1. Considere $\boldsymbol{Z} = \sum_{j=1}^{N} \alpha_j \boldsymbol{Y}_j$. De este modo,[1]

$$\varphi_Z(\boldsymbol{t}) = \mathsf{E}\{\exp(i\boldsymbol{t}^\top \boldsymbol{Z})\} = \mathsf{E}\Big\{\exp\Big(i\boldsymbol{t}^\top \sum_{j=1}^{N} \alpha_j \boldsymbol{Y}_j\Big)\Big\} = \mathsf{E}\Big\{\prod_{j=1}^{N} \exp(i\,\alpha_j \boldsymbol{t}^\top \boldsymbol{Y}_j)\Big\},$$

por la independencia entre $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_N$, sigue que

$$\varphi_Z(\boldsymbol{t}) = \prod_{j=1}^{N} \mathsf{E}\{\exp(i\,\boldsymbol{h}_j^\top \boldsymbol{Y}_j)\},$$

con $\boldsymbol{h}_j = \alpha_j \boldsymbol{t}$, para $j = 1, \dots, N$. Así,

$$\begin{aligned}
\varphi_Z(\boldsymbol{t}) &= \prod_{j=1}^{N} \varphi_{Y_j}(\boldsymbol{h}_j) = \prod_{j=1}^{N} \exp(i\,\boldsymbol{h}_j^\top \boldsymbol{\mu}_j - \tfrac{1}{2}\boldsymbol{h}_j^\top \boldsymbol{\Sigma}_j \boldsymbol{h}_j) = \prod_{j=1}^{N} \exp(i\,\alpha_j \boldsymbol{t}^\top \boldsymbol{\mu}_j - \tfrac{\alpha_j}{2}\boldsymbol{t}^\top \boldsymbol{\Sigma}_j \boldsymbol{t}) \\
&= \exp\Big\{i\boldsymbol{t}^\top\Big(\sum_{j=1}^{N} \alpha_j \boldsymbol{\mu}_j\Big) - \frac{1}{2}\boldsymbol{t}^\top\Big(\sum_{j=1}^{N} \alpha_j^2 \boldsymbol{\Sigma}_j\Big)\boldsymbol{t}\Big\}.
\end{aligned}$$

Es decir,

$$\boldsymbol{Z} \sim \mathsf{N}_k\Big(\sum_{j=1}^{N} \alpha_j \boldsymbol{\mu}_j, \sum_{j=1}^{N} \alpha_j^2 \boldsymbol{\Sigma}_j\Big).$$

Para una muestra aleatoria $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_N$, tenemos $\boldsymbol{X}_i = \boldsymbol{Y}_i - \boldsymbol{\mu} \sim \mathsf{N}_k(\boldsymbol{0}, \boldsymbol{\Sigma})$ para $i = 1, \dots, N$, y

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\boldsymbol{Y}_i - \boldsymbol{\mu}) = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \boldsymbol{X}_i \sim \mathsf{N}_k(\boldsymbol{0}, \boldsymbol{\Sigma}).$$

2. Podemos escribir $Q_i = \boldsymbol{X}^\top \boldsymbol{A}_i \boldsymbol{X}$, para $i = 1, 2$, con

$$\boldsymbol{A}_1 = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Además, note que

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

[1] Este resultado, puede ser mostrado de diversas maneras. Aquí hemos calculado la función característica de $\boldsymbol{Z}$. Alternativamente, podríamos haber usado, por ejemplo, que $\boldsymbol{Z}$ es una combinación lineal de vectores normales.

2.a. De este modo, notando que

$$\boldsymbol{A}_1\boldsymbol{\Sigma} = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix},$$

que es matrix idempotente. Además,

$$\lambda = \boldsymbol{\mu}^\top \boldsymbol{A}_1 \boldsymbol{\mu}/2 = 4/2 = 2, \qquad \operatorname{rg}(\boldsymbol{A}_1) = \operatorname{tr}(\boldsymbol{A}_1) = 2.$$

Es decir, $Q_1 \sim \chi^2(2;2)$ una chi-cuadrado con 2 grados de libertad y parámetro de no-centralidad $\lambda = 2$.

Por otro lado,

$$\boldsymbol{A}_2\boldsymbol{\Sigma} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3/2 & 3/2 & 1 & 1 \\ 3/2 & 3/2 & 1 & 1 \\ 1 & 1 & 3/2 & 3/2 \\ 1 & 1 & 3/2 & 3/2 \end{pmatrix},$$

que no es idempotente y por tanto Q_2 no sigue una distribución chi-cuadrado.

2.b. Es fácil notar que

$$\boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_2 = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} = \boldsymbol{0}$$

y portanto Q_1 es independiente de Q_2.[2]

3.a. Podemos escribir,

$$\begin{aligned} \|\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}\|^2 &= (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{X}\widehat{\boldsymbol{\beta}})^\top (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{X}\widehat{\boldsymbol{\beta}}) \\ &= (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}})^\top (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}) - (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}})^\top \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \\ &\quad - (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top \boldsymbol{X}^\top (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}) + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top \boldsymbol{X}^\top \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}). \end{aligned}$$

Sabemos que $\boldsymbol{H}\boldsymbol{X} = \boldsymbol{X}$ de ahí que $(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top \boldsymbol{X} = \boldsymbol{Y}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{X} = \boldsymbol{0}$ y portanto,

$$\|\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}\|^2 = \|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top \boldsymbol{X}^\top \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$

como ambos términos son positivos, es evidente que el mínimo de la función

$$Q(\boldsymbol{\beta}) = \|\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}\|^2,$$

se obtiene cuando $\boldsymbol{\beta} = \widehat{\boldsymbol{\beta}}$.

[2] Note que en la demostración de la independencia entre formas cuadráticas esencialmente hemos usado la normalidad de $\boldsymbol{X}$.

3.b. Sabemos que $\boldsymbol{e} = \boldsymbol{Y} - \widehat{\boldsymbol{Y}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$ y $\widehat{\boldsymbol{Y}} = \boldsymbol{X}\widehat{\boldsymbol{\beta}} = \boldsymbol{H}\boldsymbol{Y}$. De este modo,

$$\boldsymbol{e}^\top \widehat{\boldsymbol{Y}} = \boldsymbol{Y}^\top (\boldsymbol{I} - \boldsymbol{H})^\top \boldsymbol{H}\boldsymbol{Y} = 0,$$

pues $\boldsymbol{H}^2 = \boldsymbol{H}$ y de ahí que $(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{H} = \boldsymbol{0}$.

3.c. Note que $\boldsymbol{Z} \sim \mathsf{N}_{n-p}(\boldsymbol{A}^\top \boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{A}^\top \boldsymbol{A})$, como $\boldsymbol{A}^\top \boldsymbol{X} = \boldsymbol{0}$, tenemos

$$\boldsymbol{Z} \sim \mathsf{N}_{n-p}(\boldsymbol{0}, \sigma^2 \boldsymbol{A}^\top \boldsymbol{A}),$$

con función de log-verosimilitud,

$$\ell_*(\sigma^2) = -\frac{n-p}{2} \log 2\pi\sigma^2 - \frac{1}{2} \log |\boldsymbol{A}^\top \boldsymbol{A}| - \frac{1}{2\sigma^2} \boldsymbol{Z}^\top (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{Z}.$$

Así,

$$\frac{\mathsf{d}\, \ell_*(\sigma^2)}{\mathsf{d}\, \sigma^2} = -\frac{n-p}{2\sigma^2} + \frac{1}{2\sigma^2} \boldsymbol{Z}^\top (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{Z}.$$

De este modo,

$$\widehat{\sigma}^2_* = \frac{1}{n-p} \boldsymbol{Z}^\top (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{Z} = \frac{1}{n-p} \boldsymbol{Y}^\top \boldsymbol{A} (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{Y} = \frac{1}{n-p} \boldsymbol{Y}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{Y}.$$

4. El modelo

$$Y_i \sim \mathsf{N}(\alpha + \theta z_i, \sigma^2), \qquad i = 1, \dots, n,$$

puede ser escrito en forma lineal como

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

con $\boldsymbol{Y} = (Y_1, \dots, Y_n)^\top$, $\boldsymbol{\beta} = (\alpha, \theta)^\top$,

$$\boldsymbol{X} = \begin{pmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_n \end{pmatrix}, \qquad \boldsymbol{\epsilon} \sim \mathsf{N}_n(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n).$$

Tenemos que

$$\boldsymbol{X}^\top \boldsymbol{X} = \begin{pmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_n \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n z_i \\ \sum_{i=1}^n z_i & \sum_{i=1}^n z_i^2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & \|\boldsymbol{z}\|^2 \end{pmatrix}$$

$$\boldsymbol{X}^\top \boldsymbol{Y} = \begin{pmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_n \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n z_i Y_i \end{pmatrix} = \begin{pmatrix} n\overline{Y} \\ \boldsymbol{z}^\top \boldsymbol{Y} \end{pmatrix},$$

con $\boldsymbol{z} = (z_1, \dots, z_n)^\top$. De ahí que

$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\theta} \end{pmatrix} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y},$$

donde

$$\widehat{\alpha} = \overline{Y}, \qquad \widehat{\theta} = \frac{\boldsymbol{z}^\top \boldsymbol{Y}}{\|\boldsymbol{z}\|^2}.$$

Sabemos que

$$\mathsf{cov}(\widehat{\boldsymbol{\beta}}) = \sigma^2 \begin{pmatrix} 1/n & 0 \\ 0 & 1/\|\boldsymbol{z}\|^2 \end{pmatrix},$$

por la normalidad de $\widehat{\boldsymbol{\beta}}$ sigue que $\widehat{\alpha}$ y $\widehat{\theta}$ son independientes.