MAT-266, 2° Semestre 2014 **Pauta de Corrección**
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1. a. Considere la transformación $\boldsymbol{Z} = \boldsymbol{H}_n \boldsymbol{Y}$, con $\boldsymbol{Y} \sim \mathcal{N}_n(\theta \boldsymbol{1}, \sigma^2 \boldsymbol{I})$. De este modo,

$$\boldsymbol{Z} \sim \mathcal{N}_n(\theta \boldsymbol{H}_n \boldsymbol{1}, \sigma^2 \boldsymbol{H}_n \boldsymbol{H}_n^\top).$$

Como $\boldsymbol{H}_n$ es matriz ortogonal tenemos que $\boldsymbol{Z} \sim \mathcal{N}_n(\theta \boldsymbol{H}_n \boldsymbol{1}, \sigma^2 \boldsymbol{H}_n \boldsymbol{H}_n^\top)$. Además

$$\mathrm{E}(\boldsymbol{Z}) = \theta \boldsymbol{H}_n \boldsymbol{1} = \theta \begin{pmatrix} \frac{1}{\sqrt{n}} \boldsymbol{1}^\top \\ \boldsymbol{A}^\top \end{pmatrix} \boldsymbol{1} = \theta \begin{pmatrix} \sqrt{n} \\ \boldsymbol{A}^\top \boldsymbol{1} \end{pmatrix}. \tag{1}$$

1. b. Considere la partición $\boldsymbol{Z} = (Z_1, \boldsymbol{Z}_2^\top)^\top$, donde

$$Z_1 = \frac{1}{\sqrt{n}} \boldsymbol{1}^\top \boldsymbol{Y} = \sqrt{n}\, \overline{Y}, \qquad \boldsymbol{Z}_2 = \boldsymbol{A}^\top \boldsymbol{Y}.$$

Sabemos que $\operatorname{Cov}(\boldsymbol{Z}) = \sigma^2 \boldsymbol{H}_n \boldsymbol{H}_n^\top$, con

$$\boldsymbol{H}_n \boldsymbol{H}_n^\top = \begin{pmatrix} \frac{1}{\sqrt{n}} \boldsymbol{1}^\top \\ \boldsymbol{A}^\top \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}} \boldsymbol{1} & \boldsymbol{A} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} \boldsymbol{1}^\top \boldsymbol{1} & \frac{1}{\sqrt{n}} \boldsymbol{1}^\top \boldsymbol{A} \\ \frac{1}{\sqrt{n}} \boldsymbol{A}^\top \boldsymbol{1} & \boldsymbol{A}^\top \boldsymbol{A} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-1} \end{pmatrix}. \tag{2}$$

En nuestro caso podemos escribir $\boldsymbol{Z} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ donde $\boldsymbol{\mu}$ y $\boldsymbol{\Sigma}$ son particionadas como en (1) y (2), respectivamente. Se desea calcular la distribución condicional de $\boldsymbol{Z}_2 | Z_1$, la cual sigue una distribución normal $(n-1)$-dimensional con media y covarianza condicionales dadas por

$$\boldsymbol{\mu}_{2\cdot 1} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} (z_1 - \mu_1), \qquad \boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}.$$

Desde la Ecuación (2) sigue que $\boldsymbol{\Sigma}_{21} = \boldsymbol{0}$ y por tanto $\boldsymbol{Z}_2 | Z_1 \stackrel{\mathsf{d}}{=} \boldsymbol{Z}_2 \sim \mathcal{N}_{n-1}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$. Es decir, $\sqrt{n}\, \overline{Y}$ y $\boldsymbol{Z}_2 = \boldsymbol{A}^\top \boldsymbol{Y}$ son independientes.

Observación: Por la ortogonalidad de $\boldsymbol{H}_n$ tenemos también que

$$\boldsymbol{H}_n^\top \boldsymbol{H}_n = \begin{pmatrix} \frac{1}{\sqrt{n}} \boldsymbol{1} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}} \boldsymbol{1}^\top \\ \boldsymbol{A}^\top \end{pmatrix} = \frac{1}{n} \boldsymbol{1}\boldsymbol{1}^\top + \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}_n,$$

es decir $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}_n - \frac{1}{n} \boldsymbol{1}\boldsymbol{1}^\top$ que corresponde a la matriz de centrado de orden n y el resultado anterior es una forma alternativa para mostrar que $\overline{Y}$ y S^2 son independientes.

2. Primeramente, note que

$$\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{g} \end{pmatrix} \sim \mathcal{N}_{n+q} \left(\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{G} \end{pmatrix} \boldsymbol{\beta}, \sigma^2 \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega} \end{pmatrix} \right).$$

es decir $\boldsymbol{Y}$ y $\boldsymbol{g}$ son independientes. Ahora,

$$\mathrm{E}(\widehat{\boldsymbol{\beta}}) = \mathrm{E}(\boldsymbol{b}) + (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{G}^\top (\boldsymbol{\Omega} + \boldsymbol{G} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{G}^\top)^{-1} \mathrm{E}(\boldsymbol{g} - \boldsymbol{G}\boldsymbol{b}).$$

Sabemos que

$$\begin{aligned}
\mathrm{E}(\boldsymbol{b}) &= (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\,\mathrm{E}(\boldsymbol{Y}) = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{\beta}\\
\mathrm{E}(\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b}) &= \mathrm{E}(\boldsymbol{g}) - \boldsymbol{G}\,\mathrm{E}(\boldsymbol{b}) = \boldsymbol{G}\boldsymbol{\beta} - \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{0}.
\end{aligned}$$

De este modo, $\mathrm{E}(\widehat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ y $\widehat{\boldsymbol{\beta}}$ es un estimador insesgado. Por otro lado, note que

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{b},\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b}) &= \mathrm{Cov}(\boldsymbol{b},\boldsymbol{g}) - \mathrm{Cov}(\boldsymbol{b},\boldsymbol{G}\boldsymbol{b}) = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\,\mathrm{Cov}(\boldsymbol{Y},\boldsymbol{g}) - \mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top\\
&= -\,\mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top,
\end{aligned}$$

mientras que

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b}) &= \mathrm{Cov}(\boldsymbol{g}) + \mathrm{Cov}(\boldsymbol{G}\boldsymbol{b}) - 2\,\mathrm{Cov}(\boldsymbol{g},\boldsymbol{G}\boldsymbol{b})\\
&= \mathrm{Cov}(\boldsymbol{g}) + \boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top - 2\,\mathrm{Cov}(\boldsymbol{g},\boldsymbol{G}\boldsymbol{b})\\
&= \mathrm{Cov}(\boldsymbol{g}) + \boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top.
\end{aligned}$$

Considere $\boldsymbol{M} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{\Omega} + \boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}$, de este modo

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{b}+\boldsymbol{M}(\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b})) &= \mathrm{Cov}(\boldsymbol{b}) + \mathrm{Cov}(\boldsymbol{b},\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b})\boldsymbol{M}^\top + \boldsymbol{M}\,\mathrm{Cov}(\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b},\boldsymbol{b})\\
&+ \boldsymbol{M}\,\mathrm{Cov}(\boldsymbol{g}-\boldsymbol{G}\boldsymbol{b})\boldsymbol{M}^\top\\
&= \mathrm{Cov}(\boldsymbol{b}) - \mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top\boldsymbol{M}^\top - \boldsymbol{M}\boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b}) + \boldsymbol{M}\,\mathrm{Cov}(\boldsymbol{g})\boldsymbol{M}^\top\\
&+ \boldsymbol{M}\boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top\boldsymbol{M}^\top.
\end{aligned}$$

Además,

$$\begin{aligned}
&\mathrm{Cov}(\boldsymbol{b}) - \mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top\boldsymbol{M}^\top - \boldsymbol{M}\boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b}) + \boldsymbol{M}\boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b})\boldsymbol{G}^\top\boldsymbol{M}^\top\\
&\quad= \mathrm{Cov}(\boldsymbol{b})(\boldsymbol{I}-\boldsymbol{G}^\top\boldsymbol{M}^\top) - \boldsymbol{M}\boldsymbol{G}\,\mathrm{Cov}(\boldsymbol{b})(\boldsymbol{I}-\boldsymbol{G}^\top\boldsymbol{M}^\top)\\
&\quad= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{G})\,\mathrm{Cov}(\boldsymbol{b})(\boldsymbol{I}-\boldsymbol{G}^\top\boldsymbol{M}^\top).
\end{aligned}$$

Finalmente,

$$\begin{aligned}
\mathrm{Cov}(\widehat{\boldsymbol{\beta}}) &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{G})\,\mathrm{Cov}(\boldsymbol{b})(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{G})^\top + \boldsymbol{M}\,\mathrm{Cov}(\boldsymbol{g})\boldsymbol{M}^\top\\
&= \sigma^2\{(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{G})(\boldsymbol{X}^\top\boldsymbol{X})^{-1}(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{G})^\top + \boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{M}^\top\}.
\end{aligned}$$

3. Sea $Q_1 = \boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}$ y $Q_2 = \boldsymbol{X}^\top\boldsymbol{B}\boldsymbol{X}$. Se desea calcular:

$$\mathrm{Cov}(Q_1,Q_2) = \mathrm{E}(Q_1Q_2) - \mathrm{E}(Q_1)\,\mathrm{E}(Q_2).$$

Es bien sabido que la esperanza de las formas cuadráticas Q_1 y Q_2 están dadas por

$$\mathrm{E}(Q_1) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top\boldsymbol{A}\boldsymbol{\mu}, \qquad \mathrm{E}(Q_2) = \mathrm{tr}(\boldsymbol{B}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top\boldsymbol{B}\boldsymbol{\mu},$$

respectivamente. Además

$$\begin{aligned}
\mathrm{E}(Q_1Q_2) &= \mathrm{E}(\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}\boldsymbol{X}^\top\boldsymbol{B}\boldsymbol{X}) = \mathrm{E}\{\mathrm{tr}(\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}\boldsymbol{X}^\top\boldsymbol{B}\boldsymbol{X})\}\\
&= \mathrm{E}[\mathrm{tr}\{(\boldsymbol{X}^\top\otimes\boldsymbol{X}^\top)(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{X}\otimes\boldsymbol{X})\}]\\
&= \mathrm{E}[\mathrm{tr}\{(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{X}\otimes\boldsymbol{X})(\boldsymbol{X}^\top\otimes\boldsymbol{X}^\top)\}]\\
&= \mathrm{tr}[(\boldsymbol{A}\otimes\boldsymbol{B})\,\mathrm{E}\{(\boldsymbol{X}\otimes\boldsymbol{X})(\boldsymbol{X}^\top\otimes\boldsymbol{X}^\top)\}]\\
&= \mathrm{tr}\{(\boldsymbol{A}\otimes\boldsymbol{B})\,\mathrm{E}(\boldsymbol{X}\boldsymbol{X}^\top\otimes\boldsymbol{X}\boldsymbol{X}^\top)\},
\end{aligned}$$

desde donde sigue el resultado.

4. Para mostrar la independencia conjunta, considere

$$\boldsymbol{A} + \boldsymbol{B} + (\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{I}_n,$$

que es matriz idempotente de rango n. Ahora, como $\boldsymbol{A}$, $\boldsymbol{B}$ son idempotentes y $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{0}$, sigue que

$$(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + \boldsymbol{AB} + \boldsymbol{BA} = \boldsymbol{A} + \boldsymbol{B},$$

y

$$\begin{aligned}(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B})^2 &= (\boldsymbol{I}_n - (\boldsymbol{A} + \boldsymbol{B}))^2 = \boldsymbol{I}_n + (\boldsymbol{A} + \boldsymbol{B})^2 - (\boldsymbol{A} + \boldsymbol{B}) - (\boldsymbol{A} + \boldsymbol{B}) \\ &= \boldsymbol{I}_n - (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B},\end{aligned}$$

es decir $(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B})$ es matriz idempotente, y de este modo $\operatorname{rg}(\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{A})$, $\operatorname{rg}(\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B})$ y

$$\operatorname{rg}(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B}) = \operatorname{tr}(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B}) = n - \operatorname{tr}(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{B}),$$

que evidentemente satisface

$$n = \operatorname{rg}(\boldsymbol{A}) + \operatorname{rg}(\boldsymbol{B}) + \operatorname{rg}(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B}).$$

Así, usando el Teorema de Cochran sigue el resultado deseado.