QUANTIFIERS AND ABSTRACTION IN THEOREM PROVING AND ITS AUTOMATION

FAN ZHENG

1. Inverse image of open set under continuous functions

Analysis is known for its concreteness and delicacy, as opposed to more abstract fields like algebra. In particular there is the issue of dependence of parameters, This is manifest in the interlacing pattern of the universal and existential quantifiers involved in probably the first definition in freshman analysis courses, that of continuity. It is also probably why there is a worked out example on continuity in the manifesto [2]. In the following I shall expand the final steps in more detail, paying special attention to the quantifiers involved in the proof.

Let's start from this state:

$$\boxed{\begin{array}{c} r > 0 \\ \forall \epsilon > 0\ \exists \delta > 0\ \forall w \in X\ (d_X(x,w) < \delta \implies d_Y(f(x), f(w)) < \epsilon) \\ \forall z \in Y\ (d_Y(f(x), z) < r \implies z \in U) \end{array}}$$
$$\boxed{\exists s > 0\ \forall y \in X\ (d_X(x,y) < s \implies f(y) \in U)}$$

I have renamed some of the bound variables to avoid collisions, so that the process does not rely on coincidence of the variable names. Moreover, in some more involved proofs, we need the $\epsilon/2$ (or $\epsilon/3$) trick which cannot be produced by exact matches.

Note that all the quantifiers over the points (as opposed to over the reals) are: $\forall w \in X$ and $\forall z \in Y$ in the hypotheses, and $\forall y \in X$ in the conclusion. The last one is thought as given, while the first two are to be instantiated. Since among the terms appearing in the conclusion, the only points in X are x and y, and the only points in Y that can be made from them are $f(x)$ and $f(y)$, the variable w had better be instantiated with x or y, and the variable z with $f(x)$ or $f(y)$. Let's pretend that we are lucky enough to make the right choices (after all there are only four of them altogether.) Then the problem state becomes

$$\boxed{\begin{array}{c} r > 0 \\ \forall \epsilon > 0\ \exists \delta > 0\ (d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon) \\ (d_Y(f(x), f(y)) < r \implies f(y) \in U) \end{array}}$$
$$\boxed{\exists s > 0\ \forall y \in X\ (d_X(x,y) < s \implies f(y) \in U)}$$

Now let's put the whole state in prenex normal form [3]. There are essentially two rules:

$$(\exists x\ P(x) \implies Q) \text{ becomes } \forall x\ (P(x) \implies Q)$$

(informally, this is the argument corresponding to "let x be such that $P(x)$") and

$$(\forall x\ P(x) \implies Q) \text{ becomes } \exists x\ (P(x) \implies Q)$$

(informally, this corresponds to finding an appropriate x to instantiate and apply the hypothesis.) Note that the first hypothesis $r > 0$ corresponds to a "global" universal quantifier, which

I put at the beginning. Thus we get

$$\forall r > 0\ \exists \epsilon > 0\ \forall \delta > 0\ \exists s > 0\ \forall y \in X\ \begin{array}{|c|}\hline (d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon) \\ (d_Y(f(x), f(y)) < r \implies f(y) \in U) \\ \hline (d_X(x,y) < s \implies f(y) \in U) \\ \hline \end{array}$$

Now all the quantifiers are over the reals, except for one which is over X. Let's do an "abstraction" step, and pretend that $d_X(x,y)$ and $d_Y(f(x), f(y))$ are arbitrary functions $A(y)$ and $B(y)$ of y, and $f(y) \in U$ is an arbitrary predicate $P(y)$ of y. Then we get

$$\forall r > 0\ \exists \epsilon > 0\ \forall \delta > 0\ \exists s > 0\ \forall y \in X\ \begin{array}{|c|}\hline (A(y) < \delta \implies B(y) < \epsilon) \\ (B(y) < r \implies P(y)) \\ \hline (A(y) < s \implies P(y)) \\ \hline \end{array}$$

In fact we can abstract more: since y is prefixed with a universal quantifier, and A and B are assumed to be arbitrary (and independent) functions of y, we can replace them by quantified real variables. Similarly we can replace $P(y)$ by an arbitrary predicate P. There is one more trick: we can set P to false, for if P is true, "there is nothing to prove." Then we get

$$\forall r > 0\ \exists \epsilon > 0\ \forall \delta > 0\ \exists s > 0\ \forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \begin{array}{|c|}\hline (a < \delta \implies b < \epsilon) \\ \neg(b < r) \\ \hline \neg(a < s) \\ \hline \end{array}$$

Now all the variables are real numbers, and it is crystal clear what choices we should make for ϵ and s for it to hold. If it is still not so, it suffices to note that linear inequalities over the reals can be solved mechanically, so, we are done!

2. Limit of uniformly convergent functions

In this section we work out the proof that the limit of uniformly continuous functions is continuous, using the method I developed previously. We start from the following state:

$$\begin{array}{|c|}\hline d > 0 \\ \forall \epsilon > 0\ \exists N\ \forall n > N\ \forall z \in X\ |f_n(z) - f(z)| < \epsilon \\ \forall m\ \forall r > 0\ \exists s > 0\ \forall w \in X\ (d(x,w) < s \implies |f_m(x) - f_m(w)| < r) \\ \hline \exists c > 0\ \forall y \in X\ (d(x,y) < c \implies |f(x) - f(y)| < d) \\ \hline \end{array}$$

Let's first focus on the variables quantified over the integers. There are three of them: N existentially quantified in the second hypotheses, n universally quantified in the same hypothesis, and m universally quantified in the last hypothesis. The obvious instantiation is then $n = M = N$, which leads to the following state:

$$\begin{array}{|c|}\hline d > 0 \\ \forall \epsilon > 0\ \exists N\ \forall z \in X\ |f_N(z) - f(z)| < \epsilon \\ \forall r > 0\ \exists s > 0\ \forall w \in X\ (d(x,w) < s \implies |f_N(x) - f_N(w)| < r) \\ \hline \exists c > 0\ \forall y \in X\ (d(x,y) < c \implies |f(x) - f(y)| < d) \\ \hline \end{array}$$

Next we look at the variables quantified over the points in X. There are also three of them: y universally quantified in the conclusion, and z and w universally quantified in the hypotheses. The latter two can both be instantiated with x and y. The tricky thing is that (with hindsight) the variable z needs to be instantiated with *both* x and y. A systematic way of doing this is to instantiate it as follows:

$$\forall \epsilon > 0\ \exists N\ (|f_N(x) - f(x)| < \epsilon \wedge |f_N(y) - f(y)| < \epsilon).$$

This is superior then separately instantiating z with x and y and taking the conjunction, i.e., the following instantiation:

$$(\forall \epsilon > 0\ \exists N\ |f_N(x) - f(x)| < \epsilon \wedge \forall \delta > 0\ \exists M\ |f_M(y) - f(y)| < \delta)$$

because, first, the former is a consequence of the latter, and hence is stronger, and second, the latter loses the uniformity of N with respect to x, and may well not lead to a successful proof. Now we turn to w. In principle it can also be instantiate with both x and y in the same way, but in practice the instantiation $w = x$ is trivial and does not help with the proof. Therefore in the following we only instantiate it with y to get the following state:

$$\boxed{\begin{array}{c} d > 0 \\ \forall \epsilon > 0\ \exists N\ (|f_N(x) - f(x)| < \epsilon \wedge |f_N(y) - f(y)| < \epsilon) \\ \forall r > 0\ \exists s > 0\ (d(x,y) < s \implies |f_N(x) - f_N(y)| < r) \end{array}}$$
$$\boxed{\exists c > 0\ \forall y \in X\ (d(x,y) < c \implies |f(x) - f(y)| < d)}$$

Now we transform it into the prenex normal form, with quantifiers suitably ordered:

$$\forall x \in X\ \forall d > 0\ \exists \epsilon > 0\ \forall N\ \exists r > 0\ \forall s > 0\ \exists c > 0\ \forall y \in X$$
$$\boxed{\begin{array}{c} (|f_N(x) - f(x)| < \epsilon \wedge |f_N(y) - f(y)| < \epsilon) \\ (d(x,y) < s \implies |f_N(x) - f_N(y)| < r) \end{array}}$$
$$\boxed{(d(x,y) < c \implies |f(x) - f(y)| < d)}$$

Observe that all variables that are not quantified over the real numbers are x, N and y, all of which universally quantified, and that all the terms involving at least one of them are $d(x,y)$, $f(x)$, $f(y)$, $f_N(x)$ and $f_N(y)$. We associate each such term to a variable in the following way:

Term	x	N	y	Associated variable (last one with +)	Abstracted with
$d(x,y)$	+	-	+	y	y
$f(x)$	+	-	-	x	x
$f(y)$	-	-	+	y	z
$f_N(x)$	+	+	-	N	N
$f_N(y)$	-	+	+	y	w

Then we abstract each of the terms with a new variable, quantified in the same place as its associated variable. We have reused some of the names in order to save letters in the alphabet. Thus we get

$$\forall x \in \mathbb{R}\ \forall d > 0\ \exists \epsilon > 0\ \forall N \in \mathbb{R}\ \exists r > 0\ \forall s > 0\ \exists c > 0\ \forall y, z, w \in \mathbb{R}\ \begin{array}{c} \boxed{\begin{array}{c} (|N - x| < \epsilon \wedge |w - z| < \epsilon) \\ (y < s \implies |N - w| < r) \end{array}} \\ \boxed{(y < c \implies |x - z| < d)} \end{array}$$

As a sanity check, note that $\epsilon = r = d/3$ and $c = s$ verifies the assertion above. Since quantifier elimination over the real numbers is decidable, the assertion above can be proved algorithmically, and we are done.

References

[1] Metamath, https://us.metamath.org/mpeuni/mmtheorems1.html#mm3b
[2] Timothy Gowers, How can it be feasible to find proofs? https://drive.google.com/file/d/1-FFa6nMVg18m1zPtoAQrFalwpx2YaGK4/view
[3] Wikipedia, prenex normal form, https://en.wikipedia.org/wiki/Prenex_normal_form