SCHATTEN-LORENTZ CHARACTERIZATION OF RIESZ TRANSFORM COMMUTATOR ASSOCIATED WITH BESSEL OPERATOR

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ABSTRACT.

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1. Introduction

For $\lambda > 0$, we consider the Bessel operator $\Delta_\lambda^{(1)}$ on $\mathbb{R}_+$ ([13]) which is defined by

$$\Delta_\lambda^{(1)} = -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}.$$

It is a densely-defined non-negative self-adjoint operator in $L^2(\mathbb{R}_+, dm_\lambda)$, where $dm_\lambda^{(1)}(x) = x^{2\lambda}dx$. Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$. In high dimension, we also consider the $(n+1)$-dimensional Bessel operator from a seminal work of Huber [7], which, for $\lambda > 0$, is defined by

$$\Delta_\lambda^{(n+1)} = -\frac{d^2}{dx_1^2} \cdots - \frac{d^2}{dx_n^2} - \frac{d^2}{dx_{n+1}^2} - \frac{2\lambda}{x_{n+1}}\frac{d}{dx_{n+1}}. \tag{1.1}$$

The operator $\Delta_\lambda^{(n+1)}$ is densely-defined non-negative self-adjoint operator $L^2(\mathbb{R}_+^{n+1}, dm_\lambda)$, where

$$dm_\lambda^{(n+1)}(x) := \prod_{j=1}^n dx_j x_{n+1}^{2\lambda} dx_{n+1}.$$

For simplicity, we shall use the following notations in the sequel.

$$\Delta_\lambda := \begin{cases} \Delta_\lambda^{(1)}, & n = 0, \\ \Delta_\lambda^{(n+1)}, & n \geq 1, \end{cases} \quad \text{and} \quad m_\lambda := \begin{cases} m_\lambda^{(1)}, & n = 0, \\ m_\lambda^{(n+1)}, & n \geq 1. \end{cases}$$

For $\lambda > 0$ and $n \geq 0$, the j-th Riesz transform associated with Bessel operator is defined by

$$R_{\lambda,j} = \frac{d}{dx_j}\Delta_\lambda^{-\frac{1}{2}}, \quad j = 1, \ldots, n+1.$$

For simplicity, if $n = 0$, then we write $R_\lambda = R_{\lambda,1}$.

For an operator T, we consider the commutator with T defined as follows.

$$[b, T](f)(x) := b(x)T(f)(x) - T(bf)(x).$$

In [13], Muckenhoupt-Stein introduced and obtained the $L^p(\mathbb{R}_+, dm_\lambda)-$boundedness of R_λ for $\lambda \in (0,\infty)$. Under this condition, the commutator theorem for R_λ was obtained in [5] via weak factorisation.

To continue, we recall the definitions of the Lebesgue-Lorentz sequence space $\ell^{p,q}$ and Schatten class $S^{p,q}(L^2(\mathbb{R}_+^{n+1}, dm_\lambda))$. A sequence $\{a_k\}$ is in the $\ell^{p,q}$ for some $0 < p < \infty$ and $0 < q < \infty$ provided the non-increasing rearrangement of $\{a_k\}$, denoted by $\{a_k^*\}$, satisfies $\sum_k a_k^{*q} k^{q/p-1} < +\infty$. For the endpoint $q = \infty$, a sequence $\{a_k\}$ is in the $\ell^{p,\infty}$ for some $0 < p < \infty$ provided $\sup_k \{k^{1/p} a_k^*\} < +\infty$. Furthermore, note that if T is any compact operator on $L^2(\mathbb{R}_+^{n+1}, dm_\lambda)$, then T^*T is compact, positive and therefore diagonalizable. For $0 < p < \infty$ and $0 < q \leq \infty$, we say that $T \in S^{p,q}(L^2(\mathbb{R}_+^{n+1}, dm_\lambda))$ if $\{\lambda_k\} \in \ell^{p,q}$, where $\{\lambda_k\}$ is the sequence of square roots of eigenvalues of T^*T (counted according to multiplicity). In the sequel, for simplicity, we denote $S_\lambda^{p,q} := S^{p,q}(L^2(\mathbb{R}_+^{n+1}, dm_\lambda))$.

For any locally integrable function f, we define its mean oscillation over a cube Q by

$$MO_Q(f) := \fint_Q |f(x) - (f)_Q| dm_\lambda(x).$$

Def of $\mathcal{D}^0$

Definition 1.1. *Suppose $0 < p < \infty$, $0 < q \leq \infty$, $\lambda > 0$ and $n \geq 0$. Then we say that f belongs to oscillation space $\mathrm{OSC}_{p,q}(\mathbb{R}^{n+1}_+, dm_\lambda)$ if*

$$\|f\|_{\mathrm{OSC}_{p,q}(\mathbb{R}^{n+1}_+, dm_\lambda)} := \|\{MO_Q(f)\}_{Q\in\mathcal{D}^0}\|_{\ell^{p,q}} < +\infty.$$

Our first main theorem can be stated as follows.

Theorem 1.2. *Suppose $1 < p < \infty$, $0 < q \leq \infty$, $\lambda > 0$, $n \geq 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Then for any $\ell \in \{1, 2, ..., n+1\}$, one has*

$$\|[b, R_{\lambda,\ell}]\|_{S^{p,q}_\lambda} \approx \|b\|_{\mathrm{OSC}_{p,q}(\mathbb{R}^{n+1}_+, dm_\lambda)}.$$

Definition 1.3. *Suppose $0 \leq p, q < \infty$, $0 < \alpha < 1$, $\lambda > 0$ and $n \geq 0$. Then we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, dm_\lambda)$ belongs to weighted Besov space $B^\alpha_{p,q}(\mathbb{R}^{n+1}_+, dm_\lambda)$ if*

$$\|f\|_{B^\alpha_{p,q}(\mathbb{R}^{n+1}_+, dm_\lambda)} := \left(\int_{\mathbb{R}^{n+1}_+}\left(\int_{\mathbb{R}^{n+1}_+}|f(x)-f(y)|^p\frac{dm_\lambda(x)}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^{\frac{p}{q}+\frac{\alpha p}{n+1}}}\right)^{q/p} dm_\lambda(y)\right)^{1/q} < +\infty.$$

Our main theorem about one-dimensional Bessel operator is the following.

How to show the second statement of the following theorem is still unclear since Lemma 3.1 doesn't work for $p = 1$.

Theorem 1.4. *Suppose $0 < p < \infty$, $\lambda > 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Then one has $[b, R_\lambda] \in S^p_\lambda$ if and only if*

1. *$b \in B^{\frac{1}{p}}_{p,p}(\mathbb{R}_+, dm_\lambda)$, if $p > 1$; in this case we have $\|[b, R_\lambda]\|_{S^p_\lambda} \approx \|b\|_{B^{\frac{1}{p}}_{p,p}(\mathbb{R}_+, dm_\lambda)}$;*
2. *b is a constant, if $0 < p \leq 1$.*

Our main theorem about higher-dimensional ($n \geq 2$) Bessel operator is the following. We need to impose an extra assumption $b \in C^2(\mathbb{R}^{n+1}_+)$ in the second statement of the following theorem.

Theorem 1.5. *Suppose $0 < p < \infty$, $\lambda > 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Then for any $\ell \in \{1, 2, ..., n+1\}$, one has $[b, R_{\lambda,\ell}] \in S^p_\lambda$ if and only if*

1. *$b \in B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda)$, if $p > n+1$; in this case we have $\|[b, R_{\lambda,\ell}]\|_{S^p_\lambda} \approx \|b\|_{B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda)}$;*
2. *b is a constant, if $0 < p \leq n+1$.*

The paper is organized as follows.

2. Preliminaries

2.1. Preliminaries on space of homogeneous type.

In the proof of necessity (upper bound) for the case $p > n+1$, we will regard $\mathbb{R}^{n+1}_+$ as a space of homogeneous type, in the sense of Coifman and Weiss ([3]), with Euclidean metric and weighted measure dm_λ. Specifically, for any $x \in \mathbb{R}^{n+1}_+$ and $r > 0$, a set $B_{\mathbb{R}^{n+1}_+}(x, r) := B(x, r) \cap \mathbb{R}^{n+1}_+$, where $B(x, r)$ is a Euclidean ball with centre x and radius r, is considered as a ball in $\mathbb{R}^{n+1}_+$. It can be deduced from [4] that for every $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+$ and $r > 0$,

$$m_\lambda(B_{\mathbb{R}^{n+1}_+}(x, r)) \sim r^{n+1}x^{2\lambda}_{n+1} + r^{n+1+2\lambda}. \tag{2.1}$$

Therefore, $(\mathbb{R}^{n+1}_+, dm_\lambda)$ satisfies the following doubling inequality: for every $x \in \mathbb{R}^{n+1}_+$ and $r > 0$,

$$\min\{2^{n+1}, 2^{2\lambda+n+1}\} m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,r)) \leq m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,2r)) \leq 2^{2\lambda+n+1} m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,r)). \tag{2.2}$$

In the following subsections, for the convenience of the readers, we collect some properties about system of dyadic cubes on homogeneous space and adapt it to $\mathbb{R}^{n+1}_+$.

2.2. **A System of Dyadic Cubes.** A countable family $\mathcal{D} := \cup_{k\in\mathbb{Z}} \mathcal{D}_k$, $\mathcal{D}_k := \{Q^k_\alpha : \alpha \in \mathcal{A}_k\}$, of Borel sets $Q^k_\alpha \subseteq \mathbb{R}^{n+1}_+$ is called *a system of dyadic cubes with parameters* $\delta \in (0,1)$ if it has the following properties:

(I) $\mathbb{R}^{n+1}_+ = \bigcup_{\alpha\in\mathcal{A}_k} Q^k_\alpha$ (disjoint union) for all $k \in \mathbb{Z}$;

(II) If $\ell \geq k$, then either $Q^\ell_\beta \subseteq Q^k_\alpha$ or $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$;

(III) For each (k,α) and each $\ell \leq k$, there exists a unique β such that $Q^k_\alpha \subseteq Q^\ell_\beta$;

(IV) For each (k,α) there exists at most M (a fixed geometric constant) β such that

$$Q^{k+1}_\beta \subseteq Q^k_\alpha, \text{ and } Q^k_\alpha = \bigcup_{\substack{Q\in\mathcal{D}_{k+1} \\ Q\subseteq Q^k_\alpha}} Q;$$

(V) For each (k,α), one has

$$B_{\mathbb{R}^{n+1}_+}(x^k_\alpha, \frac{1}{12}\delta^k) \subseteq Q^k_\alpha \subseteq B_{\mathbb{R}^{n+1}_+}(x^k_\alpha, 4\delta^k) =: B_{\mathbb{R}^{n+1}_+}(Q^k_\alpha); \tag{2.3}$$

(VI) If $\ell \geq k$ and $Q^\ell_\beta \subseteq Q^k_\alpha$, then

$$B_{\mathbb{R}^{n+1}_+}(Q^\ell_\beta) \subseteq B_{\mathbb{R}^{n+1}_+}(Q^k_\alpha).$$

The set Q^k_α is called a *dyadic cube of generation* k with centre point $x^k_\alpha \in Q^k_\alpha$ and sidelength δ^k.

It can be deduce from the properties of the dyadic system above that there exists a positive constant C_0, such that for any Q^k_α and Q^{k+1}_β with $Q^{k+1}_\beta \subset Q^k_\alpha$,

$$m_\lambda(Q^{k+1}_\beta) \leq m_\lambda(Q^k_\alpha) \leq C_0 \delta^{-(2\lambda+n+1)} m_\lambda(Q^{k+1}_\beta). \tag{2.4}$$

A system of dyadic cubes on $\mathbb{R}^{n+1}_+$ can be constructed in a standard way illustrated as follows: let $\mathcal{D}^0 := \cup_{k\in\mathbb{Z}} \mathcal{D}^0_k$, where $\mathcal{D}^0_k$ is the standard dyadic partition of $\mathbb{R}^{n+1}_+$ into cubes with vertices at the sets $\{(2^{-k}m_1, \ldots, 2^{-k}m_{n+1}) : (m_1, \ldots, m_{n+1}) \in \mathbb{Z}^n \times \mathbb{N}\}$.

The notion of *nearly weakly orthogonal (NWO)* sequences of functions proposed by Rochberg and Semmes [15] plays a crucial role in establishing the below inequality (see [15, (1.10), §3]): for any bounded compact operator A on $L^2(\mathbb{R}^{n+1}_+, dm_\lambda)$:

$$\Big[\sum_{Q\in\mathcal{D}} |\langle Ae_Q, f_Q\rangle|^p\Big]^{1/p} \lesssim \|A\|_{S^p_\lambda}, \tag{2.5}$$

where $\{e_Q\}_Q$ and $\{f_Q\}_Q$ are function sequences satisfying $|e_Q|, |f_Q| \leq m_\lambda(Q)^{-1/2}\chi_{cQ}$ for some $c > 0$.

2.3. **Adjacent Systems of Dyadic Cubes.** A finite collection $\{\mathcal{D}^\nu\colon \nu = 1, 2, \ldots, \kappa\}$ of the dyadic families is called a collection of adjacent systems of dyadic cubes over $\mathbb{R}^{n+1}_+$ with parameters $\delta \in (0, 1)$ and $1 \le C_{\rm adj} < \infty$ if it satisfies the following properties:

- Each $\mathcal{D}^\nu$ is a system of dyadic cubes with parameters $\delta \in (0, 1)$;
- For each ball $B_{\mathbb{R}^{n+1}_+}(x, r) \subseteq \mathbb{R}^{n+1}_+$ with $\delta^{k+3} < r \le \delta^{k+2}, k \in \mathbb{Z}$, there exist $\nu \in \{1, 2, \ldots, \kappa\}$ and $Q \in \mathcal{D}^\nu_k$ of generation k and with centre point ${}^\nu x^k_\alpha$ such that $|x - {}^\nu x^k_\alpha| < 2\delta^k$ and

$$B_{\mathbb{R}^{n+1}_+}(x, r) \subseteq Q \subseteq B_{\mathbb{R}^{n+1}_+}(x, C_{\rm adj} r). \tag{2.6}$$

We adapt the construction in [8] to our setting, which can be stated as follows.

Lemma 2.1. *On $\mathbb{R}^{n+1}_+$ with Euclidean metric and weighted measure dm_λ, there exists a collection $\{\mathcal{D}^\nu\colon \nu = 1, 2, \ldots, \kappa\}$ of adjacent systems of dyadic intervals with parameters $\delta \in (0, \frac{1}{96})$ and $C_{\rm adj} := 8\delta^{-3}$ such that the centre points ${}^\nu x^k_\alpha$ of the cubes $Q \in \mathcal{D}^\nu_k$ satisfy, for each $\nu \in \{1, 2, \ldots, \kappa\}$,*

$$|{}^\nu x^k_\alpha - {}^\nu x^k_\beta| \ge \frac{1}{4}\delta^k \quad (\alpha \ne \beta), \qquad \min_\alpha |x - {}^\nu x^k_\alpha| < 2\delta^k \quad \text{for all } x \in \mathbb{R}^{n+1}_+.$$

Furthermore, these adjacent systems can be constructed in such a way that each $\mathcal{D}^\nu$ satisfies the distinguished centre point property: given a fixed point $x_0 \in \mathbb{R}^{n+1}_+$, for every $k \in \mathbb{Z}$, there exists $\alpha \in \mathcal{A}_k$ such that $x_0 = x^k_\alpha$, the centre point of $I^k_\alpha \in \mathcal{D}^\nu_k$.

2.4. **An Explicit Haar Basis.** We shall adapt the explicit construction in [11] of a Haar basis associated to the dyadic intervals $Q \in \mathcal{D}$ to our Bessel setting. Denote $M_Q := \#\mathcal{H}(Q) = \#\{R \in \mathcal{D}_{k+1}\colon R \subseteq Q\}$ be the number of dyadic sub-cubes ("children"); namely $\mathcal{H}(Q)$ is the collection of dyadic children of Q. Then for any $Q \in \mathcal{D}_k$, we let $h^1_Q, h^2_Q, \ldots, h^{M_Q-1}_Q$ be a family of Haar functions which satisfy some properties collected in the following two lemmas.

Lemma 2.2 ([11])**.** *For each $f \in L^p(\mathbb{R}^{n+1}_+, dm_\lambda)$, we have*

$$f(x) = \sum_{Q\in\mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle f, h^\epsilon_Q\rangle h^\epsilon_Q(x),$$

where the sum converges (unconditionally) both in the $L^p(\mathbb{R}^{n+1}_+, dm_\lambda)$-norm and pointwise almost everywhere.

Lemma 2.3 ([11])**.** *The Haar functions h^ϵ_Q, where $Q \in \mathcal{D}$, and $\epsilon \in \{1, 2, \ldots, M_Q - 1\}$, have the following properties:*

(i) *h^ϵ_Q is a simple Borel-measurable real function on $\mathbb{R}^{n+1}_+$;*
(ii) *h^ϵ_Q is supported on Q;*
(iii) *h^ϵ_Q is constant on each $R \in \mathcal{H}(Q)$;*
(iv) $\int_{\mathbb{R}^{n+1}_+} h^\epsilon_Q(x)\, dm_\lambda(x) = 0$ *(cancellation);*
(v) $\langle h^\epsilon_Q, h^{\epsilon'}_Q\rangle = 0$ *for $\epsilon \ne \epsilon'$, $\epsilon, \epsilon' \in \{1, \ldots, M_Q - 1\}$;*
(vi) *the collection $\{m_\lambda(Q)^{-1/2}\chi_Q\} \cup \{h^\epsilon_Q : \epsilon = 1, \ldots, M_Q - 1\}$ is an orthogonal basis for the vector space $V(Q)$ of all functions on Q that are constant on each sub-interval $R \in \mathcal{H}(Q)$;*
(vii) *if $h^\epsilon_Q \not\equiv 0$ then $\|h^\epsilon_Q\|_{L^p(\mathbb{R}^{n+1}_+, dm_\lambda)} \approx m_\lambda(Q)^{\frac{1}{p}-\frac{1}{2}}$ for $1 \le p \le \infty$;*
(viii) $$\|h^\epsilon_Q\|_{L^1(\mathbb{R}^{n+1}_+, dm_\lambda)} \cdot \|h^\epsilon_Q\|_{L^\infty(\mathbb{R}^{n+1}_+, dm_\lambda)} \approx 1.$$

2.5. **Basic estimates of the Bessel Riesz transform kernel.** Denote by $K_{\lambda,\ell}(x,y)$ the kernel of the ℓ-th Riesz transform $R_{\lambda,\ell}$. The following Lemma establishes an non-degenerate lower bound of Riesz transform kernel associated with Bessel operator, which can be regarded as a suitable substitution of homogeneity in the case of classical Euclidean Riesz transform kernel.

Lemma 2.4. *Given $\ell \in \{1, 2, \dots, n+1\}$ and a system of dyadic cubes $\mathcal{D} := \cup_{k\in\mathbb{Z}}\mathcal{D}_k$ on $\mathbb{R}^{n+1}_+$ with parameter $\delta \in (0,1)$. Then there exists a constant $A > 0$ such that for any $Q \in \mathcal{D}_k$ with center x_0 and satisfying $Q \subset \mathbb{R}^{n+1}_+$, one can find a ball $\hat{Q} := B_{\mathbb{R}^{n+1}_+}(y_0, \frac{1}{12}\delta^k) \subset \mathbb{R}^{n+1}_+$ such that $|x_0 - y_0| = A\delta^k$, and for all $(x,y) \in Q \times \hat{Q}$, $K_{\lambda,\ell}(x,y)$ does not change sign and satisfies*

$$|K_{\lambda,\ell}(x,y)| \geq \frac{C}{m_\lambda(Q)}$$

for some constant $C > 0$.

Proof. □

3. The proof of necessary condition when $p > n+1$

This section is devoted to showing that $b \in B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda)$ under the assumption $[b, R_{\lambda,\ell}] \in S^p_\lambda$ for some $p > n+1$ and $\ell \in \{1, 2, ..., n+1\}$.

To begin with, given a system of dyadic cubes $\mathcal{D} := \cup_{k\in\mathbb{Z}}\mathcal{D}_k$ with parameter $\delta \in (0,1)$ over $\mathbb{R}^{n+1}_+$, we define the conditional expectation of locally integrable function f by the expression:

$$E_k(f)(x) = \sum_{\substack{Q\in\mathcal{D}_k\\ Q\subseteq\mathbb{R}^{n+1}_+}} (f)_Q \chi_Q(x), \ \ x \in \mathbb{R}^{n+1}_+,$$

where we denote by $(f)_Q$ the average of f over Q, that is,

$$(f)_Q := \fint_Q f(x) dm_\lambda(x) := \frac{1}{m_\lambda(Q)} \int_Q f(x) dm_\lambda(x).$$

For any $Q \in \mathcal{D}_k$, we let $h^1_Q, h^2_Q, \dots, h^{M_Q-1}_Q$ be a family of Haar functions associated to Q and then, we choose h_Q among these Haar functions such that $\left|\int_Q b(x) h^\epsilon_Q(x)\, dm_\lambda(x)\right|$ is maximal with respect to $\epsilon \in \{1, 2, \dots, M_Q - 1\}$. Observe that the function $(E_{k+1}(b)(x) - E_k(b)(x))\chi_Q(x)$ is a sum of $M_Q - 1$ Haar functions, which implies the following inequality:

$$\left(\fint_Q |E_{k+1}(b)(x) - E_k(b)(x)|^p\, dm_\lambda(x)\right)^{1/p} \leq C m_\lambda(Q)^{-1/2} \left|\int_Q b(x) h_Q(x)\, dm_\lambda(x)\right|, \tag{3.7}$$

where C is a constant only depending on p and n.

Lemma 3.1. *Given a system of dyadic cubes $\mathcal{D} := \cup_{k\in\mathbb{Z}}\mathcal{D}_k$ with parameters $\delta \in (0,1)$. Let $1 < p < \infty$ and suppose that $b \in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ satisfying $\|[b, R_{\lambda,\ell}]\|_{S^p_\lambda} < \infty$ for some $\ell \in \{1, 2, ..., n+1\}$, then there exists a constant $C = C(\delta, \lambda, p) > 0$ such that*

$$\sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{D}_k\\ Q\subseteq\mathbb{R}^{n+1}_+}} \fint_Q |E_{k+1}(b)(x) - E_k(b)(x)|^p dm_\lambda(x) \leq C \|[b, R_{\lambda,\ell}]\|^p_{S^p_\lambda}. \tag{3.8}$$

Proof. To begin with, we recall from (3.7) that

$$\fint_Q |E_{k+1}(b)(x) - E_k(b)(x)|^p dm_\lambda(x) \le C m_\lambda(Q)^{-p/2} \left| \int_Q b(x) h_Q(x) dm_\lambda(x) \right|^p . \tag{3.9}$$

To continue, for any $Q \in \mathcal{D}_k$ with $Q \subseteq \mathbb{R}^{n+1}_+$, let $\hat{Q}$ be the ball chosen in Lemma 2.4, then $K_{\lambda,\ell}(x,y)$ does not change sign and

$$|K_{\lambda,\ell}(x,y)| \gtrsim \frac{1}{m_\lambda(Q)}, \tag{3.10}$$

for all $(x,y) \in Q \times \hat{Q}$. Now we define $\alpha_S(f)$ be the median value of f over a given set S, which means $\alpha_S(f)$ is a real number satisfying

$$m_\lambda(\{x \in S : f(x) > \alpha_S(f)\}) \le \frac{1}{2} m_\lambda(S) \text{ and } m_\lambda(\{x \in S : f(x) < \alpha_S(f)\}) \le \frac{1}{2} m_\lambda(S).$$

Recall from [10] that median value of a given function always exists, but may not be unique. Next, we set

$$E_1^Q := \left\{x \in Q : b(x) \le \alpha_{\hat{Q}}(b)\right\} \text{ and } E_2^Q := \left\{x \in Q : b(x) > \alpha_{\hat{Q}}(b)\right\}. \tag{3.11}$$

Next we decompose Q into a union of disjoint sub-cubes by writing $Q = \bigcup_{i=1}^{M_Q} P_i$ with $P_i \in \mathcal{D}_{k+1}$ and $P_i \subseteq Q$. Applying the cancellation property of h_Q, we deduce that

$$\begin{aligned}
& m_\lambda(Q)^{-1/2} \left| \int_Q b(x) h_Q(x) dm_\lambda(x) \right| \\
&= m_\lambda(Q)^{-1/2} \left| \int_Q (b(x) - \alpha_{\hat{Q}}(b)) h_Q(x)\, dm_\lambda(x) \right| \\
&\le \frac{1}{m_\lambda(Q)} \int_Q \left| b(x) - \alpha_{\hat{Q}}(b) \right| dm_\lambda(x) \\
&\le \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i} \left| b(x) - \alpha_{\hat{Q}}(b) \right| dm_\lambda(x) \\
&\le \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i \cap E_1^Q} \left| b(x) - \alpha_{\hat{Q}}(b) \right| dm_\lambda(x) + \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i \cap E_2^Q} \left| b(x) - \alpha_{\hat{Q}}(b) \right| dm_\lambda(x) \\
&=: \mathrm{A}_1^Q + \mathrm{A}_2^Q. \qquad (3.12)
\end{aligned}$$

Now we denote

$$F_1^Q := \{\hat{x} \in \hat{Q} : b(\hat{x}) \ge \alpha_{\hat{Q}}(b)\} \text{ and } F_2^Q := \{\hat{x} \in \hat{Q} : b(\hat{x}) \le \alpha_{\hat{Q}}(b)\}.$$

Then the definition of $\alpha_{\hat{Q}}(b)$ implies that $m_\lambda(F_1^Q) = m_\lambda(F_2^Q) \sim m_\lambda(\hat{Q})$ and $F_1^Q \cup F_2^Q = \hat{Q}$. Furthermore, for $s = 1, 2$, if $x \in E_s^Q$ and $y \in F_s^Q$, then

$$\begin{aligned}
\left| b(x) - \alpha_{\hat{Q}}(b) \right| &\le \left| b(x) - \alpha_{\hat{Q}}(b) \right| + \left| \alpha_{\hat{Q}}(b) - b(y) \right| \\
&= \left| b(x) - \alpha_{\hat{Q}}(b) + \alpha_{\hat{Q}}(b) - b(y) \right| = |b(y) - b(x)|.
\end{aligned}$$

This implies that, for $s = 1, 2$,

$$\begin{aligned}
\mathrm{A}_s^Q &\lesssim \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i \cap E_s^Q} \left|b(x) - \alpha_{\hat{Q}}(b)\right| dm_\lambda(x) \frac{m_\lambda(F_s^Q)}{m_\lambda(Q)} \\
&\lesssim \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i \cap E_s^Q} \int_{F_s^Q} \left|b(x) - \alpha_{\hat{Q}}(b)\right| \left|K_{\lambda,\ell}(x,y)\right| dm_\lambda(y) dm_\lambda(x) \\
&\lesssim \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \int_{P_i \cap E_s^Q} \int_{F_s^Q} |b(y) - b(x)| \left|K_{\lambda,\ell}(x,y)\right| dm_\lambda(y) dm_\lambda(x) \\
&= \frac{1}{m_\lambda(Q)} \sum_{i=1}^{M_Q} \left| \int_{P_i \cap E_s^Q} \int_{F_s^Q} (b(y) - b(x)) K_{\lambda,\ell}(x,y) dm_\lambda(y) dm_\lambda(x) \right|, \qquad (3.13)
\end{aligned}$$

where in the last equality we used the fact that $K_{\lambda,\ell}(x,y)$ and $b(y) - b(x)$ do not change sign for $(x,y) \in (P_i \cap E_s^Q) \times F_s^Q$, $s = 1, 2$. This, in combination with the inequalities (3.9) and (3.12), implies that

$$\begin{aligned}
\sum_{\substack{Q \in \mathcal{D}_k \\ Q \subseteq \mathbb{R}_+^{n+1}}} m_\lambda(Q)^{-p/2} \left| \int_Q b(x) h_Q(x) dx \right|^p &\lesssim \sum_{s=1}^{2} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subseteq \mathbb{R}_+^{n+1}}} \left|\mathrm{A}_s^Q\right|^p \\
&\lesssim \sum_{s=1}^{2} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subseteq \mathbb{R}_+^{n+1}}} \left(\sum_{i=1}^{M_Q} \left| \left\langle [b, R_{\lambda,\ell}] \frac{m_\lambda(P_i)^{1/2} \chi_{F_s^Q}}{m_\lambda(Q)}, \frac{\chi_{P_i \cap E_s^Q}}{m_\lambda(P_i)^{1/2}} \right\rangle \right| \right)^p. \qquad (3.14)
\end{aligned}$$

Note that the functions $e_Q := \frac{m_\lambda(P_i)^{1/2} \chi_{F_s^Q}}{m_\lambda(Q)} \subset \hat{Q}$ and $f_Q := \frac{\chi_{P_i \cap E_s^Q}}{m_\lambda(P_i)^{1/2}} \subset Q$ satisfy $|e_Q|, |f_Q| \leq C m_\lambda(Q)^{-\frac{1}{2}} \chi_{cQ}$, where C and c are absolute constants independent of Q. Summing this last inequality over $k \in \mathbb{Z}$ and then using (2.5), we finish the proof of Lemma 3.1. □

This is an immediate corollary.

Corollary 3.2. *Given a system of dyadic cubes $\mathcal{D} := \cup_{k \in \mathbb{Z}} \mathcal{D}_k$ with parameters $\delta \in (0,1)$. Let $1 < p < \infty$ and suppose that $b \in L^1_{\rm loc}(\mathbb{R}_+^{n+1})$ satisfying $\|[b, R_{\lambda,\ell}]\|_{S_\lambda^p} < \infty$ for some $\ell \in \{1, 2, ..., n+1\}$, then there exists a constant $C = C(\delta, \lambda, p) > 0$ such that for any $k \in \mathbb{Z}$,*

$$\left(\int_{\mathbb{R}_+^{n+1}} \frac{|b(x) - E_k b(x)|^p}{m_\lambda(B_{\mathbb{R}_+^{n+1}}(x, \delta^k))} dm_\lambda(x) \right)^{1/p} \leq C \|[b, R_{\lambda,\ell}]\|_{S_\lambda^p}. \qquad (3.15)$$

Proof. It follows from Lemma 2.2 that $E_k(b) \to b$ a.e. as $k \to \infty$. Moreover, we note that if $x \in Q \in \mathcal{D}_k$, then by the doubling inequality (2.4),

$$m_\lambda(Q) \simeq m_\lambda(B_{\mathbb{R}_+^{n+1}}(x, \delta^k)).$$

This, in combination with Lemma 3.1, implies that for any $k \in \mathbb{Z}$,

$$\left(\int_{\mathbb{R}_+^{n+1}} \frac{|E_{k+1} b(x) - E_k b(x)|^p}{m_\lambda(B_{\mathbb{R}_+^{n+1}}(x, \delta^k))} dm_\lambda(x) \right)^{1/p} \leq C \|[b, R_{\lambda,\ell}]\|_{S_\lambda^p}.$$

This, in combination with the doubling inequality (2.2), yields

$$\begin{aligned}\left(\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^k))}dm_\lambda(x)\right)^{1/p} &\le \sum_{j\ge k}\left(\int_{\mathbb{R}^{n+1}_+}\frac{|E_{j+1}b(x)-E_jb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^k))}dm_\lambda(x)\right)^{1/p}\\ &\le \sum_{j\ge k}\delta^{\frac{(n+1)(j-k)}{p}}\left(\int_{\mathbb{R}^{n+1}_+}\frac{|E_{j+1}b(x)-E_jb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^j))}dm_\lambda(x)\right)^{1/p}\\ &\le C\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}.\end{aligned}$$

This ends the proof of Lemma 3.2. □

Lemma 3.3. *Given a system of dyadic cubes $\mathcal{D}:=\cup_{k\in\mathbb{Z}}\mathcal{D}_k$ with parameter $\delta\in(0,1)$. Let $1<p<\infty$ and suppose that $b\in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ satisfying $\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}<\infty$ for some $\ell\in\{1,2,...,n+1\}$, then there exists a constant $C=C(\delta,\lambda,p)>0$ such that*

$$\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}\le C\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}. \tag{3.16}$$

Proof. It suffices to show that

$$\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}\le C\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda} \tag{3.17}$$

for some constant $C>0$ independent of $L<M\in\mathbb{N}$. To show this inequality, we denote by $\mathfrak{I}$ the term in the left-hand side above and then observe that

$$\begin{aligned}\mathfrak{I}&\le\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_{k+1}b(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}+\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|E_{k+1}b(x)-E_kb(x)|^p}{m_\lambda(Q(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}\\ &=\left(\sum_{k=L+1}^{M+1}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k}))}dm_\lambda(x)\right)^{1/p}+\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|E_{k+1}b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}\\ &=:\mathrm{Term}_1+\mathrm{Term}_2.\end{aligned}$$

By Lemma 3.1, Term_2 is bounded by the required norm. Moreover, it follows from Corollary 3.2 and inequality (2.2) that Term_1 is dominated by

$$\begin{aligned}&\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k}))}dm_\lambda(x)\right)^{1/p}+\left(\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_{M+1}b(x)|^p}{m_\lambda(Q(x,\delta^{M+1}))}dm_\lambda(x)\right)^{1/p}\\ &\le\delta^{\frac{n+1}{p}}\left(\sum_{k=L}^{M}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-E_kb(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)\right)^{1/p}+C_p\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}.\end{aligned} \tag{3.18}$$

Since the first term of the right-hand side in (3.18) can be absorbed into $\mathfrak{I}$, we finish the proof of Lemma 3.3. □

Proposition 3.4. *Let $1<p<\infty$ and suppose that $b\in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ satisfying $\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}<\infty$ for some $\ell\in\{1,2,...,n+1\}$, then there exists a constant $C=C(\lambda,p)>0$ such that*

$$\|b\|_{B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)}\le C\|[b,R_{\lambda,\ell}]\|_{S^p_\lambda}.$$

Proof. By the definition, we need to show

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(x)dm_\lambda(y)\lesssim \|[b,R_{\lambda,\ell}]\|^p_{S^p_\lambda}. \tag{3.19}$$

To this end, we first observe that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(x)dm_\lambda(y)\\&\leq\sum_{k\in\mathbb{Z}}\iint_{\substack{\delta^{k+1}<|x-y|\leq\delta^k\\ x,y\in\mathbb{R}^{n+1}_+}}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(x)dm_\lambda(y)\\&\leq\sum_{k\in\mathbb{Z}}\iint_{\substack{\delta^{k+1}<|x-y|\leq\delta^k\\ x,y\in\mathbb{R}^{n+1}_+}}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y).\end{aligned}$$

To estimate the term on the right-hand side, we first recall from Section 2.3 that there exists a collection $\{\mathcal{D}^\nu\colon \nu=1,2,\ldots,\kappa\}$ of adjacent systems of dyadic cubes on $\mathbb{R}^{n+1}_+$ with parameters $\delta\in(0,\frac{1}{96})$ and $C_{\rm adj}:=8\delta^{-3}$ such that the properties in Section 2.3 hold. With these collection of adjacent systems, for any $f\in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$, we define

$$E^\nu_k(f)(x)=\sum_{Q\in\mathcal{D}^\nu_k}(f)_Q\chi_Q(x),\ \ x\in\mathbb{R}^{n+1}_+.$$

Moreover, we denote by $d(y,Q)$ the distance from a point $y\in\mathbb{R}^{n+1}_+$ to a set Q and the notation x_0 to denote the centre of Q. Then, we observe that there exists an absolute constant $k_0>0$ such that for any $Q\in\mathcal{D}^0_k$, the following inclusion holds:

$$Q_{\delta^{k+1}}:=\{y\in\mathbb{R}^{n+1}_+:d(y,Q)\leq\delta^{k+1}\}\subset B_{\mathbb{R}^{n+1}_+}(x_0,\delta^{k-k_0}),$$

Next, applying Lemma 2.1 to deduce that there exist $\nu\in\{1,2,\ldots,\kappa\}$ and $Q'\in\mathcal{D}^\nu_{k-k_0-2}$ such that

$$B_{\mathbb{R}^{n+1}_+}(x,\delta^{k-k_0})\subseteq Q'\subseteq B_{\mathbb{R}^{n+1}_+}(x,C_{\rm adj}\delta^{k-k_0}). \tag{3.20}$$

Combining with all the above facts, we conclude that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\iint_{\substack{\delta^{k+1}<|x-y|\leq\delta^k\\ x,y\in\mathbb{R}^{n+1}_+}}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y)\\&\lesssim\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^0_k}\int_Q\int_{Q_{\delta^{k+1}}}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y)\\&\lesssim\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^0_k}\int_{Q'}\int_{Q'}\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y)\\&\lesssim\sum_{\nu=1}^{\kappa}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^\nu_{k-k_0-2}}\int_Q\int_Q\frac{|b(x)-b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y),\end{aligned} \tag{3.21}$$

where the last inequality applied the fact that each $Q'\in\mathcal{D}^\nu_{k-k_0-2}$ contains at most c (an absolute constant) $Q\in\mathcal{D}^0_k$. To continue, we apply the doubling inequality (2.4) to deduce that if $x,y\in Q\in\mathcal{D}^\nu_{k-k_0-2}$, then

$$m_\lambda(Q)\simeq m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))\simeq m_\lambda(B_{\mathbb{R}^{n+1}_+}(y,\delta^{k+1})).$$

Thus, the right-hand side in (3.21) is bounded by (up to a harmless constant)

$$\sum_{\nu=1}^{\kappa}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^{\nu}_{k-k_0-2}}\int_Q\int_Q \frac{|b(x)-E^{\nu}_{k-k_0-2}b(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}+\frac{|b(y)-E^{\nu}_{k-k_0-2}b(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))^2}dm_\lambda(x)dm_\lambda(y)$$
$$=\sum_{\nu=1}^{\kappa}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}^{\nu}_{k-k_0-2}}\int_Q \frac{|b(x)-E^{\nu}_{k-k_0-2}b(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x)$$
$$=\sum_{\nu=1}^{\kappa}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}_+} \frac{|b(x)-E^{\nu}_{k}b(x)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\delta^{k+1}))}dm_\lambda(x).$$

Applying Lemma 3.3 to the last term, we complete the proof of Proposition 3.4. □

4. Proof of the sufficient condition

At the beginning of this section, we will establish an interpolation theorem for the Bessel-Besov space $B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)$, which plays a key role in the proof of sufficiency (the upper bound of Theorem 1.5).

Lemma 4.1. *Let $1<p_1<p<p_2<\infty$ and $0<\theta<1$, then the spaces $B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)$ and $(B^{\frac{n+1}{p}}_{p_1,p_1}(\mathbb{R}^{n+1}_+,dm_\lambda),B^{\frac{n+1}{p}}_{p_2,p_2}(\mathbb{R}^{n+1}_+,dm_\lambda))_{\theta_p}$ coincide, with equivalent norms, where θ_p satisfies $\frac{1-\theta_p}{p_1}+\frac{\theta_p}{p_2}=\frac{1}{p}$.*

Proof. Define a quasi-distance function by

$$d(x,y):=m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|)),\ \text{for any } x,\ y\in\mathbb{R}^{n+1}_+.$$

Then it can be verified that $d(x,y)$ is equivalent to the Macias-Segovia distance function given by

$$\rho(x,y):=\inf\{m_\lambda(B): B \text{ are balls containing } x \text{ and } y\}.$$

Denote by $B_d(x,r):=\{y\in\mathbb{R}^{n+1}_+ : d(x,y)<r\}$ the ball in $\mathbb{R}^{n+1}_+$ with respect to the distance function d. By [12, Theorems 2 and 3] $(\mathbb{R}^{n+1}_+,d,m_\lambda)$ is a 1-dimensional space of homogeneous type, satisfying $m_\lambda(B_d(x,r))\sim r$. Moreover, for $f\in L^p_{\rm loc}(\mathbb{R}^{n+1}_+,dm_\lambda)$,

$$\left(\sum_{\nu\in\mathbb{Z}}2^\nu\int_{\mathbb{R}^{n+1}_+}\fint_{B_d(x,2^{-\nu})}|f(x)-f(y)|^p dm_\lambda(y)dm_\lambda(x)\right)^{1/p}$$
$$=\left(\sum_{\nu\in\mathbb{Z}}2^\nu\int_{\mathbb{R}^{n+1}_+}\frac{1}{m_\lambda(B_d(x,2^{-\nu}))}\sum_{\ell\geq\nu}\int_{B_d(x,2^{-\ell})\setminus B_d(x,2^{-\ell-1})}|f(x)-f(y)|^p dm_\lambda(y)dm_\lambda(x)\right)^{1/p}$$
$$\sim\left(\sum_{\ell\in\mathbb{Z}}\sum_{\nu\leq\ell}2^{2(\nu-\ell)}\int_{\mathbb{R}^{n+1}_+}\int_{B_d(x,2^{-\ell})\setminus B_d(x,2^{-\ell-1})}\frac{|f(x)-f(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(y)dm_\lambda(x)\right)^{1/p}$$
$$\lesssim\left(\sum_{\ell\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}_+}\int_{B_d(x,2^{-\ell})\setminus B_d(x,2^{-\ell-1})}\frac{|f(x)-f(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(y)dm_\lambda(x)\right)^{1/p}$$
$$=\left(\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}\frac{|f(x)-f(y)|^p}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,|x-y|))^2}dm_\lambda(y)dm_\lambda(x)\right)^{1/p}.$$

Note that the expression in the left-hand side above is exactly the Lipschitz-type norm $\dot{L}_t(1, p, p, \mathbb{R}^{n+1}_+)$ of f given in [14, Definition 3.2]. This, in combination with [14, Theorem 4.1] and [17, Theorem 3.1], finishes the proof of Lemma 4.1. $\square$

Lemma 4.2. *For any $p > 2$, $q > 1$ satisfying $\frac{1}{q} = 1 - \frac{2}{p}$, one has*

$$\left\| \frac{1}{m_\lambda(B(x, |x-y|))^{1-2/p}} \right\|_{L^\infty, L^{q,\infty}} < +\infty. \tag{4.22}$$

Proof. We apply inequality (2.1) to deduce that

$$\begin{aligned}
&\left\| \frac{1}{m_\lambda(B(x, |x-y|))^{1-2/p}} \right\|_{L^\infty, L^{q,\infty}} \\
&= \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : \frac{1}{m_\lambda(B(x, |x-y|))^{1-2/p}} > \alpha \right\}^{1/q} \\
&\leq \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : \frac{c}{(|x-y|^{n+1} x_{n+1}^{2\lambda} + |x-y|^{n+1+2\lambda})^{1-2/p}} > \alpha \right\}^{1/q} \\
&\leq \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| < x_{n+1} \text{ and } \frac{c}{(|x-y|^{n+1} x_{n+1}^{2\lambda})^{1-2/p}} > \alpha \right\}^{1/q} \\
&+ \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| \geq x_{n+1} \text{ and } \frac{c}{(|x-y|^{n+1+2\lambda})^{1-2/p}} > \alpha \right\}^{1/q} \\
&=: \mathrm{I} + \mathrm{II},
\end{aligned}$$

for some $c > 0$ depending only on the implicit doubling constant.

For the term I, we apply inequality (2.1) and the condition $\frac{1}{q} = 1 - \frac{2}{p}$ to deduce that

$$\begin{aligned}
\mathrm{I} &= \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| < \min\left\{ x_{n+1}, c_{n,p} \alpha^{-\frac{1}{(n+1)(1-2/p)}} x_{n+1}^{-\frac{2\lambda}{n+1}} \right\} \right\}^{1/q} \\
&\leq \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{0<\alpha<x_{n+1}^{-(n+1+2\lambda)(1-2/p)}} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| < x_{n+1} \right\}^{1/q} \\
&+ \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha\geq x_{n+1}^{-(n+1+2\lambda)(1-2/p)}} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| < c_{n,p} \alpha^{-\frac{1}{(n+1)(1-2/p)}} x_{n+1}^{-\frac{2\lambda}{n+1}} \right\}^{1/q} \\
&\leq C \left(\sup_{x\in\mathbb{R}^{n+1}_+} \sup_{0<\alpha<x_{n+1}^{-(n+1+2\lambda)(1-2/p)}} \alpha x_{n+1}^{\frac{n+1+2\lambda}{q}} + \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha\geq x_{n+1}^{-(n+1+2\lambda)(1-2/p)}} \alpha \left(\left(c_{n,p} \alpha^{-\frac{1}{(n+1)(1-2/p)}} x_{n+1}^{-\frac{2\lambda}{n+1}} \right)^{n+1} x_{n+1}^{2\lambda} \right)^{1/q} \right) \\
&\leq C,
\end{aligned}$$

where we denote $c_{n,p} = c^{\frac{1}{(1-2/p)(n+1)}}$ for simplicity.

Similarly, for the term II, we have

$$\begin{aligned}
\mathrm{II} &= \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{\alpha>0} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : x_{n+1} \leq |x-y| < c_{n,p,\lambda} \alpha^{-\frac{1}{(1-2/p)(n+1+2\lambda)}} \right\}^{1/q} \\
&\leq \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{0<\alpha<c_{n,p,\lambda} x_{n+1}^{-(1-2/p)(n+1+2\lambda)}} \alpha m_\lambda \left\{ y \in \mathbb{R}^{n+1}_+ : |x-y| < c_{n,p,\lambda} \alpha^{-\frac{1}{(1-2/p)(n+1+2\lambda)}} \right\}^{1/q}
\end{aligned}$$

$$\leq C \sup_{x\in\mathbb{R}^{n+1}_+} \sup_{0<\alpha<c_{n,p,\lambda}x_{n+1}^{-(1-2/p)(n+1+2\lambda)}} \alpha\Big(c_{n,p,\lambda}\alpha^{-\frac{1}{(1-2/p)(n+1+2\lambda)}}\Big)^{\frac{n+1+2\lambda}{q}}$$
$$\leq C,$$

where we denote $c_{n,p,\lambda} = c^{\frac{1}{(1-2/p)(n+1+2\lambda)}}$ for simplicity. □

Proposition 4.3. *Suppose $\ell \in \{1, 2, \ldots, n+1\}$, $n+1 < p < \infty$ and $b \in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$. If $b \in B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda)$, then $[b, R_{\lambda,\ell}] \in S^p_\lambda$.*

Proof. To begin with, we note that $B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda) \subset \mathrm{VMO}(\mathbb{R}^{n+1}_+, dm_\lambda)$. Then, the condition $b \in B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+, dm_\lambda)$ guarantees the compactness of $[b, R_{\lambda,\ell}]$ (see [2, Theorem 5.3]). Next, we shall adapt Russo's principle ([16]) to our Bessel setting, which is useful to dominate the Schatten norms of integral operators. The principle can be formulated as follows: for a general measure space (X, μ), if $K(x, y) \in L^2(X \times X)$, then the integral operator T associated to the kernel $K(x, y)$ satisfies for any $p > 2$,

$$\|T\|_{S^p} \leq \|K\|^{1/2}_{L^p,L^{p'}} \|K^*\|^{1/2}_{L^p,L^{p'}},$$

where p' is the conjugate index of p such that $1/p + 1/p' = 1$, $K^*(x, y) = \overline{K(y, x)}$, and $\|\cdot\|_{L^p,L^{p'}}$ denotes the mixed-norm: $\|K\|_{L^p,L^{p'}} := \Big\|\|K(x, y)\|_{L^p(d\mu(x))}\Big\|_{L^{p'}(d\mu(y))}$. Later, Goffeng ([6]) proved that the condition $K(x, y) \in L^2(X \times X)$ in the above statement can be removed.

Furthermore, the above principle was extended to the corresponding weak-type version (see [9, Lemma 1 and Lemma 2]): if $p > 2$ and $1/p + 1/p' = 1$, then

$$\|T\|_{S^{p,\infty}} \leq \|K\|^{1/2}_{L^p,L^{p',\infty}} \|K^*\|^{1/2}_{L^p,L^{p',\infty}}, \tag{4.23}$$

where $\|\cdot\|_{L^p,L^{p',\infty}}$ denotes the mixed-norm: $\|K\|_{L^p,L^{p',\infty}} := \Big\|\|K(x, y)\|_{L^p(d\mu(x))}\Big\|_{L^{p',\infty}(d\mu(y))}$.

Next, back to our Bessel setting, we apply Lemma 4.2 and weak-type Young's inequality to deduce that that for $1/q = 1 - 2/p$,

$$\begin{aligned}\|(b(x) - b(y))K_{\lambda,\ell}(x, y)\|_{L^p,L^{p',\infty}} &\leq \left\|\frac{b(x) - b(y)}{m_\lambda(B(x, |x - y|))}\right\|_{L^p,L^{p',\infty}} \\ &\leq \left\|\frac{b(x) - b(y)}{m_\lambda(B(x, |x - y|))^{2/p}}\right\|_{L^p,L^p} \left\|\frac{1}{m_\lambda(B(x, |x - y|))^{1-2/p}}\right\|_{L^\infty,L^{q,\infty}} \\ &\leq C\|b\|_{B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)}.\end{aligned}$$

Therefore,

$$\Big\|(b(x) - b(y))K_{\lambda,\ell}(x, y)\Big\|_{L^p,L^{p',\infty}} \leq C\|b\|_{B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)}. \tag{4.24}$$

Combining the inequality (4.24) and then applying the weak-type Russo's inequality (4.23), we see that

$$\|[b, R_{\lambda,\ell}]\|_{S^{p,\infty}} \leq C\|b\|_{B^{\frac{n+1}{p}}_{p,p}(\mathbb{R}^{n+1}_+,dm_\lambda)}.$$

Since this inequality holds for all $n+1 < p < \infty$, we can apply the interpolation $(S_\lambda^{p_1,\infty}, S_\lambda^{p_2,\infty})_{\theta_p} = S_\lambda^p$ and $(B_{p_1,p_1}^{\frac{n+1}{p_1}}(\mathbb{R}_+^{n+1}, dm_\lambda), B_{p_2,p_2}^{\frac{n+1}{p_2}}(\mathbb{R}_+^{n+1}, dm_\lambda))_{\theta_p} = B_{p,p}^{\frac{n+1}{p}}(\mathbb{R}_+^{n+1}, dm_\lambda)$, where $\frac{1-\theta_p}{p_1} + \frac{\theta_p}{p_2} = \frac{1}{p}$, to obtain that

$$\|[b, R_{\lambda,\ell}]\|_{S_\lambda^p} \leq C\|b\|_{B_{p,p}^{\frac{n+1}{p}}(\mathbb{R}_+^{n+1}, dm_\lambda)}.$$

This finishes the proof of sufficient condition for the case $n+1 < p < \infty$. □

5. Theorem 1.5: $0 < p \leq n+1$

This section is devote to providing a proof for the second case of Theorem 1.5. That is, for each $\ell \in \{1, 2, \ldots, n+1\}$ and for $0 < p \leq n+1$, we will show that the commutator $[b, R_{\lambda,\ell}] \in S_\lambda^p$ if and only if b is a constant on $\mathbb{R}_+^{n+1}$. The key difficulty is to show the necessary part of the Theorem 1.5. To show this, it suffices to consider the endpoint case $p = n+1$ since one has the inclusion $S^p \subset S^q$ for $p < q$.

To complete our proof we will use the following lemmas.

Lemma 5.1. *For any $k \in \mathbb{Z}$ and cube $Q \in \mathcal{D}_k^0$ and $a_j = \pm 1 (j = 1, 2, \ldots, n+1)$, there are cubes $Q' \in \mathcal{D}_{k+2}^0, Q'' \in \mathcal{D}_{k+2}^0$ such that $Q' \subset Q, Q'' \subset Q$ and if $x = (x_1, x_2, \ldots, x_{n+1}) \in Q'$, $y = (y_1, y_2, \ldots, y_{n+1}) \in Q''$, then $a_j(x_j - y_j) \geq 2^{-k}$ for $j = 1, 2, \ldots, n+1$.*

Now, we provide a lower bound for a local pseudo oscillation of the symbol b in the commutator.

Lemma 5.2. *Let $b \in C^2(\mathbb{R}_+^{n+1})$. Suppose that there is a point $x_0 \in \mathbb{R}_+^{n+1}$ such that $\nabla b(x_0) \neq 0$. Then there exist constants $C > 0, \epsilon > 0$ and $N > 0$ such that if $k > N$, then for any cube $Q \in \mathcal{D}_k^0$ satisfying $|\mathrm{center}(Q) - x_0| < \epsilon$, one has*

$$\left| \fint_{Q'} b(x) dm_\lambda(x) - \fint_{Q''} b(x) dm_\lambda(x) \right| \geq C 2^{-k} |\nabla b(x_0)|, \tag{5.25}$$

where Q' and Q'' are the cubes chosen in Lemma 5.1.

Proof. We denote by $c_Q := (c_Q^1, c_Q^2, \ldots, c_Q^{n+1})$ the center of Q and $x = (x_1, x_2, \ldots, x_{n+1})$, then it follows from Taylor's formula that

$$b(x) = b(c_Q) + \sum_{j=1}^{n+1} (\partial_{x_j} b)(c_Q)(x_j - c_Q^j) + R(x, c_Q), \tag{5.26}$$

where the remainder term $R(x, c_Q)$ satisfies

$$|R(x, c_Q)| \leq C \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \sup_{\theta \in [0,1]} \left| (\partial_{x_j} \partial_{x_k} b)(x + \theta(c_Q - x)) \right| |c_Q - x|^2, \tag{5.27}$$

for some $\theta \in [0, 1]$. Note that if $x \in Q$, then

$$|x + \theta(c_Q - x) - c_Q| \lesssim 2^{-k}.$$

By Lemma 5.1, for $x' = (x_1', ..., x_{n+1}') \in Q'$ and $x'' = (x_1'', ..., x_{n+1}'') \in Q''$, we have

$$\mathrm{sgn}(\partial_{x_j} b)(c_Q)(x_j' - x_j'') \geq 2^{-k}, \quad j = 1, 2, \ldots, n+1.$$

Combining the above facts, we deduce that

$$\begin{aligned}
&\left|\fint_{Q'} b(x')dm_\lambda(x') - \fint_{Q''} b(x'')dm_\lambda(x'')\right| \\
&\geq \left|\fint_{Q'}\fint_{Q''}\sum_{j=1}^{n+1}(\partial_{x_j}b)(c_Q)(x_j^{'} - x_j^{''})dm_\lambda(x'')dm_\lambda(x')\right| \\
&\qquad - \fint_{Q'}|R(x',c_Q)|dm_\lambda(x') - \fint_{Q''}|R(x'',c_Q)|dm_\lambda(x'') \\
&\geq C\sum_{j=1}^{n+1}|(\partial_{x_j}b)(c_Q)|2^{-k} - C2^{-2k}\|\nabla^2 b\|_{L^\infty(B(x_0,1))} \\
&\geq C2^{-k}|\nabla b(x_0)|,
\end{aligned}$$

where in the last inequality we choose $\epsilon = \epsilon_b$ be a sufficiently small constant. This completes the proof of Lemma 5.2. □

Define the conditional expectation of a locally integrable function f on $\mathbb{R}_+^{n+1}$ with respect to the increasing family of σ−algebras $\sigma(\mathcal{D}_k^0)$ by the expression:

$$E_k^0(f)(x) = \sum_{Q\in\mathcal{D}_k^0}(f)_Q\chi_Q(x),\ x\in\mathbb{R}_+^{n+1},$$

We have the following Lemma.

Lemma 5.3. *A function $b \in C^2(\mathbb{R}_+^{n+1})$ is a constant on $\mathbb{R}_+^{n+1}$ if there exist constants $C > 0$ and $\ell \in \{1, 2, ..., n+1\}$ such that*

$$\left\|\left\{\fint_Q\fint_Q |E_{k+2}^0(b)(x') - E_{k+2}^0(b)(x'')|dm_\lambda(x')dm_\lambda(x'')\right\}_{\substack{Q\in\mathcal{D}^0\\ Q\subseteq\mathbb{R}_+^{n+1}}}\right\|_{l^{n+1}} \leq C\|[b, R_{\lambda,\ell}]\|_{S_\lambda^{n+1}}. \tag{5.28}$$

(In the display $Q \in \mathcal{D}_k^0$ and $Q \subseteq \mathbb{R}_+^{n+1}$, and both Q and k vary.)

Proof. If not, then observe that there exists a point $x_0 = (x_0^{(1)}, \ldots, x_0^{(n+1)}) \in \mathbb{R}_+^{n+1}$ such that $\nabla b(x_0) \neq 0$. By Lemma 5.2, there exist some $\epsilon \in (0, x_0^{(n+1)})$ and $N > 0$ such that if $k > N$, then for any cube $Q \in \mathcal{D}_k^0$ satisfying $|\mathrm{center}(Q) - x_0| < \epsilon$,

$$\fint_Q\fint_Q |E_{k+2}^0(b)(x') - E_{k+2}^0(b)(x'')|dm_\lambda(x')dm_\lambda(x'') \gtrsim 2^{-k}|\nabla b(x_0)|.$$

Denote $\mathcal{A}_k(x_0)$ be the set consisting of $Q \in \mathcal{D}_k^0$ satisfying $|\mathrm{center}(Q) - x_0| < \epsilon$. Then observe that for any $k > N$, the number of $\mathcal{A}_k(x_0)$ is at least $2^{k(n+1)}$, which implies that

$$\sum_{k>N}\sum_{Q\in\mathcal{A}_k(x_0)}\left(\fint_Q\fint_Q |E_{k+2}^0(b)(x') - E_{k+2}^0(b)(x'')|dm_\lambda(x')dm_\lambda(x'')\right)^{n+1} = +\infty.$$

However, the left hand side above is dominated by $\|[b, R_{\lambda,\ell}]\|_{S_\lambda^{n+1}}^{n+1}$, which is a contradiction. This ends the proof of Lemma 5.3. □

Proposition 5.4. *Suppose $b \in C^2(\mathbb{R}_+^{n+1})$. Then for any $\ell \in \{1, 2, \ldots, n+1\}$, the commutator $[b, R_{N,\ell}] \in S_\lambda^{n+1}$ if and only if b is a constant on $\mathbb{R}_+^{n+1}$.*

Proof. It is obvious that if b is a constant on $\mathbb{R}^{n+1}_+$, then $[b, R_{\lambda,\ell}] = 0$. Hence it remains to consider the direction in which we assume $[b, R_{\lambda,\ell}] \in S^{n+1}_\lambda$. To this end, by Lemma 3.1, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\left\|\left\{ \fint_Q \fint_Q |E^0_{k+2}(b)(x') - E^0_{k+2}(b)(x'')| dm_\lambda(x') dm_\lambda(x'')\right\}_{\substack{Q\in\mathcal{D}^0\\ Q\subseteq\mathbb{R}^{n+1}_+}}\right\|_{l^{n+1}} \\
&\leq C\left\|\left\{ \fint_Q |E^0_{k+2}(b)(x') - E^0_{k}(b)(x')| dm_\lambda(x')\right\}_{\substack{Q\in\mathcal{D}^0\\ Q\subseteq\mathbb{R}^{n+1}_+}}\right\|_{l^{n+1}} \\
&\leq C\|[b, R_{\lambda,\ell}]\|_{S^{n+1}_\lambda}. \qquad (5.29)
\end{aligned}$$

[Comment: the red part doesn't work for $n = 0$ since the endpoint equal to 1 while we require $p > 1$ in Lemma 3.1]

(In the display $Q \in \mathcal{D}^0_k$, $Q \subseteq \mathbb{R}^{n+1}_+$, and both Q and k vary.) This, together with Lemma 5.3, finishes the proof of Proposition 5.4. □

6. Weak-type endpoint Schatten estimate

Unify the notation D_k in the above sections

For any $Q \in \mathcal{D}^0$, let $\hat{Q}$ be the cube chosen in Lemma 2.4 We need to make sure that $\hat{Q} \in \mathcal{D}^0_k$. Define

$$J_Q(x, y) := m_\lambda(Q)^{-1} m_\lambda(\hat{Q})^{-1} K_{\lambda,\ell}(x, y)^{-1} \chi_Q(x)\chi_{\hat{Q}}(y).$$

Here we omit the dependence on λ and ℓ since these two parameters play no role in the proof below.

Lemma 6.1. *There is a finite index set $\mathcal{I}$ such that for any $\ell \in \{1, 2, ..., n + 1\}$, $Q \in \mathcal{D}^0$ and $k \in \mathbb{Z}^2$, one can construct functions $f_{Q,k,v}$, $g_{Q,k,v}$ and a sequence $\{C_{Q,k,v}\}$ satisfying for any $v \in \mathcal{I}$,*

1. $\mathrm{supp}(f_{Q,k,v}) \subset Q$, $\mathrm{supp}(g_{Q,k,v}) \subset \hat{Q}$,
2. $\|f_{Q,k,v}\|_\infty \leq m_\lambda(Q)^{-1/2}$, $\|g_{Q,k,v}\|_\infty \leq m_\lambda(Q)^{-1/2}$,
3. $|C_{Q,k,v}| \leq C_N(1 + |k|)^{-N}$ *for any* $N \in \mathbb{Z}_+$ *and for some* $C_N > 0$ *independent of* Q *and* k,
4. J_Q *admits the following factorization*
$$J_Q(x, y) = \sum_{v\in\mathcal{I}} \sum_{k\in\mathbb{Z}^2} C_{Q,k,v} f_{Q,k,v}(x) g_{Q,k,v}(y).$$

Proof. To begin with, define the conditional expectation of a locally integrable function f on $\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$ by

$$E^{(1)}_k(f)(x, y) = \sum_{Q\in\mathcal{D}^0_k} \fint_Q f(z, y) dm_\lambda(z)\chi_Q(x), \;\; x, y \in \mathbb{R}^{n+1}_+,$$

$$E^{(2)}_k(f)(x, y) = \sum_{Q\in\mathcal{D}^0_k} \fint_Q f(x, z) dm_\lambda(z)\chi_Q(y), \;\; x, y \in \mathbb{R}^{n+1}_+.$$

We also define partial martingale difference of a locally integrable function f on $\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$ by

$$D^{(i)}_k(f)(x, y) = E^{(i)}_k(f)(x, y) - E^{(i)}_{k-1}(f)(x, y), \;\; i = 1, 2.$$

Expanding in terms of Haar functions yields an equivalent form (see [11]):

$$D_k^{(1)}(f)(x,y)=\sum_{P\in\mathcal{D}_{k-1}^0}\sum_{\epsilon=1}^{2^n-1}\langle f(\cdot,y),h_P^\epsilon\rangle h_P^\epsilon(x),\tag{6.30}$$

$$D_k^{(2)}(f)(x,y)=\sum_{P\in\mathcal{D}_{k-1}^0}\sum_{\epsilon=1}^{2^n-1}\langle f(x,\cdot),h_P^\epsilon\rangle h_P^\epsilon(y).\tag{6.31}$$

Since $E_k(f)\to f$, a.e as $k\to-\infty$, one has

$$\begin{aligned}J_Q(x,y)&=\left(-E_{k_Q}^{(1)}+\sum_{k_1=-\infty}^{k_Q}D_{k_1}^{(1)}\right)\left(-E_{k_Q}^{(2)}+\sum_{k_2=-\infty}^{k_Q}D_{k_2}^{(2)}\right)(J_Q)(x,y)\chi_Q(x)\chi_{\hat{Q}}(y)\\&=E_{k_Q}^{(1)}E_{k_Q}^{(2)}(J_Q)(x,y)\chi_Q(x)\chi_{\hat{Q}}(y)-\sum_{k_1=-\infty}^{k_Q}D_{k_1}^{(1)}E_{k_Q}^{(2)}(J_Q)(x,y)\chi_Q(x)\chi_{\hat{Q}}(y)\\&\quad-\sum_{k_2=-\infty}^{k_Q}E_{k_Q}^{(1)}D_{k_2}^{(2)}(J_Q)(x,y)\chi_Q(x)\chi_{\hat{Q}}(y)+\sum_{k_1=-\infty}^{k_Q}\sum_{k_2=-\infty}^{k_Q}D_{k_1}^{(1)}D_{k_2}^{(2)}(J_Q)(x,y)\chi_Q(x)\chi_{\hat{Q}}(y)\\&=:\sum_{u=1}^{4}\mathrm{I}_u.\end{aligned}\tag{6.32}$$

where we choose k_Q be the integer such that $2^{-k_Q}\le\ell(Q)<2^{-k_Q+1}$. Next we deal with these four terms separately.

For the term I_1, by the definition of conditional expectation, we see that

$$\mathrm{I}_1=m_\lambda(Q)^{-1}m_\lambda(\hat{Q})^{-1}\fint_Q\fint_{\hat{Q}}K_{\lambda,\ell}(z,w)^{-1}dm_\lambda(z)dm_\lambda(w)\chi_Q(x)\chi_{\hat{Q}}(y).$$

Let $f_{Q,0,0}(x):=m_\lambda(Q)^{-1/2}\chi_Q(x)$, $g_{Q,0,0}(y):=m_\lambda(Q)^{-1/2}\chi_{\hat{Q}}(y)$ and

$$C_{Q,0,0}:=m_\lambda(\hat{Q})^{-1}\fint_Q\fint_{\hat{Q}}K_{\lambda,\ell}(z,w)^{-1}dm_\lambda(z)dm_\lambda(w).$$

It follows from Lemma 2.4 that $|C_{Q,0,0}|\lesssim1$. Moreover,

$$\mathrm{I}_1=C_{Q,0,0}f_{Q,0,0}(x)g_{Q,0,0}(y).$$

For the term I_2, it follows from the definition of conditional expectation and formula (6.30) that

$$\begin{aligned}\mathrm{I}_2&=\sum_{k_1=-\infty}^{k_Q}\sum_{P\in\mathcal{D}_{k_1-1}^0}\sum_{\epsilon=1}^{2^n-1}\langle E_{k_Q}^{(2)}(J_Q)(\cdot,y),h_P^\epsilon\rangle h_P^\epsilon(x)\chi_Q(x)\chi_{\hat{Q}}(y)\\&=m_\lambda(Q)^{-1}m_\lambda(\hat{Q})^{-1}\sum_{k_1=-\infty}^{k_Q}\sum_{P\in\mathcal{D}_{k_1-1}^0}\sum_{\epsilon=1}^{2^n-1}\int_{P\cap Q}\fint_{\hat{Q}}K_{\lambda,\ell}(z,w)^{-1}dm_\lambda(w)h_P^\epsilon(z)dm_\lambda(z)h_P^\epsilon(x)\chi_Q(x)\chi_{\hat{Q}}(y).\end{aligned}$$

Note that for any $k \in \mathbb{Z}$, there is a unique $P \in \mathcal{D}^0_{k-1}$ such that $P \cap Q \neq \emptyset$. We denote this cube by Q_{k-1}. Then

$$\mathrm{I}_2 = m_\lambda(Q)^{-1} m_\lambda(\hat{Q})^{-1} \sum_{k_1=-\infty}^{k_Q} \sum_{\epsilon=1}^{2^n-1} \int_Q \fint_{\hat{Q}} K_{\lambda,\ell}(z,\, w)^{-1} dm_\lambda(w) h^\epsilon_{Q_{k_1-1}}(z) dm_\lambda(z) h^\epsilon_{Q_{k_1-1}}(x)\chi_Q(x)\chi_{\hat{Q}}(y).$$

For any integer $k_1 \le k_Q$, we let

$$f^\epsilon_{Q,k_Q-k_1,0}(x) := \Big(\frac{m_\lambda(Q_{k_1-1})}{m_\lambda(Q)}\Big)^{1/2} h^\epsilon_{Q_{k_1-1}}(x)\chi_Q(x), \quad g^\epsilon_{Q,k_Q-k_1,0}(y) := m_\lambda(Q)^{-1/2}\chi_{\hat{Q}}(y)$$

and

$$C^\epsilon_{Q,k_Q-k_1,0} := m_\lambda(\hat{Q})^{-1} m_\lambda(Q_{k_1-1})^{-1/2} \int_Q \fint_{\hat{Q}} K_{\lambda,\ell}(z,\, w)^{-1} dm_\lambda(w) h^\epsilon_{Q_{k_1-1}}(z) dm_\lambda(z).$$

It follows from Lemma 2.4 and inequality (2.2) that

$$|C^\epsilon_{Q,k_Q-k_1,0}| \lesssim \frac{m_\lambda(Q)}{m_\lambda(Q_{k_1-1})} \lesssim \delta^{(n+1)(k_Q-k_1)} \le C_N(1+|k_Q-k_1|)^{-N}.$$

A change of variable yields that

$$\mathrm{I}_2 = \sum_{\epsilon=1}^{2^n-1} \sum_{k_1\ge 0} C^\epsilon_{Q,k_1,0} f^\epsilon_{Q,k_1,0}(x) g^\epsilon_{Q,k_1,0}(y).$$

For the term I_3, note that $E^{(1)}_{k_Q}$ commutes with $D^{(2)}_{k_2}$. Therefore, I_3 can be factorized in a similar way as term I_2 by changing the roles of x and y.

For the term I_4, we first note that for any $k \in \mathbb{Z}$, there is a unique $P \in \mathcal{D}^0_{k-1}$ such that $P \cap \hat{Q} \neq \emptyset$. We denote this cube by $\hat{Q}_{k-1}$. Then it follows from formulas (6.30) and (6.31) that

$$\begin{aligned}
\mathrm{I}_4 = &\sum_{k_1=-\infty}^{k_Q} \sum_{k_2=-\infty}^{k_Q} \sum_{P_1\in\mathcal{D}^0_{k_1-1}} \sum_{P_2\in\mathcal{D}^0_{k_2-1}} \sum_{\epsilon_1=1}^{2^n-1} \sum_{\epsilon_2=1}^{2^n-1} \\
&\quad \fint_Q \fint_{\hat{Q}} K_{\lambda,\ell}(z,\, w)^{-1} h^{\epsilon_1}_{P_1}(z) h^{\epsilon_2}_{P_2}(w) dm_\lambda(z) dm_\lambda(w) h^{\epsilon_1}_{P_1}(x)\chi_Q(x) h^{\epsilon_2}_{P_2}(y)\chi_{\hat{Q}}(y) \\
= &\sum_{k_1=-\infty}^{k_Q} \sum_{k_2=-\infty}^{k_Q} \sum_{\epsilon_1=1}^{2^n-1} \sum_{\epsilon_2=1}^{2^n-1} \\
&\quad \fint_Q \fint_{\hat{Q}} K_{\lambda,\ell}(z,\, w)^{-1} h^{\epsilon_1}_{Q_{k_1-1}}(z) h^{\epsilon_2}_{\hat{Q}_{k_2-1}}(w) dm_\lambda(z) dm_\lambda(w) h^{\epsilon_1}_{Q_{k_1-1}}(x)\chi_Q(x) h^{\epsilon_2}_{\hat{Q}_{k_2-1}}(y)\chi_{\hat{Q}}(y).
\end{aligned}$$

For any integer $k_1 \le k_Q$ and any integer $k_2 \le k_Q$, we let

$$f^{\epsilon_1,\epsilon_2}_{Q,k_Q-k_1,k_Q-k_2}(x) := \Big(\frac{m_\lambda(Q_{k_1-1})}{m_\lambda(Q)}\Big)^{1/2} h^{\epsilon_1}_{Q_{k_1-1}}(x)\chi_Q(x), \quad g^{\epsilon_1,\epsilon_2}_{Q,k_Q-k_1,k_Q-k_2}(y) := \Big(\frac{m_\lambda(\hat{Q}_{k_2-1})}{m_\lambda(Q)}\Big)^{1/2} h^{\epsilon_2}_{\hat{Q}_{k_2-1}}(y)\chi_{\hat{Q}}(y)$$

and

$$C^{\epsilon_1,\epsilon_2}_{Q,k_Q-k_1,k_Q-k_2} : = m_\lambda(Q_{k_1-1})^{-1/2} m_\lambda(\hat{Q}_{k_2-1})^{-1/2} \int_Q \fint_{\hat{Q}} K_{\lambda,\ell}(z,\, w)^{-1} h^{\epsilon_1}_{Q_{k_1-1}}(z) h^{\epsilon_2}_{\hat{Q}_{k_2-1}}(w) dm_\lambda(z) dm_\lambda(w).$$

It follows from Lemma 2.4 and inequality (2.2) that

$$|C^{\epsilon_1,\epsilon_2}_{Q,k_Q-k_1,k_Q-k_2}| \lesssim \left(\frac{m_\lambda(Q)}{m_\lambda(Q_{k_1-1})}\right)\left(\frac{m_\lambda(Q)}{m_\lambda(Q_{k_2-1})}\right)$$

$$\lesssim \delta^{(n+1)(k_Q-k_1)}\delta^{(n+1)(k_Q-k_2)}$$
$$\leq C_N(1+|k_Q-k_1|)^{-N}(1+|k_Q-k_2|)^{-N}.$$

A change of variable yields that

$$\mathrm{I}_4 = \sum_{\epsilon_1=1}^{2^n-1}\sum_{\epsilon_2=1}^{2^n-1}\sum_{k_1\geq 0}\sum_{k_2\geq 0} C^{\epsilon_1,\epsilon_2}_{Q,k_1,k_2} f^{\epsilon_1,\epsilon_2}_{Q,k_1,k_2}(x) g^{\epsilon_1,\epsilon_2}_{Q,k_1,k_2}(y).$$

Therefore, we have factorized each term I_u, $u = 1, 2, 3, 4$, into a required form, respectively. The proof of Lemma 8.2 is completed. □

Proposition 6.2. *Suppose* $1 < p < \infty$, $0 < q \leq \infty$, $\lambda > 0$, $n \geq 0$ *and* $b \in L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$. *Then there is a constant* $C > 0$ *such that for any* $\ell \in \{1, 2, ..., n+1\}$,

$$\left\|\{MO_Q(b)\}_{Q\in\mathcal{D}^0}\right\|_{\ell^{p,q}} \leq C\|[b, R_{\lambda,\ell}]\|_{S^{p,q}_\lambda}.$$

Proof. By Lemma 2.4, for any $Q \in \mathcal{D}^0_k$ with center c_Q, one can find a ball $\hat{Q} := B_{\mathbb{R}^{n+1}_+}(c_{\hat{Q}}, \frac{1}{12}\delta^k) \subset \mathbb{R}^{n+1}_+$ such that $|x_0 - y_0| = A\delta^k$, and for all $(x, y) \in Q\times\hat{Q}$, $K_{\lambda,\ell}(x, y)$ does not change sign and satisfies

$$|K_{\lambda,\ell}(x, y)| \geq \frac{C}{m_\lambda(Q)}.$$

To continue, we let $s_Q(x) := \mathrm{sgn}(b(x) - (b)_{\hat{Q}})\chi_Q(x)$ and deduce that

$$MO_Q(b) \lesssim ⨍_Q |b(x) - (b)_{\hat{Q}}| dm_\lambda(x)$$
$$= C ⨍_Q (b(x) - (b)_{\hat{Q}}) s_Q(x) dm_\lambda(x). \tag{6.33}$$

Furthermore, we let

$$L_Q(f)(x) := \int_Q s_Q(x) J_Q(x, y) f(y) dm_\lambda(y),$$

where J_Q is given in Lemma 8.2. Then a direct calculation yields that

$$[b, R_{\lambda,\ell}]L_Q(f)(x) = ⨍_Q ⨍_{\hat{Q}} (b(x) - b(z)) K_{\lambda,\ell}(x, z) s_Q(z) K_{\lambda,\ell}(z, y)^{-1} f(y) dm_\lambda(y) dm_\lambda(z).$$

This implies that

$$\mathrm{Trace}([b, R_{\lambda,\ell}]L_Q) = ⨍_Q ⨍_{\hat{Q}} (b(x) - b(y)) s_Q(x) dm_\lambda(y) dm_\lambda(x). \tag{6.34}$$

Combining inequality (6.33) with equality (6.34), we see that

$$MO_Q(b) \lesssim |\mathrm{Trace}([b, R_{\lambda,\ell}]L_Q)|.$$

Using the duality between Lorentz spaces $\ell^{p',q'}$ and $\ell^{p,q}$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, we deduce that

$$\left\|\{MO_Q(b)\}_{Q\in\mathcal{D}^0}\right\|_{\ell^{p,q}} \lesssim \left\|\{\mathrm{Trace}([b, R_{\lambda,\ell}]L_Q)\}_{Q\in\mathcal{D}^0}\right\|_{\ell^{p,q}}$$
$$= C \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \left| \sum_{Q\in\mathcal{D}^0} \mathrm{Trace}([b, R_{\lambda,\ell}]L_Q) a_Q \right|$$

$$
\begin{aligned}
&= C \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \Big\| [b, R_{\lambda,\ell}]\Big(\sum_{Q\in\mathcal{D}^0} L_Q a_Q \Big) \Big\|_{S^1_\lambda} \\
&\lesssim \|[b, R_{\lambda,\ell}]\|_{S^{p,q}_\lambda} \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \Big\| \sum_{Q\in\mathcal{D}^0} L_Q a_Q \Big\|_{S^{p',q'}_\lambda}. \qquad (6.35)
\end{aligned}
$$

To continue, we apply Lemma 8.2 to see that $\sum_{Q\in\mathcal{D}^0} L_Q a_Q$ can be written as

$$
\sum_{Q\in\mathcal{D}^0} L_Q a_Q = \sum_{\nu\in\mathcal{I}} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}^0} a_Q C_{Q,k,\nu} h_{Q,k,\nu}(x) \langle f, g_{Q,k,\nu}\rangle,
$$

where $h_{Q,k,\nu}(x) := s_Q(x) f_{Q,k,\nu}(x)$ and $g_{Q,k,\nu}(x)$ are two *nearly weakly orthogonal (NWO)* sequences of functions. Therefore, it follows from [15, Corollary 2.7] that

$$
\begin{aligned}
\sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \Big\| \sum_{Q\in\mathcal{D}^0} L_Q a_Q \Big\|_{S^{p',q'}_\lambda} &\leq \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \sum_{\nu\in\mathcal{I}} \sum_{k\in\mathbb{Z}} \Big\| \sum_{Q\in\mathcal{D}^0} a_Q C_{Q,k,\nu} h_{Q,k,\nu}(x) \langle f, g_{Q,k,\nu}\rangle \Big\|_{S^{p',q'}_\lambda} \\
&\lesssim \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \sum_{\nu\in\mathcal{I}} \sum_{k\in\mathbb{Z}} \|\{a_Q C_{Q,k,\nu}\}_{Q\in\mathcal{D}^0}\|_{\ell^{p',q'}} \\
&\lesssim \sup_{\|\{a_Q\}\|_{\ell^{p',q'}}=1} \sum_{\nu\in\mathcal{I}} \sum_{k\in\mathbb{Z}} (1+|k|)^{-N} \|\{a_Q\}_{Q\in\mathcal{D}^0}\|_{\ell^{p',q'}} \\
&\lesssim 1.
\end{aligned}
$$

Substituting the above inequality into (6.35), we complete the proof of Proposition 6.2. □

7. Schatten-Lorentz estimate

Let Γ and J_s denote the Gamma function and the Bessel function of the first kind of order s with $s \in (-1/2, \infty)$, respectively. For any $f, g \in L^1((0,\infty), dm_\lambda)$, we define their Hankel convolution by

$$
f \sharp_\lambda g(x) := \int_0^\infty f(y) \tau_x^{[\lambda]} g(y) dm_\lambda(y), \text{ for all } x \in (0,\infty),
$$

where $\tau_x^{[\lambda]}$ denotes the Hankel translation of g, which can be expressed as

$$
\tau_x^{[\lambda]} g(y) := \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\sqrt{\pi}} \int_0^\pi g(\sqrt{x^2+y^2-2xy\cos\theta})(\sin\theta)^{2\lambda-1} d\theta.
$$

For any function ϕ, we denote $\phi_t(y) := t^{-2\lambda-1}\phi(y/t)$. Then recall from [1] that

$$
e^{-t\Delta_\lambda} f = f \sharp_\lambda W^{[\lambda]}_{\sqrt{2t}} \qquad (7.36)
$$

for all $t \in \mathbb{R}_+$, where we denote

$$
W^{[\lambda]}(x) = 2^{(1-2\lambda)/2} \exp(-x^2/2)/\Gamma(\lambda+1/2).
$$

1-dim?$m_\lambda(B_{\mathbb{R}^{n+1}_+}(x, t))$?

Lemma 7.1. *There exist constants $C, c > 0$ such that*

$$
(1)\ |K_{e^{-t^2\Delta_\lambda}}(x,y)| \leq \frac{C}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,t))} \exp\left(-c\frac{|x-y|^2}{t^2}\right)
$$

$$
(2)\ |K_{\partial_t e^{-t^2\Delta_\lambda}}(x,y)| \leq \frac{C}{t\, m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,t))} \exp\left(-c\frac{|x-y|^2}{t^2}\right)
$$

(3) $|K_{\partial_x e^{-t^2\Delta_\lambda}}(x,y)| \le \frac{C}{t m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,t))}\exp\left(-c\frac{|x-y|^2}{t^2}\right)$

(4) $\int_0^\infty K_{e^{-t^2\Delta_\lambda}}(x,y)dm_\lambda(y) = 1.$ *Multiplied by a normalized constant?*

for all $x, y \in \mathbb{R}^{n+1}_+$ and $t \in \mathbb{R}_+$.

Proof. By (7.36), we see that

$$
\begin{aligned}
K_{e^{-t^2\Delta_\lambda}}(x,y) &= \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\sqrt{\pi}}\int_0^\pi W^{[\lambda]}_{\sqrt{2}t}(\sqrt{x^2+y^2-2xy\cos\theta})(\sin\theta)^{2\lambda-1}d\theta \\
&= 2^{(1-2\lambda)/2}\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\sqrt{\pi}}\int_0^\pi(\sqrt{2}t)^{-2\lambda-1}\exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right)(\sin\theta)^{2\lambda-1}d\theta \qquad (7.37)\\
&\sim t^{-2\lambda-1}\int_0^\pi \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right)(\sin\theta)^{2\lambda-1}d\theta. \qquad (7.38)
\end{aligned}
$$

To continue, we divide the proof into two cases.

Case (i). If $t \ge x$ or $|x-y| \ge x/2$, then it follows from (7.38) and the doubling condition (2.2) that

$$
\begin{aligned}
K_{e^{-t^2\Delta_\lambda}}(x,y) &\lesssim t^{-2\lambda-1}\exp\left(-\frac{|x-y|^2}{4t^2}\right)\\
&\lesssim \frac{1}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,t))}\exp\left(-\frac{|x-y|^2}{8t^2}\right).
\end{aligned}
$$

Case (ii). If $t < x$ and $|x-y| < x/2$, then observe that $x \sim y$, $\sin\theta \sim \theta$ for all $\theta \in (0,\pi/2)$ and $1-\cos\theta \ge 2(\theta/\pi)^2$. Now we note that the right-hand side of (7.38) can be written as

$$
\begin{aligned}
&Ct^{-2\lambda-1}\int_0^{\pi/2}\exp\left(-\frac{|x-y|^2+2xy(1-\cos\theta)}{4t^2}\right)(\sin\theta)^{2\lambda-1}d\theta\\
&\quad + Ct^{-2\lambda-1}\int_{\pi/2}^{\pi}\exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right)(\sin\theta)^{2\lambda-1}d\theta =: \mathrm{A}+\mathrm{B}.
\end{aligned}
$$

For the term A, by a change of variable, we have

$$
\begin{aligned}
\mathrm{A} &\lesssim t^{-2\lambda-1}\int_0^{\pi/2}\exp\left(-\frac{|x-y|^2+4xy\theta^2/\pi^2}{4t^2}\right)\theta^{2\lambda-1}d\theta\\
&\lesssim t^{-2\lambda-1}\exp\left(-\frac{|x-y|^2}{4t^2}\right)\int_0^\infty\exp\left(-\frac{4xy\theta^2/\pi^2}{4t^2}\right)\theta^{2\lambda-1}d\theta\\
&\lesssim (tx^{2\lambda}y^{2\lambda})^{-1}\exp\left(-\frac{|x-y|^2}{4t^2}\right)\\
&\lesssim \frac{1}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,t))}\exp\left(-\frac{|x-y|^2}{4t^2}\right).
\end{aligned}
$$

For the term B, we note that $\cos\theta < 0$ for all $\theta \in (\pi/2,\pi)$. Thus,

$$
\mathrm{B} \lesssim t^{-2\lambda-1}\exp\left(-\frac{x^2}{4t^2}\right)\exp\left(-\frac{|x-y|^2}{4t^2}\right)\int_{\pi/2}^{\pi}(\sin\theta)^{2\lambda-1}d\theta
$$

$$\lesssim (tx^{2\lambda})^{-1} \exp\left(-\frac{|x-y|^2}{4t^2}\right)$$
$$\lesssim \frac{1}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\, t))} \exp\left(-\frac{|x-y|^2}{4t^2}\right).$$

This ends the proof of (1).

Now we show (2). It follows from (7.37) that

$$K_{\partial_t e^{-t^2\Delta_\lambda}}(x,\, y) \sim t^{-2\lambda-2} \int_0^\pi \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right) (\sin\theta)^{2\lambda-1} d\theta$$
$$+ \quad t^{-2\lambda-4}(x^2+y^2-2xy\cos\theta) \int_0^\pi \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right) (\sin\theta)^{2\lambda-1} d\theta$$
$$\lesssim t^{-2\lambda-2} \int_0^\pi \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{8t^2}\right) (\sin\theta)^{2\lambda-1} d\theta. \tag{7.39}$$

Note that the integral term above is exactly the integral term on the right-hand side of (7.38), up a harmless constant in the exponential term. Therefore, we obtain (2).

Finally, we show (3). It follows from (7.37) that

$$K_{\partial_x e^{-t^2\Delta_\lambda}}(x,\, y) = -C_\lambda \frac{x-y\cos\theta}{t^2} \int_0^\pi t^{-2\lambda-1} \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right) (\sin\theta)^{2\lambda-1} d\theta$$
$$\lesssim \frac{|x-y|}{t^2} \int_0^\pi t^{-2\lambda-1} \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{4t^2}\right) (\sin\theta)^{2\lambda-1} d\theta$$
$$\lesssim t^{-2\lambda-2} \int_0^\pi \exp\left(-\frac{x^2+y^2-2xy\cos\theta}{8t^2}\right) (\sin\theta)^{2\lambda-1} d\theta.$$

Note that up an absolute constant, the term above is equal to the term on the right-hand side of (7.39). Therefore, we obtain (3). This finishes the proof of Lemma 7.1. □

Lemma 7.2. *Suppose that $1 < p < \infty$. Let $A = (a_{ij})$ be an infinite matrix with non-negative entries. Suppose that there is a constant $K > 0$ and a non-negative sequence $\{b_i\}$ such that*

$$\sum_j a_{ij} b_j^{p'} \le K b_i^{p'}, \text{ for any } j = 1, 2, \ldots,$$
$$\sum_i a_{ij} b_i^{p} \le K b_j^{p}, \text{ for any } j = 1, 2, \ldots.$$

Then the map $T:\ (f_i)_i \to (\sum_j a_{ij} f_j)_i$ is bounded on ℓ^p with norm bounded by K.

For any $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, dm_\lambda)$, we define the heat maximal function associated with Bessel operator over Carleson box centred at $(x,\, t) \in \mathbb{R}^{n+1}_+$ by

$$S_\lambda(b)(x,\, t) := \sup\left\{ s\left|\nabla e^{-s^2\Delta_\lambda} b(y)\right| :\ y \in B_{\mathbb{R}^{n+1}_+}(x,\, t),\ t/2 \le s \le t \right\},$$

where ∇ is the full gradient given by $\nabla = (\partial_{x_1}, \ldots, \partial_{x_{n+1}}, \partial_t)$. Moreover, for any $1 \le r < \infty$ and any $b \in L^r_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, dm_\lambda)$, we define its r-mean oscillation over a cube Q by

$$MO^r_Q(b) := \left(⨍_Q |b(x) - (b)_Q|^r dm_\lambda(x) \right)^{1/r}.$$

Lemma 7.3. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $b \in L^1_{\rm loc}(\mathbb{R}^{n+1}_+, dm_\lambda)$. *Then for any* $\nu \in \{1, 2, \ldots, \kappa\}$, *one has*

$$\left\| \{\sup\{S_\lambda(b)(x,t) : x \in Q,\ \delta\ell(Q) \le t \le \ell(Q)\}\}_{Q\in\mathcal{D}^\nu} \right\|_{\ell^{p,q}} \lesssim \sum_{\iota=1}^{\kappa} \left\| \left\{ MO^1_Q(b) \right\}_{Q\in\mathcal{D}^\iota} \right\|_{\ell^{p,q}},$$

where $\ell(Q)$ *is denoted by the sidelength of* Q.

Proof. To begin with, recall from (2.3) that for any $Q \in \mathcal{D}^\nu_j$ centred at $c_Q \in \mathbb{R}^{n+1}_+$, there is a ball $B_Q := B_{\mathbb{R}^{n+1}_+}(c_Q, 4\delta^j)$ such that $Q \subset B_Q$. Then by Lemma 7.1, for any sufficient constant $N > 0$ to be chosen later,

$$\begin{aligned}
& s\left|\nabla e^{-s^2\Delta_\lambda} b(y)\right| \\
&= s\left|\nabla \int_{\mathbb{R}^{n+1}_+} (b(y) - (b)_{B_Q}) K_{e^{-s^2\Delta_\lambda}}(x,\ y) dm_\lambda(y)\right| \\
&\le \int_{\mathbb{R}^{n+1}_+} |b(y) - (b)_{B_Q}||K_{s\nabla e^{-s^2\Delta_\lambda}}(x,\ y)| dm_\lambda(y) \\
&\lesssim \sum_{k\ge0} \frac{1}{m_\lambda(B_{\mathbb{R}^{n+1}_+}(x,\ s))} \int_{2^{k+1}B_Q\setminus 2^k B_Q} |b(y) - (b)_{B_Q}| \left(\frac{s}{s + |x-y|}\right)^N dm_\lambda(y) \\
&\lesssim \sum_{k\ge0} 2^{-kN} \fint_{2^{k+1}B_Q} |b(y) - (b)_{B_Q}| dm_\lambda(y)
\end{aligned}$$

$$\lesssim \sum_{k\ge0} 2^{-kN} \fint_{2^{k+1}B_Q} |b(y) - (b)_{2^{k+1}B_Q}| dm_\lambda(y) \tag{7.40}$$

for some implicit constant $C > 0$ independent of $x \in Q \in \mathcal{D}^\nu$, $\delta\ell(Q) \le t \le \ell(Q)$, $y \in B_{\mathbb{R}^{n+1}_+}(x,\ t)$ and $\delta t \le s \le t$, where the last two inequalities used the doubling inequality (2.2). To continue, we apply Lemma 2.1 to see that for any $k \in \mathbb{Z}$ and $Q \in \mathcal{D}^\nu_j$, there is a $\iota \in \{1, 2, \ldots, \kappa\}$ and a cube $P^k_Q \in \mathcal{D}^\iota_{j-k-1}$ such that

$$2^{k+1}B_Q \subseteq P^k_Q \subseteq C_{\rm adj} 2^{k+1} B_Q. \tag{7.41}$$

Therefore,

$$\text{RHS of (7.40)} \lesssim \sum_{k\ge0} 2^{-kN} \fint_{P^k_Q} |f(y) - (f)_{P^k_Q}| dm_\lambda(y). \tag{7.42}$$

Note that for any $k \in \mathbb{Z}$ and $Q \in \mathcal{D}^\nu_j$, one has $Q \subset P^k_Q$. Therefore, the number of $Q' \in \mathcal{D}^\nu_j$ such that $P^k_Q = P^k_{Q'}$ is at most $c2^{k(n+1)}$. This, in combination with inequality (7.42), yields

$$\begin{aligned}
& \left\| \{\sup\{S_\lambda(b)(x,t) : x \in Q,\ \delta\ell(Q) \le t \le \ell(Q)\}\}_{Q\in\mathcal{D}^\nu} \right\|_{\ell^{p,q}} \\
&\lesssim \sum_{k\ge0} 2^{-kN} \left\| \left\{ \fint_{P^k_Q} |f(y) - (f)_{P^k_Q}| dm_\lambda(y) \right\}_{Q\in\mathcal{D}^\nu} \right\|_{\ell^{p,q}} \\
&\lesssim \sum_{\iota=1}^{\kappa} \sum_{k\ge0} 2^{-k(N-n-1)} \left\| \left\{ \fint_{Q} |f(y) - (f)_{Q}| dm_\lambda(y) \right\}_{Q\in\mathcal{D}^\iota} \right\|_{\ell^{p,q}}
\end{aligned}$$

$$\lesssim \sum_{\iota=1}^{\kappa} \|\{MO_Q^1(b)\}_{Q\in\mathcal{D}^\iota}\|_{\ell^{p,q}},$$

where the last inequality holds since N can be chosen to be larger than $n+1$. This ends the proof of Lemma 7.3. □

Lemma 7.4. *Let $1 \le r < \infty$, $0 < p < \infty$ and $0 < q \le \infty$. Then the following statements are equivalent:*

(1) *For any $r \in [1,\infty)$ and $\nu \in \{1,2,\dots,\kappa\}$, one has $\{MO_Q^r(b)\}_{Q\in\mathcal{D}^\nu} \in \ell^{p,q}$.*
(2) *For any $\nu \in \{1,2,\dots,\kappa\}$, one has $\{MO_Q^1(b)\}_{Q\in\mathcal{D}^\nu} \in \ell^{p,q}$.*

Proof. It is direct that (1) implies (2). Now we show that (2) implies (1). To begin with, we have

$$\begin{aligned}\|\{MO_Q^r(b)\}_{Q\in\mathcal{D}^\nu}\|_{\ell^{p,q}} &\lesssim \left\|\left\{\left(\fint_Q |b(x)-e^{-\ell(Q)^2\Delta_\lambda}b(c_Q)|^r dm_\lambda(x)\right)^{1/r}\right\}_{Q\in\mathcal{D}^\nu}\right\|_{\ell^{p,q}}\\ &\lesssim \left\|\left\{\sup_{x\in Q}\left|e^{-\ell(Q)^2\Delta_\lambda}b(x)-e^{-\ell(Q)^2\Delta_\lambda}b(c_Q)\right|\right\}_{Q\in\mathcal{D}^\nu}\right\|_{\ell^{p,q}}\\ &\quad+\left\|\left\{\left(\fint_Q\left(\int_0^{\ell(Q)}\left|\frac{\partial}{\partial t}e^{-t^2\Delta_\lambda}b(x)\right|dt\right)^r dm_\lambda(x)\right)^{1/r}\right\}_{Q\in\mathcal{D}^\nu}\right\|_{\ell^{p,q}}\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

For the term I, we apply the mean value theorem and Lemma 7.3 to conclude that

$$\mathrm{I} \lesssim \left\|\left\{\sup_{x\in Q}\ell(Q)|\nabla e^{-\ell(Q)^2\Delta_\lambda}b(x)|\right\}_{Q\in\mathcal{D}^\nu}\right\|_{\ell^{p,q}} \lesssim \sum_{\iota=1}^{\kappa}\|\{MO_Q^1(b)\}_{Q\in\mathcal{D}^\iota}\|_{\ell^{p,q}}.$$

For the term II, we first apply Jensen's inequality to deduce that

$$\begin{aligned}&\left(\fint_Q\left(\int_0^{\ell(Q)}\left|\frac{\partial}{\partial t}e^{-t^2\Delta_\lambda}b(x)\right|dt\right)^r dm_\lambda(x)\right)^{1/r}\\ &\le\left(\fint_Q\left(\sum_{j=-\infty}^{\log_{\delta^{-1}}\ell(Q)}\sup_{\delta^{-j+1}\le t\le\delta^{-j}} t|\nabla e^{-t^2\Delta_\lambda}b(x)(x,\,t)|\right)^r dm_\lambda(x)\right)^{1/r}\\ &\le\left(\fint_Q\left(\sum_{j=-\infty}^{\log_{\delta^{-1}}\ell(Q)} S_\lambda(b)(x,\,\delta^{-j})\right)^r dm_\lambda(x)\right)^{1/r}\\ &=\left(\fint_Q\left(\sum_{j=-\infty}^{\log_{\delta^{-1}}\ell(Q)}(\log_{\delta^{-1}}\ell(Q)-j+1)^{-2}\times(\log_{\delta^{-1}}\ell(Q)-j+1)^2 S_\lambda(b)(x,\,\delta^{-j})\right)^r dm_\lambda(x)\right)^{1/r}\\ &\lesssim\left(\fint_Q\sum_{j=-\infty}^{\log_{\delta^{-1}}\ell(Q)}(\log_{\delta^{-1}}\ell(Q)-j+1)^{-2}\times\left((\log_{\delta^{-1}}\ell(Q)-j+1)^2 S_\lambda(b)(x,\,\delta^{-j})\right)^r dm_\lambda(x)\right)^{1/r}\end{aligned}$$

$$= C\left(\sum_{j=-\infty}^{\log_{\delta^{-1}}\ell(Q)}(\log_{\delta^{-1}}\ell(Q)-j+1)^{2r-2}\fint_Q S_\lambda(b)(x,\,\delta^{-j})^r dm_\lambda(x)\right)^{1/r}. \tag{7.43}$$

To continue, for each $\nu \in \{1, 2, \dots, \kappa\}$, $j \in (-\infty, \log_{\delta^{-1}} \ell(Q)] \cap \mathbb{Z}$ and $Q \in \mathcal{D}^\nu$, we let

$$\mathcal{D}^\nu_Q := \{R \in \mathcal{D}^\nu : \ R \subseteq Q\},$$
$$\mathcal{D}^\nu_{Q,j} := \{R \in \mathcal{D}^\nu_j : \ R \subseteq Q\}.$$

Then we have

$$\text{RHS of (7.43)} \lesssim \left(\sum_{j=-\infty}^{\log_{\delta^{-1}} \ell(Q)} \sum_{R \in \mathcal{D}^\nu_{Q,-j}} (\log_{\delta^{-1}} \ell(Q) - j + 1)^{2r-2} m_\lambda(Q)^{-1} \int_R S_\lambda(b)(x, \delta^{-j})^r dm_\lambda(x) \right)^{1/r}$$

$$\lesssim \left(\sum_{R \in \mathcal{D}^\nu_Q} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{2r-2} \sup_{x \in R} S_\lambda(b)(x, \ell(R))^r \right)^{1/r}. \tag{7.44}$$

To continue, for any sequence $\{a_Q\}_{Q \in \mathcal{D}^\nu}$, we let

$$M_r(a)(Q) = \sum_{R \in \mathcal{D}^\nu_Q} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{2r-2} |a_R|.$$

We **claim** that M_r is a bounded operator on $\ell^{p,q}$ for any $0 < p < \infty$ and $0 < q \le \infty$. Before providing its proof, we first illustrate how it implies our desired inequality. we apply the $\ell^{p,q}$ boundedness of M_r and inequality (7.44), together with Lemma 7.3, to conclude that

$$\begin{aligned}
\mathrm{II} &\le C \left\| \left\{ \left(\sum_{R \in \mathcal{D}^\nu_Q} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{2r-2} \sup_{x \in R} S_\lambda(b)(x, \ell(R))^r \right)^{1/r} \right\}_{Q \in \mathcal{D}^\nu} \right\|_{\ell^{p,q}} \\
&= C \left\| \left\{ \sum_{R \in \mathcal{D}^\nu_Q} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{2r-2} \sup_{x \in R} S_\lambda(b)(x, \ell(R))^r \right\}_{Q \in \mathcal{D}^\nu} \right\|_{\ell^{p/r, q/r}} \\
&\le C \left\| \{ S_\lambda(b)(x, \ell(Q))^r \}_{Q \in \mathcal{D}^\nu} \right\|_{\ell^{p/r, q/r}} \\
&= C \left\| \{ S_\lambda(b)(x, \ell(Q)) \}_{Q \in \mathcal{D}^\nu} \right\|_{\ell^{p,q}} \\
&\le C \sum_{\iota=1}^{\kappa} \| \{ MO^1_Q(b) \}_{Q \in \mathcal{D}^\iota} \|_{\ell^{p,q}}.
\end{aligned}$$

Now we go back to the proof of the claim. By interpolation, it suffices to show that it is a bounded operator on ℓ^p for any $0 < p < \infty$. To show this, we let

$$\mathcal{E}^\nu_R := \{Q \in \mathcal{D}^\nu : \ Q \supseteq R\},$$

and then divide the proof into two cases:

Case (i). If $0 < p \le 1$, then we apply inequality (2.2) to get that

$$\begin{aligned}
\| \{ M_r(a)(Q) \}_{Q \in \mathcal{D}^\nu} \|_{\ell^p} &\le \left(\sum_{Q \in \mathcal{D}^\nu} \sum_{R \in \mathcal{D}^\nu_Q} \left(\frac{m_\lambda(R)}{m_\lambda(Q)} \right)^p \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{(2r-2)p} |a_R|^p \right)^{1/p} \\
&= \left(\sum_{R \in \mathcal{D}^\nu} \sum_{Q \in \mathcal{E}^\nu_R} \left(\frac{m_\lambda(R)}{m_\lambda(Q)} \right)^p \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1 \right)^{(2r-2)p} |a_R|^p \right)^{1/p}
\end{aligned}$$

$$\lesssim \left(\sum_{R\in\mathcal{D}^\nu} \sum_{Q\in\mathcal{E}_R^\nu} \left(\frac{\ell(R)}{\ell(Q)}\right)^{(n+1)p} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1\right)^{(2r-2)p} |a_R|^p \right)^{1/p}$$

$$= C\left(\sum_{R\in\mathcal{D}^\nu} \sum_{j:\delta^j\geq\ell(R)} \left(\frac{\ell(R)}{\delta^j}\right)^{(n+1)p} \left(\log_{\delta^{-1}} \frac{\delta^j}{\ell(R)} + 1\right)^{(2r-2)p} |a_R|^p \right)^{1/p},$$

where in the last equality we used the fact that for any $R \in \mathcal{D}^\nu$ and $j \in \mathbb{Z}$ satisfying $\delta^j \geq \ell(R)$, there exists a unique $Q \in \mathcal{D}_j^\nu$ such that $Q \supseteq R$. This deduces that the right-hand side above is dominated by $C\|\{a_Q\}_{Q\in\mathcal{D}^\nu}\|_{\ell^p}$.

Case (ii). If $1 < p < \infty$, then we shall verify that

$$\sum_{R\in\mathcal{D}_Q^\nu} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1\right)^{2r-2} b_R^{p'} \lesssim b_Q^{p'}, \tag{7.45}$$

$$\sum_{Q\in\mathcal{E}_R^\nu} \frac{m_\lambda(R)}{m_\lambda(Q)} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1\right)^{2r-2} b_Q^{p'} \lesssim b_R^{p'}, \tag{7.46}$$

where we choose $b_Q := m_\lambda(Q)^\epsilon$ for some $\epsilon > 0$ small enough. Indeed, to obtain (7.45), by changing the order of the sum and then applying inequality (2.2), we see that

$$\begin{aligned}
\text{LHS of (7.45)} &= m_\lambda(Q)^{\epsilon p'} \sum_{R\in\mathcal{D}_Q^\nu} \left(\frac{m_\lambda(R)}{m_\lambda(Q)}\right)^{1+\epsilon p'} \left(\log_{\delta^{-1}} \frac{\ell(Q)}{\ell(R)} + 1\right)^{2r-2} \\
&\leq Cm_\lambda(Q)^{\epsilon p'} \sum_{R\in\mathcal{D}_Q^\nu} \left(\frac{m_\lambda(R)}{m_\lambda(Q)}\right)^{1+\epsilon p'/2} \\
&= Cm_\lambda(Q)^{\epsilon p'} \sum_{j:\delta^j\leq\ell(Q)} \sum_{R\in\mathcal{D}_{Q,j}^\nu} \left(\frac{m_\lambda(R)}{m_\lambda(Q)}\right)^{1+\epsilon p'/2},
\end{aligned} \tag{7.47}$$

where the constant C depends only on r and p. Since we are in the Bessel setting, in which the upper dimension is not equal to the lower one, one need to deal with the volumn term more delicately. To this end, we apply inequality (2.2) to see that

$$\begin{aligned}
\sum_{j:\delta^j\leq\ell(Q)} \sum_{R\in\mathcal{D}_{Q,j}^\nu} \left(\frac{m_\lambda(R)}{m_\lambda(Q)}\right)^{1+\epsilon p'/2} &\leq C \sum_{j:\delta^j\leq\ell(Q)} \left(\frac{\delta^j}{\ell(Q)}\right)^{(n+1)\epsilon p'/2} \sum_{R\in\mathcal{D}_{Q,j}^\nu} \left(\frac{m_\lambda(R)}{m_\lambda(Q)}\right) \\
&= C \sum_{j:\delta^j\leq\ell(Q)} \left(\frac{\delta^j}{\ell(Q)}\right)^{(n+1)\epsilon p'/2} \left(\frac{m_\lambda(\bigcup_{R\in\mathcal{D}_{Q,j}^\nu} R)}{m_\lambda(Q)}\right) \\
&\leq C.
\end{aligned}$$

Substituting the above inequality into (7.47), we deduce that the right-hand side of (7.47) is dominated by $m_\lambda(Q)^{\epsilon p'}$. This verifies (7.45). Next, we verify (7.46). By inequality (2.2),

$$\begin{aligned}
\text{LHS of (7.46)} &\lesssim m_\lambda(R)^{\epsilon p'} \sum_{j:\delta^j\geq\ell(R)} \left(\frac{\ell(R)}{\delta^j}\right)^{(n+1)(1-\epsilon p')} \left(\log_{\delta^{-1}} \frac{\delta^j}{\ell(R)} + 1\right)^{2r-2} \\
&\lesssim m_\lambda(R)^{\epsilon p'},
\end{aligned}$$

where in the first inequality we used the fact again that for any $R \in \mathcal{D}^\nu$ and $j \in \mathbb{Z}$ satisfying $\delta^j \geq \ell(R)$, there exists a unique $Q \in \mathcal{D}^\nu_j$ such that $Q \supseteq R$. Combining (7.45), (7.46) with Lemma 7.2, we see that M_r is a bounded operator on ℓ^p. This completes the proof of Lemma 7.4. □

8. ???

Let $\Xi_+ = \{(x, y) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ : x = y\}$. Notation of $\ell(Q)$

Lemma 8.1. *There exists a family of closed standard dyadic cubes $\mathcal{P} = \{P_j\}_j$ on $(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$ such that*

(1) $\cup_j P_j = (\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$;
(2) $\ell(P_j) \approx d(P_j, (\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+)$;
(3) *If the boundaries of two cubes P_j and P_k touch, then $\ell(P_j) \approx \ell(P_k)$;*
(4) *For a given P_j, there are at most c_n cubes P_k that touch it;*
(5) *Let $0 < \epsilon < 1/4$ and P^*_j be the cube with the same center as P_j and with the sidelength $(1+\epsilon)\ell(P_j)$. Then each point of $(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$ is contained in at most c_n of the cubes P^*_j.*

Proof. Since the boundary of $(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$ is $\Xi_+ \cup \{(x, 0) : x \in \mathbb{R}^{n+1}_+\} \cup \{(0, y) : y \in \mathbb{R}^{n+1}_+\}$, one may not apply the standard dyadic Whitney decomposition on $(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$ directly to get the desired result. Instead, we apply the standard dyadic Whitney decomposition on the region $(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\backslash\Xi$, where $\Xi = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : x = y\}$, to see that there exists a family of closed standard dyadic cubes $\mathcal{P}' = \{P'_j\}_j$ on $(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\backslash\Xi$ such that (1)–(5) holds with P_j and $(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+$ replaced by P'_j and $(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\backslash\Xi$, respectively. Since each P'_j is contained in or disjoint with $\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\Xi_+$, this allows us to choose $\mathcal{P} = \{P_j\}_j$ be a sub-family of $\mathcal{P}'$ such that

$$\cup_j P_j = (\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+.$$

For these $\{P_j\}_j$, we see that (2) holds since

$$d(P_j, (\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)\backslash\Xi_+) \approx d(P_j, (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})\backslash\Xi).$$

Moreover, (3)–(5) hold directly due to the construction of $\{P'_j\}_j$. This ends the proof of Lemma 8.1. □

Let $\mathcal{P} = \{P_i\}$ be a dyadic Whitney decomposition of $\Omega := \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \backslash \{(x, y) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ : x = y\}$. This means that $\Omega = \cup P_i$, each P_i is a dyadic cube in $\mathbb{R}^{n+1}_+$, $P_i^\circ \cap P_j^\circ = \emptyset$ if $i \neq j$, and the sidelength of P_i is comparable to the distance from P_i to the diagonal $\{(x, y) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ : x = y\}$. Moreover, for any $P \in \mathcal{P}$, one can write $P = Q^1 \times Q^2$, where $Q^1, Q^2 \in \mathcal{D}^0$, the collection of standard dyadic cubes in $\mathbb{R}^{n+1}_+$.

For any $Q \in \mathcal{D}^0$, let $\hat{Q}$ be the cube chosen in Lemma 2.4 We need to make sure that $\hat{Q} \in \mathcal{D}^0_k$. Define

$$U_Q(x, y) := K_{\lambda,\ell}(x, y)\chi_Q(x)\chi_{\hat{Q}}(y).$$

Here we omit the dependence on λ and ℓ since these two parameters play no role in the proof below.

Lemma 8.2. *There is a finite index set $\mathcal{I}$ such that for any $\ell \in \{1, 2, ..., n+1\}$, $Q \in \mathcal{D}^0$ and $k \in \mathbb{Z}^2$, one can construct functions $F_{Q,k,\nu}$, $G_{Q,k,\nu}$ and a sequence $\{B_{Q,k,\nu}\}$ satisfying for any $\nu \in \mathcal{I}$,*

(1) $\mathrm{supp}(F_{Q,k,v}) \subset Q$, $\mathrm{supp}(G_{Q,k,v}) \subset \hat{Q}$,
(2) $\|F_{Q,k,v}\|_\infty \leq m_\lambda(Q)^{-1/2}$, $\|G_{Q,k,v}\|_\infty \leq m_\lambda(Q)^{-1/2}$,
(3) $|B_{Q,k,v}| \leq C_N(1+|k|)^{-N}$ *for any* $N \in \mathbb{Z}_+$ *and for some* $C_N > 0$ *independent of* Q *and* k,
(4) U_Q *admits the following factorization*

$$U_Q(x, y) = \sum_{v\in\mathcal{I}} \sum_{k\in\mathbb{Z}^2} C_{Q,k,v} F_{Q,k,v}(x) G_{Q,k,v}(y).$$

Proof. By Lemma ???, the kernel of $[b, R_{\lambda,\ell}]$ can be factorized as

$$\sum_{v\in\mathcal{I}} \sum_{k\in\mathbb{Z}^2} C_{Q,k,v} F_{Q,k,v}(x) G_{Q,k,v}(y)$$

□

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