Se desea verificar que la matriz de covarianza,

$$\operatorname{cov}(\boldsymbol{X}) = \begin{pmatrix} 1/12 & 1/24 \\ 1/24 & 7/144 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 7/12 \end{pmatrix},$$

es semidefinida positiva, consideraremos diversos métodos. A saber:

(a) Considere $\boldsymbol{a} = (a_1, a_2)^\top \in \mathbb{R}^2$. Entonces,

$$\begin{aligned} \boldsymbol{a}^\top \operatorname{cov}(\boldsymbol{X})\boldsymbol{a} &= \frac{1}{12}(a_1, a_2)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 7/12 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{12}\Big(a_1^2 + \frac{a_1 a_2}{2} + \frac{a_1 a_2}{2} + \frac{7a_2^2}{12}\Big) \\ &= \frac{1}{12}\Big(a_1^2 + 2\frac{a_1 a_2}{2} + \frac{a_2^2}{2} - \frac{a_2^2}{4} + \frac{7a_2^2}{12}\Big) = \frac{1}{12}\Big\{\Big(a_1 + \frac{a_2^2}{2}\Big)^2 + \Big(\frac{7-3}{12}\Big)a_2^2\Big\} \\ &= \frac{1}{12}\Big(a_1 + \frac{a_2^2}{2}\Big)^2 + \frac{a_2^2}{36} \geq 0, \end{aligned}$$

es decir, $\operatorname{cov}(\boldsymbol{X})$ es semidefinida positiva.

(b) Deseamos obtener las raíces del polinomio característico $|\operatorname{cov}(\boldsymbol{X}) - \lambda \boldsymbol{I}_2| = 0$. Primeramente,

$$\begin{aligned} \det\begin{pmatrix} 1/12 - \lambda & 1/2 \\ 1/2 & 7/144 - \lambda \end{pmatrix} &= \Big(\frac{1}{12} - \lambda\Big)\Big(\frac{7}{144} - \lambda\Big) - \frac{1}{24^2} \\ &= \lambda^2 - \frac{19}{144}\lambda + \frac{1}{144}\Big(\frac{1}{3}\Big), \end{aligned}$$

es decir, tenemos que obtener las raíces de la ecuación cuadrática

$$144\lambda^2 - 19\lambda + 1/3 = 0.$$

Notando que

$$\Delta = 19^2 - 4 \cdot 144/3 = 19^2 - 12 \cdot 4 = 361 - 192 = 169 = 13^2.$$

De ahí que

$$\lambda_1 = \frac{32}{2 \cdot 144} = \frac{1}{9}, \qquad \lambda_2 = \frac{6}{2 \cdot 144} = \frac{1}{48},$$

como $\lambda_1 > \lambda_2 > 0$, sigue que $\operatorname{cov}(\boldsymbol{X})$ es definida positiva.

(c) Los dos cofactores principales de la matriz $\operatorname{cov}(\boldsymbol{X})$ están dados por $1/12$ y $7/144$, que son positivos. Mientras que,

$$\begin{aligned} \det\begin{pmatrix} 1/12 & 1/24 \\ 1/24 & 7/144 \end{pmatrix} &= \Big(\frac{1}{12}\Big)^2 \det\begin{pmatrix} 1 & 1/2 \\ 1/2 & 7/12 \end{pmatrix} = \Big(\frac{1}{12}\Big)^2\Big\{\frac{7}{12} - \Big(\frac{1}{2}\Big)^2\Big\} \\ &= \Big(\frac{1}{12}\Big)^2\Big(\frac{1}{3}\Big) > 0, \end{aligned}$$

es decir, $\operatorname{cov}(\boldsymbol{X})$ es definida positiva.

(d) Sea $\boldsymbol{A}$ una matriz 2×2, es decir,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Luego, la descomposición LU es dada por

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{pmatrix}, \qquad \boldsymbol{U} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{21}a_{12}/a_{11} \end{pmatrix}.$$

De ahí que, para $\operatorname{cov}(\boldsymbol{X})$ tenemos:

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}, \qquad \boldsymbol{U} = \begin{pmatrix} 1/12 & 1/24 \\ 0 & 4/144 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 1 & 1/2 \\ 0 & 1/3 \end{pmatrix}.$$

Como $u_{11} = 1/12$ y $u_{22} = 1/36$ son positivos, sigue que $\operatorname{cov}(\boldsymbol{X})$ es definida positiva.

(e) Sea $\boldsymbol{A} = \boldsymbol{G}^\top \boldsymbol{G}$ con $\boldsymbol{G}$ matriz triangular superior. Luego, para $\boldsymbol{A}$ matriz 2×2, obtenemos

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} g_{11}^2 & g_{11}g_{12} \\ g_{21}g_{11} & g_{12}g_{21} + g_{22}^2 \end{pmatrix}.$$

De ahí que,

$$a_{11} = g_{11}^2, \quad a_{12} = g_{11}g_{12}, \quad a_{21} = g_{21}g_{11}, \quad a_{22} = g_{12}g_{21} + g_{22}^2.$$

Es decir,

$$g_{11} = \sqrt{a_{11}}, \qquad g_{12} = \frac{a_{12}}{g_{11}} = \frac{a_{12}}{\sqrt{a_{11}}}, \qquad g_{21} = \frac{a_{21}}{g_{11}} = \frac{a_{21}}{\sqrt{a_{11}}},$$
$$g_{22} = \sqrt{a_{22} - g_{12}g_{21}} = \sqrt{a_{22} - a_{12}^2/a_{11}}.$$

La descomposición Cholesky de $\operatorname{cov}(\boldsymbol{X})$ es dada por:

$$g_{11} = \frac{1}{\sqrt{12}}, \qquad g_{12} = \frac{1/24}{\sqrt{1/12}} = \frac{\sqrt{12}}{24} = \frac{1}{2\sqrt{12}},$$
$$g_{22} = \sqrt{\frac{7}{144} - \frac{1/24^2}{1/12}} = \sqrt{\frac{1}{12^2}\Big(7 - \frac{12}{4}\Big)} = \frac{\sqrt{4}}{12} = \frac{1}{6}.$$

Esto lleva a,

$$\boldsymbol{G} = \frac{1}{2}\begin{pmatrix} 2/\sqrt{12} & 1/\sqrt{12} \\ 0 & 1/3 \end{pmatrix}.$$

Así, $\operatorname{cov}(\boldsymbol{X}) = \boldsymbol{G}^\top \boldsymbol{G}$ es matriz definida positiva.