Game theory

a course for the
MSc in ICT for Internet and multimedia

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Bayesian Games

From common knowledge to “just an idea”

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Bayesian games

- Equilibrium concepts were developed earlier as modeling players that are forming both **consistent** and **correct** beliefs
  - e.g., the payoff of other players must be known
- Harsanyi (1960) proposed that this can be incorporated into the concept of beliefs
  - We had beliefs about other players' moves (with certainty over their preferences and costs)
  - Now, we include beliefs about these characteristics as well!

Bayesian games

- Game of **incomplete information**: beliefs over the characteristics of other players are captured by their **types**
- Players can be of different types, which implies them to behave differently and also the other players to have beliefs about it
- We will develop an equilibrium concept that still requires these beliefs to be consistent and correct (so we use the same procedures)

Preliminary Nature's move

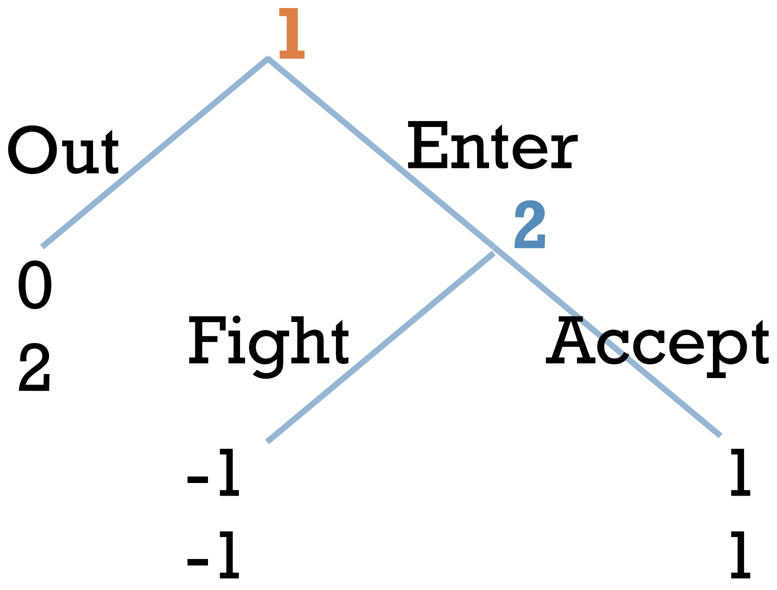
- Players may have different payoffs (each represented by a type)
- The timing of the game is as follows
    - Nature draws a type vector $(t_1,\ldots,t_n)$ among all possibilities for all the players
    - Nature reveals type t_i to player i only
    - Players choose their actions
    - Payoffs are computed
- This is a dynamic game where the players do not know Nature's move in its entirety

Remark on types

- Actually, types can represent more than just different preferences
  - Types may not differ in the player's preferences but in the **knowledge** that a player has about the types of the other players or some other characteristics of the game
  - More subtle variant: we keep it for a later stage

Example: Entry Game

- Player 1 is a newcomer (e.g., in a market or network); he may (E)nter or stay (O)ut
- Player 2 is incumbent, if 1 enters, 2 may (A)ccept or (F)ight
- SPE outcome is (E,A)
  - there is another NE outcome that is not SPE and it is (O,F); check it!



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Example: Entry Game

- Now, assume player 2 can be of two types
  - Rational: behaves as already discussed
  - Crazy: enjoys fighting and his/her payoff for (Enter, Fight) is actually 2 instead of -1
- Nature sets the type of 2: with probability *p* to be "Rational" (with 1-*p* he/she is "Crazy")
- Player 1 does not know the type of 2, and has non-singleton information set about it
  - Player 2 instead knows his/her own type

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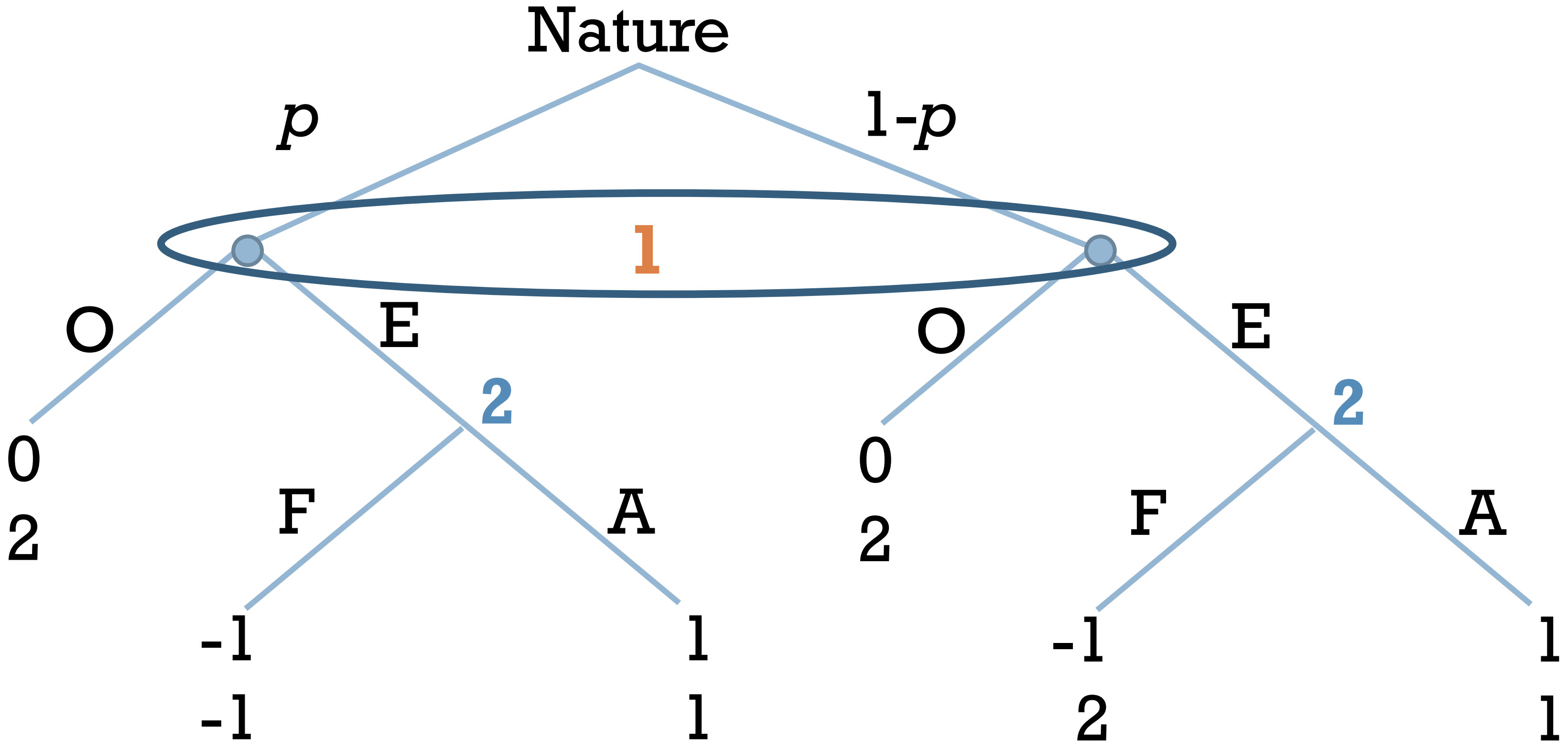
Entry game, with types



Entry game, beliefs

- Players know their type, not the opponents'
- How can they well form a best response?
  - They need to create beliefs about these types
  - For this to be possible, we assume that they do not precisely know the types of their opponents, but know how they are set
  - Thus, they know the **probability distribution** of the opponents' types: it is common knowledge!
  - This is called **common prior assumption**

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Entry game, strategies

- Also strategies need to be expanded
  - Apparently, player 2 does not have much choice even though he/she moves only if 1 chooses E
- In normal form, 2 has **four** pure strategies: two actions × two information sets
  - Because of Nature's move, 2 can have two types (1's move is irrelevant instead: check yourself)
- In other words, 2's strategy is (xy) where:
  - x describes what a rational player 2 does
  - y describes what a crazy player 2 does

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Bayesian strategies

- In incomplete information games, strategies are further expanded by stating what each type of a player does
  - The number of combinations explodes soon!
- In our example of Entry game:
  - player 1 has two strategies (O,E)
  - player 2 has four strategies (AA, AF, FA, FF)
- Each pair of pure strategies sets the path of play, which also depends on Nature's choice

Bayesian game, normal form

- For example, if the game is played as (E, AF)
  - player **1** gets: $p \times 1 + (1-p) \times (-1) =$ $\mathbf{2p-1}$
  - player **2** gets: $p \times 1 + (1-p) \times 2 =$ $\mathbf{2-p}$

player **2**

player **1**	AA	AF	FA	FF
O	0, 2	0, 2	0, 2	0, 2
E	1, 1	$2p$-1, 2-p	1-2p, 1-2p	-1, 2-3p

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Bayesian game, normal form

- The value of *p* matters to determine the NEs
- For example, set p = ⅔

player 2

player 1

	AA	AF	FA	FF
O	0, 2	0, 2	0, 2	0, 2
E	1, 1	⅓, 4/3	-⅓, -⅓	-1, 0

- If we are agnostic on the matrix and just seek NEs, we found three: (O,FA), (O,FF), and (E,AF)

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Bayesian games, discussion

- Actually the reasoning can be extended to mixed strategies in a straightforward way
- The reasoning still applies expected payoffs
  - but now derived from the probability of **types**!
- We change incomplete information into imperfect information
  - Specifically, uncertainty is now on Nature's choice of players' types at the beginning

Bayesian games, discussion

- In this example, player 1 just has one type
  - Clearly he wants to average on all possibilities of his opponent's (player 2) types
- For player 2, things are a bit more bizarre
  - He needs to take expectation on his own types!
  - Real players (**Rational 2** or **Crazy 2**) are replaced by a meta-player **MP2** playing for both sometimes called "type agent" representation
- Do the best responses of **MP2** = reality?
  - Yes, due to equivalence normal→extensive form

Bayesian games, discussion

- Evaluating expectations means that the probability distribution of types (not the types themselves) is common knowledge
- This is a strong assumption that is required to compute equilibria, because players must be able to conjecture on the game outcome
- The realism of this assumption may be critical and needs to be checked every time

Bayesian game: definition

Formal definition of the terms

Representing Bayesian games

- Our formalization of games used up to now a normal-form representation including:
  - set of players $\mathcal{N} = 1, .. , n$
  - strategy space of each player S_i (for $i = 1..n$)
  - utilities of players $u_i : (S_1, S_2 , .. S_n) \longrightarrow \mathbb{R}$ (for $i = 1..n$)
- Bayesian games add three more ingredients:

① **type** & type **space** of each player T_i (for $i = 1..n$)

② also, utilities are now **type-dependent**

③ finally, we need **beliefs** about other players types

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Static Bayesian game

- We consider a static game: n players, each player's strategy is just an action a_i in set A_i

① Player *i*'s type is $t_i \in T_i$, chosen by Nature for each player from 1 to *n* through the **prior probability distribution** $\phi(t_1,\ldots,t_n)$

  - the prior is **common knowledge** among players

② About the influence on types over payoffs, we make the assumption of **private values**

  - This means that $u_i = u_i(a_1, a_2, \ldots, a_n, t_i)$
  - If $u_i = u_i(a_1, a_2, \ldots, a_n, t_1, t_2, \ldots, t_n)$: **common values**

Type of a player

- For example, player i can have two different payoff functions $u_{i,a}(a_i, a_{-i})$, $u_{i,b}(a_i, a_{-i})$
  - we represent this by setting type space $T_i = \{t_a, t_b\}$ and imposing $u_i(a_i, a_{-i}; t_j) = u_{i,j}(a_i, a_{-i})$
- Types can be used to limit available actions
  - If a player has actions {F,G,H}, but H is permitted only with probability q, we define types t_a and t_b
  - t_a and t_b have respective probabilities q and $1 - q$
- In both cases {F,G,H} are feasible actions, but all payoffs of move H under type t_b are $-\infty$

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Beliefs on types

③ Because players know their own types, they can form **beliefs** on the other opponent by simply applying conditional probability:

$$\boxed{\phi(t_{-i} \mid t_i) = \phi(t_1,\ldots,t_n) / \phi(t_i)}$$

because players know their own types!

- **Comment**. Actually, we should include the entire hierarchy of "beliefs about other players' beliefs" that soon gets complicated
  - It can be shown that it is equivalent to "merge" down the hierarchy just to 1st-order beliefs

Beliefs on types (comments)

- Types are correlated; they are independent if

$$\boxed{\phi(t_1,\ldots,t_n) = \phi(t_1) \cdot \ldots \cdot \phi(t_n)}$$

- *i* knows its own type, but not others' (t_{-i}); he estimates them via $\boxed{\textbf{belief } \phi_i(t_{-i} \mid t_i)}$
  - Prior versus posterior probabilities
- Our assumption of the prior being common knowledge equals to **perfect information**
  - In the case of incomplete and imperfect info., belief $\phi_i(t_{-i} \mid t_i)$ may even be wrong and have nothing to do with the true prior

Static Bayesian game

- Static Bayesian needs
  - players ①
  - action spaces ②
  - type spaces ③
  - beliefs ④
  - (type-dep.) payoffs ⑤
- $G = \{\mathcal{N} \overset{①}{;} A_1, \overset{②}{\ldots}, A_n ; T_1, \overset{③}{\ldots}, T_n ; \phi_1, \overset{④}{\ldots}, \phi_n ; u_1, \overset{⑤}{\ldots}, u_n \}$
  - where $u_i = u_i(a_1, \ldots, a_n ; t_i)$.
- A **pure strategy** for *i* is a map $s_i : T_i \longrightarrow A_i$, i.e., it tells what *i* plays as his/her type is known
- A **mixed strategy** for *i* is a probability distribution over pure strategies

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Strategies of Bayesian games

- Type-contingent definition of pure/mixed strategies → reminiscent of dynamic games
  - We can think of a general strategy as being defined before the type of *i* is even set!
  - Player *i* decides a strategy $\boxed{s_i : T_i \longrightarrow A_i}$: then, if her type is $t_i \in T_i$, she will play $s_i(t_i)$
  - This is useful because it allows **other players** to create beliefs over the strategy of a player *i* who can be of different types

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Strategies of Bayesian games

take p = $\frac{2}{3}$

		player 2			
		AA	AF	FA	FF
player 1	O	0, 2	0, 2	0, 2	0, 2
	E	1, 1	$\frac{1}{3}$, $\frac{4}{3}$	-$\frac{1}{3}$, -$\frac{1}{3}$	-1, 0

- 2's strategies depend on Nature setting $t_2 \in \{r, c\}$
- If 1 believes 2 uses pure strategy s_2=AF: $\begin{cases} s_2(r)=A \\ s_2(c)=F \end{cases}$
- 1's expected payoff when playing E is $\mathbb{E}[u_1(E, s_2)] = p\, u_1(E, s_2(r)) + (1-p)\, u_1(E, s_2(c))$ that for $p=\frac{2}{3}$ is exactly the matrix entry for (E,AF)

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Strategies of Bayesian games

- **Comment 1**. Assume a typed player i is using a type-dependent pure strategy, and Nature randomly chooses his/her type t_i
  - from the point of view of the opponents $-i$, they are facing a player using mixed strategies
  - in both cases an expectation is taken: for the opponents $-i$ it does not matter if it is i's own decision to randomize the action or just a consequence of its (Nature-chosen) type

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Strategies of Bayesian games

- **Comment 2.** When specifying a strategy of a typed player, must we be type-dependent?
  - Why specify what *i* will do if Nature chooses any of *i*'s types, if in the end only one is real?
- No need for *i* itself.. but for the opponents –*i* !
  - This way, they can form beliefs over *i*'s behavior
  - And, combined with the posterior (giving the condition on *i*'s type), compute expected payoffs
- We did a similar thing for dynamic games!

Bayesian Nash equilibrium

- It is a Nash equilibrium in Bayesian games
- In $G=\{\mathcal{N}; A_1,..,A_n; T_1,..,T_n; \phi_1,..,\phi_n; u_1,..,u_n\}$, joint strategy $s^* = (s^*_1,\dots, s^*_n)$ is said to be a **Bayesian Nash Equilibrium** if for each player *i* and each type $t_i \in T_i$, $s_i(t_i)$ maximizes the expected payoff.

  That is

$$\max_{s_i \in S_i} \sum_{t_{-i}} u_i(s^*_1(t_1),..,s^*_{i-1}(t_{i-1}), s_i, s^*_{i+1}(t_{i+1}),..,s^*_n(t_n), t_i)\, \phi_i(t_{-i} \mid t_i)$$

Bayesian Nash equilibrium

- This can be rewritten as:

$$\mathbb{E}[u_i(s^*_i(t_i), s^*_{-i}(t_{-i}), t_i) \mid t_i] \geq \mathbb{E}[u_i(\mathbf{s}_i , , s^*_{-i}(t_{-i}), t_i) \mid t_i]$$

for every $\mathbf{s}_i \in S_i$

- In other words, *i* does not want to change strategy given the information he/she knows
  - strategy of *i* = a choice of action for each type
  - what *i* does not know, he just estimates!
- This definition can be generalized according to the type space (if continuous → integrals)

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mixed strategies / types

Harsanyi's interpretation of beliefs

back to Battle of the Sexes

		Brian	
		R	S
Ann	R	2, 1	0, 0
	S	0, 0	1, 2

- Mixed strategy = probabilities to play R
- NEs of the game: (0,0), (1,1) and (⅔,⅓)
- The mixed NE can be seen as **pure** BNE of a related game with a bit of incomplete info

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back to Battle of the Sexes

Brian

		R	S
Ann	R	2 + **c** , 1	0, 0
	S	0, 0	1, 2 + **d**

- Ann and Brian do not know each other well
- Increase Ann’s payoffs at (R,R) and Brian’s at (S,S) by **c** and **d**, both (**c** and **d**) falling in [0,x]
  - Think of x as a “perturbation”
  - The exact amount of **c** (or **d**) is privately known by Ann (or Brian) only: type of the player

back to Battle of the Sexes

- Ann's strategy: play R if $c > C$, otherwise play S
  - Same for Brian, choose S if $d > D$, else play R
- This strategy is in fact a Bayesian NE
- Ann does not know d! Her expected payoff is

  $D/x\ (2+c) + (1- D/x) \cdot 0 \ = (2+c)\ D/x$

  if she plays R

  $D/x\ \cdot 0 + (1- D/x) \cdot 1 = 1- D/x$

  if she plays S
- Thus, Ann plays R if $c \geq \boxed{x/D - 3} = C$

back to Battle of the Sexes

- We have $x / \mathbf{D} - 3 = \mathbf{C}$
- Similarly, Brian's expected payoff is

  $(1- \mathbf{C}/x) \cdot 0 + \mathbf{C}/x\,(2+\mathbf{d}) = (2+\mathbf{d})\,\mathbf{C}/x$

  if he plays S

  $(1- \mathbf{C}/x) \cdot 1 + \mathbf{C}/x\,(2+\mathbf{d}) \cdot 0 = 1- \mathbf{C}/x$

  if he plays R
- Brian plays S if $\mathbf{d} \geq x / \mathbf{C} - 3$. Thus, $x / \mathbf{C} - 3 = \mathbf{D}$.
- Combining these two conditions we have

  $x / \mathbf{D} - 3 = \mathbf{C}$, $x / \mathbf{C} - 3 = \mathbf{D}$

  Thus, $\mathbf{C} = \mathbf{D}$ and $\mathbf{C}^2 + 3\,\mathbf{C} - x = 0$

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back to Battle of the Sexes

- Solving $\mathbf{C}^2 + 3\,\mathbf{C} - x = 0$
- $\mathbf{C} = \dfrac{-3 + \sqrt{9+4x}}{2}$
- The probability of playing R for Ann is thus 1-**C**/x, i.e.,

$$\frac{2x + 3 - \sqrt{9+4x}}{2x} \xrightarrow[x \to 0]{} \tfrac{2}{3}$$

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