016

Larger (7-9 vertices) *induced* graphlets in large graphs:

- M. Bressan, S. Leucci, and A. Panconesi. Proc. VLDB Endowment, 2019

Experimental Evaluation

M. Bressan, S. Leucci, A. Panconesi. MOTIVO: fast motif counting via succinct color coding and adaptive sampling. Proc. VLDB Endow., 2019

"We ran all our experiments on a commodity machine equipped with 64GB of main memory and 48 Intel Xeon E5-2650v4 cores at 2.5GHz with 30MB of L3 cache. We allocated 880GB of secondary storage on a Samsung SSD850 solid-state drive, dedicated to the treelet count tables of MOTIVO."

Table 1: our graphs (* = with biased coloring)

graph	M nodes	M edges	source	k
FACEBOOK	0.1	0.8	MPI-SWS	9
BERKSTAN	0.7	6.6	SNAP	9
AMAZON	0.7	3.5	SNAP	9
DBLP	0.9	3.4	SNAP	9
ORKUT	3.1	117.2	MPI-SWS	7
LIVEJOURNAL	5.4	49.5	LAW	8
YELP	7.2	26.1	YLP	8
TWITTER	41.7	1202.5	LAW	6 (7*)
FRIENDSTER	65.6	1806.1	SNAP	6 (7*)

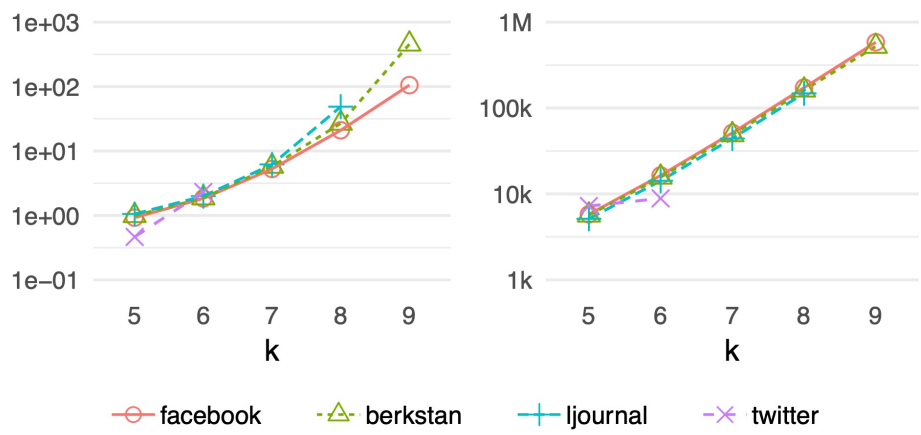


Figure 6: MOTIVO's build-up time (seconds per million edge) and space usage (bits per input node).

Motifs

Definition

A *motif* for a graph G is a graphlet that is *significantly overrepresented* in G.

How do we measure if a graphlet is *significantly overrepresented*?

Comparison with appropriate random graphs:

- large z-score
- small p-value

Motifs (continue)

Sometimes the *normalized* z-scores of all graphlets of size up to k are used as graph-level features.

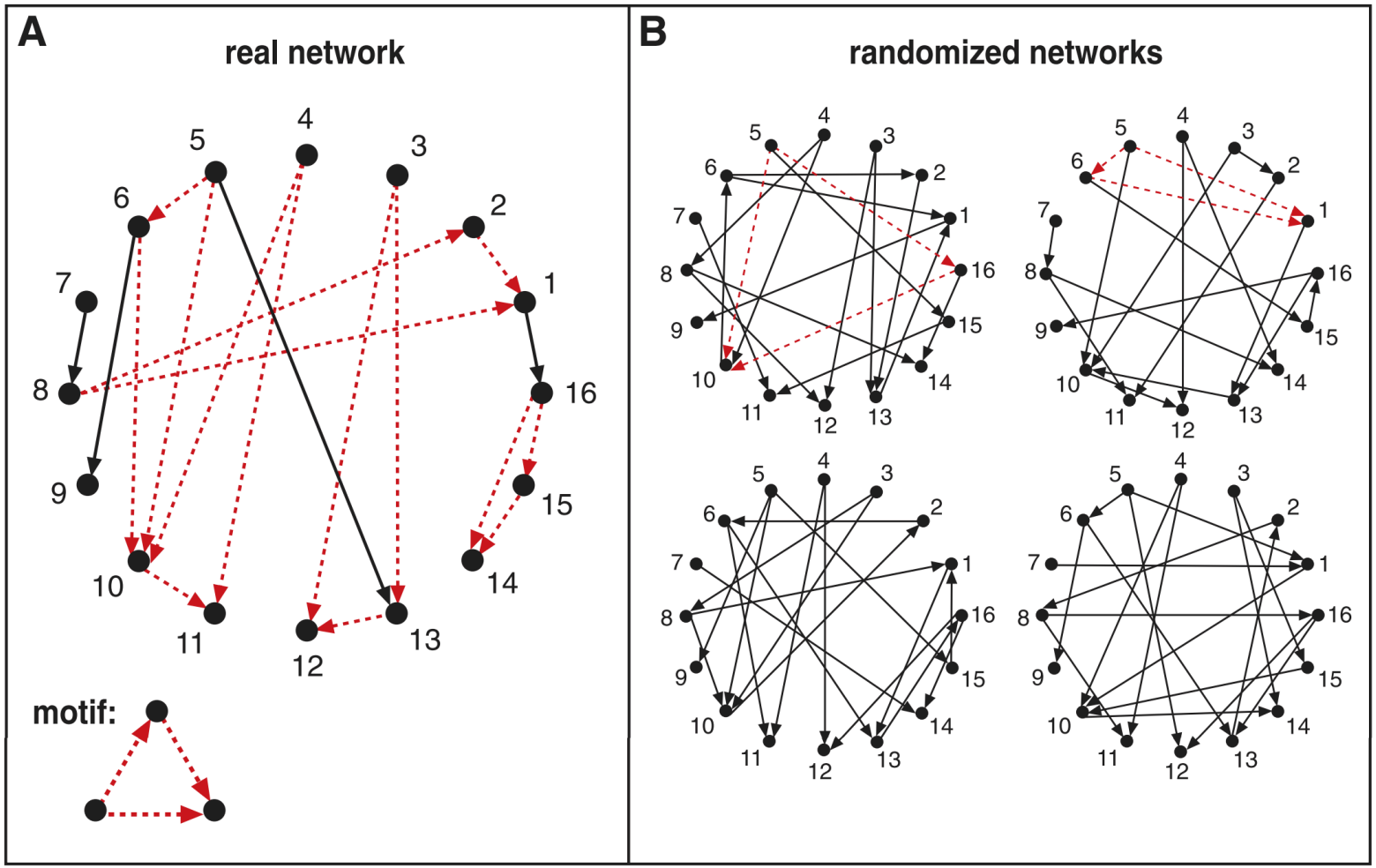
Definition

Given a graph G and $k \in \mathbb{N}^+$, the *network significance profile* of G is the vector of the normalized z-scores of the graphlets of size $\leq k$ in G. If $\mathcal{G}(k)$ is the set of graphlets of size $\leq k$ and $z_G(H)$ is the z-score for graphlet H in G, then the *normalized z-score* of H in G is

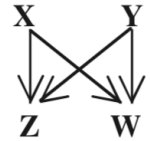
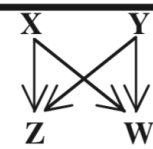
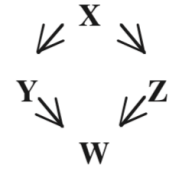
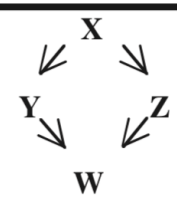
$$\frac{z_G(H)}{\sum_{H' \in \mathcal{G}(k)} z_G(H')}$$

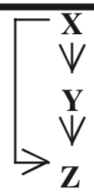
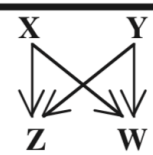
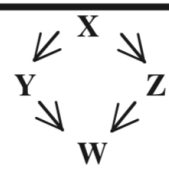
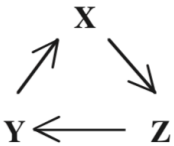
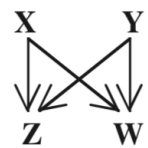
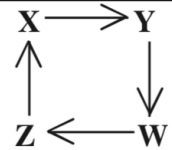
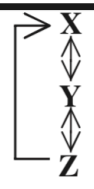
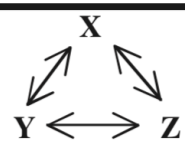
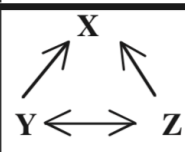
Motifs: Example

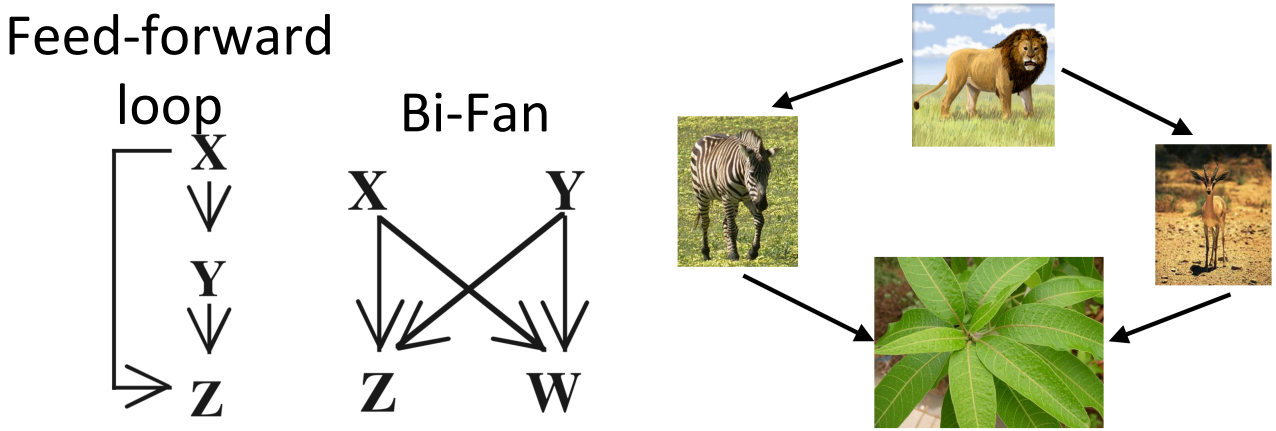
Milo et al. (2002) *Network Motifs: Simple Building Blocks of Complex Networks*. Science.



Random graph model: each node has same indegree and same outdegree.

Network	Nodes	Edges	N_{real}	$N_{rand} \pm SD$	Z score	N_{real}	$N_{rand} \pm SD$	Z score	N_{real}	$N_{rand} \pm SD$	Z score
Gene regulation (transcription)					**Feed-forward loop**			**Bi-fan**			
E. coli	424	519	40	7 ± 3	10	203	47 ± 12	13			
*S. cerevisiae**	685	1,052	70	11 ± 4	14	1812	300 ± 40	41			
Neurons					**Feed-forward loop**			**Bi-fan**			**Bi-parallel**
C. elegans†	252	509	125	90 ± 10	3.7	127	55 ± 13	5.3	227	35 ± 10	20
Food webs					**Three chain**			**Bi-parallel**			
Little Rock	92	984	3219	3120 ± 50	2.1	7295	2220 ± 210	25			
Ythan	83	391	1182	1020 ± 20	7.2	1357	230 ± 50	23			
St. Martin	42	205	469	450 ± 10	NS	382	130 ± 20	12			
Chesapeake	31	67	80	82 ± 4	NS	26	5 ± 2	8			
Coachella	29	243	279	235 ± 12	3.6	181	80 ± 20	5			
Skipwith	25	189	184	150 ± 7	5.5	397	80 ± 25	13			
B. Brook	25	104	181	130 ± 7	7.4	267	30 ± 7	32			

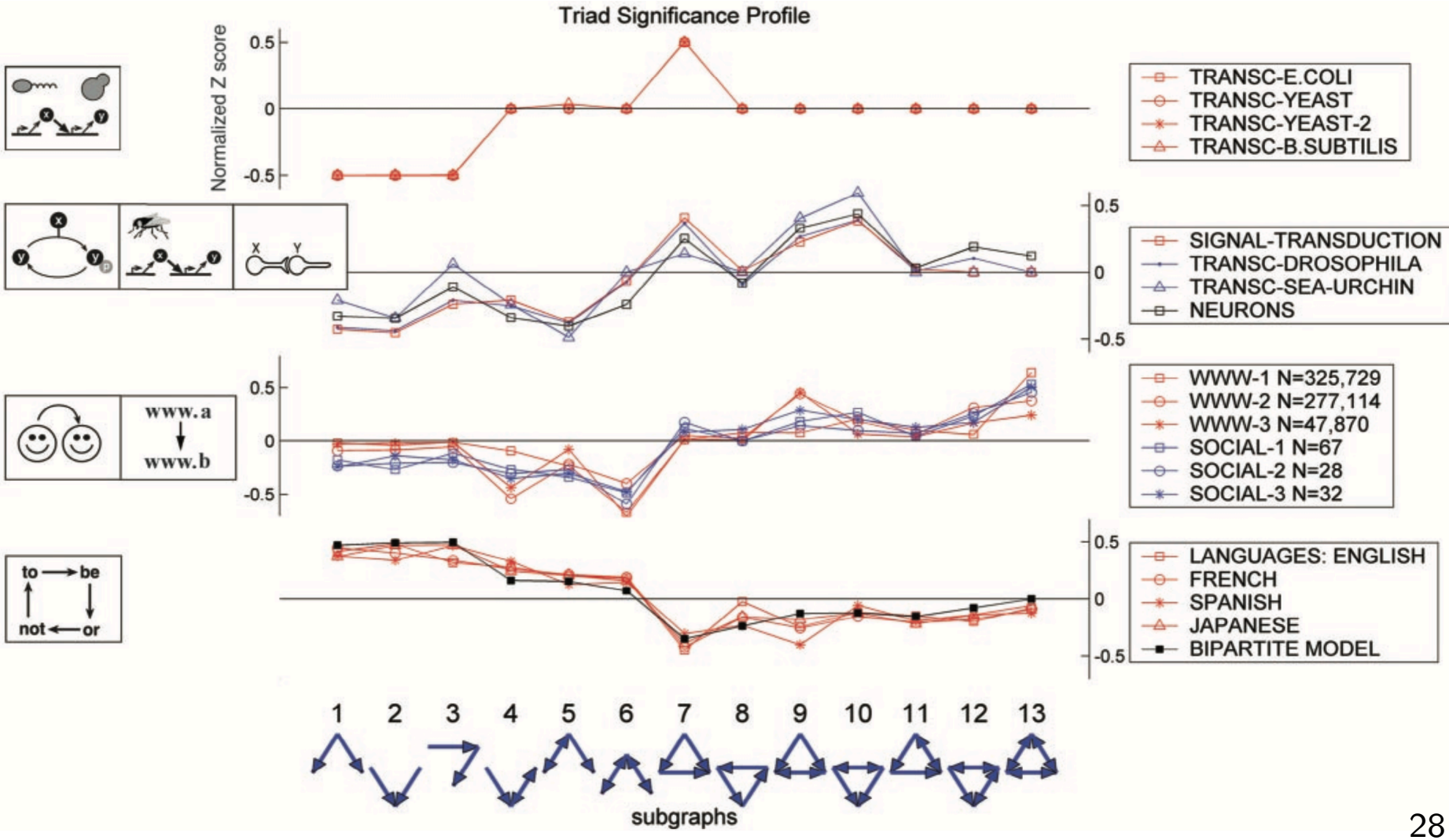
Electronic circuits (forward logic chips)					**Feed-forward loop**			**Bi-fan**			**Bi-parallel**
s15850	10,383	14,240	424	2 ± 2	285	1040	1 ± 1	1200	480	2 ± 1	335
s38584	20,717	34,204	413	10 ± 3	120	1739	6 ± 2	800	711	9 ± 2	320
s38417	23,843	33,661	612	3 ± 2	400	2404	1 ± 1	2550	531	2 ± 2	340
s9234	5,844	8,197	211	2 ± 1	140	754	1 ± 1	1050	209	1 ± 1	200
s13207	8,651	11,831	403	2 ± 1	225	4445	1 ± 1	4950	264	2 ± 1	200
Electronic circuits (digital fractional multipliers)					**Three-node feedback loop**			**Bi-fan**			**Four-node feedback loop**
s208	122	189	10	1 ± 1	9	4	1 ± 1	3.8	5	1 ± 1	5
s420	252	399	20	1 ± 1	18	10	1 ± 1	10	11	1 ± 1	11
s838‡	512	819	40	1 ± 1	38	22	1 ± 1	20	23	1 ± 1	25
World Wide Web					**Feedback with two mutual dyads**			**Fully connected triad**			**Uplinked mutual dyad**
nd.edu§	325,729	1.46e6	1.1e5	2e3 ± 1e2	800	6.8e6	5e4±4e2	15,000	1.2e6	1e4 ± 2e2	5000

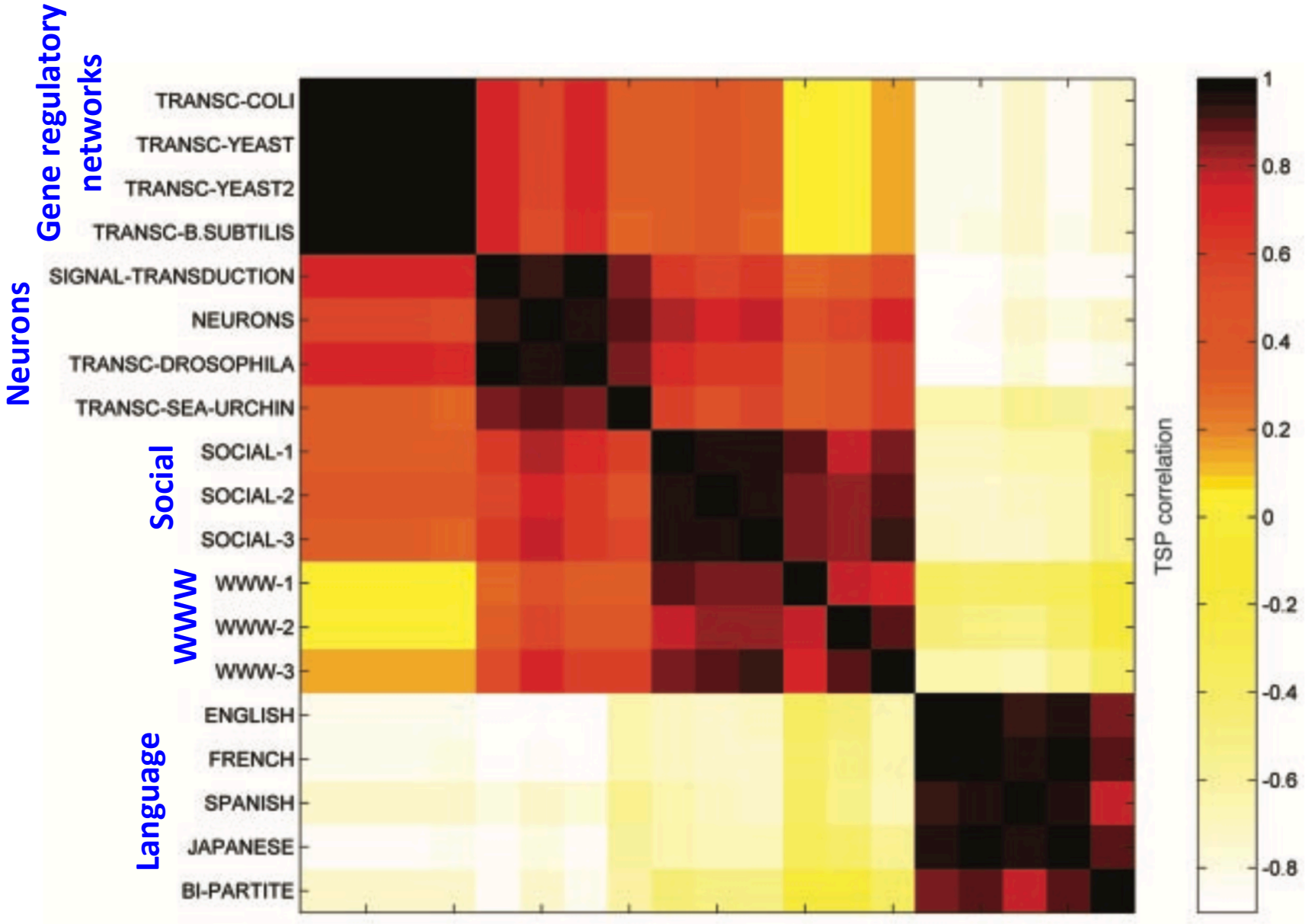


- networks of neurons and gene networks contain similar motifs:
  - feed-forward loops and bi-fan structures
  - both are information processing networks with sensory and acting components
- food webs have parallel loops: prey of a particular predator share prey
- WWW network has bidirectional links: design that allows the shortest path between sets of related pages

Motifs: Example

Milo et al. (2004) *Superfamilies of Evolved and Designed Networks*. Science.



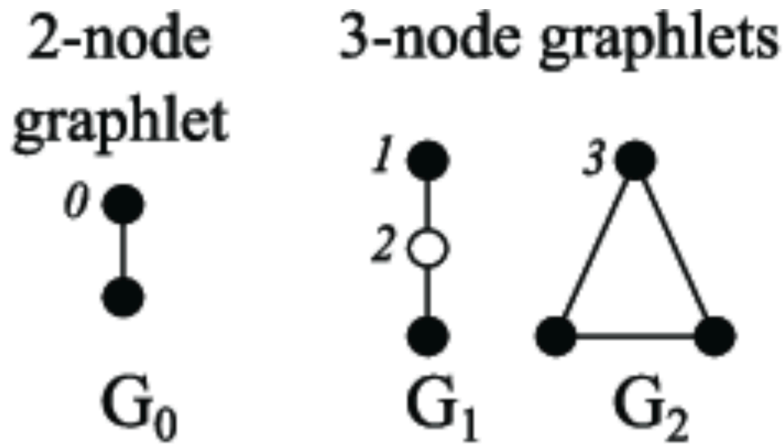

Gene regulatory networks
Neurons
Social
WWW
Language
TRANSC-COLI
TRANSC-YEAST
TRANSC-YEAST2
TRANSC-B.SUBTILIS
SIGNAL-TRANSDUCTION
NEURONS
TRANSC-DROSOPHILA
TRANSC-SEA-URCHIN
SOCIAL-1
SOCIAL-2
SOCIAL-3
WWW-1
WWW-2
WWW-3
ENGLISH
FRENCH
SPANISH
JAPANESE
BI-PARTITE
TSP correlation
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Graphlets and Motifs: Node Level

Graphlets and motifs can be used to define node level features as well.

Definition

Given a graphlet H, its **orbits** are the different positions in which a node can appear (up to automorphism).

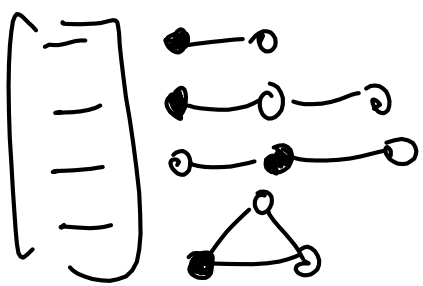


Graphlets and Motifs: Node Level (continue)

Definition

Given a graph $G = (V, E)$ $k \in \mathbb{N}^+$, and a node $v \in V$ the *graphlet degree vector* $GDV(v, G)$ of v in G is the vector where each component is the number of occurrences of v in a given orbit of a given graphlet of size $\leq k$.

One can define a similar vector based on *motifs* (i.e., z-scores) instead.



1 componente per orbita per graphlet

Algorithm

How can we obtain the $GDV(v, G)$ for a node v?

Simple algorithm:

1. based on the graphlet size k, compute the maximum distance r between v and a node in a graphlet H of size $\leq k$
2. collect the subgraph G' of G containing v and all its neighbours up to distance r
3. use the first phase of the ESU algorithm to enumerate all appearances of graphlets in G'
4. compute $GDV(v, G)$

Extensions

There are several extensions of the problems we discussed, for example

- Graphlet/motif definition:
  - directed
  - node colors
  - temporal networks
  - ...
- Variants of graphlet *scores*:
  - different concepts of *appearing as a subgraph*
  - different measures for significance
  - ...
- ...

Open Problems

There are several open problems that are the focus of recent research:

- efficient sampling algorithms with guarantees for graphlets
- efficient algorithms to compute p-values for motifs
- efficient algorithms for large values of k
- ...