Game theory

A course for the
MSc in ICT for Internet and multimedia

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Static games of complete information

The simplest form of multi-player games

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Game with multiple players

- How do multiple players interact? una per giocatore
  - We assume they have a payoff (utility) function
- Remember that rational players move to maximize of their own payoffs
- What is the simplest interaction like this?
  - certainly not sports or dices where most moves are random – we will not see them in the course
  - not even board games - they are closer, still they require some extensions

Static games of complete information

- **Static** (all players move together; they do not necessarily play simultaneously, but without knowledge of everybody else's move)
- **Complete information** (meaning anybody's payoff function is known)
  - most games within this class are actually "artificial" games (theoretical models)
  - examples of actual games: Odds & Evens, Matching pennies, Rock/paper/scissors

Static games of complete information

- Each player i in the game **simultaneously** and **independently** chooses an action from its own set of available actions A_i
- The combination of actions chosen by the n players determines the **outcome** of the game
- Outcome $(a_1, a_2, \ldots a_n)$ determines a payoff for each player through an individual utility function of player i: $u_i = u_i(a_1, a_2, \ldots a_n)$
- 3 ingredients = actions + outcome + utility

Action versus strategy

- As will be seen later, it is useful to think of **strategies** instead of actions
- A strategy is a **plan of action** a seconda di condizioni
  - e.g.: if these conditions are met, then my action is a, otherwise is either a' or a''
  - this plan can even be random (we will see why)
- Right now, we just need **certain** plans
- These are called **pure strategies**, i.e.: a pure strategy is a deterministic plan of action

Normal-form representation

- Each player i simultaneously chooses a strategy from a set of pure strategies S_i
- This results in a given action chosen by each of the n players that ultimately determines a payoff for each player
- If any player i plays strategy $s_i \in S_i$, the combination of moves is $(s_1, s_2, \ldots, s_i, \ldots, s_n)$
- Player i gets payoff $u_i(s_1, s_2, \ldots, s_i, \ldots, s_n) \in \mathbb{R}$
- The **normal form** of the game is specified by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ ⇒ di norma, si specificherebbero anche giocatori ⇒ si capisce già da n

Simultaneous and independent

- Simultaneous moves do not really need to happen at the same time
  - it is just that strategies are chosen without knowledge of everybody else's actions
- These two versions are both "simultaneous"
  - version A: two players are writing their strategy on opposite sides of a board at the same time
  - version B: player 1 is asked to write first, while he writes, player 2 is blindfolded; then the board is turned and player 2 writes

Common knowledge

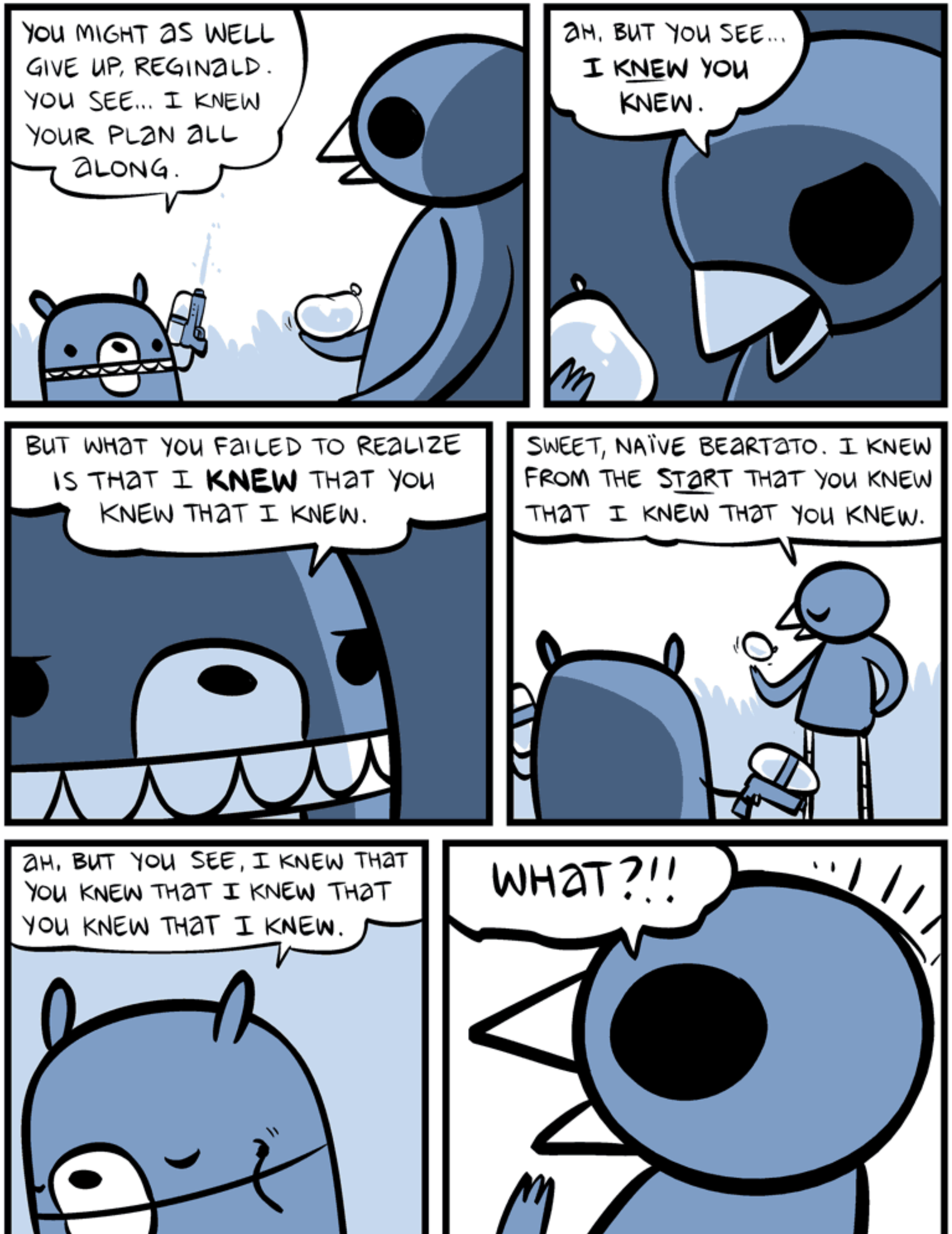
- We say that E is common knowledge if:
  - everybody knows E
  - everybody knows that everybody knows E
  - and so on, ad infinitum

- This is a powerful but not obvious assumption
  - it requires not only full knowledge on information pertinent to myself, but also on what should be everybody else's concern

Common knowledge

- "Complete information" means that
  - all possible actions of all players
  - all possible outcomes resulting from these actions
  - the individual preferences of all players about these outcomes (i.e., their utilities about them)

  are common knowledge among the players
- Player rationality is common knowledge
  - which means that everybody is maximizing their own payoff and everybody **knows** that everybody is maximizing their payoff!


YOU MIGHT AS WELL GIVE UP, REGINALD. YOU SEE... I KNEW YOUR PLAN ALL ALONG.
AH, BUT YOU SEE... I KNEW YOU KNEW.
BUT WHAT YOU FAILED TO REALIZE IS THAT I **KNEW** THAT YOU KNEW THAT I KNEW.
SWEET, NAÏVE BEARTATO. I KNEW FROM THE START THAT YOU KNEW THAT I KNEW THAT YOU KNEW.
AH, BUT YOU SEE, I KNEW THAT YOU KNEW THAT I KNEW THAT YOU KNEW THAT I KNEW.
WHAT?!!

Matrix representation

- n-player games can be represented as functions in $\boxed{S_1 \times S_2 \times \ldots \times S_n}$
- **If S_is are discrete sets**, a n-dimensional matrix can be used, where each cell contains again a n-dimensional value in $\mathbb{R}^n$
- Usually we have $n=2$, so the representation is simply an ordinary matrix where we put a pair of real numbers in each cell (therefore called a **bi-matrix**)

Example of bi-matrix

Player 2 strategies

Player 1 strategies

	$s_2^{(1)}$	$s_2^{(2)}$	$s_2^{(j)}$
$s_1^{(1)}$	$u_1(s_1^{(1)}, s_2^{(1)})$, $u_2(s_1^{(1)}, s_2^{(1)})$	$u_1(s_1^{(1)}, s_2^{(2)})$, $u_2(s_1^{(1)}, s_2^{(2)})$	……
$s_1^{(2)}$	$u_1(s_1^{(2)}, s_2^{(1)})$, $u_2(s_1^{(2)}, s_2^{(1)})$	……	……
$s_1^{(i)}$	……	……	$u_1(s_1^{(i)}, s_2^{(j)})$, $u_2(s_1^{(i)}, s_2^{(j)})$

payoff of user 1

payoff of user 2

Example 1

- Player A has three strategies: {U, M, D}
- Player B has two strategies: {L, R}

		player B: L	player B: R
player A	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

Example 2

- Player A has three strategies: {U, M, D}
- Player B has three strategies: {L, C, R}

		player B		
		L	C	R
player A	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

Example 3: Odds & Evens

- Players Odd and Even bet 4 euros
- Player Odd has two strategies: {0,1}
- Player Even has two strategies: {0,1}

		Even: 0	Even: 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

- "Similar" to: matching pennies (head/tails), penalty kick (left/right), poker (bluff/bet)

Example 4: rock/paper/scissors

- Players A and B bet 4 euros
- Player A has two strategies: {R,P,S}
- Player B has two strategies: {R,P,S}

		player B		
		R	P	S
player A	R	0, 0	-4, 4	4, -4
	P	4, -4	0, 0	-4, 4
	S	-4, 4	4, -4	0, 0

Example 5: Battle of Sexes

- Ann and Brian are partners who agreed to meet at a movie theater not knowing that 2 movies are available: romance (R) or sci-fi (S)
- Main goal of both is to see the other, but Ann prefers movie R and Brian prefers S

Ann \ Brian	R	S
R	2, 1	0, 0
S	0, 0	1, 2

Example 6: Prisoners' Dilemma

- Simple version: both Al and Bob can choose between (M) "lose 1$" or (F) "let the other pay 20$"

Al \ Bob	M	F
M	-1, -1	-21, 0
F	0, -21	-20, -20

Example 6: Prisoners' Dilemma

- Original version: involves a theft Al and Bob committed together. Caught by police, they can choose between (M)um and (F)ink
- Their (negative) payoff is the number of months they will spend in jail

		Bob: M	Bob: F
Al	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Pareto efficiency

- A joint strategy $\boxed{s}$ is **Pareto dominated** by another joint strategy $\boxed{s'}$ if

$$\boxed{\begin{array}{ll} u_i(s') \geq u_i(s) & \text{for } \textbf{every} \text{ player } i \\ u_i(s') > u_i(s) & \text{for } \textbf{some} \text{ player } i \end{array}}$$

no altra strategia strettamente migliore sotto ogni aspetto

- A joint strategy s not Pareto dominated by any joint strategy s', is said to be **Pareto efficient**
- There may be more than one Pareto efficient strategy, none of which dominates the others

Strict dominance

a comparison of strategies

Strictly dominated strategy

- Consider game G = $\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$
- If $s_i , s_i' \in S_i$, we say that s_i is **strictly dominated** by s_i' if i 's payoff when playing s_i' is greater than when playing s_i for any other move of the other players, i.e.

$$u_i(s_1,s_2,\ldots,s_i',\ldots,s_n) > u_i(s_1,s_2,\ldots,s_i,\ldots,s_n)$$

$$\forall\ (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n) \in S_1\times\ldots S_{i-1}\times S_{i+1}\times\ldots\times S_n$$

- Rational players do not play such strategies

back to Example 1

- Strategy D is strictly dominated by M

player B

player A		L	R
	U	6, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

M sempre superato dato altra mossa

back to Example 1

- Other strictly dominated strategies are found:
  first L for player B (dominated by R)
  then U for player A (dominated by M)

		player B	
		L	R
player A	U	6, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

rational players play (M, R), with result (4,3)

back to Example 6

- F dominates M for both Al and Bob.
- The only playable strategies give outcome = F,F
- This justifies the "Dilemma" name

Al \ Bob	M	F
M	-1, -1	-21, 0
F	0, -21	-20, -20

- The result does not seem that efficient

razionalità rispettata, ma risultato rimasto non è Pareto efficiente

Solving problems via IESDS

- This procedure is called "iterated elimination of strictly dominated strategies" (IESDS)
- Sometimes can find the outcome of a game, and is useful to obtain a "smaller" game by relying on common knowledge
- However, in several cases, it does not provide any solution

back to Example 2

- Here, no dominated strategy can be found.
- However, (D,R) seems to be a good choice.

player B

player A		L	C	R
	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

back to Example 3

- Neither 0 or 1 strictly dominates the other
- There seems not to be any “better” strategy

		Even 0	Even 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

back to Example 5

- Here, there are **two** strategies that seem to be "good" for rational players, (R,R) and (S,S)
- But again, no dominated strategy to eliminate

		Brian R	Brian S
Ann	R	2, 1	0, 0
Ann	S	0, 0	1, 2