**1.a.** Tenemos que $\widehat{\boldsymbol{\beta}} = (\widehat{\alpha}, \widehat{\theta})^\top$ tiene covarianza $\mathsf{Cov}(\widehat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1}$, con

$$\boldsymbol{X}^\top\boldsymbol{X} = \begin{pmatrix} \mathbf{1}^\top \\ \boldsymbol{z}^\top \end{pmatrix}(\mathbf{1}, \boldsymbol{z}) = \begin{pmatrix} \mathbf{1}^\top\mathbf{1} & \mathbf{1}^\top\boldsymbol{z} \\ \boldsymbol{z}^\top\mathbf{1} & \boldsymbol{z}^\top\boldsymbol{z} \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & \|\boldsymbol{z}\|^2 \end{pmatrix},$$

pues $\mathbf{1}^\top\boldsymbol{z} = 0$. Notando que $\widehat{\boldsymbol{\beta}} \sim \mathsf{N}_p(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1})$, sigue que $\widehat{\alpha}$ y $\widehat{\theta}$ son independientes, pues $\mathsf{Cov}(\widehat{\alpha}, \widehat{\theta}) = 0$.

**1.b.** Evidentemente podemos escribir $b^* = \boldsymbol{a}^\top\boldsymbol{Y}$, donde $\boldsymbol{a} = (a_1, \dots, a_n)^\top$, con

$$\begin{aligned} a_1 &= -\frac{1}{n-1}\frac{Y_1}{\triangle z_2}, \\ a_i &= -\frac{1}{n-1}\Big(\frac{1}{\triangle z_{i-1}} - \frac{1}{\triangle z_i}\Big)Y_{i-1}, \qquad i = 2, \dots, n-1, \\ a_n &= \frac{1}{n-1}\frac{Y_n}{\triangle z_n}, \end{aligned}$$

es decir, es una función lineal. Además,

$$\mathsf{E}(b^*) = \frac{1}{n-1}\sum_{i=2}^n \frac{\mathsf{E}(Y_i) - \mathsf{E}(Y_{i-1})}{\triangle z_i},$$

como $\mathsf{E}(Y_i) = \alpha + \theta z_i$, $(i = 1, \dots, n)$. Así,

$$\mathsf{E}(b^*) = \frac{1}{n-1}\sum_{i=2}^n \frac{\alpha + \theta z_i - (\alpha + \theta z_{i-1})}{\triangle z_i} = \frac{1}{n-1}\sum_{i=2}^n \frac{\theta(z_i - z_{i-1})}{\triangle z_i} = \theta,$$

es decir, $b^*$ es insesgado. Dado que $b^*$ **no es el estimador LS** de $\theta$. Sigue que no es $\mathsf{BLUE}$.

**2.a.** Considere $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. así,

$$\boldsymbol{X}^\top\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1^\top \\ \boldsymbol{x}_2^\top \\ \boldsymbol{x}_3^\top \end{pmatrix}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \begin{pmatrix} \boldsymbol{x}_1^\top\boldsymbol{x}_1 & \boldsymbol{x}_1^\top\boldsymbol{x}_2 & 0 \\ \boldsymbol{x}_2^\top\boldsymbol{x}_1 & \boldsymbol{x}_2^\top\boldsymbol{x}_2 & 0 \\ 0 & 0 & x_{13}^2 \end{pmatrix},$$

como $\boldsymbol{X}^\top\boldsymbol{X}$ es diagonal en bloques, tenemos que $(\widehat{\beta}_1, \widehat{\beta}_2)^\top$ es independiente de $\widehat{\beta}_3$. Además,

$$\boldsymbol{X}^\top\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{x}_1^\top\boldsymbol{Y} \\ \boldsymbol{x}_2^\top\boldsymbol{Y} \\ \boldsymbol{x}_3^\top\boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1^\top\boldsymbol{Y} \\ \boldsymbol{x}_2^\top\boldsymbol{Y} \\ x_{13}Y_1 \end{pmatrix}.$$

De este modo, $\widehat{\beta}_3 = x_{13}Y_1/x_{13}^2 = Y_1/x_{13}$.

**2.b.** Sabemos que

$$Q(\widehat{\boldsymbol{\beta}}) = \|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 = \|\boldsymbol{Y}\|^2 - \|\boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2.$$

Ahora,

$$\begin{aligned}\widehat{\boldsymbol{\beta}}^\top \boldsymbol{X}^\top \boldsymbol{X}\widehat{\boldsymbol{\beta}} &= (\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)\begin{pmatrix} \boldsymbol{x}_1^\top \boldsymbol{x}_1 & \boldsymbol{x}_1^\top \boldsymbol{x}_2 & 0 \\ \boldsymbol{x}_2^\top \boldsymbol{x}_1 & \boldsymbol{x}_2^\top \boldsymbol{x}_2 & 0 \\ 0 & 0 & x_{13}^2 \end{pmatrix}\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \end{pmatrix} = (\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)\begin{pmatrix} \boldsymbol{x}_1^\top \boldsymbol{x}_1\widehat{\beta}_1 + \boldsymbol{x}_1^\top \boldsymbol{x}_2\widehat{\beta}_2 \\ \boldsymbol{x}_2^\top \boldsymbol{x}_1\widehat{\beta}_1 + \boldsymbol{x}_2^\top \boldsymbol{x}_2\widehat{\beta}_2 \\ x_{13}^2\widehat{\beta}_3 \end{pmatrix} \\ &= \widehat{\beta}_1^2 \boldsymbol{x}_1^\top \boldsymbol{x}_1 + \widehat{\beta}_1\widehat{\beta}_2 \boldsymbol{x}_1^\top \boldsymbol{x}_2 + \widehat{\beta}_1\widehat{\beta}_2 \boldsymbol{x}_2^\top \boldsymbol{x}_1 + \widehat{\beta}_2^2 \boldsymbol{x}_2^\top \boldsymbol{x}_2 + x_{13}^2\widehat{\beta}_3^2.\end{aligned}$$

Esto permite escribir,

$$\|\boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 = \widehat{\beta}_1^2\|\boldsymbol{x}_1\|^2 + \widehat{\beta}_2^2\|\boldsymbol{x}_2\|^2 + 2\widehat{\beta}_1\widehat{\beta}_2 \boldsymbol{x}_1^\top \boldsymbol{x}_2 + x_{13}^2\widehat{\beta}_3^2 = \|\boldsymbol{x}_1\widehat{\beta}_1 + \boldsymbol{x}_2\widehat{\beta}_2\|^2 + Y_1^2.$$

De ahí que

$$s^2 = \frac{1}{n-3}\|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 = \frac{1}{n-3}\Big(\|\boldsymbol{Y}\|^2 - \|\boldsymbol{x}_1\widehat{\beta}_1 + \boldsymbol{x}_2\widehat{\beta}_2\|^2 - Y_1^2\Big).$$

**2.c.** Considere la hipótesis $H_0 : \beta_1 = \beta_2 = 0$. En efecto, puede ser escrita como $H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}$, con

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad q = 2.$$

Ahora, bajo $H_0$ tenemos el modelo,

$$\boldsymbol{Y} = \boldsymbol{x}_3\beta_3 + \boldsymbol{\epsilon},$$

Tenemos $\widetilde{\beta}_3 = \boldsymbol{x}_3^\top \boldsymbol{Y}/\|\boldsymbol{x}_3\|^2 = Y_1/x_{13} = \widehat{\beta}_3$, lo que lleva a escribir

$$Q(\widetilde{\boldsymbol{\beta}}) = Q_R(\widetilde{\beta}_3) = \|\boldsymbol{Y} - \boldsymbol{x}_3\widehat{\beta}_3\|^2 = \|\boldsymbol{Y}\|^2 - \|\boldsymbol{x}_3\widehat{\beta}_3\|^2 = \|\boldsymbol{Y}\|^2 - Y_1^2.$$

Luego, el estadístico $F$ adopta la forma:

$$F = \frac{\{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})\}/2}{s^2} = \frac{\|\boldsymbol{x}_1\widehat{\beta}_1 + \boldsymbol{x}_2\widehat{\beta}_2\|^2/2}{s^2} \sim \mathsf{F}(2, n-3).$$

De este modo, rechazamos $H_0 : \beta_1 = \beta_2 = 0$, si

$$F \geq \mathsf{F}_{1-\alpha}(2, n-3).$$

**2.d.** La hipótesis $H_0 : \beta_3 = 0$ puede ser escrita como $H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}$, con $\boldsymbol{G} = (0, 0, 1)$, $g = 0$ y $q = 1$. Así,

$$\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g} = \widehat{\beta}_3, \qquad \boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top = 1/x_{13}^2,$$

y

$$F = \frac{(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})}{s^2} = \frac{x_{13}^2\widehat{\beta}_3^2}{s^2} = \frac{Y_1^2}{s^2} \sim \mathsf{F}(1, n-3),$$

o equivalentemente,[1]

$$T = \frac{Y_1}{s} \sim t(n-3),$$

y rechazamos $H_0 : \beta_3 = 0$, si

$$T \geq t_{1-\alpha/2}(n-3).$$

[1] En efecto, $F(1, \nu) \stackrel{\mathsf{d}}{=} t^2(\nu)$.