1.a. Deseamos calcular,

$$\begin{aligned} M_X(t) = \mathsf{E}(e^{tX}) &= \int_0^\infty e^{tx}\, f(x;\theta)\,\mathrm{d}x = \frac{1}{2\theta^3}\int_0^\infty x^2 e^{-x(1/\theta - t)}\,\mathrm{d}x \\ &= \frac{1}{2\theta^3}\int_0^\infty x^{3-1} e^{-x(1/\theta-t)}\,\mathrm{d}x = \frac{1}{2\theta^3}\frac{\Gamma(3)}{(1/\theta - t)^3} \\ &= \Big(\frac{1}{1-\theta t}\Big)^3, \qquad t < 1/\theta. \end{aligned}$$

1.b. Como $M_X(t)$ es la MFG de la distribución $\mathsf{Gama}(3, 1/\theta)$. De ahí que,

$$\mathsf{E}(X) = 3\theta, \qquad \mathsf{var}(X) = 3\theta^2.$$

2. Por el teorema de probabilidad total sigue que

$$F_X(x) = P(X \le x) = pF_1(x) + qF_2(x),$$

derivando obtenemos que la función de densidad de X es dada por

$$f_X(x) = pf_1(x) + qf_2(x).$$

Ahora

$$\begin{aligned} \mathsf{E}(X) &= \int_{-\infty}^\infty x f_X(x)\,\mathrm{d}x = p\int_{-\infty}^\infty x f_1(x)\,\mathrm{d}x + q\int_{-\infty}^\infty x f_2(x)\,\mathrm{d}x \\ &= p\,\mathsf{E}_1(X) + q\,\mathsf{E}_2(X) = p\,\mu_1 + q\,\mu_2, \end{aligned}$$

mientras que

$$\begin{aligned} \mathsf{E}(X^2) &= \int_{-\infty}^\infty x^2 f_X(x)\,\mathrm{d}x = p\int_{-\infty}^\infty x^2 f_1(x)\,\mathrm{d}x + q\int_{-\infty}^\infty x^2 f_2(x)\,\mathrm{d}x \\ & p\,\mathsf{E}_1(X^2) + q\,\mathsf{E}_2(X^2) = p\,\phi_1 + q\,\phi_2. \end{aligned}$$

Luego

$$\mathsf{var}(X) = \mathsf{E}(X^2) - \{\mathsf{E}(X)\}^2 = p\,\phi_1 + q\,\phi_2 - (p\,\mu_1 + q\,\mu_2)^2.$$

3. Note que

$$\exp\{-\tfrac{1}{2}(x-\theta)^2\} = \exp(-x^2/2 + x\theta - \theta^2/2) = \exp(-x^2/2)\exp(x\theta - \theta^2/2),$$

luego,

$$\begin{aligned} f(x;\theta) &= \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)\exp\{x\theta - \theta^2/2 - \log\Phi(\theta)\} \\ &= \phi(x)\exp\{\theta x - \theta^2/2 - \log\Phi(\theta)\}. \end{aligned}$$

Es decir X pertenece a la FE (1-paramétrica), con $T(x) = x$, $\eta = \theta$, $b(\theta) = \theta^2/2 + \log \Phi(\theta)$ y $h(x) = \phi(x)$. De este modo, obtenemos

$$K_X(t) = b(\theta + t) - b(\theta) = (\theta + t)^2/2 + \log \Phi(\theta + t) - \theta^2/2 - \log \Phi(\theta).$$

Tenemos que $(\theta + t)^2 - \theta^2 = 2t\theta + t^2$. De este modo,

$$K_X(t) = t\theta + \theta^2/2 + \log\Big(\frac{\Phi(\theta + t)}{\Phi(\theta)}\Big).$$

Luego, sigue que

$$\mathsf{E}(X) = b'(\theta) = \theta + \frac{\Phi'(\theta)}{\Phi(\theta)} = \theta + \frac{\phi(\theta)}{\Phi(\theta)}.$$