RECAP

Weighted Vertex Cover $\langle G, w \rangle$

- determine VC V^* of minimum weight $\sum_{v \in V^*} w(v)$
  - FAIR PRICING extends the maximal matching algorithm (unweighted case)
  - LP rounding:

$$\begin{cases} \min \sum_{v \in V} w(v) x_v \\ \text{s.t.} \\ \quad x_u + x_v \geq 1 \; : \; \{u,v\} \in E \\ \quad 0 \leq x_u \leq 1 \; : \; u \in V \end{cases}$$

$$V' = \{ v \in V : x_v^* \geq \tfrac{1}{2} \}$$

CURIOSITY!

DUAL:

(Ignore constraints $x_u \leq 1$)

REDUNDANT

$$\begin{cases} \max \sum_{e \in E} p_e \\ \text{s.t.} \\ \quad \sum_{e \ni u} p_e \leq w(u) \quad u \in V \\ \quad p_e \geq 0 \qquad e \in E \end{cases}$$

Linear relaxation of fair pricing!

THE TRAVELLING SALESMAN PROBLEM (TSP)

$\to$ SALESPERSON

Decision version:

TSP

I: $\langle G=(V,E), c, K \rangle$:

G complete, undirected graph

$c : V \times V \to \mathbb{N}$, symmetric

$K \in \mathbb{N} - \{0\}$

$(c(u,v) = c(v,u))$

$(c(u,u) = 0)$

Q: Is there a tour (hamiltonian cycle) $\gamma = \langle v_1, v_2, \ldots, v_{|V|}, v_{|V|+1} = v_1 \rangle$

$(v_i \neq v_j \quad 1 \le i < j \le |V|)$

of cost

$$c(\gamma) = \sum_{i=1}^{|V|} c(v_i, v_{i+1}) \le K \,?$$

OBSERVATION: There are $\sim |V|!$ tours in G

We can prove TSP $\in$ NPH by reducing from HAMILTON:

An istance of HAMILTON is an arbitrary undirected graph $G=(V,E)$.

Reduction:

$\Rightarrow$ complexity: $O(|V|^2)$

$$\langle G=(V,E) \rangle \overset{f}{\leadsto} \langle G'=(V,E'), c, K \rangle$$

with $E' = \{ \{u,v\} : u,v \in V \}$ (complete set)

$$c(u,v) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 2 & \text{otherwise} \end{cases} \; ; \quad K = |V|$$

If $\boxed{\langle G \rangle \in \text{HAMILTON}}$, then the cost of a hamiltonian cycle of G under $c(\cdot)$ is exactly $|V| \Rightarrow \boxed{f(\langle G \rangle) = \langle G', c, |V| \rangle \in \text{TSP}}$

$c(\cdot) = 1$

If $\boxed{f(\langle G \rangle) = \langle G, c, |V| \rangle \in \text{TSP}}$ then there is a tour γ (made of $|V|$ edges) of cost $c(\gamma) = |V|$

$\Rightarrow$ ALL edges in the tour must cost 1

$\Rightarrow$ ALL edges in the tour belong to E

$\Rightarrow$ the tour is a hamiltonian cycle of G

$\Rightarrow$ $\boxed{\langle G \rangle \in \text{HAMILTON}}$

Two interesting results on TSP:

1. Strong inapproximability result of general TSP:
   if $P \neq NP$ there cannot exist a poly-time $\rho(n)$-approximation for many functions $\rho(n)$

2. Good approximation algorithms for a "natural" (still hard) restriction of TSP (metric cost function)

   2-approximation (based on MST)

   $\frac{3}{2}$-approximation (Christofides' algorithm) (based on MST and min-cost matching)

INAPPROXIMABILITY RESULT FOR GENERAL TSP

THEOREM If $P \neq NP$ there cannot exist a polynomial $\rho(|V|)$-approximation algorithm for TSP, for ANY polynomial-time computable function $\rho(|V|)$.

Very strong result: observe that even $\rho(|V|) = 2^{|V|}$ is poly-time computable (generate 1 followed by $|V|$ 0's)

STANDARD TECHNIQUE: by contradiction: I show that if a $\rho(|V|)$-approximation algorithm exists, I can use it to solve an NPC problem in polynomial time (against the hypothesis $P \neq NP$)

PROOF We show that if A^{S}_{TSP} exists, I can solve HAMILTON in polynomial time;

Let $\boxed{\langle G = (V, E) \rangle}$ be an instance of HAMILTON. I create $\boxed{\langle G' = (V, E'), c : V \times V \to \mathbb{N} \rangle}$ an instance of the optimization TSP problem following the idea of the reduction above:

$$\boxed{E' = \{\{u,v\} : u \neq v \in V\};\ c(u,v) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ |V| \cdot \rho(|V|) + 1 & \text{otherwise} \end{cases}}$$

Observe that $\langle G', c\rangle$ can be computed in polynomial time since $\rho(|V|)$ is ptc.

Thus, $\boxed{|\langle G', c\rangle| = poly(|\langle G\rangle|)}$

Let us see what happens if I run $A_{TSP}^{\rho}(G', c)$:

1) If $\langle G\rangle \in HAMILTON$, then G' contains a tour γ^* of cost $c(\gamma) = |V|$. But then $A_{TSP}^{\rho}(G')$ has to return a tour γ_A of cost $c(\gamma_A) \leq \rho(|V|) \cdot |V|$:

$\Rightarrow$ γ_A cannot use any edge $\notin E$!

$\Rightarrow$ γ_A is a hamiltonian cycle of G of cost $\boxed{c(\gamma_A) = c(\gamma^*) = |V|}$

2) If $\langle G\rangle \notin HAMILTON$, then any tour γ of G must use an edge $e \notin E$ $\Rightarrow$ $\boxed{\forall \gamma : c(\gamma) > |V| \cdot \rho(|V|)}$

1) + 2) imply that

$$\boxed{\langle G\rangle \in HAMILTON \Leftrightarrow c\left(\gamma_A = A_{TSP}^{\rho}(G', c)\right) \leq |V|\rho(|V|)}$$

Then :

```
DECIDE_HAMILTON (<G=(V,E)>)
  * build <G'=(V,E'), c> *
  γA ← A^ρ_TSP(G', c)
  if (c(γA) ≤ |V| ρ(|V|))
     then return 1
     else return 0
```

is a poly-time decision algorithm for HAMILTON, which contradicts $P \neq NP$

The proof implements the general GAP technique to prove inapproximability results:

GIVEN: minimization problem Π_m for which I want to prove $\rho(n)$-inapproximability:

DETERMINE: decision problem $\boxed{\Pi_d \in NPC}$

- ptc function $\boxed{f(x)}$ transforming instances of Π_d into instances of Π_m such that:

- Let $\boxed{c: \mathcal{S} \to \mathbb{N}^+}$ be the cost function of Π_m: then exist a function $k(n)$ such that

$$\boxed{\begin{array}{l} 1.\ x \in L_{\Pi_d} \Rightarrow c(s^*(f(x)) \leq k(|f(x)|) \\ 2.\ x \notin L_{\Pi_d} \Rightarrow c(s^*(f(x)) > \rho(|f(x)|) \cdot k(|f(x)|) \end{array}}$$

- Then if $P \neq NP$ there cannot be a poly-time $\rho(n)$-approximation algorithm for Π_m

PROOF If $A^{\rho}_{\Pi_m}$ existed:

```
DECIDE(x)
  s ← A^ρ_Πm (A_f(x))
  if c(s) ≤ ρ(|f(x)|) · k(f(x))
    then return 1
    else return 0
```

would decide Π_d in polynomial time

RESTRICTION: METRIC COST FUNCTION

DEF: A symmetric function $\boxed{c: V \times V \to \mathbb{N}}$ is $\boxed{\text{metric}}$ if it satisfies the triangle inequality: $\boxed{\forall u,v,w: c(u,v) \leq c(u,w) + c(w,v)}$

Triangle ($\triangle$) inequality states that it is always more convenient to go directly from u to v rather than via intermediate nodes. E.g.: nodes u, v are points in $\mathbb{R}^d$ and $c(u,v)$ is their Euclidean distance

The decision problem $\boxed{\text{TRIANGLE-TSP}}$ is defined as TSP: the instance is $\langle G, c, K \rangle$ with the additional constraint that c is metric

THEOREM $\boxed{\text{TRIANGLE-TSP} \in \text{NPH}}$

PROOF We reduce from TSP as follows:

$$\boxed{\langle G=(V,E), c, K \rangle \xrightarrow{f} \langle G=(V,E), c', K' \rangle}$$

Let $\boxed{W = \max\{c(u,v) : u,v \in V\}}$:

$$\boxed{\begin{aligned} c'(u,v) &= \begin{cases} c(u,v) + W, & u \neq v \in V \\ 0 & u = v \in V \end{cases} \\ K' &= K + |V| \cdot W \end{aligned}}$$

The reduction is correct, since c' satisfies $\triangle$:

$$\boxed{\begin{aligned}\forall u \neq v \in V : c'(u,v) &= c(u,v) + W \leq W + W \\ &\leq c(u,w) + W + c(w,v) + W \\ &= c'(u,w) + c'(w,v)\end{aligned}}$$

Clearly, f is ptc

$\boxed{\langle G, c, K\rangle \in TSP} \iff \exists$ tour γ in G with $c(\gamma) \leq K$

$\iff$ (since γ uses $|V|$ edges)

$\exists$ tour γ in G' with $c'(\gamma) \leq \overbrace{K + |V|W}^{K'}$

$\iff \boxed{\langle G, c', K'\rangle \in \text{TRIANGLE-TSP}}$

A 2-APPROXIMATION ALGORITHM FOR $\triangle$-TSP

We approximated VC by relating $|V^*|$ to $|A|$ a (maximal) matching of G. For $\triangle$-TSP we use a similar approach and relate the optimal tour γ^* to a Minimum Spanning Tree (MST) of G.

DEF: Given an undirected, connected, weighted $\boxed{G = (V,E),\ c : V \times V \to \mathbb{N}^+}$, a $\boxed{\text{Spanning Tree}}$ $\boxed{T = (V, E_T)}$ is any connected subgraph of G, with $\boxed{|E_T| = |V| - 1}$. We set $\boxed{c(T) = \sum_{\{u,v\} \in E_T} c(u,v)}$

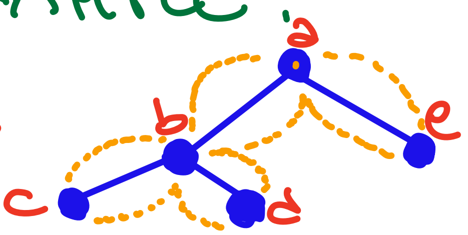
FACT A Minimum Spanning Tree (MST) of G can be determined in time $O(|E| \log |V|)$ (e.g. Prim's algorithm, 572 CLRS)

IDEA: I determine an MST T^* of G and obtain a TOUR γ_{T^*} from it.

HOW: γ_{T^*} is obtained as the preorder visit of T^* (closed on the first node)

EXAMPLE:

$T^* =$

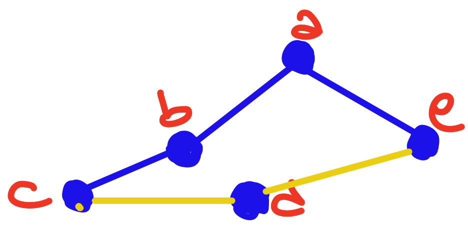


$\longrightarrow$ PREORDER VISIT: $\langle a, b, c, d, e \rangle$

$\downarrow$

TOUR: $\langle a, b, c, d, e, a \rangle$

$\gamma_{T^*} =$



We will prove that $\boxed{c(\gamma_{T^*}) \leq 2\, c(\gamma^*)\,!}$

Here is the algorithm:

APPROX_T_TSP(G, c)

$*\ V = \{v_1, v_2, \ldots, v_{|V|}\}\ *$

$T^* = (V, E_T) \leftarrow \text{PRIM}(G, c)$

$r = v_1$

$\gamma_{T^*} \leftarrow \langle \text{PREORDER}(T^*, r), r \rangle$

return γ_{T^*}

PREORDER(T, r)

$P \leftarrow \langle r \rangle$

if (not leaf(r))

then

for each $(s \in \text{child}(r))$

do

$P \leftarrow \langle P, \text{PREORDER}(T, s) \rangle$

return P

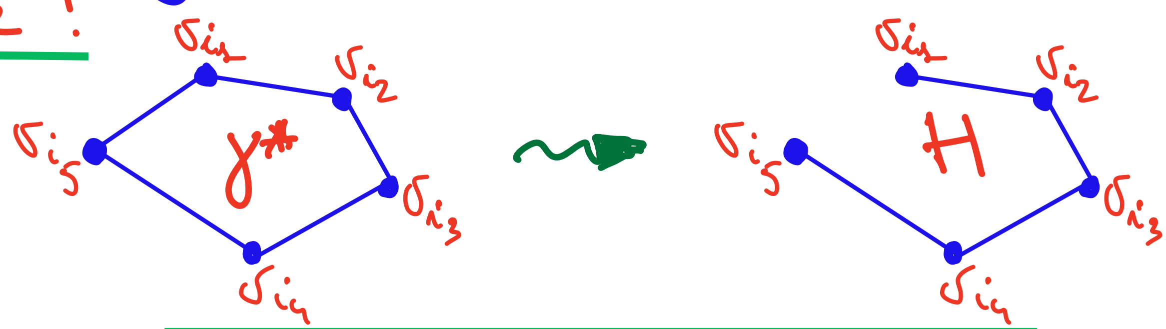
$$\boxed{T_{APP}(|\langle G, c\rangle|) = \underbrace{O(|E|\log|V|)}_{T_{PRIM}} = O(|\langle G, c\rangle| \log |\langle G, c\rangle|)}$$

To prove that A-T-T is a 2-approximation, we must relate OPT $= c(\gamma^*)$ and $c(T^*)$

LEMMA $\boxed{c(\gamma^*) \geq c(T^*)}$

PROOF Let $\boxed{\gamma^* = \langle v_{i_1}, v_{i_2}, \dots, v_{i_{|V|}}, v_{i_1} \rangle}$

Remove edge $\{v_{i_{|V|}}, v_{i_1}\}$: we obtain a hamiltonian path $\boxed{H = \langle v_{i_1}, v_{i_2}, \dots, v_{i_{|V|}} \rangle}$ touching all nodes. H is a Spanning Tree!



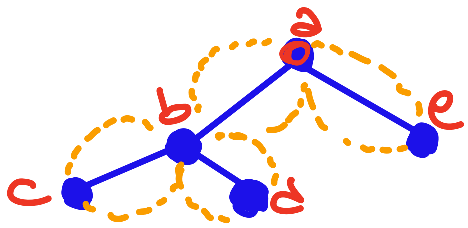
We have $\boxed{c(\underbrace{T^*}_{MST}) \leq c(\underbrace{H}_{ST}) \leq c(\gamma^*)}$!

The cost $c(T^*)$ is a lower bound to $c(\gamma^*)$ BUT:

observe that γ_{T^*} uses edges outside T^*!
We have to upper bound $c(\gamma_{T^*})$ as a function of $c(T^*)$

DEF A Full Preorder Walk (FPW) is the nonsimple cycle obtained by considering the sequence of active calls of procedure PREORDER(T, r)

EXAMPLE:



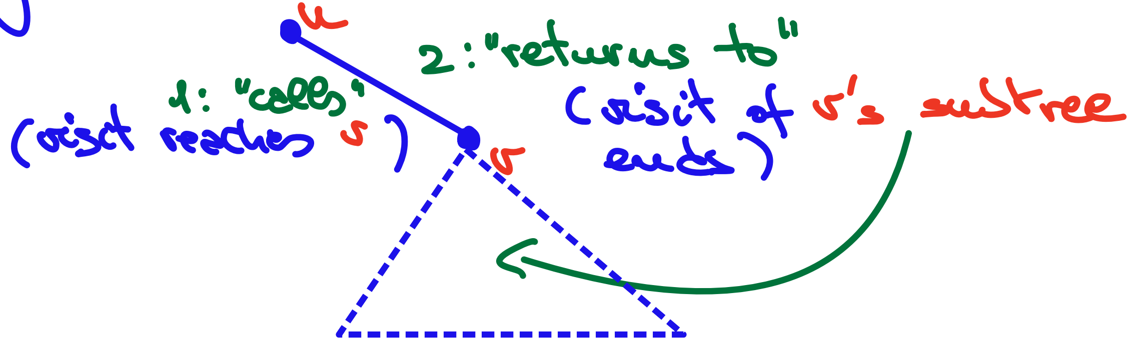
PREORDER(T, a) calls
PREORDER(T, b) calls
PREORDER(T, c) returns to
PREORDER(T, b) calls
PREORDER(T, d) returns to
PREORDER(T, b) returns to
PREORDER(T, a) calls
PREORDER(T, e) returns to PREORDER(T, a)

FPW = <a, b, c, b, d, b, a, e, a>

PROPERTIES OF A FPW

1. The first occurrence of each node gives the preorder visit of T

2. Each edge of T is used $\boxed{2 \text{ times}}$ by FPW:



Therefore: $\boxed{c(FPW) = 2c(T^*)}$