Causal Inference Workshop

Week 2 - Potential Outcomes Framework

Causal Inference Workshop

January 30, 2024

Workshop outline

A. Causal inference fundamentals
   - Modeling assumptions matter too
   - Conceptual framework (potential outcomes framework)
B. Design stage: common identification strategies
   - IV + RDD [coding]
   - DiD, DiDiD, Event Studies, New TWFE Lit [coding]
   - Synthetic Control / Synthetic DiD [coding]
C. Analysis stage: strengthening inferences
   - Limitations of identification strategies, pre-estimation steps
   - Estimation [controls] and post-estimation steps [supporting assumptions]
D. Other topics in causal inference and sustainable development
   - Inference (randomization inference, bootstrapping)
   - Weather data regressions, other common/fun SDev topics [coding]
   - Remote sensing data, other common/fun SDev topics

Causal inference roadmap

- *Potential outcomes* [framework] [today]
    - Causal effect is difference between two potential outcomes
- *Identification* [application/implementation]
    - Identifying assumptions needed for a statistical estimate to have causal interpretation
    - Removing selection bias in regressions
    - E.g., RD, IV, ...
- *Estimation* [application/implementation] [last week]
    - (Usually) use linear regression model

Outline

Workshop outline

Potential outcomes framework

An alternative framework, the DAG

Summary

Causal inference roadmap

- *Potential outcomes (PO)* [framework]
  - Causal effect is difference between two potential outcomes
  - Lingua franca for expressing causal statements in economics / social sciences
  - This is one “approach to causality” (Imbens 2020)
    - Builds on Neyman 1923
    - Extended to observational studies by Rubin 1974
  - PO framework is not the *only* approach
    - Directed Acyclic Graph (DAG) approach is another alternative

- *Identification* [application/implementation]

- *Estimation* [application/implementation]

Outline

Workshop outline

Potential outcomes framework

- The original selection bias problem
- Treatment effects as a linear regression
- When does IA/CIA not hold?

An alternative framework, the DAG

- A very basic overview of DAGs
- Comparative strengths and weaknesses of the PO and DAG approaches

Summary

Potential outcomes framework and treatment effects

- We have:
  - A population, of which we observe sample of units $i = 1, ..., N$
  - A binary treatment of interest $D_i \in \{0, 1\} \rightarrow$ want to estimate the causal effect of D on Y
  - Let unit i's potential outcomes be: Y_i^1 if received treatment, Y_i^0 otherwise
  - Let unit i's observable outcome be: Y_i

Potential outcomes framework and treatment effects

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- Note the difference between *potential* outcomes (Y_i^1, Y_i^0) and *observable* or "actual" outcomes (Y_i); can relate them according to: $Y_i = D_i Y_i^1 + (1 - D_i) Y_i^0$

Potential outcomes framework and treatment effects

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  - A population, of which we observe sample of units $i = 1, \ldots, N$
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- Define:

individual treatment effects (TEs)	$Y_i^1 - Y_i^0 \; \forall i$	*ideally estimate; unknowable*
average treatment effect (ATE)	$\mathbb{E}[Y_i^1 - Y_i^0]$	*reasonably estimate; unknowable, but might be identifiable*
average treatment effect on the treated (ATT)	$\mathbb{E}[Y_i^1 - Y_i^0 \mid D_i = 1]$	*reasonably estimate; unknowable, but might be identifiable*
difference in average observed outcomes	$\mathbb{E}[Y_i \mid D_i = 1] - \mathbb{E}[Y_i \mid D_i = 0]$	*directly identifiable, straightforward estimator*

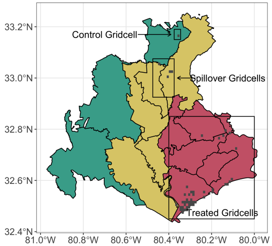
Potential outcomes framework and treatment effects: assumptions

- Assume additive treatment effects and no interference between units
- **Stable Unit Treatment Value Assumption (SUTVA)**: treatment received by one unit does not affect potential outcomes for other units
  - Each unit has only two possible potential outcomes Y_i^1, Y_i^0, which implies:
    - No spillovers
    - No general equilibrium effects

Potential outcomes framework and treatment effects: assumptions

Example of possible SUTVA violation:

- What are the effects of plastic bag laws on plastic litter in the environment?
- Use data on $\sim$100k shoreline cleanups
- Aggregate outcome data to 0.01 lat/lon gridcells
- Treatment at the zip code level (is there a policy in zip code?)
- Why may SUTVA be violated?



Potential outcomes framework and treatment effects: assumptions

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  - Each unit has only two possible potential outcomes Y_i^1, Y_i^0, which implies:
    - No spillovers
    - No general equilibrium effects
  - Often not realistic in economics studies
  - Many papers on SUTVA as nuisance
    - Can change how treatment is defined (e.g., within-household spillover)
    - Change level at which you interpret results
  - Some papers on SUTVA as substance (modeling the impact of the interference between units), e.g., spillovers:
    - (Hong and Raudenbush 2006; Hudgens and Halloran 2008; Aronow and Samii 2017; Rosenbaum 2007)

Potential outcomes framework and the selection bias problem

- Back to the various parameters:

individual treatment effects (TEs)	$Y_i^1 - Y_i^0 \ \forall i$	*ideally estimate; unknowable*
average treatment effect (ATE)	$\mathbb{E}[Y_i^1 - Y_i^0]$	*reasonably estimate; unknowable, but might be identifiable*
average treatment effect on the treated (ATT)	$\mathbb{E}[Y_i^1 - Y_i^0 \vert D_i = 1]$	*reasonably estimate; unknowable, but might be identifiable*
difference in average observed outcomes	$\mathbb{E}[Y_i \vert D_i = 1] - \mathbb{E}[Y_i \vert D_i = 0]$	*directly identifiable, straightforward estimator*

- We never observe causal effects

- What we can do is compute the difference in average observed outcomes:

$$\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] = \dots$$

$$= \underbrace{\mathbb{E}[Y_i^1 - Y_i^0|D_i = 1]}_{\text{ATT}} + \underbrace{\mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]}_{\text{selection bias}}$$

▸ math details

Potential outcomes framework and the selection bias problem

- What we can do is compute the difference in average observed outcomes:

$$\begin{aligned}\mathbb{E}[Y_i|D_i=1]-\mathbb{E}[Y_i|D_i=0] &= \ldots \\ &= \underbrace{\mathbb{E}[Y_i^1-Y_i^0|D_i=1]}_{\text{ATT}}+\underbrace{\mathbb{E}[Y_i^0|D_i=1]-\mathbb{E}[Y_i^0|D_i=0]}_{\text{selection bias}}\end{aligned}$$

- Selection bias is the average difference in Y_i^0 between the treated and untreated
- Necessary and sufficient condition to remove selection bias and thus identify ATT: $\mathbb{E}[Y_i^0|D_i=1]=\mathbb{E}[Y_i^0|D_i=0]$
- $(Y_i^0, Y_i^1)\perp\!\!\!\perp D_i \implies Y_i^0 \perp\!\!\!\perp D_i \implies \mathbb{E}[Y_i^0|D_i=1]=\mathbb{E}[Y_i^0|D_i=0]$

▸ math details

Independence assumption and selection bias

- When treatment is independent of POs $\rightarrow$ no selection bias *in expectation*
  - $(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i$; **independence assumption** (IA)
  - Selection bias is eliminated and $\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] = \mathbb{E}[Y_i^1 - Y_i^0|D_i = 1]$ or difference in average observed outcomes equals the ATT. Note that IA is sufficient but not necessary.
  - Holds in expectation for experiments, not for (virtually any) observational study

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  - Holds in expectation for experiments, not for (virtually any) observational study

- Let's consider various assignment mechanisms:
  - Random assignment (e.g., experiments)
  - Selection on observables
  - Selection on unobservables

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  - Holds in expectation for experiments, not for (virtually any) observational study

- Let's consider various assignment mechanisms:
  - Random assignment (e.g., experiments)
    - If treatment is randomly assigned, IA holds and identifies ATT (no selection bias in expectation, **NOT** for any single trial)
  - Selection on observables
  - Selection on unobservables

Independence assumption and selection bias

- When treatment is independent of POs $\rightarrow$ no selection bias *in expectation*
  - $(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i$; **independence assumption** (IA)
  - Selection bias is eliminated and $\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] = \mathbb{E}[Y_i^1 - Y_i^0|D_i = 1]$ or difference in average observed outcomes equals the ATT. Note that IA is sufficient but not necessary.
  - Holds in expectation for experiments, not for (virtually any) observational study

- Let's consider various assignment mechanisms:
  - Random assignment (e.g., experiments)
  - Selection on observables
    - If conditional on some pre-treatment characteristic X_i, we have $(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i|X_i$, we can once again eliminate selection bias in expectation (**conditional independence assumption**, CIA)
    - Compare outcomes within each stratum of X_i
  - Selection on unobservables

Independence assumption and selection bias

- When treatment is independent of POs → no selection bias *in expectation*
  - $(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i$; **independence assumption** (IA)
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  - Holds in expectation for experiments, not for (virtually any) observational study

- Let's consider various assignment mechanisms:
  - Random assignment (e.g., experiments)
  - Selection on observables
  - Selection on unobservables
    - Will need other identification strategies to eliminate selection bias

Identifying assumptions

- We can recover an **unbiased** estimator of a causal effect if **(conditional) independence assumption** holds:
  - if IA holds ($(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i$) $\rightarrow$ estimate ATT and ATE
  - if $\cancel{IA}$, but CIA ($(Y_i^0, Y_i^1) \perp\!\!\!\perp D_i | X_i$) $\rightarrow$ can estimate ATT and ATE in each stratum (and then combine)
  - if $\cancel{CIA}$, need relevant exogenous source of variation in D_i (e.g., $(Y_i^0, Y_i^1) \perp\!\!\!\perp Z_i,\ Z_i \not\!\perp\!\!\!\perp D_i$) $\rightarrow$ estimate a LATE

- Need an **identification strategy** that convinces us that IA holds

- Bottom-line:
  - Econometrics / regression controls won't bring causality $\rightarrow$ need identification strategy
  - BUT, even with good identification strategy, no reason to expect balance for all relevant pre-treatment characteristics $\rightarrow$ control for relevant pre-treatment variables

Outline

Treatment effects as a linear regression

- *Potential outcomes (PO)* [framework] [just now]
- *Identification* [application/implementation]
- *Estimation* [application/implementation] [last week]

Treatment effects as a linear regression

- Suppose heterogeneous TE: $Y_i^1 - Y_i^0 = \beta_i$
    - Denote ATT by β; that is, $\beta = \mathbb{E}[\beta_i | D_i = 1] = \mathbb{E}[Y_i^1 - Y_i^0 | D_i = 1]$

- Then we can write

$$
\begin{aligned}
Y_i &= Y_i^0 + (Y_i^1 - Y_i^0) D_i \\
&= Y_i^0 + \beta_i D_i \\
&= Y_i^0 + (\beta_i - \beta + \beta) D_i \\
&= \mathbb{E}[Y_i^0] + \beta D_i + Y_i^0 - \mathbb{E}[Y_i^0] + (\beta_i - \beta) D_i \\
&= \alpha + \beta D_i + e_i
\end{aligned}
$$

where $\alpha \equiv Y_i^0$ and $e_i \equiv Y_i^0 - \mathbb{E}[Y_i^0] + (\beta_i - \beta) D_i$

Treatment effects as a linear regression

- Think about OLS regression of Y_i on D_i and a constant:

$$\hat{\beta}_{ols} = \bar{Y}_1 - \bar{Y}_0$$

  - $\mathbb{E}[\hat{\beta}_{ols}] = \mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0]$

- By $Y = \alpha + \beta D_i + e_i$ and the definition of e_i, we have
  - $\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] = \beta + \mathbb{E}[e_i|D_i = 1] - \mathbb{E}[e_i|D_i = 0]$
  - $\mathbb{E}[e_i|D_i = 1] - \mathbb{E}[e_i|D_i = 0] = \mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]$

math details, part 1 | math details, part 2 | math details, part 3

Treatment effects as a linear regression

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- Thus:

$$\begin{aligned}\mathbb{E}[\hat{\beta}_{ols}] &= \mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] \\ &= \beta + \mathbb{E}[e_i|D_i = 1] - \mathbb{E}[e_i|D_i = 0] \\ &= \underbrace{\beta}_{\text{ATT}} + \underbrace{\mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]}_{\text{selection bias}}\end{aligned}$$

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Treatment effects as a linear regression

- Thus:

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&= \underbrace{\beta}_{\text{ATT}} + \underbrace{\mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]}_{\text{selection bias}}
\end{aligned}$$

- Which means that:
  - $\mathbb{E}[\hat{\beta}_{ols}]$ is equal to ATT iff:
    - there is no selection bias (identification problem; independence)

Outline

Endogeneity

- In simple linear regression model $y_i = \alpha + \beta x_i + e_i$, variable x_i is:
  - **endogenous** if it is correlated with the error term, or $cov[x_i, e_i] \neq 0$
  - **exogenous** otherwise, if $cov[x_i, e_i] = 0$ (A3. of CLRM)
- If x is endogenous, then OLS estimator of β will be biased and inconsistent for β
- In our setting (potential outcomes framework), if treatment D_i is endogenous ($cov[D_i, e_i] \neq 0$), there is imbalance in potential outcomes across treatment groups
  - $\rightarrow$ CIA doesn't hold (again, identification problem $\leftrightarrow$ regression problem)

Sources of endogeneity

- **Reverse causality or simultaneity**
  - If y also affects D, this is captured by e, making e correlated with D
- **Measurement error** in D that is **correlated** with y
- **Omitted variable bias (OVB)**
  - If omitted variable w is correlated with D, e is correlated with D (w is a "confounding variable")

→ in observational studies, excluding confounder creates bias, so must adjust for all confounders; but we can rarely be certain to have measured all confounders, which is why we turn to alternative **"identification" strategies**

Outline

Outline

Directed acyclic graphs (DAGs)

- An alternative to the potential outcomes framework is the causal graph framework or work on directed acyclic graphs (DAGs) (Pearl 2009)
  - PO and DAG frameworks are *not* contradicting; both define causality using counterfactuals
  - Each framework has its own benefits (see Imbens 2020 for a review of these) and are therefore complementary perspectives

Directed acyclic graphs (DAGs)

- Relationships between random variables are encoded with nodes and directed edges
  - Nodes are random variables (solid for observed variables, hollow for unobserved)
  - Arrows represent possible direct causal relationships
  - Paths are sequences of edges
  - DAG is a *complete* encoding of assumptions about causal relationships

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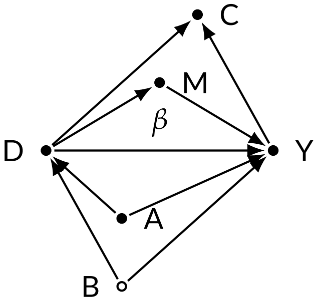
  - DAG is a *complete* encoding of assumptions about causal relationships

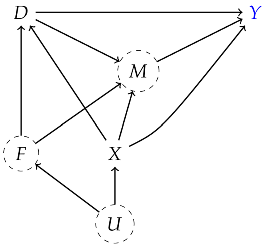
  - Types of elementary paths:
    - **Mediating path:** $D \rightarrow M \rightarrow Y$
    - **Confounding paths:** $D \leftarrow A \rightarrow Y$ (closed); $D \leftarrow\!-\!- B -\!-\!\rightarrow Y$ (open)
    - **Colliding path:** $D \rightarrow C \leftarrow Y$

  - Identification strategies:
    - Blocking back-door paths (adjusting for all confounders)
    - Instruments (alternative identification strategies)
      $\rightarrow$ same conclusion as with potential outcomes framework



401(k) eligibility and financial wealth example (Chernozhukov et al. 2025)

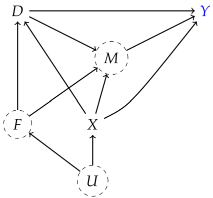
- **Question:** What is the causal effect of 401(k) *eligibility* on household financial wealth?
- **Variables:**
  - D: eligibility for a 401(k) plan
  - Y: net financial assets
  - X: observed worker characteristics (income, age)
  - F: unobserved firm characteristics
  - M: employer match
  - U: latent factors



- Eligibility may affect wealth:
  - directly (saving incentives, tax advantages)
  - indirectly through employer matching
- Selection into eligible jobs depends on X and unobserved F

Backdoor blocking and identification

- **Goal:** identify the causal effect of D on Y.
- **Backdoor criterion:** choose a set Z such that
  1. no variable in Z is a **descendant** of D, and
  2. Z **blocks every backdoor path** from D to Y (a backdoor path starts with an arrow into D).



- **Why?**
  - Rule (1) prevents blocking causal paths (e.g. $D \to M \to Y$).
  - Rule (2) removes non-causal association due to confounding.
- **In the 401(k) DAG, backdoor paths include:**
  - $D \leftarrow X \to Y$
  - $D \leftarrow F \to M \to Y$
  - $D \leftarrow F \leftarrow U \to X \to Y$
- **Valid adjustment set:** $Z = \{F, X\}$ blocks all backdoor paths and implies

$$Y(d) \perp\!\!\!\perp D \mid F, X.$$

Outline

Strengths and weaknesses of the PO and DAG approaches

- See Imbens 2020 for a review of the relevance of DAGs for empirical economics
  - **Experiments and manipulability:**
    $\rightarrow$ PO framework elevates randomized experiments as “gold standard”, while DAG doesn’t deem experiments special ($\sim$ notion of manipulability)
  - **Parts of causal analysis addressed: (pre-identification, identification, post-identification)**
    $\rightarrow$ DAGs only consider step 2, while steps 2 and 3 are considered jointly in PO
  - **Representation of identifying assumptions and identification strategies**
    $\rightarrow$ Identifying assumptions explicit in graphical versions and often much clearer than algebraic versions, BUT many other assumptions not easily captures in DAG framework; accounting for treatment heterogeneity difficult with DAGs

- Bottom-line:
  - Can be very helpful for thinking about or communicating research designs
  - May be helpful to know how to represent your analysis in both frameworks

Outline

Causal inference roadmap

- *Potential outcomes* [framework] [today]
  - Causal effect is the difference between two potential outcomes
  - We can't observe this difference, but can see differences in average observed outcomes
  - If **(conditional) independence assumption** holds, can estimate unbiased ATT
- *Identification* [application/implementation] [up next!]
  - In most empirical settings, IA and CIA do not hold, which is why we need an **identification strategy**
  - Want to eliminate selection bias (identification problem)
- *Estimation* [application/implementation] [last week]
  - (Usually) use linear regression model
  - $\hat{\beta}_{OLS}$ unbiased estimator for ATT if e is uncorrelated with treatment (regression problem)

Questions? Comments?

Thank you!

References

Heavily based on Claire Palandri's 2022 version and Anna Papp's 2024 version of the Causal Inference Workshop.

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Appendix

$$
\begin{aligned}
\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] \\
&= \mathbb{E}[Y_i^1|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0] \\
&= \mathbb{E}[Y_i^1|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 1] + \mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0] \\
&= \underbrace{\mathbb{E}[Y_i^1 - Y_i^0|D_i = 1]}_{\text{ATT}} + \underbrace{\mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]}_{\text{selection bias}}
\end{aligned}
$$

back

Appendix

$$Y_i = \lambda + \gamma D_i + \varepsilon_i, \quad \mathbb{E}[D_i \varepsilon_i] = 0$$

Projection ceofficient of this projection model:

$$\gamma = \frac{cov[Y_i, D_i]}{Var[D_i]} = \frac{\mathbb{E}[Y_i D_i] - \mathbb{E}[Y_i]\mathbb{E}[D_i]}{\mathbb{E}[D_i^2] - \mathbb{E}[D_i]^2} =$$

$$= \frac{\mathbb{E}[Y_i | D_i = 1] P(D_i = 1) - \big(\mathbb{E}[Y_i | D_i = 0] P(D_i = 0) + \mathbb{E}[Y_i | D_i = 1] P(D_i = 1)\big) P(D_i = 1)}{P(D_i = 1) - P(D_i = 1)^2}$$

$$= \frac{\big(\mathbb{E}[Y_i | D_i = 1] - \mathbb{E}[Y_i | D_i = 0]\big)\big(P(D_i = 1) - P(D_i = 1)^2\big)}{P(D_i = 1) - P(D_i = 1)^2}$$

$$= \mathbb{E}[Y_i | D_i = 1] - \mathbb{E}[Y_i | D_i = 0]$$

back

OLS with a binary regressor (1/2)

$$\hat{\beta}_{\text{OLS}} = \frac{\sum_{i=1}^{n} Y_i(D_i - \bar{D})}{\sum_{i=1}^{n} D_i(D_i - \bar{D})}, \qquad \bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i.$$

Define

$$N_1 = \sum_{i=1}^{n} D_i, \quad N_0 = n - N_1 \quad \left(\text{and thus } \bar{D} = \frac{N_1}{n}\right)$$

$$\bar{Y}_1 = \frac{1}{N_1}\sum_{i:D_i=1} Y_i, \quad \bar{Y}_0 = \frac{1}{N_0}\sum_{i:D_i=0} Y_i.$$

Denominator:

$$\sum_{i=1}^{n} D_i(D_i - \bar{D}) = \sum_{i=1}^{n}(D_i - \bar{D}D_i) = (1 - \bar{D})\sum_{i=1}^{n} D_i = \frac{N_0 N_1}{n}.$$

OLS with a binary regressor (2/2)

Numerator:

$$\sum_{i=1}^{n} Y_i(D_i - \bar{D}) = \sum_{i=1}^{n} Y_i D_i - \bar{D} \sum_{i=1}^{n} Y_i.$$

$$= N_1 \bar{Y}_1 - \bar{D}(N_1 \bar{Y}_1 + N_0 \bar{Y}_0) = \frac{N_1 N_0}{n}(\bar{Y}_1 - \bar{Y}_0).$$

Combining numerator and denominator:

$$\hat{\beta}_{\text{OLS}} = \frac{\frac{N_1 N_0}{n}(\bar{Y}_1 - \bar{Y}_0)}{\frac{N_1 N_0}{n}} = \bar{Y}_1 - \bar{Y}_0.$$

$$\boxed{\hat{\beta}_{\text{OLS}} = \bar{Y}_1 - \bar{Y}_0.}$$

back

Appendix

Since:

- $\mathbb{E}[Y_i|D_i = 1] = \alpha + \beta + \mathbb{E}[e_i|D_i = 1]$ and
- $\mathbb{E}[Y_i|D_i = 0] = \alpha + \mathbb{E}[e_i|D_i = 0]$

We then have:

$$\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0] = \beta + \mathbb{E}[e_i|D_i = 1] - \mathbb{E}[e_i|D_i = 0]$$

back

Appendix

Since:

$$e_i = Y_i^0 - \mathbb{E}[Y_i^0] + (\beta_i - \beta)D_i$$

We have:

$$\begin{aligned}
\mathbb{E}[e_i|D_i = 1] - \mathbb{E}[e_i|D_i = 0] \\
&= \mathbb{E}[\beta_i - \beta|D_i = 1] + e\mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0] - \mathbb{E}[Y_i^0|D_i = 0] + \mathbb{E}[Y_i^0] \\
&= \mathbb{E}[\beta_i|D_i = 1] - \beta + \mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0] \\
&= \mathbb{E}[Y_i^0|D_i = 1] - \mathbb{E}[Y_i^0|D_i = 0]
\end{aligned}$$

back