Game theory

a course for the
MSc in ICT for Internet and multimedia

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No Nash Equilibrium?

- In general, finding Nash Equilibria is tricky
- Sometimes, we do not seem to have one
  - For example, in the Odds&Evens game…

		Even 0	Even 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

which feels somehow incomplete (especially if we want to use Nash Equilibrium as solution/prediction)

Mixed strategies

Uncertainty makes the games interesting

Missing outcome

- Expand Odds&Evens to find the outcome

forniamo NE

		Even		
		0	½	1
Odd	0	-4, 4	0, 0	4, -4
	½	0, 0	0, 0	0, 0
	1	4, -4	0, 0	-4, 4

scelta intermedia

- It seems that (½ ,½) is a NE. Let formalize this.

Mixed strategies

- If A is a non-empty discrete set, a **probability distribution** over A is a function $p : A \rightarrow [0,1]$, such that $\sum_{x \in A} p(x) = 1$
- The set of possible probability distributions over A is called the **simplex** and denoted as ΔA
- For a normal form game $(S_1,\ldots,S_n ; u_1,\ldots,u_n)$, a **mixed strategy** for player i is a probability distribution m_i over set S_i
- That is, i chooses strategies in $S_i = (s_{i,1},\ldots,s_{i,n})$ with probabilities $(m_i(s_{i,1}),\ldots,\ m_i(s_{i,n})\ )$

Expected payoff

- Utility u_i can be extended to the expected utility, which is a real function over $\Delta S_1 \times \Delta S_2 \times \ldots \times \Delta S_n$
- If players choose mixed strategies $(m_1,\ldots,m_n)$, $=: s$ compute player i 's payoff by weighing on m_i's

$$u_i(m_1,\ldots,m_n) = \sum_{s \in S} m_1(s_1) \cdot m_2(s_2) \cdot \ldots \cdot m_n(s_n) \cdot u_i(s)$$

- In other words:
  - fix a global strategy s
  - compute its probability
  - weigh the utility of s on this probability and sum

Intuition

- Consider Odds&Evens game and assume Odd decides to play 0 with probability q, while Even plays 0 with probability r
  - Consequently 1 is played by Odd and Even with probability $1-q$ and $1-r$, respectively

		Even: 0 (prob r)	Even: 1 (prob $1-r$)
Odd	0 (prob q)	$-4qr, 4qr$	$4q(1-r), -4q(1-r)$
	1 (prob $1-q$)	$4(1-q)r, -4(1-q)r$	$-4(1-q)(1-r), 4(1-q)(1-r)$

this is a **single** global strategy $\boxed{m = (m_1, m_2) = (q, r)}$

Intuition

- In other words, we revise the game so that each player can choose not only either 0 or 1, but also a value between them: q for Odd, r for Even
- Odd's payoff is $-16qr + 8q + 8r - 4 = -4(2q-1)(2r-1)$

		Even: 0 (prob r)	Even: 1 (prob $1-r$)
Odd	0 (prob q)	$-4qr, 4qr$	$4q(1-r), -4q(1-r)$
	1 (prob $1-q$)	$4(1-q)r, -4(1-q)r$	$-4(1-q)(1-r), 4(1-q)(1-r)$

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		Even		
		0	r	1
Odd	0			
	q		$-16qr + 8q + 8r - 4$, $16qr - 8q - 8r + 4$	
	1			

Pure strategies

- Given a mixed strategy $m_i \in \Delta S_i$ we define the **support** of m_i as $\{s_i \in S_i : m_i(s_i) > 0\}$
- Each strategy $s_i \in S_i$ (an element of S_i) can be identified with the mixed strategy p (which is an element of ΔS_i) such that $p(s_i) = 1$
  - Hence, $p(s'_i) = 0$ if $s'_i \neq s_i$ and also support(p)=$\{s_i\}$
- Thereafter, we identify p with s_i: Pure strategy s_i is seen as a degenerate probability distribution
  - Previous definitions of dominance and NE only refer to the pure strategy case

Strict/weak dominance

- Consider game $G = \{S_1,\dots,S_n ; u_1,\dots,u_n\}$.
- If $m_i', m_i \in \Delta S_i$, m_i' **strictly dominates** m_i if
  $u_i(m_i', m_{-i}) > u_i(m_i, m_{-i})$ for every m_{-i}
- We say that m_i' **weakly dominates** m_i if
  $u_i(m_i', m_{-i}) \geq u_i(m_i, m_{-i})$ for **every** m_{-i}
  $u_i(m_i', m_{-i}) > u_i(m_i, m_{-i})$ for **some** m_{-i}
- Note: there are infinitely (and continuously) many m_{-i} in the set: $\Delta S_1 \times \dots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \dots \times \Delta S_n$

Strict/weak dominance

visto che considero solo i, posso togliere $m_i(s_i)$ dal sommatorio di u_i

- However, it is possible to prove that:

  If m_i', $m_i \in \Delta S_i$, m_i' **strictly dominates** m_i if

  $$u_i(m_i', s_{-i}) > u_i(m_i, s_{-i}) \qquad \text{for every } s_{-i} \in S_{-i}$$

- Similarly, m_i' **weakly dominates** m_i if

  $$u_i(m_i', s_{-i}) \geq u_i(m_i, s_{-i}) \qquad \text{for every } s_{-i} \in S_{-i}$$

  $$u_i(m_i', s_{-i}) > u_i(m_i, s_{-i}) \qquad \text{for some } s_{-i} \in S_{-i}$$

- That is, we can limit our search to pure strategies of the opponents.

Nash equilibrium

- Consider game $G = \{S_1,\ldots,S_n\,;\, u_1,\ldots,u_n\}$.
- A joint mixed strategy $m \in \Delta S_1 \times \ldots \times \Delta S_n$ is said to be a Nash equilibrium if for all i :

$$u_i(m) \geq u_i(m_i', m_{-i}) \text{ for every } m_i' \in \Delta S_i$$

- This reprise the same concept of NE in pure strategies: no player has an incentive to change his/her move (which is a mixed strategy now)

back to Example 3

- In the Odds&Evens game, the payoff for Odd is $-4(2q - 1)(2r - 1)$, the opposite for Even.
- If $q = ½$, or $r = ½$, **both** players have payoff 0.
- If $q = r = ½$ no player has incentive to change.

Even

Odd	0	½	1
0		0, 0 0, 0	
½	0, 0 0, 0	0, 0	0, 0 0, 0
1		0, 0 0, 0	

Nash equilibrium

back to Example 3

- As an exercise, prove that (½ , ½) is the **only** Nash Equilibrium of the Odds&Evens game
- How to proceed
  - Consider three cases, where the payoff of player Odd is <0, >0, =0 but joint strategy is not (½ , ½)
  - Show that in each case there is a player (who?) having an incentive in changing strategy
  - None of this is a NE. (½ , ½) is the only one

Using mixed strategies

and introducing the Nash theorem

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IESDS vs mixed strategies

player B

player A	L	C	R
T	7, 4	5, 0	8, 1
D	6, 0	3, 4	9, 1

- R is not dominated by L or C. But mixed strategy $\boxed{m = ½\,L + ½\,C}$ gets $u_B = 2$ regardless of A's move
- Pure strategy R is strictly dominated by *m*
  - R can be eliminated
  - Further eliminations are possible

IESDS vs mixed strategies

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Nash equilibrium

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IESDS vs mixed strategies

- Similar theorems to the pure strategy case hold for IESDS in mixed strategies $\boxed{\text{(IESDSm)}}$.
- **Theorem.** Nash equilibria survive IESDSm.
- **Theorem.** The order of IESDSm is irrelevant.

- **Note:** Use strict (not weak) dominance!
  A weakly dominated strategy can be a NE, or belong to the support of a NE

Characterization

- **Theorem**. Take a game $\boxed{G = \{S_1,\ldots,S_n\,;\,u_1,\ldots,u_n\}}$ and a joint mixed strategy $\boxed{m}$ for game G
  The following statements are equivalent:

(1) Joint mixed strategy m is a Nash equilibrium

(2) For each i:

$$\boxed{\begin{array}{l} u_i(m) = u_i(s_i, m_{-i}) \text{ for every } s_i \in \text{support}(m_i) \\ u_i(m) \geq u_i(s_i, m_{-i}) \text{ for every } s_i \notin \text{support}(m_i) \end{array}}$$

- **Corollary**. If a pure strategy is a NE, it is such also as a mixed strategy

back to Example 5

		Brian R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- This game had two pure NEs: (R,R) and (S,S)
- We show now that there is also a mixed NE
- Ann (or Brian) plays R with probabilities q (or r)
- A mixed strategy is uniquely identified by (q,r)
  - Ann's payoff is $\boxed{u_A(q,r) = 2qr + (1-q)(1-r)}$
  - Brian's is $\boxed{u_B(q,r) = qr + 2(1-q)(1-r)}$

back to Example 5

- Assume (a,b) is a mixed NE.
  - Note: support (a) = support (b) = {R,S}. Pure strategies R/S correspond with q (or r) being 0/1
- Due to the Theorem, $\boxed{u_A(a,b) = u_A(0,b) = u_A(1,b)}$
- Now, use: $u_A(q,r) = 2qr + (1-q)(1-r)$
- $\boxed{2ab + (1-a)(1-b) = 1-b = 2b}$
- Solution: $\boxed{b = 1/3}$
- Similarly, $\boxed{u_B(a,0) = u_B(a,1)}$
- Solution: $\boxed{a = 2/3}$

Nash theorem (intro)

- The reasoning we used to find the third (mixed) NE of the Battle of Sexes is more general
- Every two-player games with two strategies has a NE in mixed strategies
- This is easy to prove and is part of the more general Nash theorem
- **Theorem** (Nash 1950). Every game with finite S_i's has at least one Nash equilibrium (possibly involving mixed strategies)

teorema di esistenza ⇒ non dice come trovare NE

Understanding mixed strategy

- Mixed strategies are key for Nash Theorem
  - What does "mixed strategies as probabilities" mean?
  - In the end, players take pure strategies.
- Possible interpretations
  - Large numbers: If the game is played M times, mixed strategy q = to choose a pure strategy qM times (note: each of the M times is one-shot memoryless)
  - Fuzzy values: Unsure actions: players do not know
  - **Beliefs**: The probability q reflects the uncertainty that my opponent has about my choice (which is pure)

Beliefs

- A **belief** of player i is a possible profile of opponents' strategies: an element of set ΔS_{-i}
  - Same definition of pure strategies (but here ↑)
- As before, a best-response-correspondence BR: $\Delta S_{-i} \rightarrow \wp(\Delta S_i)$ associates to $m_{-i} \in \Delta S_{-i}$ a subset of ΔS_i such that each $m_i \in \mathrm{BR}(m_{-i})$ is a best response to m_{-i}
  - Also, best responses are still not unique

NE as best responses

- Using beliefs, we can speak of **best response** to an opponent's (mixed) strategy
  - Intuition

		Bea: F	Bea: G
Art	U	6, 1	0, 4
Art	D	2, 5	4, 0

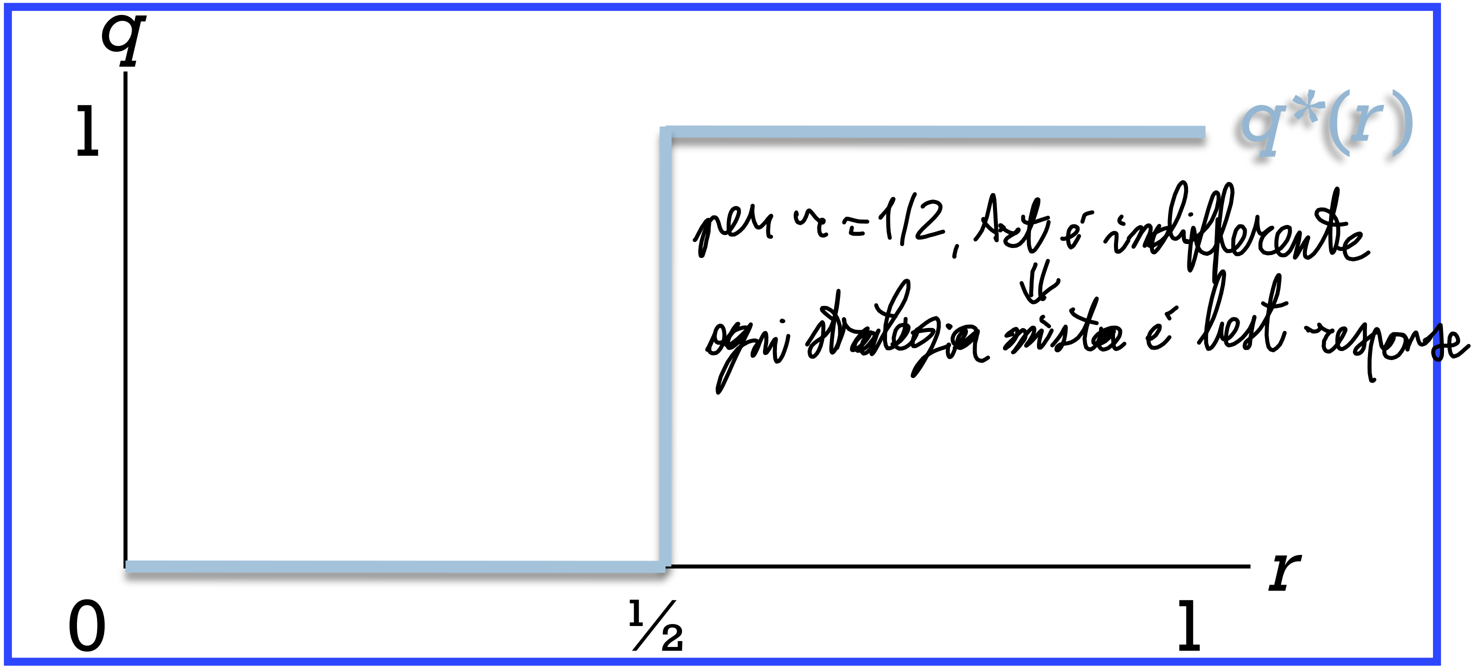
  - Bea ignores what Art will play
  - So she assumes he will play U with probability q
  - And, Art thinks Bea will play F with probability r

NE as best responses

Art \ Bea	F	G
U	6, 1	0, 4
D	2, 5	4, 0

- E.g., if Bea is known for always playing F ($r=1$), Art's best response is to play U ($q=1$). In general?
- It holds: $\boxed{u_A(\mathrm{D},r) = 2r + 4(1-r),\ u_A(\mathrm{U},r) = 6r}$
- U is actually Art's best response as long as $r > ½$, else it is D. If $r = ½$ they are equivalent
- Denote Art's best response with $\boxed{q^*(r)}$

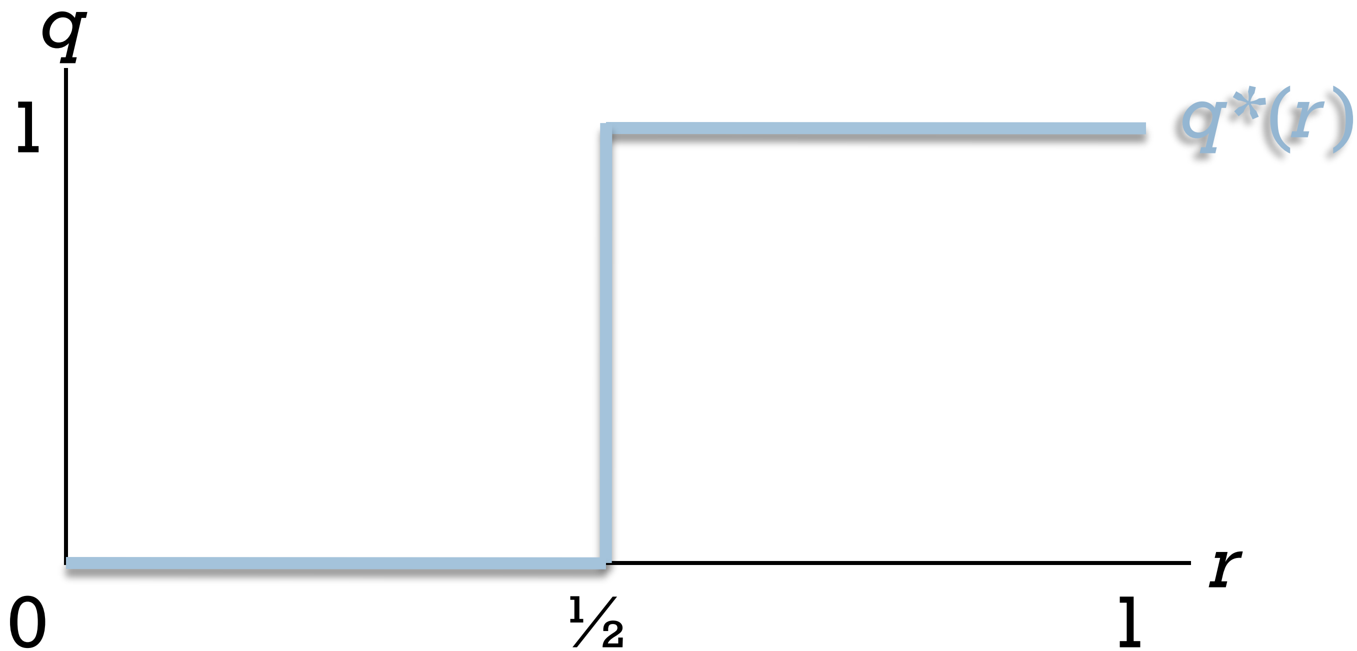
NE as best responses



- Art's best response is either U or D means that $q^*(r) = 1, 0$, respectively; then, $q^*(r)$ is step-wise

$$\boxed{q^*(r) = 0 \text{ if } r < ½ \ , \qquad q^*(r) = 1 \text{ if } r > ½}$$

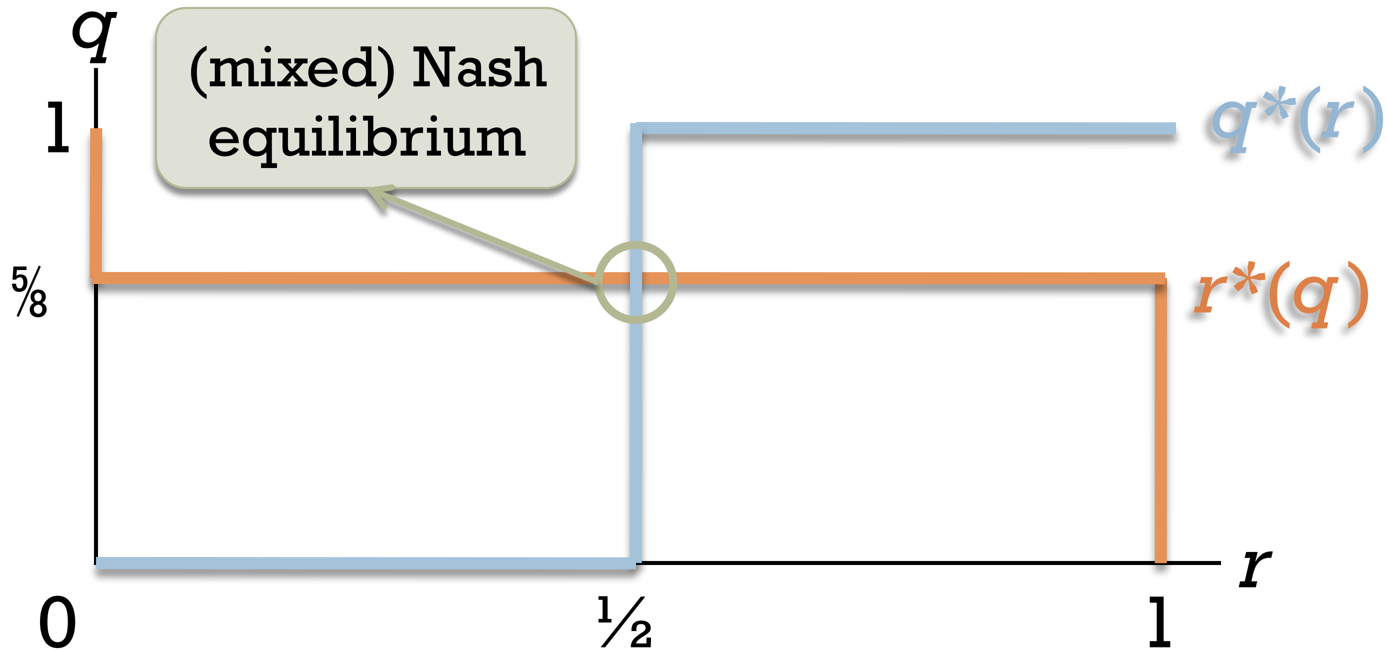
NE as best responses



- For Bea: $\boxed{u_B\,(q,\mathrm{F}) = q + 5(1\text{-}q),\, u_B(q,\mathrm{G}) = 4q}$
- Thus, Bea's best response $\boxed{r^*(q)}$ is step-wise

$$\boxed{r^*(q) = 1 \text{ if } q < 5/8 \,, \qquad r^*(q) = 0 \text{ if } q > 5/8}$$

NE as best responses



- Joint strategy m = (q = ½, r = ⅝) is a NE.
- NE are points were the choice of each player is the best response to the other player's choice.

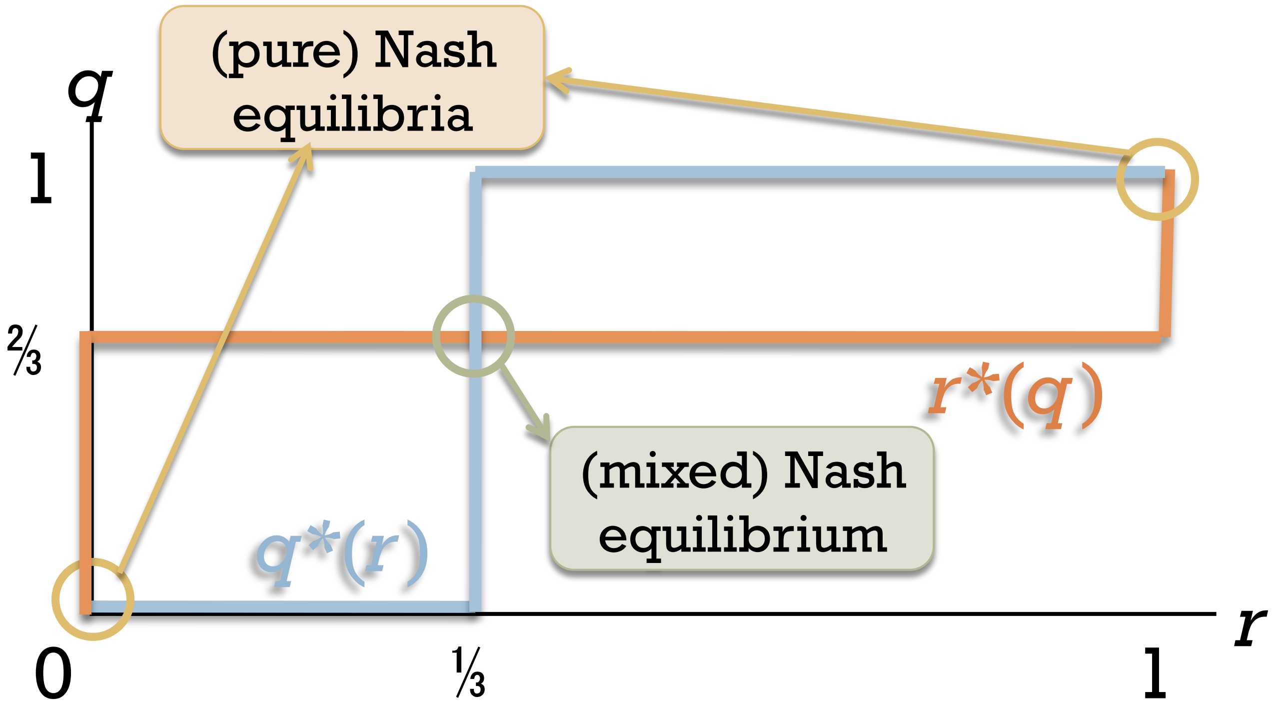
Existence of NE

- Clearly, the existence of at least one NE is guaranteed by topological reasons.
- There may be more NEs (e.g. Battle of Sexes).

		Brian: R	Brian: S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- $u_A(\mathrm{R},r) = 2r$, $u_A(\mathrm{S},r) = 1 - r$, $q^*(r) = 1 - \mathrm{h}(r - ⅓)$
- $u_B(q,\mathrm{R}) = q$, $u_B(q,\mathrm{S}) = 2(1 - q)$, $r^*(q) = 1 - \mathrm{h}(q - ⅔)$

Existence of NE



- Anyway, $q^*(r)$ must intersect $r^*(q)$ at least once.
- The Nash theorem generalizes this reasoning.

Tadelis:

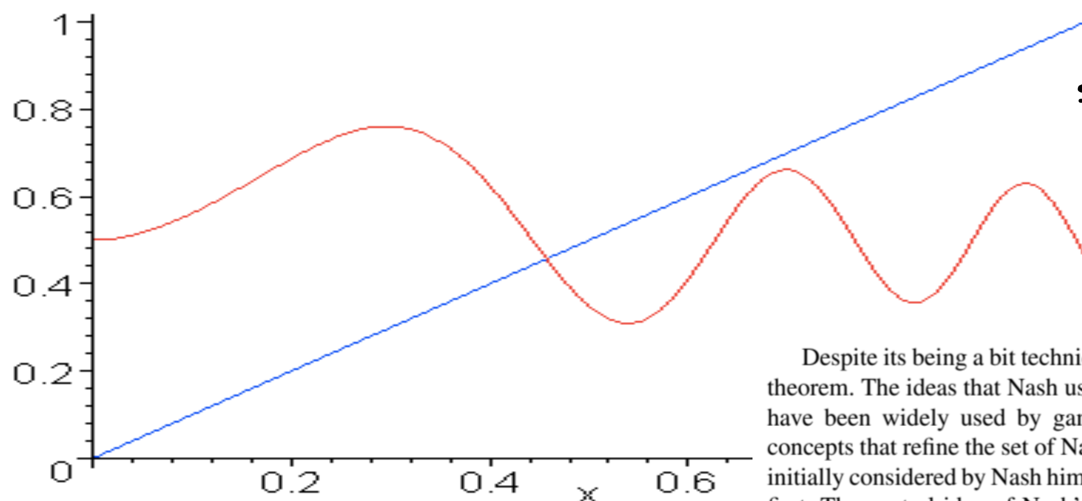
Theorem (Nash's Existence Theorem) *Any n-player normal-form game with finite strategy sets S_i for all players has a (Nash)* equilibrium in mixed strategies.[8]

The Nash theorem

- For game $\boxed{G = \{S_1,\ldots,S_n; u_1,\ldots,u_n\}}$, define:
  $\boxed{BR_i : \Delta S_1 \times \ldots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \ldots \times \Delta S_n \to \wp\, \Delta S_i}$
  $\boxed{BR_i(m_{-i}) = \{m_i \in \Delta S_i : u_i(m_i, m_{-i}) \text{ is maximal}\}}$
- Then define $\boxed{\mathbf{BR}: \Delta S \to \wp\, \Delta S}$ as
  $\boxed{\mathbf{BR}(m) = BR_1(m_{-1}) \times \ldots \times BR_n(m_{-n})}$
- $BR_i(m_{-i})$ is the set of best responses of *i* to what others may do (m_{-i}); **BR** is their aggregate.
  - m is a NE if $\boxed{m \in \mathbf{BR}(m)}$
  - Properties of $BR_i(m_{-i})$: (1) is always non-empty (2) always contains at least a pure strategy

The Nash theorem

- $\boxed{\textbf{Brouwer's Fixed Point Theorem}}$
- $\boxed{\text{If } f(x) \text{ is a continuous function from a closed real interval } \mathcal{J} \text{ to itself, } \exists\, x^* \in \mathcal{J} \text{ such that } f(x^*) = x^*}$
- **Proof:** consider $\boxed{\mathcal{J} = [0,1]}$. If f(0) > 0 and f(1) < 1, apply Bolzano-Weierstrass theorem to $f(x) - x$



⇒ noi non abbiamo funzioni continue

Tadelis:
Despite its being a bit technical, we will actually prove a restricted version of this theorem. The ideas that Nash used to prove the existence of his equilibrium concept have been widely used by game theorists, who have developed related solution concepts that refine the set of Nash equilibria, or generalize it to games that were not initially considered by Nash himself. It is illuminating to provide some basic intuition first. The central idea of Nash's proof builds on what is known in mathematics as a *fixed-point theorem*. The most basic of these theorems is known as Brouwer's fixed-point theorem:

The Nash theorem

- **Kakutani's Fixed Point Theorem**
- If A is a non-empty, compact, convex subset of $\mathbb{R}^n$
- If correspondence $F : A \rightrightarrows A$ is such that
  - For all $x \in A, F(x)$ is non-empty and convex
  - If $\{x_i\}, \{y_i\}$ are sequences in $\mathbb{R}^n$ converging to x and y, respectively: $y_i \in F(x_i) \Rightarrow y \in F(x)$ (F has **closed graph**)
- Then there exists $x^* \in A$ such that $x^* \in F(x^*)$.

proof

- **Nash theorem.** Nothing but Kakutani theorem applied to the global best-response **BR**

Adding a time dimension

Still "static" games?

Fictitious Play

- In fictitious play (G.W. Brown, 1951), regrets become actual changes of moves
  - Each player *i* assumes the (possibly mixed) strategies played by –*i* as fixed
  - If *i* gets a chance to play again, it best responds to what see the other players just did
  - Somehow, "full rationality" is denied (we acknowledge predictions may be incorrect)
- How does fictitious game evolve? se si è a NE, nessuno vuole uscirci
  - Nash equilibrium points are **absorbing** states. So, are they always convergence points?

Fictitious Play

- Not always! Players can also keep "cycling" (we will see examples of this)
  - In Rock/Paper/Scissors, FP does not converge.
- FP converges to a NE in some relevant cases:
  - The game can be solved by IESDS
  - **Potential games**
  - (also other cases such as 2xN games with generic payoffs – which means every outcome has a different payoff for all the players)

Potential games

- Take $G = \{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$. $S = S_1 \times \ldots \times S_n$
- Function $\Omega: S \rightarrow \mathbb{R}$ is an **(exact) potential** for G if:

$$\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \Delta u_i$$

- $\Omega: S \rightarrow \mathbb{R}$ is a **weighted potential** with weight vector $\mathbf{w} = \{w_i > 0\}$ if: $\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = w_i \Delta u_i$
- $\Omega: S \rightarrow \mathbb{R}$ is an **ordinal potential** for G if:

$$\Omega(s_i', s_{-i}) > \Omega(s_i, s_{-i}) \Leftrightarrow u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

- If G admits a potential (ordinal potential), it is called a **potential (ordinal potential) game**.

Potential games

- Potential games have nice properties
- If $\boxed{G = \{S_1, S_2, .., S_n ; u_1, u_2, .., u_n\}}$ has an ordinal potential $\boxed{\Omega}$, it is immediate that its set of NEs is the same of $\boxed{G' = \{S_1, S_2, .., S_n ; \Omega, \Omega, .., \Omega\}}$
- I.e., all the players want to max the potential
  - Multi-person reduces to single-goal optimization
  - To some extent, enables distributed optimization
  - The physical meaning of the potential may not be always immediate

Examples of potential

- The Prisoner's Dilemma is a potential game.

Al \ Bob	M	F
M	-1, -1	-9, 0
F	0, -9	-6, -6

Al \ Bob	M	F
M	0	1
F	1	4

potential Ω

- This potential is exact
- However, the players are not very smart (they do not maximize their global welfare!)
- So, there must be some dummy somewhere

Examples of potential

- The game of Cournot oligopoly is an ordinal potential game.
  - Recall that firms choose q_1 and q_2;
  - the market clearing price is $a - q_1 - q_2$;
  - unit cost is c (so cost to produce $q_i = c\, q_i$)
- Thus $u_i(q_i, q_j) = q_i(a - q_i - q_j - c)$

  and an ordinal potential function is:

  $\Omega(q_i, q_j) = q_i\, q_j (a - q_i - q_j - c)$

Potential games

- **Theorem**. Every finite ordinal potential game has (at least) a pure strategy Nash eq.
  - This NE can be found deterministically
- **Proof**: a consequence of fictitious play
  - All players move, one at a time, to maximize their utility → they also maximize the potential
  - Repeat this until a local maximum of Ω is found

Congestion games

- Congestion games are a special case of potential game. They involve the choice of "least congested resources"
  - Especially found in network problems (finding the least congested route on a graph)
  - Or in resource allocation (minority games)
- It can actually be found that:
  - congestion games are potential games
  - for every potential game, there exists a congestion game with the same potential

Coordination game

- A **coordination game** models situations where players are required to act together
  - They give higher payoffs to the players when they make the same choice
  - An example is the Battle of sexes
  - In the historical "Stag Hunt" (proposed by Rousseau) 2 hunters may decide to hunt a deer (value 20), but they succeed only together; or, each one can hunt a hare (worth 7), even alone

Coordination game

- A coordination game has multiple pure strategy NEs
- It can be seen as a potential game, with coordination points as potential maxima
  - For the Stag Hunt:

		Grunt S	Grunt H
Brunt	S	10,10	0,7
Brunt	H	7,0	7,7

payoffs

		Grunt S	Grunt H
Brunt	S	-4	-7
Brunt	H	-7	0

potential Ω

Coordination game

- Another case is the **anti-coordination game**
  - For example the Hawk-and-Dove, Chicken
  - Players try not to select the same thing

	Hawk	Dove
Hawk	-99,-99	10,-10
Dove	-10,10	0,0

- Hawk = buy nuclear weapons
  Dove = be peaceful
- Hawk = hold the wheel; if you win, the other is a chicken
  Dove = steer the wheel

- Note: Odd/Even and similar ones (a player is for =, the other ≠) are called **discoordination games**

Potential=coordination+dummy

- Finally, a **dummy** (or pure externality) game is such that for all s_{-i}, $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$, i.e., payoff of player i only depends on s_{-i}
- Every potential game is a sum of a pure coordination and a dummy game

Bob

Al	M	F
M	-1,-1	-9,0
F	0,-9	-6,-6

=

coordination (greedy)

	M	F
M	-1,-1	0,0
F	0,0	3,3

+

dummy (externality)

	M	F
M	0,0	-9,0
F	0,-9	-9,-9

Computational complexity

Is a NE easy to find?

How easy is to find a NE?

- Since Nash equilibria are regarded as the "natural" evolution of the system, one may wonder how much it takes to find them
- We already have the Nash theorem, which is an existence theorem
- Plus, there are notable results for certain specific games

A negative result

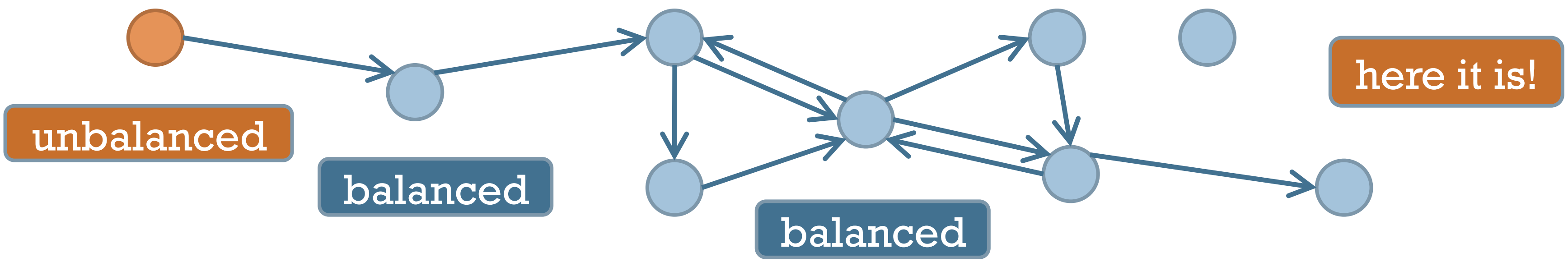
- Unfortunately, in the general case, finding a NE is **computationally hard**
- This has been proven in some recent papers by Papadimitriou et al.
- However, computationally hard does not mean NP-complete
- The search for a NE cannot be NP-complete as a solution *must* exist (there may even be multiple solutions, which complicates things)

The PPAD class

- The NASH problem is PPAD-complete
  - PPAD = Polynomial Parity Arguments on Directed graphs (Papadimitriou, 1994)
  - The PPAD class is somehow intermediate between P and NP
  - More or less, P<PPAD<NP. This means it is computationally hard, unless P=NP
  - This class includes the problem equivalent to the end-of-line problem

The PPAD class

- Consider the end-of-line problem:
  - "Take a directed graph with an unbalanced node. There must be another (at least). Find it."



- This problem is bound to have a solution
- However, finding it without exploring the whole graph is far from trivial (and in certain cases cannot be avoided)

How is NASH a PPAD problem?

- The NASH problem corresponds to find a fixed point of the **BR** function
- Finding a fixed point over a compact set can be shown to be equivalent to finding the end of a proper path on a directed graph
- There are elegant (not difficult but very long) proofs of it, involving graph coloring and compact partitioning

Consequences on NE?

- This may imply bad consequences on the practical usefulness of Nash Equilibrium
- To be optimistic:
  - Certain simple problems can be shown to have a NE which can be found through constructive steps (good for engineers)
  - one may be "close" to a NE (maybe it is enough) → relaxation: ε-**Nash Equilibrium**, i.e., instead of checking for "no unilateral improvements," ignore all improvements less than a given $\varepsilon > 0$