### Game theory

A course for the MSc in ICT for Internet and multimedia

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# Rationalizing solutions

Best responses and beliefs

meglio, non gjorare streategre rempre dominate

### Single- vs. multi-player games

- For single-agent problems, once the setup is known, the solution can be found directly
- Not so for multi-player games
  - Here the solution depends on other players
  - Sometimes rationality can help (eg. we identify a dominated strategy → we do not play it)
  - We can extend this reasoning by assuming rationality of other players, which leads to IESDS
  - But still most of the times no solution is found

#### Best response

Strategy  $s_i \in S_i$  is i's best response to the opponent moves  $(s_1,...,s_{i-1},s_{i+1},...,s_n)$  if:

$$u_i(s_1,...,s_{i-1},s_i,s_{i+1},...,s_n) \ge u_i(s_1,...,s_{i-1},s_i',s_{i+1},...,s_n)$$
  
for every  $s_i' \in S_i$ 

- Notation:  $(s_1,...,s_{i-1},s_{i+1},...,s_n) \in S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_i$
- This is often shortened to " $\mathbf{s}_{-i} \in \mathcal{S}_{-i}$ "
- Thus:  $s_i \in S_i$  is a best response to  $s_{-i} \in S_{-i}$  if  $u_i(s_i, s_{-i}) \ge u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i$

#### Best response

- There may be more than one best response!
  - All with identical value of  $u_i(s_i, s_{-i})$  of course

c the		L	player B C	R
S;=U/D	<b>₹</b> U	~3 <sub>5</sub> 3	5, 1	62
5.; = R	layer v		8,4	(MG3,6) A1
	ם D	4)0	9,6	vi 6 8

■ So, U and D are both best responses to player B's strategy to play R. While for strategy L?

#### Best response

- Claim: a rational player who believes that the opponents are playing some  $s_{-i} \in S_{-i}$ , will always choose a best response to  $s_{-i}$
- □ Theorem: if  $\underline{s_i} \subseteq S_i$  is a strictly dominated strategy, it is no best response to any  $\underline{s_{-i}} \subseteq S_{-i}$ 
  - Proof: there must be  $s_i' \subseteq S_i$  dominating it
  - It is immediate to see that the definition of best response applied to  $s_i$  is violated by  $s_i$ .

#### Beliefs

- A **belief** of player i is a possible profile of opponents' strategies, ie., an element of set  $S_{-i}$ 
  - Beliefs are connected to best responses!
- We define a best-response-correspondence BR:  $S_{-i} \to p(S_i)$  that associates to  $s_{-i} \in S_{-i}$  a subset of  $S_i$  such that each  $s_i \in BR(s_{-i})$  is a best response to  $s_{-i}$ 
  - This is not a function: but  $BR(s_{-i})$  can be a singleton (if the best response is unique)

# Nash equilibrium

the key tool of game theory

### Nash equilibrium

- We want to strengthen the dominated strategy concept with this idea in mind:
  - game theory should make predictions about the outcome of games played by a rational players
  - a prediction is correct if the players are willing to play their predicted strategy
- That is, players choose their best response to the predicted strategy of the others
   If this happens, the prediction is said to be
- If this happens, the prediction is said to be self-enforcing (or also strategically stable)

#### Formal definition

- In a n-player game  $G = \{S_1, ..., S_n; u_1, ..., u_n\}$ , strategies  $(s_1^*, ..., s_n^*)$  are a **Nash equilibrium** if, for any i,  $s_i^*$  is the best response of player i to  $(s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*)$
- □ That is,  $\forall s_i \in S_i$ :  $u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*)$   $\geq u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*)$

$$s_i^* = \operatorname{argmax}_{s_i} u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots s_n^*)$$

#### Motivation

- □ Take a possible combination  $(s_1',...,s_n')$
- If this is not a Nash equilibrium then there exist some player i, such that  $s_i$  is **not** the best response to  $(s_1', ..., s_{i-1}', s_{i+1}', ..., s_n')$ .
- $\square$  That is,  $\exists s_i'' \in S_i$  such that

$$u_{i}(s_{1}',...,s_{i-1}',s_{i}',s_{i+1}',...s_{n}')$$
  
 $< u_{i}(s_{1}',...,s_{i-1}',s_{i}'',s_{i+1}',...s_{n}')$ 

□ Thus, there exists an incentive for player i to deviate from  $(s_1',...,s_n')$ 

#### Comment

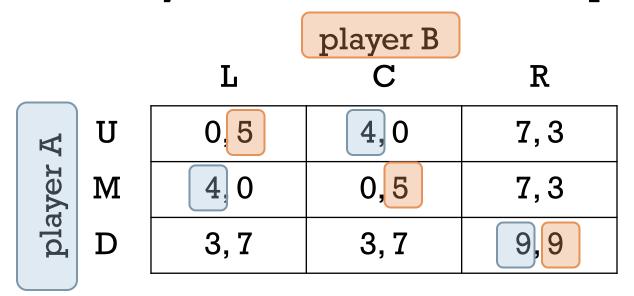
- Remember this is a static (one-shot) game
- A NE can also be seen as the case where nobody has regrets on his/her choice
  - it is intended as a forecast of the outcome, not as the final result of several moves
  - repeated games will disprove this wrong (but diffuse) misconception
- We will also discuss how useful it is to know that there is such a "natural" outcome

Combination (M,R) is a Nash equilibrium player B

	L	R		
<b>∀</b> U	6, 0	0, 5		
player A U W C	1,0	4,3		
pla D	0, 7	2,0		

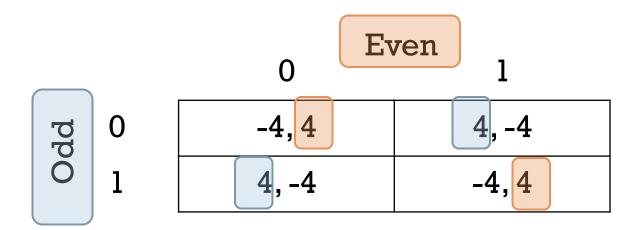
- (M,R) satisfies the NE condition
- A first way to find Nash equilibria is brute force search: here, (M,R) is the only one

Another way is to focus on "best responses"

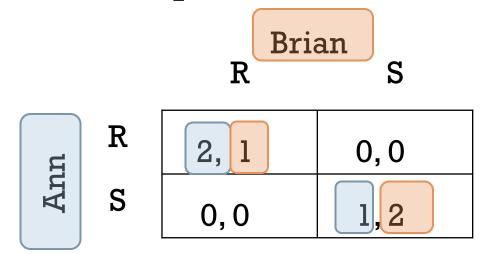


 (D,R) is the only Nash equilibrium, found by checking the cell with both entries highlighted

- Here there is no Nash equilibrium
- We will see that there is actually one, but we need to "extend" the game somehow

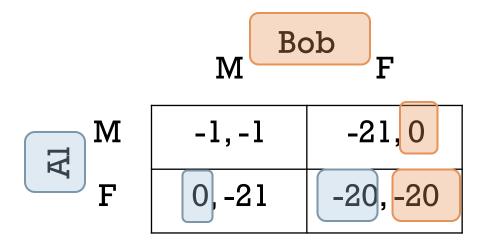


- □ (R,R) and (S,S) are both Nash equilibria
- This reflects our previous intuition



 However, here the NE concept is less useful as it cannot be used to make predictions

Combination (F,F) is a Nash equilibrium



 It seems that Nash equilibrium extends iterated elimination of strictly dominated strategies (i.e., if any exists, it is a NE)

#### Theorem

- □ In a finite game, if  $(s_1^*,...,s_n^*)$  is:
  - the only survivor of IESDS
  - or the only rationalizable profile  $\Rightarrow$  interests then  $(s_1^*, ..., s_n^*)$  is a NE supple
- □ Lemma: a  $NE(s_1^*,...,s_n^*)$  survives iterated elimination of strictly dominated strategies
- Another result: IESDS is order irrelevant

#### To sum up

- Two requirements must be satisfied by a NE
  - Everyone plays a best response to their beliefs
  - Everyone's beliefs are correct
- Actually the first requirement is quite logical and consequent from rationality, while the second requirement is quite demanding
  - It may be inferred only from some external reasoning (for example, one player being particularly "influential" in the game)

# Dominance, efficiency

further comparisons

#### Strict/weak dominance

□ For brevity, we write thereafter

$$S_{-i} = (S_j)_{j \neq i} = (S_1, S_2, ..., S_{i-1}, S_{i+1}, ..., S_n)$$

- Recall that  $s_i$  strictly dominates  $s_i$  if  $u_i(s_i',s_{-i}) > u_i(s_i,s_{-i})$  for every  $s_{-i}$
- $\square$  We say that  $s_i$  weakly dominates  $s_i$  if

$$u_{i}(s_{i}',s_{-i}) \ge u_{i}(s_{i},s_{-i})$$
 for **every**  $s_{-i}$   
 $u_{i}(s_{i}',s_{-i}) > u_{i}(s_{i},s_{-i})$  for **some**  $s_{-i}$  (\*)

 $\square$  Without (\*), we say that  $s_i$  dominates  $s_i$ 

## Dominance/Nash equilibrium

 A strategy that (strictly, weakly) dominates every other strategy of a user is said to be (strictly, weakly) dominant

#### Lemma

If every user i has a dominant strategy  $s_i^*$  then  $(s_1^*,...,s_i^*,...,s_n^*)$  is a Nash equilibrium.

- It directly follows from the definition of NE
- The reverse statement is false (only sufficient condition, not necessary)

#### Do not eliminate weakly dom.

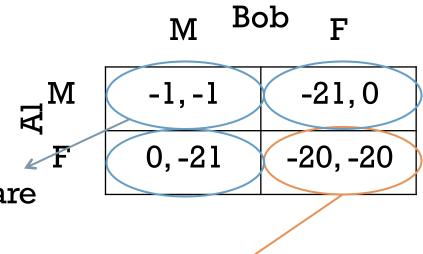
- Enlarge the Odd/Even game with a third strategy "Punch the opponent" (P)
- P is weakly dominated, yet it is a NE
- □ If we eliminate it, we lost the only NE

(a strange NE: later	ne	0	Even l	P	
course we will see similar situation)	a T	0	-5, 5	5, -5	-5, -5
, and the second	Odc	1	5, -5	-5, 5	-5, -5
		P	-5, -5	-5, -5	-5, -5

### NE vs. Pareto efficiency

- Pareto efficiency is different from NE:
  - Pareto efficiency: no way (in the whole game) a user can improve without somebody else being worse
  - Nash equilibrium: no way a user can improve with a unilateral change
  - The outcome of the Prisoner's Dilemma is not "efficient!"

These strategies are Pareto efficient



(F,F) is the only Nash equilibrium

#### NE vs. Pareto efficiency

non formulato bene

- Pareto inefficient Nash equilibria arise as we assume players are only driven by egoism
- To estimate the inefficiency of being selfish (or distributed) one can compare Nash equilibria with Pareto efficient strategies
- □ To this end, assume that a joint strategy s has a social cost K(s)
  - E.g.,  $K(s) = \sum_{i} -u(s_{i})$ , or  $K(s) = \max_{j} -u(s_{j})$  (this means overall welfare) (this is minmax tairness)

## Price of anarchy

The price of anarchy is the ratio between the social costs in the worst NE s\* and in the best Pareto efficient strategy (i.e., social optimum)

$$A = K(s^*) / (\min K(s))$$

- If the <u>best NE</u> is considered, it is sometimes spoken of **price of stability**
- For certain classes of problems, there are theoretical results on the price of anarchy

### Fun game

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game
- You are paired with an unknown classmate
- You secretly choose an integer from 18 to 30, and so does the classmate
- Then the numbers are checked
  - If they match, this is the score you both get.
  - If they don't, let L be the lower number. Who proposed L gets L+R, the other gets L-R (score <18 means rejection, >30 means honors)
- □ Play the game with R=2... Now with R=10

### Solution of the game

- If R > 1, there is a unique Nash equilibrium,
   which is to play 18 for both students
- However, cooperative behaviors may arise, even though they are not NE
  - Criticism against rationality of players
- Usually, a high R (for example 10) dampens the cooperation