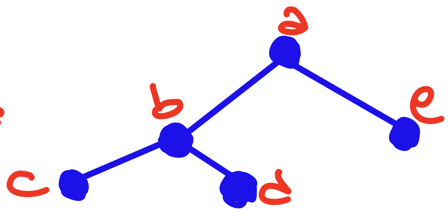
RECAP

- TSP is not $g(n)$-approximable for ANY $g(n)$ ptc
- T_TSP (NPH): c satisfies $\triangle$-inequality

MST - BASED APPROXIMATION

```
APPROX_T_TSP(G, c)
  * V = {v_1, v_2, ---, v_|V|} *
  T* = (V, E_T) ← PRIM(G, c)
  r = v_1
  γ_T* ← <PREORDER(T*, r), r>
  return γ_T*
```

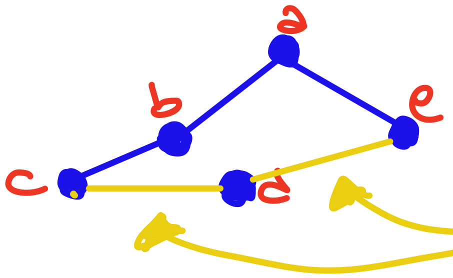
$T^* =$



PREORDER: $\langle a, b, c, d, e \rangle$

TOUR: $\langle a, b, c, d, e, a \rangle$

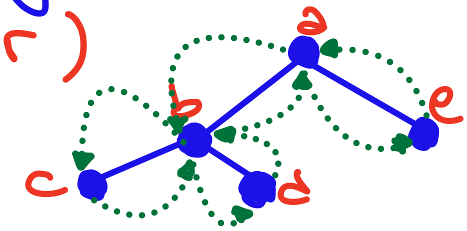
$\gamma_{T^*} =$



γ_{T^*} uses edges NOT in T^*

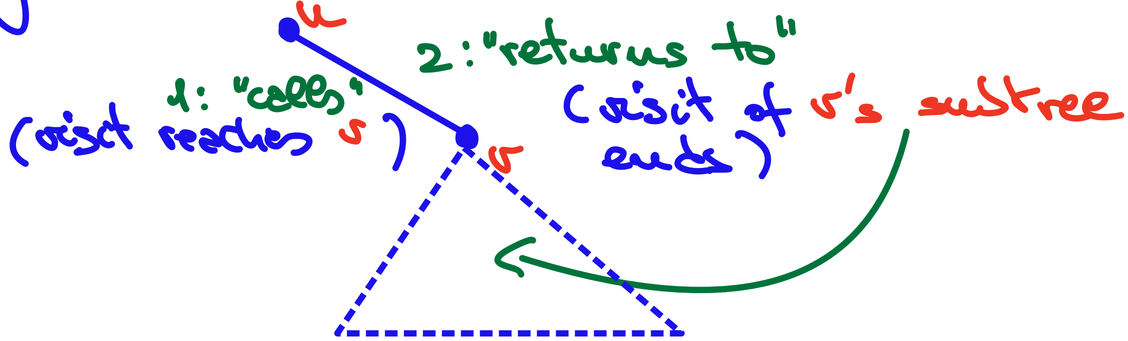
FRW: Nonsimple cycle associated to calls of PREORDER(T^*, r)

$\langle a, b, c, b, d, b, a, e, a \rangle$



PROPERTIES OF A FPW

1. The first occurrence of each node gives the preorder visit of T^*
2. Each edge of T^* is used $\boxed{\text{2 times}}$ by FPW:



Therefore: $c(FPW) = 2c(T^*)$

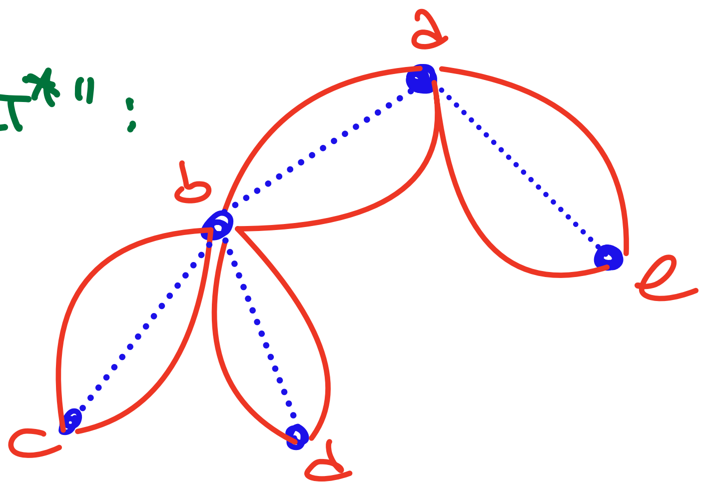
OBSERVATION (useful for later)

The preorder visit on T^* follows a nonsimple cycle (circuit) which uses each edge of T^* twice!

+ archi tra due nodi

Multigraph obtained from T^* replicating each edge twice

"$2T^*$":



$FPW = \langle a, b, c, b, d, b, a, e, a \rangle$

touches all edges of the multigraph once!

A nonsimple cycle touching ALL EDGES of a (multi-)graph ONCE is called **EULER TOUR**

We are ready to prove:

LEMMA

$$\boxed{c(\gamma_{T^*}) \le c(FPW) = 2c(T^*)}$$

PROOF

Consider the FPW associated to the call PREORDER(T^*, r) in A-T-T(G, c)

$$\boxed{FPW = \langle v_1 = r, v_2, \ldots, v_{2|V|-1} = v_1 = r \rangle}$$ ⇒ proprietà albero

(the FPW contains $2|V|-2$ edges)

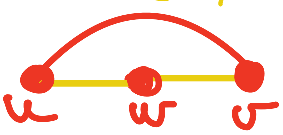
Thanks to the Δ-inequality, we can use shortcutting to eliminate all duplicate nodes (but $v_{2|V|-1} = r$) from FPW without increasing the cost:

duplicate

$FPW = \langle \ldots\ u, w, v\ \ldots \rangle \rightsquigarrow$ Δ $c(u,w) + c(w,v) \geq c(u,v)$

$FPW' = \langle \ldots\ \boxed{u, v}\ \ldots \rangle$

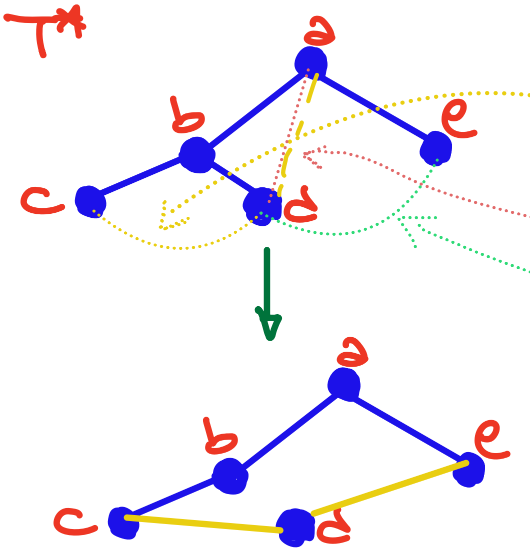
$$\boxed{c(FPW) \geq c(FPW')}\ !$$



By removing all these duplicates, we obtain $\boxed{\langle \text{PREORDER}(T^*, r), r \rangle = \gamma_{T^*}!}$

EXAMPLE

T^*



$FPW = \langle a, b, c, b, d, b, a, e, a \rangle$

$\rightsquigarrow \langle a, b, c, d, b, a, e, a \rangle$

$\rightsquigarrow \langle a, b, c, d, a, e, a \rangle$

$\rightsquigarrow \langle a, b, c, d, e, a \rangle$

$= \langle \text{PREORDER}(T^*), r \rangle$

$= \gamma_{T^*}$

Since each shortcut can only decrease $c(FPW)$, we have proved that

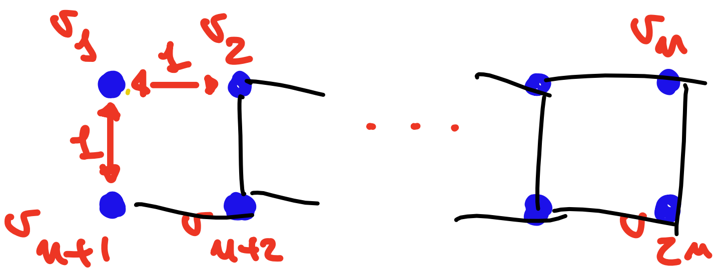
$$\boxed{c(\gamma_{T^*}) \le c(FPW) = 2c(T^*)}$$

Therefore

$$\boxed{\rho = \frac{c(\gamma_{T^*})}{c(\gamma^*)} \le \frac{2c(T^*)}{c(T^*)} = 2}$$

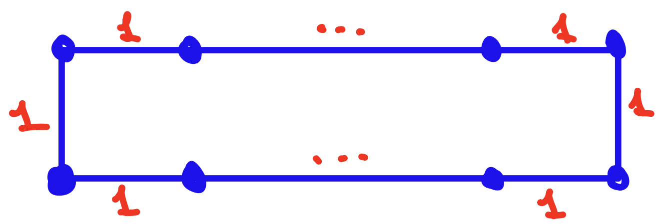
The bound on the approximation is asymptotically tight:

G: $2n$ points in $\mathbb{R}^2$ (unit grid)



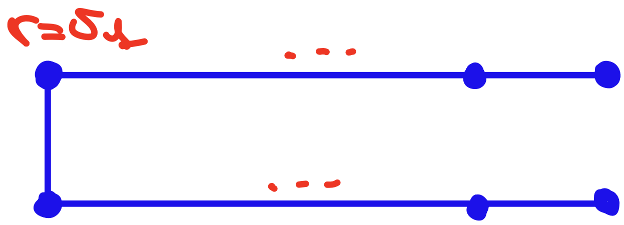
$c(u,v) = \|u-v\|_2$
Euclidean distance

Optimal tour: $c(\gamma^*) = 2n$ (perimeter)

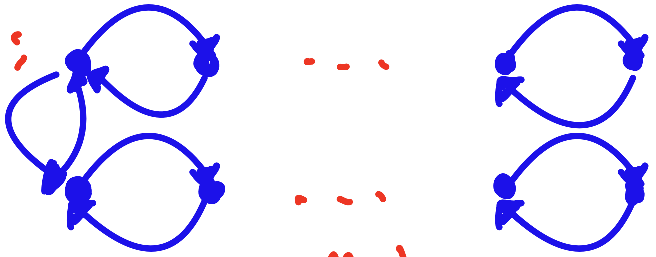


(edges have length ≥ 1)

MST:

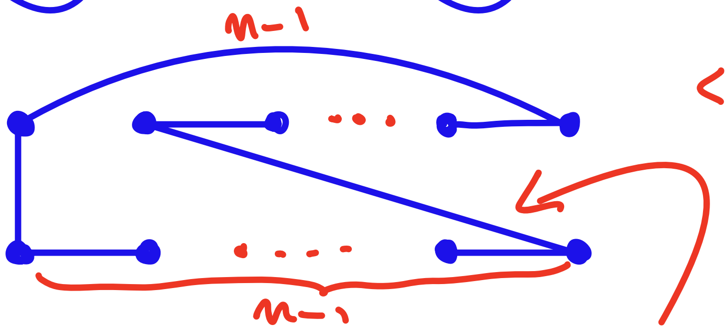


FPW :



$\langle s_1, s_{n+1}, \dots s_{2n}, s_{2n-1}, \dots, \dots, s_{n+1}, s_1, \dots, s_n, s_{n-1}, s_1 \rangle$

γ_{T^*} :



$\langle s_1, s_{n+1}, \dots, s_{2n}, s_2, \dots, s_n, s_1 \rangle$

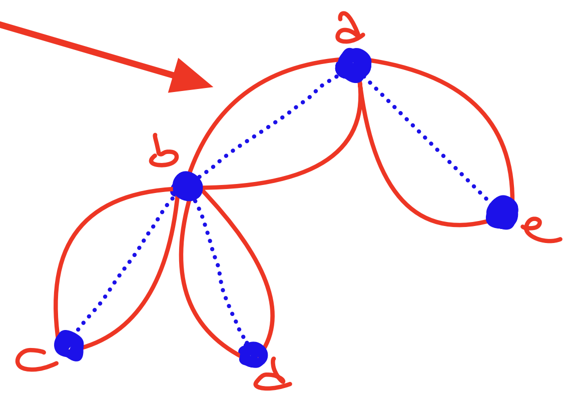
$$C(\gamma_{T^*}) = 3n-3+\sqrt{1+(n-2)^2} > 4n-5$$

Therefore $\rho(n) > \dfrac{4n-5}{2n} = 2 - \dfrac{5}{2n} \xrightarrow[n\to+\infty]{} 2$

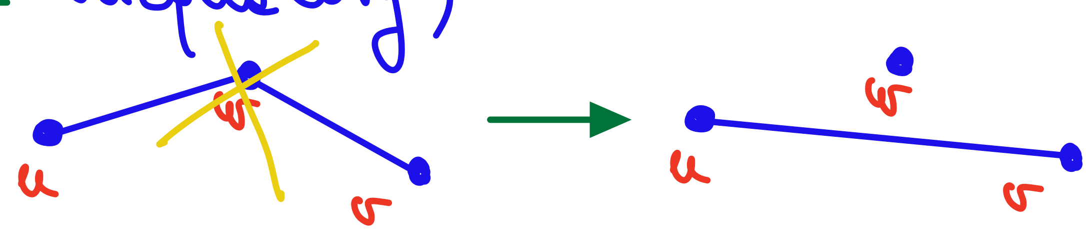
LESSON LEARNED from MST-based approximation:

- MST has been used to create the FULL-PREORDER-WALK (FPW), a nonsimple cycle of small cost ($2 \cdot c(T^*)$) touching all nodes (multiple times)

FPW: Duplicate all edges of T^* and traverse all replicas: $\boxed{\text{Euler tour}}$ of the (multi-)-subgraph "$2T^*$"



- ANY nonsimple cycle can be made simple using shortcutting without increasing the cost (thanks to Δ-inequality)

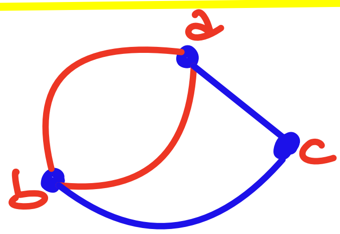


→ Any Euler tour yields a tour of lesser cost!

Christofides algorithm : determine a better subgraph than the double replica "2T*" characterized by a Euler tour of smaller cost.

DEF A $\boxed{\text{multigraph}}$ $G=(V,E)$ is such that E is a multiset (each edge is associated to a multiplicity

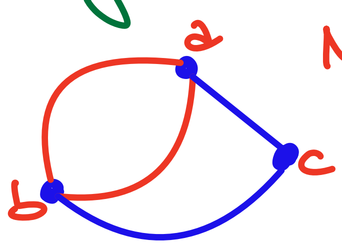
E.g.



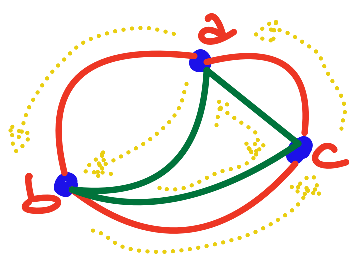
Edge $e=\{a,b\}$ has multiplicity $m(e)=2$

DEF An undirected multigraph $G=(V,E)$ is $\boxed{\text{Eulerian}}$ if $\exists$ a Euler tour (nonsimple cycle) traversing each edge $m(e)$ times

E.g.



NOT EULERIAN



EULERIAN

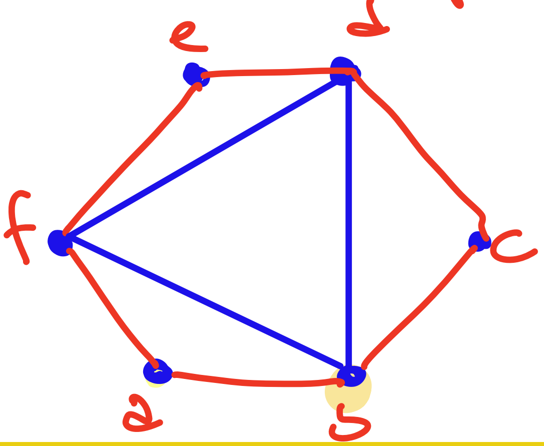
ET: $\langle a,b,c,a,b,c,a\rangle$

Observe: A Euler tour enters and exits each node the same number of times

THEOREM (Euler) A connected multi-graph $G=(V,E)$ is Eulerian if and only if all nodes have even degree

FACT A Euler tour of a Eulerian multi-graph can be found in $O(|E|)$ time:

- Find an initial cycle from an initial node v and remove the edges
- Starting from v, follow the cycle: if the degree of the current node >0:
  - Find a cycle going through node
  - Splice the cycles together, eliminating the edges



$\langle a, b, c, d, e, f, a \rangle$

$\Downarrow$

$\langle a, b, d, f, b, c, d, e, f, a \rangle$

GENERALIZATION OF THE MST APPROACH

Given an instance $\langle G=(V,E) \rangle$ let us determine a (multi)-subgraph of G, $G'=(V,E')$ that is Eulerian and such that

$$\boxed{\sum_{e \in E'} m(e) \cdot w(e)}$$

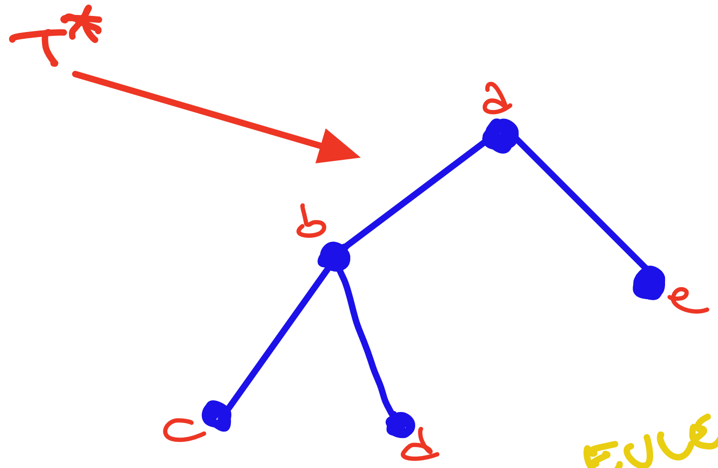
is small

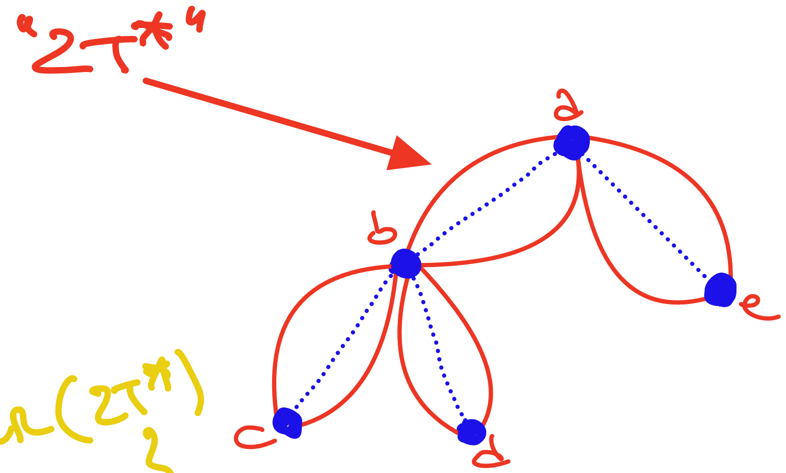
multiplicity

cost of the Euler tour

Once the Euler tour has been obtained, we can use shortcutting to obtain a simple tour whose cost is not larger.

OBSERVATION: This is exactly what we did in the MST approach:



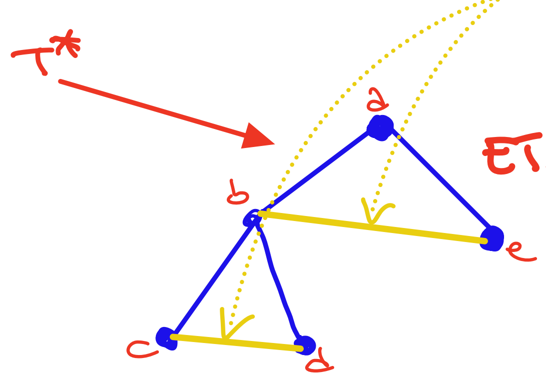


FPW: $\langle a,b,c,d,b,a,e,a \rangle$ = EULER($2T^*$) ⇝ Tour: $\langle a,b,c,d,e,a \rangle$

CHRISTOFIDES IDEA:

"Doubling" T^* is too expensive. Let us add a subset of edges of small cost to T^* to make it an Eulerian (multi-)subgraph!

EXAMPLE:



only two edges to shown

ET = $\langle a,b,c,d,b,e,a \rangle$

⇓

$\gamma = \langle a,b,c,d,e,a \rangle$

If $\boxed{c(c,d) + c(b,e) < \alpha \cdot c(\gamma^*),\ \alpha < 1}$ we get a better quality guarantee, since

$$\boxed{\begin{array}{l} c(\hat{\gamma}) \leq c(T^*) + \alpha\, c(\gamma^*) \leq (1+\alpha)\, c(\gamma^*) \\ \Rightarrow \rho \leq (1+\alpha) \end{array}}$$

Let us see how to do it!

PROPERTY Let $\boxed{G = (V, E)}$ be an arbitrary undirected graph, and let $\boxed{V_{ODD}}$ be the subset of nodes of odd degree. ⇒ problematici per Euler tour
Then $|V_{ODD}|$ is even

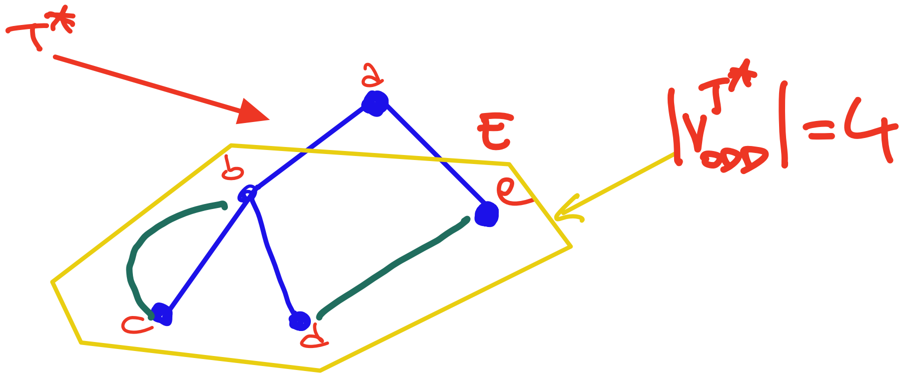
PROOF In any graph, the sum of the degrees $\boxed{\sum_{v \in V} deg(v) = 2|E|}$ (each e counted twice)

Then

$$\boxed{\underbrace{2|E|}_{\text{EVEN}} = \sum_{v \in V} deg(v) = \underbrace{\sum_{v \in V_{EVEN}} deg(v)}_{\text{EVEN}} + \underbrace{\sum_{v \in V_{ODD}} deg(v)}_{\text{it must be even!}}}$$

Since $\sum_{v \in V_{ODD}} deg(v)$ is even and the single terms are odd, the number of these terms $|V_{ODD}|$ must be even!

Let us apply the property to the MST T^*: we have that $|V_{ODD}^{T^*}|$ is even!



CRUCIAL IDEA: Let us complete T^* into a Eulerian multi-subgraph by adding $|V_{ODD}|/2$ edges between pairs of nodes in V_{ODD} and use the resulting Euler Tour instead of the FPW.

(with the new edges, all degrees become even)

DEF Given a weighted undirected complete graph $G' = (V', E')$ with $|V'|$ even, a $\boxed{\text{perfect matching}}$ is a matching $\boxed{M \subseteq E'}$ of size $|V'|/2$ (a perfect matching "touches" all nodes in V'). Its cost is

$$\boxed{c(M) = \sum_{\{u,v\} \in M} c(u,v)}$$

FACT A perfect matching M^* of minimum cost can be found in $O(|V'|^3)$ time (blossom algorithm) (extremely complex)

We are ready to specify the algorithm:

CHRISTOFIDES $(G=(V,E), c)$

$T^* = (V, E_{T^*}) \leftarrow MST(G, c)$

* let $V_{ODD}^{T^*}$ be the set of nodes of odd degree of T^* and let $E_{ODD}^{T^*} = \{e = \{u,v\} \in E : u,v \in V_{ODD}^{T^*}\}$ *

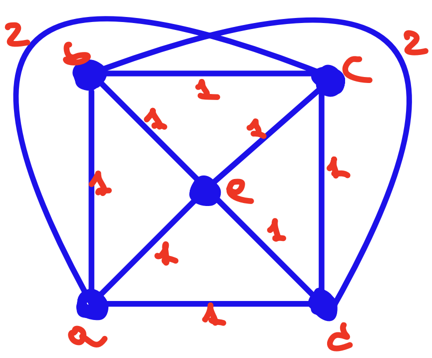
$M^* \leftarrow M\_C\_MATCHING(G' = (V_{ODD}^{T^*}, E_{ODD}^{T^*}), c)$

$W \leftarrow EULER\_TOUR(\overline{G} = (V, E_{T^*} \cup M^*))$

$\gamma \leftarrow SHORTCUT(W)$

return γ

EXAMPLE

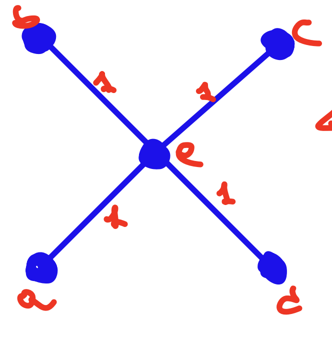


complete graph with 5 nodes

$c(u,v)$ satisfies $\triangle$

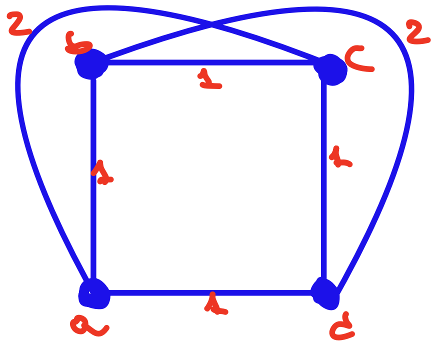
$c(\gamma^*) = 5$

$\langle a,b,c,d,e,a \rangle$



T^*: $c(T^*) = 4$

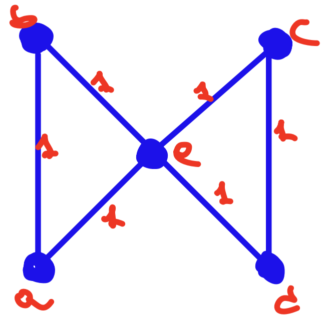
$V_{ODD}^{T^*} = \{a,b,c,d\}$



$G' = (V^{T^*}_{odd}, E^{T^*}_{odd})$

$M^* = \{\{a,b\}, \{c,d\}\}$

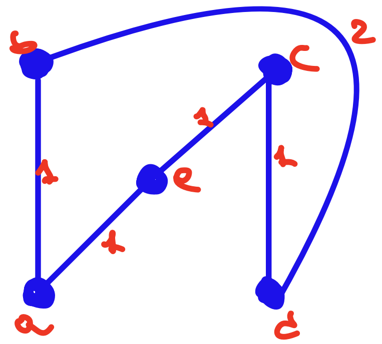
min-cost perfect matching



$\overline{G} = (V, E_{T^*} \cup M^*)$

$W = \langle d, c, e, a, b, e, d \rangle$

EULER TOUR



$\hat{\gamma} = \text{SHORTCUT}(W) =$

$= \langle d, c, e, a, b, d \rangle$

$c(\hat{\gamma}) = 6$

Correctness: the algorithm returns a tour $\hat{\gamma}$ (since it shortcuts a Euler Tour W of the Eulerian multigraph $G = (V, E_{T^*} \cup M^*)$)

Running time: $\boxed{O(|V|^3)}$ (dominated by M-C-MATCHING)

Approximation ratio: It suffices to bound $\boxed{c(W) = c(E_{T^*}) + c(M^*)}$, since $c(\hat{\gamma}) \le c(W)$ (shortcutting). We already know that $c(E_{T^*}) \le c(\gamma^*)$.

LEMMA $\boxed{c(M^*) \le c(\gamma^*)/2}$

PROOF Consider an optimal tour

$$\boxed{\gamma^* = \langle s_{i_1}, s_{i_2}, \ldots, s_{i_{|V|}}, s_{i_1} \rangle}$$

We can use shortcutting to eliminate all nodes in $V^{T^*}_{EVEN}$. We are left with a simple cycle of all nodes in $V^{T^*}_{ODD}$!

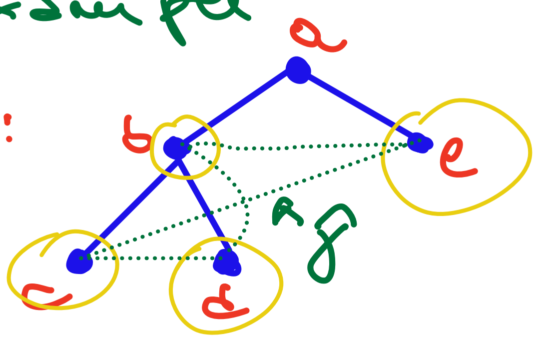
$$\boxed{\hat{\gamma} = \langle \hat{s}_{i_1}, \hat{s}_{i_2}, \ldots, \hat{s}_{i_{|V^{T^*}_{ODD}|}}, \hat{s}_{i_1} \rangle}$$

EVEN!

Clearly, $\boxed{c(\hat{\gamma}) \le c(\gamma^*)}$

Example

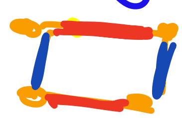
T^*:



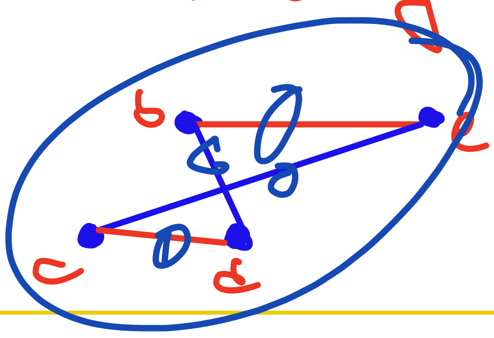
If $\gamma^* = \langle b, e, a, c, d, b \rangle$ SHORTCUT

$\hat{\gamma} = \langle b, e, c, d, b \rangle$

$\hat{\gamma}$ is a simple cycle of $|V^{T^*}_{ODD}|$ (even!) edges containing all nodes in $V^{T^*}_{ODD}$



Let us color the edges in the cycle alternating between RED and BLUE!



I obtain TWO PERFECT MATCHINGS of $V^{T^*}_{ODD}$! M_{RED} and M_{BLUE}

We have:

$$\boxed{c(\gamma) = c(M_{RED}) + c(M_{BLUE})}$$

Therefore one of the two matchings has $\boxed{\text{cost} \leq c(\gamma)/2 \leq c(\gamma^*)/2}$!

Therefore:

$$\boxed{c(M^*) \leq \min\{c(M_{RED}), c(M_{BLUE})\} \leq c(\gamma^*)/2}$$

Putting it all together:

Let $\boxed{\gamma = \text{CHRISTOFIDES}(G = (V, E), c)}$

$$\boxed{\begin{aligned} c(\gamma) &\leq c(E_{T^*}) + c(M^*) \leq c(\gamma^*) + c(\gamma^*)/2 \\ &= \tfrac{3}{2} c(\gamma^*) \end{aligned}}$$

$$\rightarrow \boxed{\rho = \frac{c(\gamma)}{c(\gamma^*)} \leq \frac{3}{2} \frac{\cancel{c(\gamma^*)}}{\cancel{c(\gamma^*)}} = \frac{3}{2}}$$