MAT-266, 2° Semestre 2014 | **Nombre:** ______________________
Certamen 1. Octubre 17, 2014 | **Rol:** ______________________
Tiempo: 90 Minutos | Profesor: Felipe Osorio.

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1. Suponga $\boldsymbol{Y} \sim \mathcal{N}_n(\theta \mathbf{1}_n, \sigma^2 \boldsymbol{I}_n)$ y denote por $\boldsymbol{H}_n$ la matriz Helmert de orden n, definida en forma particionada como

$$\boldsymbol{H}_n = \begin{pmatrix} \frac{1}{\sqrt{n}} \mathbf{1}^\top \\ \boldsymbol{A}^\top \end{pmatrix}, \qquad \text{tal que} \qquad \boldsymbol{H}_n^\top \boldsymbol{H}_n = \boldsymbol{H}_n \boldsymbol{H}_n^\top = \boldsymbol{I}_n.$$

a. (10 pts) Determine la distribución de $\boldsymbol{Z} = \boldsymbol{H}_n \boldsymbol{Y}$.

b. (15 pts) Obtenga la distribución condicional de $\boldsymbol{A}^\top \boldsymbol{Y}$ dado $\sqrt{n}\,\overline{Y}$.

2. (25 pts) (Restricciones lineales estocásticas) Considere la transformación lineal

$$\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{g} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{G} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{u} \end{pmatrix},$$

donde $\boldsymbol{Y}$ y $\boldsymbol{g}$ son vectores $n \times 1$ y $q \times 1$, respectivamente, $\boldsymbol{X} \in \mathbb{R}^{n \times p}$, $\boldsymbol{G} \in \mathbb{R}^{q \times p}$, $\boldsymbol{\beta}$ es vector p-dimensional y $\boldsymbol{\epsilon}$, $\boldsymbol{u}$ son vectores aleatorios independientes, con $\mathrm{E}(\boldsymbol{\epsilon}) = \mathbf{0}$, $\mathrm{E}(\boldsymbol{u}) = \mathbf{0}$, y

$$\mathrm{Cov}\begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{u} \end{pmatrix} = \sigma^2 \begin{pmatrix} \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{pmatrix}, \qquad \sigma^2 > 0.$$

Sea

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{b} + (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{G}^\top (\boldsymbol{\Omega} + \boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{G}^\top)^{-1} (\boldsymbol{g} - \boldsymbol{G}\boldsymbol{b}),$$

donde $\boldsymbol{b} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y}$. Muestre que $\widehat{\boldsymbol{\beta}}$ es insesgado y calcule su matriz de covarianza.

3. (25 pts) Sea $\boldsymbol{X}$ un vector aleatorio $p \times 1$ tal que $\mathrm{E}(\boldsymbol{X}\boldsymbol{X}^\top)$ y $\mathrm{E}(\boldsymbol{X}\boldsymbol{X}^\top \otimes \boldsymbol{X}\boldsymbol{X}^\top)$ existen. Considere $\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}$ y asuma que $\boldsymbol{A}$ y $\boldsymbol{B}$ son matrices simétricas de orden $p \times p$. Muestre que

$$\begin{aligned} \mathrm{Cov}(\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X}, \boldsymbol{X}^\top \boldsymbol{B}\boldsymbol{X}) &= \mathrm{tr}\{(\boldsymbol{A} \otimes \boldsymbol{B})\, \mathrm{E}(\boldsymbol{X}\boldsymbol{X}^\top \otimes \boldsymbol{X}\boldsymbol{X}^\top)\} \\ &\quad - \{\mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top \boldsymbol{A}\boldsymbol{\mu}\}\{\mathrm{tr}(\boldsymbol{B}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top \boldsymbol{B}\boldsymbol{\mu}\}. \end{aligned}$$

Sugerencia: Note que

$$\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X}\boldsymbol{X}^\top \boldsymbol{B}\boldsymbol{X} = \mathrm{tr}\{(\boldsymbol{X}^\top \otimes \boldsymbol{X}^\top)(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{X} \otimes \boldsymbol{X})\}.$$

4. (25 pts) Sea $\boldsymbol{Y} \sim \mathcal{N}_n(\mathbf{0}, \boldsymbol{I}_n)$ y considere $\boldsymbol{A}$, $\boldsymbol{B}$ matrices idempotentes, tales que $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} = \mathbf{0}$. Muestre la independencia conjunta entre $\boldsymbol{Y}^\top \boldsymbol{A}\boldsymbol{Y}$, $\boldsymbol{Y}^\top \boldsymbol{B}\boldsymbol{Y}$ y $\boldsymbol{Y}^\top (\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B})\boldsymbol{Y}$.