Abstract

Differential equations with integer order or fractional order derivatives have attracted much attention for a long time, as they are very useful in modelling phenomena that arise from various fields such as physics, chemistry, biology, mechanics, finance and engineering. Indeed, the 'memory' property of fractional derivatives enhances the accuracy of the models. However, it is a fact that finding analytical solutions of differential equations, fractional or non-fractional, is nearly impossible in many instances. The development of efficient numerical schemes for such equations is therefore very important, and this will be the focus of our work in this thesis.

In this thesis, we shall investigate the numerical treatment of differential equations in two categories: (i) ordinary differential equations; (ii) partial differential equations. In the first category, we consider two problems – a second-order boundary value problem with discontinuous second order derivative at some breakup points; and a fractional Bagley-Torvik equation. For both problems, we propose new numerical methods that give more accurate solutions than other methods in the literature. In fact, the numerical scheme for the second-order boundary value problem is based on cubic non-polynomial spline deployed over the mid-knots of a uniform mesh, while discrete cubic spline and weighted shifted Grünwald-Letnikov difference operator are used to solve the fractional Bagley-Torvik equation. We also present theoretical proofs of the stability and convergence of the numerical schemes, and confirm the theoretical results by numerical experiments.

In the second category of partial differential equations, we consider a fractional nonlinear Schrödinger equation and a *generalized* fractional diffusion equation. For the fractional nonlinear Schrödinger equation, our proposed method, which employs quintic non-polynomial spline, improves the spatial convergence achieved so far in the literature. As for the *generalized* fractional diffusion equation, here the fractional derivative is 'generalized' in the sense that it features a scale function and a weight function, which can scale up/down the time domain or assign different weights at different time points. With specific scale functions and weight functions, the generalized fractional derivative reduces to well known fractional derivatives in the literature. The approximation of the generalized fractional derivative as well as the numerical treatment of generalized fractional problems have been scarcely researched in the literature. In this thesis, we shall derive higher order approximations for the generalized fractional derivative and then apply them in the numerical schemes proposed for a generalized fractional diffusion problem. In all the problems that we considered above, both theoretical analysis and numerical simulation are presented to support and illustrate the efficiency of our methods.