Capturing Information Flow in Financial Markets

Piyush Ahuja

***Abstract*—This paper seeks to model the information flow in financial markets using some basic concepts from measure theory. We introduce the useful constructs of a filtration and an adapted stochastic process. The model developed here provides an efficient tool for capturing information for use in subsequent financial activities, like derivative pricing or hedging.**

A. *Preliminaries*

The construct of a σ-algebra is used extensively in measure theory literature. These are the sets on which a measure can suitably be defined. In probability theory (a special case of measure theory), a sigma algebra is the collection of events which can be assigned probabilities. We will find that this structure is specially suited to provide a representation of *information* , and thus will be extensively used in our modelling of the financial markets.

Given a set X , a σ-algebra is a family $\mathbb{X}$ of subsets of X, which are "well-behaved" in a certain technical sense[**?**]. This special behavior arises out of the structure it posseses - the family contains the empty set $\emptyset$ and the entire set X, and it is closed under complementation and countable unions:

Definition 1. A family $\mathbb{X}$ of subsets of a set X is said to be a σ-algebra in case :

i) $\emptyset$, X belong to $\mathbb{X}$.

ii) If A belongs to X, then $A^{\complement}$ belongs to$\mathbb{X}$.

iii) If $(A_n)$is a sequence of sets in $\mathbb{X}$, then the union $\bigcup_{n=1}^{\infty} A_n$ $\in \mathbb{X}$.

An ordered pair (X, $\mathbb{X}$) consisting of a set X and a $\sigma - algebra$ $\mathbb{X}$ is called a *measurable space.*

Another concept which we will find useful is that of measurable functions : structure preserving functions between measurable spaces.

Definition 2. A function f from a measurable space $(X, \mathbb{X})$ to another measurable space $(Y, \mathbb{Y})$ is called *measurable* in case the set $f^{-1}(E) = \{x \in X : f(x) \in E\}$ belongs to $\mathbb{X}$ for every set E belonging to $\mathbb{Y}$.

Remark 3. Measurable functions will not only help us to shift to spaces , like $(\mathbb{R},\mathscr{B}(\mathbb{R})$, which we are comfortable with, but also provide us a insights on how and what kind of information is contained in a $\sigma - algebra$.

B. *Information and $\sigma - algebras$*

In finance, we seek to make decisions and develop strategies based on some quantitative information we have.This information could be the past asset prices, past trading strategies or something more abstract, like a measure of public sentiment. We attempt to mathematically model the information on which our future strategies or actions are based. To estimate outcomes based on information, we denote the set of all outcomes of a random experiment as Ω, and a particular member of the set i.e. a particular outcome by ω[1]. We also might have some information, which may or may not us in evaluating ω. For example, the information we have might not help us calculate the precise outcome of a random experiment, but it might be enough to narrow down the possibilities.We illustrate this with the following example :

Example 4. Infinite Independent Coin Toss space

We toss a coin infinitely many times and let Ω_∞ denote the set of all possible outcomes. $\Omega_\infty =$ the set of infinite sequences of Hs and Ts. A generic element of Ω_∞ will be denoted by $\omega = \omega_1\omega_2\ldots$, where ω_n indicates the result of the nth coin toss.

We define the sets $A_{\upsilon_1\upsilon_2\ldots\upsilon_n} = \{\omega; \omega_1\omega_2\ldots\omega_n = \upsilon_1\upsilon_2\ldots\upsilon_n\}$ i.e. the set of all sequences beginning with $\upsilon_1\upsilon_2\ldots\upsilon_n$, where $\upsilon_i = H$ or T.

Now, suppose we have performed only k coin tosses, and know the the outcome of these. In such a situation, although we do not know the true value of ω precisely (because we dont know the outcomes of the future coin tosses), we can still make a list of sets which are sure to contain it and other sets that are sure not to contain it. We call these sets as having been *resolved* by the information. Note that the all the sets of the form $A_{\upsilon_1\upsilon_2\ldots\upsilon_s}$, where $s \le k$, have been resolved, so have their complements, and countable unions. The empty set $\emptyset$ and the whole space Ω are always resolved, even without any *information*; the eventual ω does not belong to $\emptyset$ and does belong to Ω. Thus, the family of all sets that have been resolved by knowing the outcome of the first k coin tosses form a $\sigma - algebra$; we denote it by $\mathscr{F}_{\mathrm{k}}$.We can think of this $\sigma - algebra$ as containing the *information* learned by observing the first k tosses. If instead of knowing the precise outcomes of the first k tosses, we know, for each set in F_{k}, whether or not the eventual (true) ω belongs to the set, we know the outcome of the first k coin tosses and nothing more. Note that $F_0 = \{\emptyset, \Omega\}$, and $F_{\mathrm{s}} \subseteq F_k$ for $s \le k$. In other words, F_{k} contains every set of $F_{\mathrm{s}}$and even more.This means that F_{k} contains more *information* than F_{s}.

The above example suggest a way by which the concept of *information* can be captured in the structure of the measure theoretic construct of a $\sigma - algebra$.We generalise the above analysis in order to define a more abstract construct, called the filtration:

Definition 5. Let Ω be a non-empty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is

Submitted as part of Mini Project (MAD 350), under the supervision of Professor S.Kundu

[1] In most real life scenarios, Ω will be uncountable infinite.

an associated $\sigma - algebra\ F(t)$. Assume further that if $s \leq t$, then $F(s) \subseteq F(t)$.Then the collection of $\sigma - algebras$, $F(t)$, $0 \leq t \leq T$, is called a *filtration.*

In continuous time models, like those characterisic of Financial Markets, a filtration tells us the information we will have at future times. When we get to time t, we will know for each set in $F(t)$ whether the eventual ω will lie in that set or not. In general, a filtration is often used to represent the change in the set of events that can be measured, through gain or loss of information. In mathematical finance, a filtration represents the information available up to and including each time t, and is more and more precise (the set of measurable events is staying the same or increasing) as more information from the evolution of the stock price becomes available.

C. Information and Random Variables

Note that we havnt introduced any measure space or measurable functions yet, and while tackling problems, we will be dealing with probability measures and random variables. The abstract space Ω is difficult to model in real life situations. It is much more convenient to associate with each outcome ω, a numerical quantity $S(\omega) \in \Re$[2]. This motivates the following definition :

Definition 6. Let $(\Omega, \mathscr{F})$ be a measurable space . A random variable S is a measurable function from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R})$, where $\mathscr{B}(\mathbb{R})$is the Borel $\sigma - algebra$ of $\mathbb{R}$.

In practical scenarios, we will acquire information about the value of ωthrough numerical quantities, or random variables. For example, in a financial market, we might possess information in terms of asset or stock price, portfolio positions etc. In such cases, we will be interested in what information we can capture through these random variables, and how are these incorporated in our filtration.

Note that, knowing the value of $S(\omega)$, instead of the true ω, does give us some information, as it resolves certain sets. Actually, every set of the form $\{\omega : S(\omega) \in B\}$, where B is a subset of $\mathbb{R}$, is resolved. Thus, the information provided by the random variable is captured in the $\sigma - algebra$ generated by S i.e. $\sigma(S)$. Also, since for each set in $\sigma(S)$, we can tell whether ω belongs to it or not, we can actually determine the value of $S(\omega)$.

Definition 7. Let S be a random variable defined on a nonemtpy sample space Ω. The $\sigma - algebra$ generated by S, denoted by $\sigma(S)$, is the collection of all subsets of Ω of the form $\{\omega : S(\omega) \in B\}$, where B ranges over the Borel subsets of $\mathbb{R}$.

If there is a $\sigma - algebra\ \mathscr{G}$ such that $\sigma(S) \subseteq \mathscr{G}$ (S is $\mathscr{G}$-measurable) , then the sets in $\sigma(S)$ are also in $\mathscr{G}$, and thus the information contained in $\mathscr{G}$ is sufficient to determing the value of S.

Having developed the relation between *information,* $\sigma - algebras$ and random variables, we are now in a position to formalise it for our purpose of modelling financial markets:

[2]Here S refers to a function of ω, defined on Ω.

Definition 8. Let Ω be a nonempty sample space equipped with a filtration $\mathscr{F}(\mathrm{t}), 0 \leq t \leq T$. Let $S(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is an *adapted stochastic process* if , for each t, the random variable $S(t)$ is $\mathscr{F}(\mathrm{t})$measurable.

D. Concluding Remarks

In Finance, we are interested in making contingency plans and developing future strategies based on current information. Thus, we needed to develop a machinery that will capture how the uncertainty between the current time and the future time is resolved. The model developed here is found to be very useful in handling this uncertainty.We introduce the notion of stochastic processes for asset prices, portfolio processes/positions, and wealth process (value of a portfolio), and these are adapted to a filtration that we regard as a model of the flow of public information. To illustrate, a portfolio positon $\triangle(t)$ taken at time t must depend only on the information available to the investor at time t.Thus $\triangle(t)$ should be $\mathscr{F}(\mathrm{t})$-measurable. We can also think of $S(t)$ as the price of some asset, and $\mathscr{F}(\mathrm{t})$ as the information obtained by watching all the prices in the market upto time t.

In general , there are three cases that arise when we have a random variable S and a $\sigma - algebra\ \mathscr{G}$:

1) The information contained in the $\mathscr{G}$ is sufficient to determine the value of S. In this case, S is $\mathscr{G}$- measurable.
2) The information contained in $\mathscr{G}$ gives us no clue about the value of S. In this case, we say that S is *independent* of $\mathscr{G}$.
3) The information contained in $\mathscr{G}$ is not sufficient to evaluate S, but we can get some estimate of S based on the information in $\mathscr{G}$.

Note that we handled the first case without using the concept of a measure. The other two cases cannot be handled without bringing the probability measure into the picture. We will be particularly interested in the last case, and the theory of Conditional Expectations will be developed to get such an estimate of the random variables.

References

[1] Robert G. Bartle.*The Elements of Integration and Lebesgue Measure* (Wiley Classics Library Edition 1995)
[2] S.E Shreve. *Stochastic Calculus for Finance - Volume 2*. Chapters 1-2. (Springer, 2004)