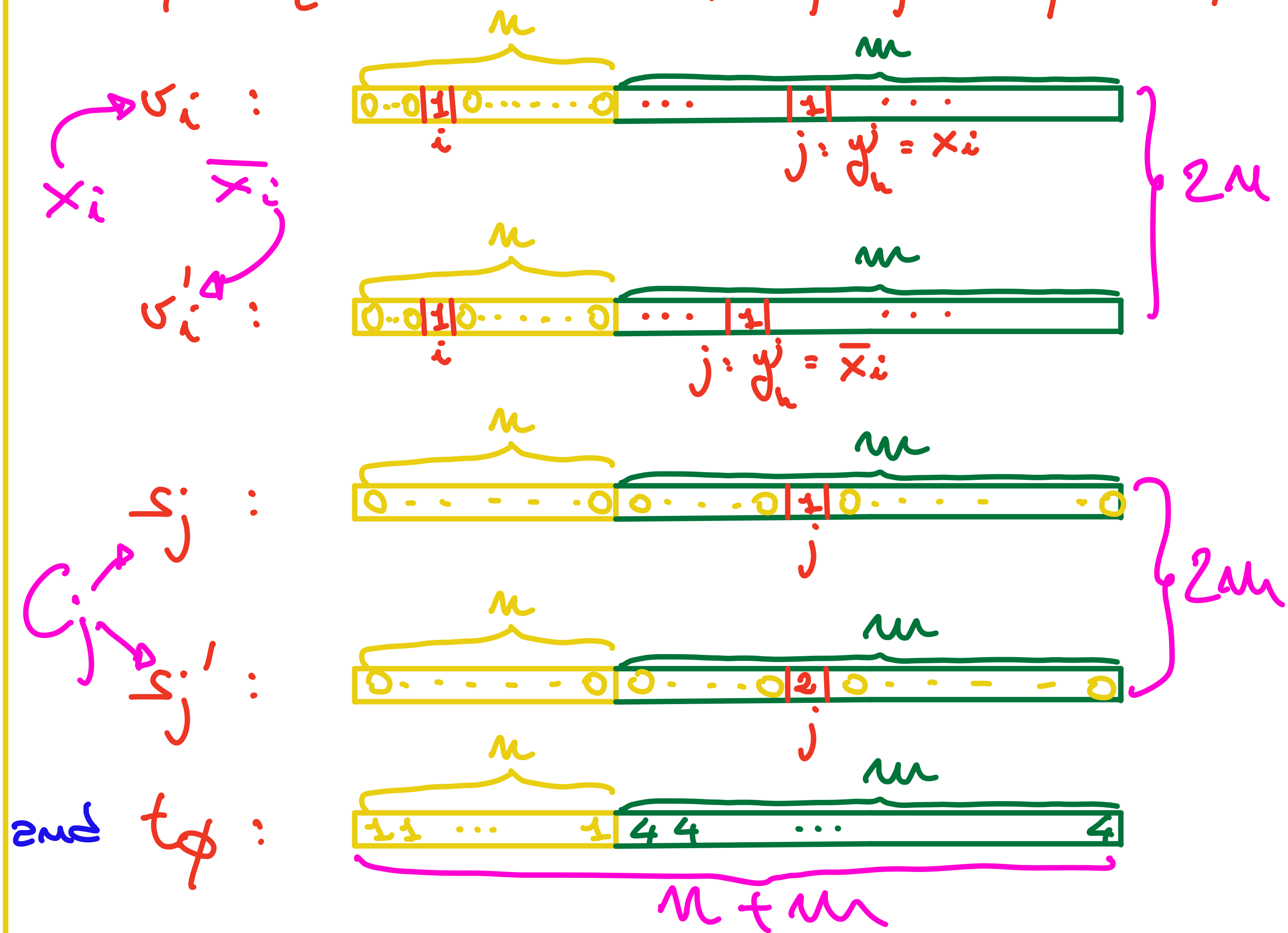
3-CNF-SAT $\le_p$ SUBSET SUM (SS)

$\langle S, t\rangle;\ \exists S' \subseteq S : \sum_{s \in S'} s = t$?

$$\langle \phi(x_1, \dots, x_n) = C_1 \wedge \dots \wedge C_m \rangle \xrightarrow{f} \langle S_\phi, t_\phi \rangle$$

$$S_\phi = \{ v_i, v_i' : 1 \le i \le n ;\ s_j, s_j' : 1 \le j \le m \}$$



Crucial property: In each position, the sum of the digits of all numbers in S_ϕ is ≤ 6. NO CARRIES!

IDEA: t_ϕ achievable if and only if each clause satisfiable under same $\vec{b}$

EXAMPLE:

$$\phi(x_1,x_2,x_3) = \underbrace{(x_1 \vee \bar{x}_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_1 \vee x_2 \vee x_3)}_{C_2}$$

S_ϕ:

	x_1	x_2	x_3	C_1	C_2	
v_1:	1	0	0	1	1	10.011
v_1':	1	0	0	0	0	10.000
v_2:	0	1	0	0	1	1.001
v_2':	0	1	0	1	0	1.010
v_3:	0	0	1	1	1	111
v_3':	0	0	1	0	0	100
s_1:	0	0	0	1	0	10
s_1':	0	0	0	2	0	20
s_2:	0	0	0	0	1	1
s_2':	0	0	0	0	2	2
t_ϕ:	1	1	1	4	4	11.144

n (variable columns), m (clause columns); $3n$ (rows v_1 to v_3'), $2m$ (rows s_1 to s_2')

there are exactly 3 1's in each clause position and exactly 2 1's in each variable position

the sum in each clause position is 3 (0 in variable positions)

NOTE:

$$\sum_{i=1}^{n}(v_i+v_i') + \sum_{j=1}^{m}(s_j+s_j') = (\underbrace{2\cdots2}_{n}\underbrace{6\cdots6}_{m})_{10}$$

Let us prove that

$$\boxed{\langle \phi(x_1,\dots,x_n)=\bigwedge_{j=1}^{m} C_j \rangle \in \text{3-CNF-SAT} \iff f(\langle\phi\rangle)=\langle S_\phi, t_\phi\rangle \in SS}$$

$\Rightarrow$ specificare subito implicazione di $x \in L_1$

$\phi \in$ 3-CNF-SAT $\Rightarrow \exists \vec{b} : \phi(\vec{b})=1$.

Let us build a subset $S'_\phi \subseteq S_\phi : \sum_{s \in S'_\phi} s = t_\phi$

For each $1 \le i \le n$:

- if $b_i = 1$ then $v_i \in S'_\phi$
- if $b_i = 0$ then $v'_i \in S'_\phi$

After this selection of n values:

1. The sum of the values in S'_ϕ will have 1's in the n most significant digits (exactly one between v_i and v'_i is selected for $1 \le i \le n$)

2. The digit of the sum in each clause column j (in least significant digits) reflects the number of true literals in C_j under $\vec{b}$ (v_i has 1's for clauses containing x_i, while v'_i has 1's for clauses containing $\overline{x}_i$)

Observe that since $\phi(\vec{S}) = 1$, thanks to 2) each sum digit in clause positions is >0! (either 1, 2, or 3).

Then $\forall j,\ 1 \le j \le n$ I can "adjust" the digit in the j-th position by adding to S'_ϕ:

1) S_j, if the sum digit is 3 (3 true literals)

2) S'_j, if the sum digit is 2 (2)

3) S_j AND S'_j, if the sum digit is 1 (1)

Now, in $\sum_{s \in S'_\phi} s$, the digit in each of the m least significant positions is 4!

Thus, $\boxed{\sum_{s \in S'_\phi} s = \underbrace{1\ldots1}_{n}\underbrace{4\ldots4}_{m} = t_\phi}$,

therefore $\boxed{f(\langle \phi \rangle) = \langle S_\phi, t_\phi \rangle \in SS}$

EXAMPLE:

$$\phi(x_1, x_2, x_3) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

Let $\vec{S} = (1, 0, 1) \Rightarrow \phi(\vec{S}) = 1$

We put s_1, s'_2, s_3 in S'_ϕ:

$\{ s_1 = 10011,\ s'_2 = 01010,\ s_3 = 00111 \}$

The sum in the column relative to

C_1 is 3 $\Rightarrow$ we add $s_1 = 00010$

C_2 is 2 $\Rightarrow$ we add $s_2 = 00002$

$$\sum_{s \in S'_\phi} s = \begin{array}{rl} 10011 + & \sigma_1 + \\ 1010 + & \sigma'_2 + \\ 111 + & \sigma_3 + \\ 10 + & s_1 + \\ 2 & s'_2 + \\ \hline 11144 & \end{array} = t_\phi$$

$\Longleftarrow$ Let $f(\langle\phi\rangle) = \langle S_\phi, t_\phi\rangle \in SS$

then $\exists\, S'_\phi : \sum_{s \in S'_\phi} s = t_\phi$

1. Since the sum digit in each of the n most significant digits is 1, there is either σ_i or σ'_i in S'_ϕ, for $1 \le i \le n$

Let $\boxed{b_i = \begin{cases} 1 & \text{if } \sigma_i \in S'_\phi \\ 0 & \text{if } \sigma'_i \in S'_\phi \end{cases}}$

In each clause column j, since the sum digit is 4, there is a contribution ≥ 1 coming from σ_i or σ'_i, for all $1 \le i \le n$

(or otherwise the sum digit would be $\leq 1+2=3$!)

$\Rightarrow$ under $\vec{b}$ there is at least one true literal for each clause

$\Rightarrow$ $\boxed{\phi(\vec{b})=1}$ $\Rightarrow$ $\boxed{\langle \phi \rangle \in \text{3-CNF-SAT}}$

EXAMPLE:

$$\phi(x_1,x_2,x_3)=(x_1 \vee \bar{x}_2 \vee x_3)\wedge(x_1 \vee x_2 \vee x_3)$$

Let $S'_\phi = \{s_1, s'_2, s_3, s_1, s'_2\}$. Then

$$\begin{array}{ccccccl}
1 & 0 & 0 & 1 & 1 & + & s_1 + \\
0 & 1 & 0 & 1 & 0 & + & s'_2 + \\
0 & 0 & 1 & 1 & 1 & + & s_3 + \\
0 & 0 & 0 & 1 & 0 & + & s_1 + \\
0 & 0 & 0 & 0 & 2 & = & s'_2 = \\
\hline
1 & 1 & 1 & 4 & 4 & = & t_\phi
\end{array}$$

We build $\vec{b}=(1,0,1)$. Under $\vec{b}$ there are 3 true literals in C_1 (only $s_1 \in S'_\phi$) and 2 true literals in C_2 (only $s'_2 \in S'_\phi$)