MAD 350

Project : Application of Measure Theory in Finance

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Content

Capturing Information Flow in Financial Markets

- Structure of a σ-algebra is specially suited to model information

Definition 1. A family $\mathbb{X}$ of subsets of a set X is said to be a σ-algebra in case :

i) $\emptyset$, X belong to $\mathbb{X}$.

ii) If A belongs to X, then $A^{\complement}$ belongs to $\mathbb{X}$.

iii) If (A_n) is a sequence of sets in $\mathbb{X}$, then the union $\bigcup_{n=1}^{\infty} A_n \in \mathbb{X}$.

An ordered pair $(X, \mathbb{X})$ consisting of a set X and a σ - *algebra* $\mathbb{X}$ is called a *measurable space.*

- Let the set of all outcomes of a random experiment be the set Ω, and let ω be a particular outcome
- We have some information, which may or may not us in evaluating ω
- If we know, for a set A , whether the eventual ω lies in it or not, then it is said to be *resolved*

i) $\emptyset$, X are always resolved.

ii) If A is resolved then $A^{\complement}$ is also resolved.

iii) If (A_n) is a sequence of resolved sets , then the union $\bigcup_{n=1}^{\infty} A_n$ is also resolved.

Capturing Information Flow in Financial Markets

Definition 5. Let Ω be a non-empty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is an associated σ - *algebra* $F(t)$. Assume further that if $s \leq t$, then $F(s) \subseteq F(t)$. Then the collection of σ - *algebras*, $F(t)$, $0 \leq t \leq T$, is called a *filtration*.

•When we get to time *t*, we will know for each set in *F(t)* whether the eventual *ω* lies in that set or not.

• In mathematical finance, a filtration represents the information available up to and including each time *t*, and is more and more precise (the set of measurable events is staying the same or increasing) as more information from the evolution of the stock price becomes available.

• Knowing the value of *S(ω)* , does give us some information

•, Resolves every set of the form *{ω : S(ω) belongs to B, where B of B(R)}*.

• Information provided by S is captured by σ(S). *

Definition 8. Let Ω be a nonempty sample space equipped with a filtration $\mathscr{F}(\mathrm{t}), 0 \leq t \leq T$. Let $S(t)$ be a collection of random variables indexed by $t \in [0,T]$. We say this collection of random variables is an *adapted stochastic process* if , for each t, the random variable $S(t)$ is $\mathscr{F}(\mathrm{t})$measurable.

• Introduce the notion of stochastic processes for asset prices, portfolio processes, and wealth process (value of a portfolio), and these are adapted to a filtration that we regard as a model of the flow of public information.

Towards Conditional Expectation

In general , there are three cases that arise when we have a random variable S and a σ - *algebra* $\mathcal{G}$:

1. The information contained in the $\mathcal{G}$ is sufficient to determine the value of S. In this case, S is $\mathcal{G}$- measurable.
2. The information contained in $\mathcal{G}$ gives us no clue about the value of S. In this case, we say that S is *independent* of $\mathcal{G}$.
3. The information contained in $\mathcal{G}$ is not sufficient to evaluate S, but we can get some estimate of S based on the information in $\mathcal{G}$.

Definition 2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$ be a $sub-\sigma-algebra$ of $\mathcal{F}$, and let S be a random variable that is either nonnegative or integrable. The conditional expectation of S given $\mathcal{G}$, denoted by $\mathbb{E}[S|\mathcal{G}]$, is any random variable that satisfies :

1) (Measurability) $\mathbb{E}[S|\mathcal{G}]$ is $\mathcal{G}$-measurable, and
2) (Partial Averaging)

$$\int_C \mathbb{E}[S|\mathcal{G}\, dP = \int_C S\, dP \; for\, all\, C \in \mathcal{G}$$

- But does it exists ?
- Proof by Radon – Nikodym Theorem
- Can also prove other properties.

Alternative View

- Closest approximation to a random variable *S* if we restrict ourselves to random variables belonging to some smaller σ-algebra . (sub- σ-algebra)

Definition 7. Let $\mathcal{G} \subseteq \mathcal{F}$ be $\sigma-algebras$ and S be a random variable on $(\Omega, \mathcal{F}, P)$. Assume $< \infty$. Then the conditional expectation is an almost surely unique $\mathcal{G}$-measurable Y_G such that $\mathbb{E}[(S - Y_G)^2] = inf_z \mathbb{E}[(S - Z)^2]$, where the infimum is over all $\mathcal{G}$-measurable random variables. Note: We denote the minimizing Y_G by $\mathbb{E}[S|\mathcal{G}]$.

- Claim 1 : The Y_G in Definition 7 satisfies, for each $C \in \mathcal{G}$, :

$$\int_C Y_G dP = \int_C S dP$$

- Claim 2 : If G = { ϕ, Ω } , then Y_G = E[S]
- Claim 3 : If $\mathcal{G} = \{\Omega,\ A\ A^{C}\ \emptyset\}$, for some event A. then

$$Y_G = \begin{cases} \mathbb{E}[SI_A]/P(A)\ for & \omega \in A \\ \mathbb{E}[SI_B]/P(B)\ for & \omega \in B \end{cases}$$

- Claim 4 : If G is generated by a finite partition $\{A_1, A_2, \ldots A_N\}$, then :

$$Y_g = \sum_1^n c_i I_{A_i}(\omega) \qquad where\, c_i = \mathbb{E}[SI_{A_i}]/P(A_i)$$

A Topological Approach (Hilbert Spaces)

- Two ways of looking at the same general, more abstract concept of Conditional Expectation
- Existence of Conditional Expectation arises as a direct consequences of the special structure of Hilbert Spaces

Theorem 14. *If $1 \leq p < \infty$, then the space $\mathcal{L}_p$ is a complete normed linear space (Banach) under the norm*

$$\| S \|_p = \left\{ \int |S|^p \right\}^{\frac{1}{p}}$$

Proof: Robert G. Bartle. *The Elements of Integration and Lebesgue Measure.* Theorem 6.14. Pg 69. ■

Lemma 16. *L_2 spaces are Hilbert Spaces with the inner product being defined by :*

$$< X,Y >= \int_C XY\, dP \text{ or } \mathbb{E}[XY]$$

Theorem 15. *Hilbert Space Projection Theorem. For every point x in a Hilbert space H and every closed convex set $C \subset H$, there exists a unique point z_0 for which $\| x - z \|$ is minimized over C. This is, in particular, true for any closed subspace M of H. In that case, a necessary and sufficient condition for y is that the vector $x - y$ be orthogonal to M.*[4]

A Topological Approach (Hilbert Spaces)

- Proof of Existence of Conditional Expectation by Hilbert Space Projection Theorem

Thus, the Conditional Expectation is :

- the best approximation of *S*
- the function in the sub-σ-algebra G closest to S, where the distance is given by the L_2 norm
- the orthogonal projection of *S* onto G.

- can extend to all non- negative random variables
- can extend all integrable random variables

The special stucture of the L_2 Hilbert Space (one-to-one correspondence between closed subspaces and orthogonal projections) and σ-algebras allow us to model public information in such a way that we can estimate any random variable (like future Stock Price) , whose Lebesgue Integral exists.

Some Useful Properties

• **Linearity of Conditional Expectation**: If X and Y are integrable random variables and c_1 and c_2 are constants, then :

$$E[(c_1X + c_2Y)|G] = c_1E[X|G] + c_2E[Y|G]$$

• **Taking out what is known** : If X and Y are integrable random variables, XY is integrable, and X is G-measurable, then :

$$E[XY|G] = XE[X|G]$$

• **Iterated Conditioning** : If H is a sub-σ-algebra of G, and X is an integrable random variable, then :

$$E[E[X|G]|H] = E[X|H]$$

• **Independence** : If X is integrable and independent of G, then:

$$E[X|G] = E[X]$$

References

1. Robert G. Bartle.The Elements of Integration and Lebesgue Measure (Wiley Classics Library Edition 1995)
2. S.E Shreve. Stochastic Calculus for Finance - Volume I and II. (Springer, 2004)
3. Robert B. Ash and Catherine A.Doleans-Dade, Probability and Measure Theory.(Elsevier)
4. http://planetmath.org

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