

RECAP: KARGER'S MIN-CUT ALGORITHM

FULL-CONTRACTION ($G = (V, E)$)

$G' = (V', E')$ & $G = (V, E)$

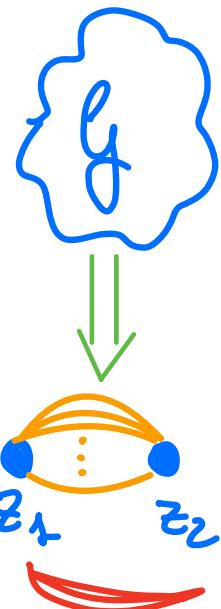
for $i \leftarrow 1$ to $|V| - 2$ do

$e \leftarrow \text{RANDOM}(E')$, {account for}
multiplicities}

$G' \leftarrow G' / e$

return $|E'|$

$O(n^2)$



KARGER (G, S)

min $\leftarrow +\infty$

repeat S times

$t \leftarrow \text{FULL-CONTRACTION}(G)$

if ($t < \text{min}$) then $\text{min} \leftarrow t$

return min

IDEA OF THE ANALYSIS If the random edge selections in FULL-CONTRACT "miss" the edges of E^* , then E^* survives and is returned
CLWY: $|E^*|$ is "small" w.r.t. $|E|$.

Conditional probabilities are important in the analysis of randomized algorithms since they allow to evaluate the probability of a sequence of choices

DEF For $G(V, E)$, $\forall v \in V$ let $d(v) = \sum_{e \in E} m_e$

$d(v)$ corresponds to the degree of v in a simple graph

PROPERTY 1 $\sum_{v \in V} d(v) = 2|E|$

Each edge multiplicity is counted twice
(if $e = \{u, v\}$: once in $d(u)$, once in $d(v)$)

PROPERTY 2 Let $G = (V, E)$, and let C^* be a min-cut of G . Then $\forall v \in V$

$$|C^*| \leq d(v)$$

The multiset of edges $\{\{v, x\} \in E\}$ disconnects v from all other nodes.

LEMMA Let $G = (V, E)$, and let C^* be a min-cut of G . Then

$$|E| \geq |C^*| \cdot |V| / 2$$

PROOF: $2|E| = \sum_{v \in V} d(v)$ (PROP 1)

$$\geq \sum_{v \in V} |C^*| \quad (\text{PROP 2})$$

$$= |C^*| |V|$$

Thus $|E| \geq |C^*| |V| / 2$

ANALYSIS

Given $G = (V, E)$, let us fix a min-cut C^* of G and study the probability that FULL-CONTRACTION (ℓ_f) does not sample any edge (corresponding to an edge) of C^* in the sequence of $|V|-2$ contractions. (Worst case: there may be many min-cuts!)

Let $|V|=n$, $|C^*|=t$. Define the event E_i : "the i -th contraction avoids C^* :

We want to lower bound

$$\Pr\left(\bigcap_{i=1}^{n-2} E_i\right)$$

Consider E_1 . By the LEMMA, we have

$$|E| \geq t \cdot n/2 \quad (|C^*||V|/2)$$

Therefore

$$\Pr(E_1) = 1 - t/|E| \geq 1 - t/(tn/2) = 1 - 2/t$$

Assume that E_1 occurs. What is the probability $\Pr(E_2 | E_1)$?

Since E_1 occurs, C^* is still the min-cut in $G_1 = (V_1, E_1) = G$ for ϵ .

Recall that $|V_1| = |V|-1 = n-1$,
thus

$$|E_1| \geq t(n-1)/2 \quad (\text{lemma})$$

Therefore

$$\Pr(E_2 | E_1) \geq 1 - t/[t(n-1)/2] = 1 - \frac{2}{n-1}$$

Inductively, at iteration i :

$$\Pr(E_i | E_1 \cap E_2 \cap \dots \cap E_{i-1}) \geq 1 - \frac{2}{n-i+1}$$

(since $|V_{i-1}| = n-i+1 \Rightarrow |E_{i-1}| \geq t(n-i+1)/2$)

Therefore, C^* survives all $n-2$ contractions with probability

$$\Pr\left(\bigcap_{i=1}^{n-2} E_i\right) = \Pr(E_1) \prod_{i=2}^{n-2} \Pr(E_i | E_1 \cap E_2 \cap \dots \cap E_{i-1})$$

$$\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{n-i+1}\right) = \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} =$$

$$\begin{aligned} & \cdot \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \cdot \frac{2}{4} \cdot \frac{1}{3} = \\ & = 2/n(n-1) \geq 2/n^2 \end{aligned} \quad \text{TELESCOPING Product}$$

REMARK The probability of correctly returning e^* is rather small.

We can simplify it by running $S = \Omega(\frac{n^2}{\epsilon})$ instances of FULL-CONTRACTION:

$$\Pr(\text{All } S \text{ instances do not return } |e^*|) \leq \left(1 - \frac{2}{n^2}\right)^{\frac{n^2}{2} \cdot S} < e^{-\frac{2}{n^2} \cdot S} = \frac{1}{n^2}$$

With $S = O(n^2 \log n)$ executions of FULL-CONTRACTION, we get the min-cut w.h.p.

RUNNING TIME: $O(n^4 \log n)$ slower than the best deterministic algorithm!

We will discuss an optimized approach (based on the same idea) due to Karger and Stein (1983) achieving running time

$$O(n^2 \log^3 n)$$

(much faster than the deterministic algorithm.)

OBSERVATION The algorithm also works for weighted graphs (min-weight edge-cut): they can be seen as multigraphs, where $\forall e \in E : m_e = w(e)$

KARGER-STEIN ALGORITHM

Recall that FULL-CONTRACTION executes WL-2 contractions and

$\Pr(\text{FC returns min-cut}) \geq$

$$\frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{n-i-1}{n-i+1} \cdots \left(\frac{2}{4} \right) \cdot \left(\frac{1}{3} \right) \geq \frac{2}{n^2}$$

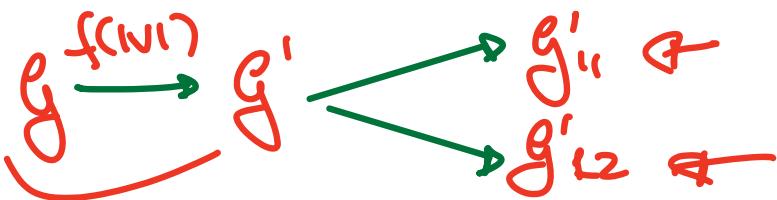
$\Pr(C^* \text{"survives" } i\text{-th contraction} \mid C^* \text{ survived previous } i-1)$

To get high probability we need $O(n^2 \log n)$ Full-contractions!

OBSERVATION: In FULL-CONTRACTION the early contractions work C^* with very high probability ($1 - \frac{2}{n}, 1 - \frac{2}{n-1}, \dots$). However the later contractions "destroy" C^* with larger and larger probability (up to $\dots \frac{1}{2}, \frac{1}{3}$)

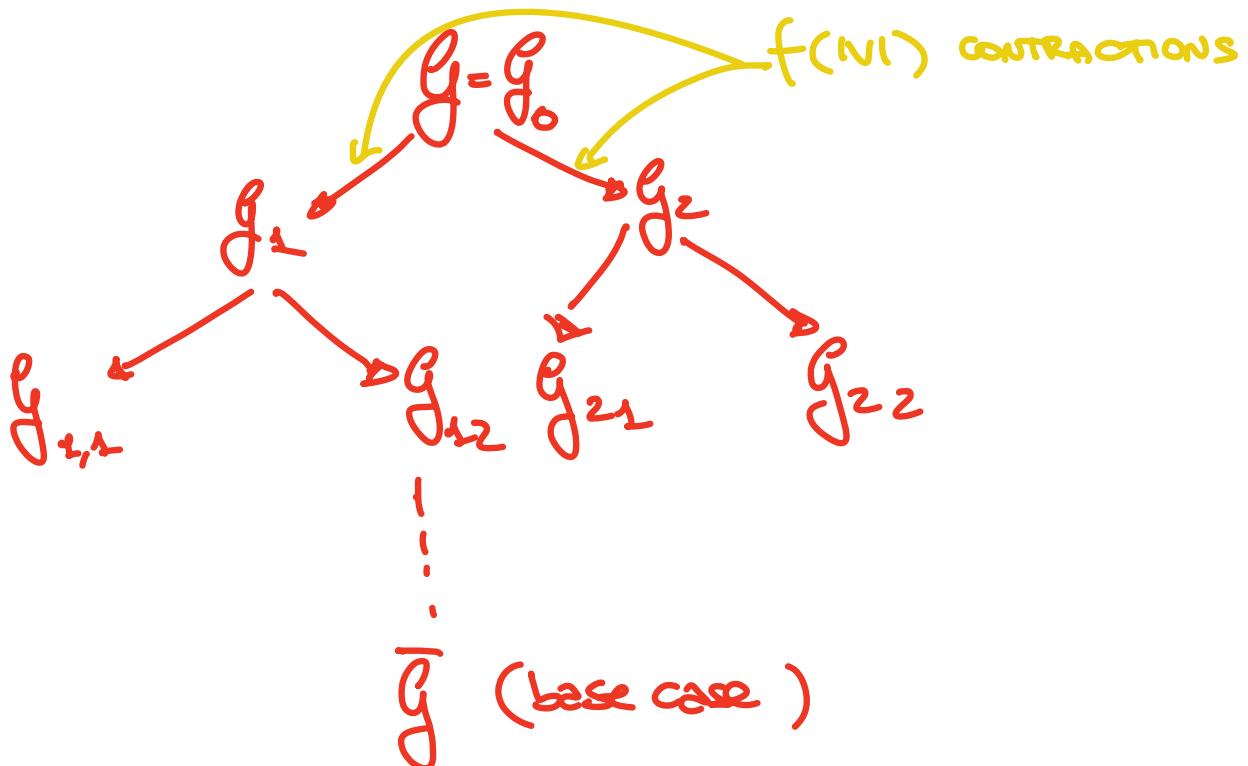
MORALE! FC is "more efficient" during the initial segment of contractions: we want to make better use of them! How: reuse them!

IDEA: "Share" an initial segment of f(WL) contractions among different instances of FC: "shrink" g_1 to g_{12} of a given size and then apply two independent sequences of contractions to g_{12} to yield fully contracted g_{11}, g_{12} .



G'_{11}, G'_{12} share the same "good" sequence
of initial contractions $G \rightarrow G'$

We can implement a recursive strategy
based on the above idea



At each level, the multigraphs "show"
different segments of contractions

Let PARTIAL-CONTRACTION(G, K) (PC) execute
 K consecutive random edge contractions
on G ($\rightarrow FC(G) = PC(G, Nl-2)$)

NOTE $PC(G, K) \rightarrow G' = (V', E')$ with $|V'| = |V| - K$
nodes and $|E'| \leq |E| - K$ edges

The recursive algorithm is the following

R-KS ($G = (V, E)$)

if ($|V| \leq 3$) then * solve directly *
 $K \in f(|V|)$ {see analysis}

$G_1 \in PC(G, K); t_1 \in R_KS(G_1)$ {T1}

$G_2 \in PC(G, K); t_2 \in R_KS(G_2)$ {T2}

{two trials, T1 and T2, to get min cost}

return $\text{MIN}(t_1, t_2)$

INDEPENDENT TRIALS T_1, T_2

We generate two instances G_1, G_2 by running two separate partial contractions and then recurse on G_1, G_2 : all instances in G_1 's (resp., G_2 's) subtree will "share" those initial $f(|V|)$ reusable contractions (that avoid edges of E^* with larger probability)

RUNNING TIME

To simplify the analysis, we upper bound the cost of $PC(G, K)$ by $O(n^2)$. Then

$$T_{RKS}(n) = \begin{cases} c & n \leq 8 \\ 2T(n-k) + O(n^2) & n > 8 \end{cases}$$

Let us set $K = n - \lceil n/\sqrt{2} + 1 \rceil \approx n \left(1 - \frac{1}{\sqrt{2}}\right) = \Theta(n)$

then

$$T_{\text{Trees}}(n) = \begin{cases} 2T\left(\lceil \frac{n}{\sqrt{2}} + 1 \rceil\right) + O(n^2) & n \geq 8 \\ c & n \leq 8 \end{cases}$$

behaves like the HT recurrence (prove it)
 $\omega(n)$

$$T(n) = 2T(n/\sqrt{2}) + O(n^2)$$

$$b = \sqrt{2}, \quad \log_b a = \log_2 2 = 2$$

$$n^{\log_b a} = n^2 = \Theta(\omega(n))$$

We are in Case 2: $T_{\text{Trees}}(n) = O(n^2 \log n)$

(which less than $O(n^2 \cdot n^2 \cdot \log n)$ of
KARGER (if, $\in \Omega(n^2 \log n)$) to get k.p.)

2-KS ($G = (V, E)$); $n \in |V|$

if ($n \leq 8$) then * solve directly *

$$k \in n - \lceil n/\sqrt{2} + 1 \rceil \quad \{ k \approx (1 - \frac{1}{\sqrt{2}})n \approx n/1.34 \}$$

$G_1 = (V_1, E_1) \in \text{PC}(G, k); T_1 \in 2\text{-KS}(G_1) \{ T_1 \}$

$G_2 = (V_2, E_2) \in \text{PC}(G, k); T_2 \in 2\text{-KS}(G_2) \{ T_2 \}$

{ two independent trials, T_1 and T_2 }

$$\{ |V_1|, |V_2| \approx n/\sqrt{2} \approx n/1.41 (> n/2) \}$$

return $\min(T_1, T_2)$

COLLECTNESS ANALYSIS

What is $\Pr(\text{RKS}(g) \text{ returns low-cut})$?

Consider a single execution of $\text{RC}(g, k)$ with $K = n - \lceil n/\sqrt{2} + 1 \rceil$:

We repeat the argument behind Karger's algorithm and evaluate $\Pr(E_1 \wedge E_2 \wedge \dots \wedge E_k)$ using conditional probabilities:

Independently:

$$\Pr(E^* \text{ "survives" in } g_1) = \\ = \Pr(E^* \text{ "survives" in } g_2) =$$

$$\geq \prod_{i=1}^{n-\lceil n/\sqrt{2} + 1 \rceil} \frac{n-i+1}{n-i} = \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \dots \cdot \frac{\cancel{n/\sqrt{2}+1}}{\cancel{n/\sqrt{2}+3}} \frac{\cancel{n/\sqrt{2}}}{\cancel{n/\sqrt{2}+2}}$$
$$= \frac{\lceil n/\sqrt{2} + 1 \rceil \lceil n/\sqrt{2} \rceil}{n(n-1)} \geq \frac{n^2}{2} \frac{1}{n(n-1)} \geq \frac{n}{2(n-1)} \geq \frac{1}{2}$$

After the initial contractions, E^* survives independently in g_1 and g_2 with probability $\geq \frac{1}{2}$

Let $p(u) = \Pr(\text{RKS}(g_j \oplus E) \text{ correct when } m=u)$

Clearly, $p(u)=1$ for $m \leq 8$ (base case)

For $m > 8$ we have,

$$p(u) = 1 - \Pr(\text{"T1 fails"} \cap \text{"T2 fails"}) =$$

independent events
with identical probability

(By "T1/2 fails" we mean that $\frac{t_1}{t_2} > 1e^{+1}$)

$$= 1 - \Pr(\text{"T1 fails"}). \Pr(\text{"T2 fails"})$$

independent events

$$= 1 - [\Pr(\text{"T1 fails"})]^2 \quad \{ \text{same Pr} \}$$

$$= 1 - [1 - \Pr(\text{"T1 successful"})]^2$$

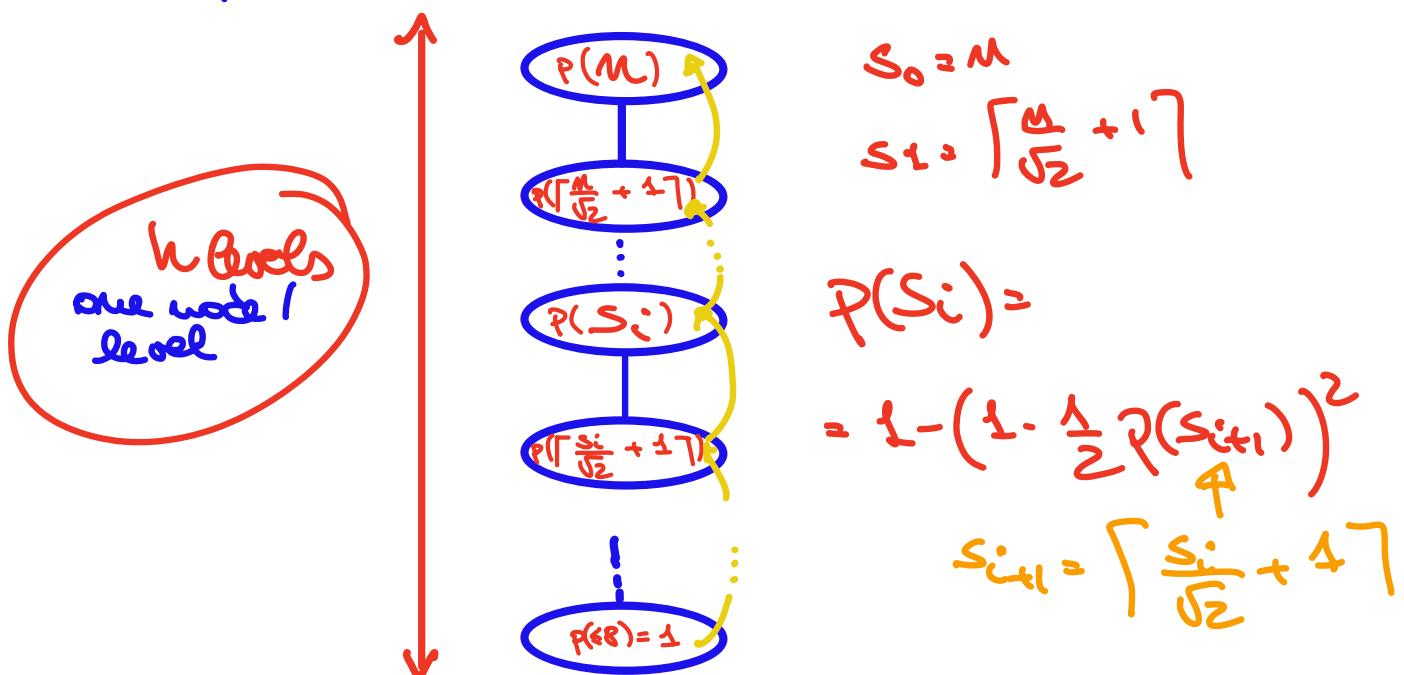
$$= 1 - [1 - \Pr(\text{"E survives in } g_i) \times \Pr(\text{RKS}(g_i) \text{ correct})]^2$$

$$\geq 1 - (1 - \frac{1}{2}) \cdot P(\lceil \frac{m}{\sqrt{2}} + 1 \rceil)^2$$

We have obtained a recurrence on $p(u)$!

$$p(u) \geq \begin{cases} 1 & u \leq 8 \\ 1 - (1 - \frac{1}{2} P(\lceil \frac{m}{\sqrt{2}} + 1 \rceil))^2 & u > 8 \end{cases}$$

We can approach the recurrence using some sort of recursion tree:



How many levels? Let us upperbound S_i , the "size" on i -th level:

LEMMA $S_i \leq \lceil \frac{M}{(\sqrt{2})^i} + 7 \rceil$

PROOF by induction on i . True for $i=0$ ($M \leq M+7$). HP: True for $k=i-1$

TH: $K=i$

$$\begin{aligned} S_i &= \lceil \frac{S_{i-1}}{\sqrt{2}} + 1 \rceil \stackrel{HP}{\leq} \lceil \frac{(\frac{M}{(\sqrt{2})^{i-1}} + 7)/\sqrt{2} + 1}{\sqrt{2}} \rceil \\ &\leq \lceil \frac{M}{(\sqrt{2})^i} + \frac{7}{\sqrt{2}} + 1 \rceil \leq \lceil \frac{M}{(\sqrt{2})^i} + 7 \rceil \end{aligned}$$

≤ 6

Note that for $i = \lceil 2 \log_2 n \rceil$ we have (base cases!)

$$S_i \leq \lceil \frac{n}{(\sqrt{2})^{2 \log_2 n}} + 7 \rceil \leq \lceil 1 + 7 \rceil = 8$$

Therefore $h \leq \lceil \log_2 u \rceil + 1$

We can rewrite the recurrence $p(n)$

as a function of the number of levels h :

$$p'(h) \geq \begin{cases} 1 & h=1 \\ 1 - \left(1 - \frac{1}{2} p'(h-1)\right)^2 & \text{otherwise} \end{cases}$$

Observe that:

$$1 - \left(1 - \frac{1}{2} p'(h-1)\right)^2 = p'(h-1) - \frac{1}{4}(p'(h-1))^2$$

We will prove:

LEMMA $p'(h) \geq \frac{1}{h}$