IECD-325: Test de hipótesis y regiones de confianza

Felipe Osorio

felipe.osorio@uv.cl

Hipótesis lineales

Problema:

Desarrollar el test de razón de verosimilitudes para probar hipótesis lineales de la forma:

$$H_0 : \boldsymbol{G\beta} = \boldsymbol{g}, \qquad \text{versus} \qquad H_1 : \boldsymbol{G\beta} \neq \boldsymbol{g} \tag{1}$$

donde $\boldsymbol{G}$ es una matriz de contrastes de orden $q \times p$ con $\operatorname{rg}(\boldsymbol{G}) = q$ y $\boldsymbol{g} \in \mathbb{R}^q$.

Observación:

H_0 es expresada como un sistema de ecuaciones mientras que H_1 indica que al menos una ecuación no se satisface.

Ejemplo (Función de producción Cobb-Douglas):

Considere la función de producción:

$$Y = AK^{\alpha}L^{\beta},$$

donde Y es el nivel de producción, K es el stock de capital (maquinaria, equipamento, construcciones), L es el trabajo (total de horas-hombre trabajadas en un año). Note que,

$$\log Y = \log A + \alpha \log K + \beta \log L,$$

donde A representa un factor de productividad (eficiencia), mientras que α y β son las elasticidades del capital y trabajo, respectivamente.

Por ejemplo, podemos estar interesados en,

$$H_0 : \alpha + \beta = 1, \qquad \text{versus} \qquad H_1 : \alpha + \beta \neq 1.$$

Para abordar hipótesis lineales, usaremos el principio de verosimilitud. Es decir, consideraremos el estadístico

$$\Lambda = \frac{\max_{G\beta=g} L(\boldsymbol{\beta}, \sigma^2)}{\max_{\Theta} L(\boldsymbol{\beta}, \sigma^2)} = \frac{L(\widetilde{\boldsymbol{\beta}}, \widetilde{\sigma}^2)}{L(\widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2)}.$$

Asumiendo que $\boldsymbol{Y} \sim \mathsf{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I})$ tenemos la función de verosimilitud

$$L(\boldsymbol{\beta}, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\Big\{ -\frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})^\top(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})\Big\}.$$

Sabemos que

$$\begin{aligned}
\max_{\Theta} L(\boldsymbol{\beta}, \sigma^2) &= L(\widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2) \\
&= (2\pi\widehat{\sigma}^2)^{-n/2} \exp\Big\{ -\frac{1}{2\widehat{\sigma}^2}\|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2\Big\} \\
&= \{2\pi Q(\widehat{\boldsymbol{\beta}})/n\}^{-n/2} \exp\Big\{ -\frac{n}{2Q(\widehat{\boldsymbol{\beta}})}\|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2\Big\} \\
&= \{2\pi Q(\widehat{\boldsymbol{\beta}})/n\}^{-n/2} \exp(-n/2).
\end{aligned}$$

Hipótesis lineales

Mientras que bajo $H_0 : \boldsymbol{G\beta} = \boldsymbol{g}$, tenemos[1]

$$\begin{aligned}\max_{G\beta=g} L(\boldsymbol{\beta}, \sigma^2) = L(\widetilde{\boldsymbol{\beta}}, \widetilde{\sigma}^2) &= (2\pi\widetilde{\sigma}^2)^{-n/2} \exp\Big\{ -\frac{1}{2\widetilde{\sigma}^2}\|\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}\|^2\Big\} \\ &= \{2\pi Q(\widetilde{\boldsymbol{\beta}})/n\}^{-n/2} \exp(-n/2).\end{aligned}$$

De este modo, el estadístico de razón de verosimilitudes

$$\begin{aligned}\Lambda = \frac{L(\widetilde{\boldsymbol{\beta}}, \widetilde{\sigma}^2)}{L(\widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2)} &= \frac{\{2\pi Q(\widetilde{\boldsymbol{\beta}})/n\}^{-n/2} \exp(-n/2)}{\{2\pi Q(\widehat{\boldsymbol{\beta}})/n\}^{-n/2} \exp(-n/2)} \\ &= \Big\{\frac{Q(\widehat{\boldsymbol{\beta}})}{Q(\widetilde{\boldsymbol{\beta}})}\Big\}^{n/2},\end{aligned}$$

y de acuerdo con el principio de verosimilitud rechazamos $H_0 : \boldsymbol{G\beta} = \boldsymbol{g}$ si Λ es pequeño.

[1] $\Theta_0 = \{\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \sigma^2)^\top : \boldsymbol{G\beta} = \boldsymbol{g}\}$.

Hipótesis lineales

Alternativamente, podemos considerar

$$\Lambda^{2/n} = \frac{Q(\widehat{\boldsymbol{\beta}})}{Q(\widetilde{\boldsymbol{\beta}})}.$$

Recuerde que $\widetilde{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}} - \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})$, así

$$\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}} = \boldsymbol{Y} - \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})) = \boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}).$$

Además,

$$\begin{aligned} Q(\widetilde{\boldsymbol{\beta}}) &= \|\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}\|^2 = \|\boldsymbol{Y} - \widehat{\boldsymbol{\beta}} - \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})\|^2 \\ &= \|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 + \|\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})\|^2 \\ &+ (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}})^\top \boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}) + (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top \boldsymbol{B}^\top \boldsymbol{X}^\top (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}). \end{aligned}$$

Sin embargo,

$$\boldsymbol{X}^\top(\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}) = \boldsymbol{X}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y} = \boldsymbol{0}$$

Luego,

$$\begin{aligned} Q(\widetilde{\boldsymbol{\beta}}) &= Q(\widehat{\boldsymbol{\beta}}) + (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top \boldsymbol{B}^\top \boldsymbol{X}^\top \boldsymbol{X} \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}) \\ &\geq Q(\widehat{\boldsymbol{\beta}}) \end{aligned}$$

Observación:

Es decir,

$$0 \leq \frac{Q(\widehat{\boldsymbol{\beta}})}{Q(\widetilde{\boldsymbol{\beta}})} \leq 1,$$

por tanto, $\Lambda^{2/n} \in [0,1]$.

Además, recordando que

$$\boldsymbol{B} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1},$$

obtenemos

$$\boldsymbol{B}^\top \boldsymbol{X}^\top \boldsymbol{X}\boldsymbol{B} = (\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}.$$

De este modo,

$$Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}}) = (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}).$$

Así

$$\frac{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})}{Q(\widehat{\boldsymbol{\beta}})} = \Lambda^{-2/n} - 1,$$

es decir, valores pequeños de Λ (en cuyo caso rechazamos H_0) implican valores grandes de la razón anterior.

Hipótesis lineales

Ahora considere

$$\begin{aligned}\mathsf{E}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g}) &= \boldsymbol{G}\,\mathsf{E}(\widehat{\boldsymbol{\beta}})-\boldsymbol{g} = \boldsymbol{G}\boldsymbol{\beta}-\boldsymbol{g}\\ \mathsf{Cov}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g}) &= \boldsymbol{G}\,\mathsf{Cov}(\widehat{\boldsymbol{\beta}})\boldsymbol{G}^\top = \sigma^2\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top.\end{aligned}$$

Note que

$$(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1} = \Big\{\frac{1}{\sigma^2}\,\mathsf{Cov}(\boldsymbol{G}\widehat{\boldsymbol{\beta}})\Big\}^{-1}.$$

De esta forma

$$\frac{(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})}{\sigma^2} \sim \chi^2(q;\delta),$$

pues $\sigma^{-2}(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}\,\mathsf{Cov}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}) = \boldsymbol{I}$, es matriz idempotente, y

$$\delta = \frac{1}{2\sigma^2}(\boldsymbol{G}\boldsymbol{\beta}-\boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\boldsymbol{\beta}-\boldsymbol{g})$$

Por otro lado, sabemos que

$$\frac{Q(\widehat{\boldsymbol{\beta}})}{\sigma^2} \sim \chi^2(n-p;0)$$

Para notar la independencia, considere $\boldsymbol{\beta}^*$ cualquier vector que satisface la condición $\boldsymbol{G}\boldsymbol{\beta}^* = \boldsymbol{g}$. Entonces,

$$\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{H})(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*),$$

pues $(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{X} = \boldsymbol{0}$ y

$$\begin{aligned} \boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g} &= \boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y} - \boldsymbol{G}\boldsymbol{\beta}^* = \boldsymbol{G}\{(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y} - \boldsymbol{\beta}^*\} \\ &= \boldsymbol{G}\{(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y} - (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{\beta}^*\} \\ &= \boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*). \end{aligned}$$

En nuestro caso,

$$\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^* \sim \mathsf{N}_n(\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\boldsymbol{\beta}^*, \sigma^2\boldsymbol{I}) \stackrel{\mathsf{d}}{=} \mathsf{N}_n(\boldsymbol{X}(\boldsymbol{\beta} - \boldsymbol{\beta}^*), \sigma^2\boldsymbol{I}).$$

De este modo,

$$Q(\widehat{\boldsymbol{\beta}}) = \|\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\|^2 = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*)^\top(\boldsymbol{I} - \boldsymbol{H})(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*),$$

mientras que

$$\begin{aligned} Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}}) = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*)^\top \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1} \\ \times \boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}^*). \end{aligned}$$

Como

$$(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top = \boldsymbol{0},$$

sigue la independencia y permite construir la estadística

$$F = \frac{\{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})\}/q}{Q(\widehat{\boldsymbol{\beta}})/(n-p)} \sim F(q, n-p; \delta).$$

Esto lleva al siguiente resultado.

Resultado 1:
Para el modelo lineal $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ con los supuestos A1 a A4*. Un test de tamaño α para probar

$$H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}, \qquad \text{versus} \qquad H_1 : \boldsymbol{G}\boldsymbol{\beta} \neq \boldsymbol{g},$$

es dado por, rechazar H_0 cuando

$$F = \frac{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})}{qs^2} \geq F_{1-\alpha}(q, n-p; 0).$$

Demostración:

Bajo $H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}$, tenemos $\delta = 0$, de ahí que $F \sim F(q, n-p; 0)$, lo que lleva al resultado deseado.

Observación:

Note que podemos escribir el estadístico F de varias formas equivalentes, a saber:

$$\begin{aligned}
F &= \Big(\frac{n-p}{q}\Big)\frac{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})}{Q(\widehat{\boldsymbol{\beta}})} \\
&= \frac{Q(\widetilde{\boldsymbol{\beta}}) - Q(\widehat{\boldsymbol{\beta}})}{qs^2} \\
&= \frac{(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top (\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})}{qs^2} \\
&\sim F(q, n-p, \delta).
\end{aligned}$$

Hemos notado la relación que existe entre el test de razón de verosimilitudes con el estadístico F para probar hipótesis lineales en el modelo de regresión lineal

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \sim \mathsf{N}_n(\boldsymbol{0}, \sigma^2 \boldsymbol{I}).$$

A continuación exploramos la relación entre el estadístico F con los test de Wald, score y gradiente para hipótesis lineales, del tipo

$$H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}, \qquad \text{versus} \qquad H_1 : \boldsymbol{G}\boldsymbol{\beta} \neq \boldsymbol{g}.$$

Primeramente, note que la matriz de información de Fisher para $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \sigma^2)^\top$, adopta la forma

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\theta}) = \frac{1}{\sigma^2} \begin{pmatrix} \boldsymbol{X}^\top \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{n}{2\sigma^2} \end{pmatrix}.$$

El estadístico de Wald para hipótesis lineales de la forma $H_0 : \boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}$, es dado por

$$\begin{aligned} W &= n(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top (\boldsymbol{G}\{\boldsymbol{\mathcal{F}}(\widehat{\boldsymbol{\beta}})\}^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}) \\ &= \frac{n(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top (\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})}{\widehat{\sigma}^2}. \end{aligned}$$

Mientras que el test score es dado por

$$\begin{aligned} R &= \frac{1}{n}\boldsymbol{U}^\top(\widetilde{\boldsymbol{\beta}})\{\boldsymbol{\mathcal{F}}(\widetilde{\boldsymbol{\beta}})\}^{-1}\boldsymbol{U}(\widetilde{\boldsymbol{\beta}}) \\ &= \frac{1}{n\widetilde{\sigma}^2}(\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}})^\top \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top(\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}). \end{aligned}$$

Como,

$$\begin{aligned}\boldsymbol{X}^\top(\boldsymbol{Y}-\boldsymbol{X}\widetilde{\boldsymbol{\beta}}) &= \boldsymbol{X}^\top(\boldsymbol{Y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}}+\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g}))\\ &= \boldsymbol{X}^\top(\boldsymbol{Y}-\boldsymbol{X}\widehat{\boldsymbol{\beta}})+\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})\\ &= \boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g}).\end{aligned}$$

Así

$$\begin{aligned}&(\boldsymbol{Y}-\boldsymbol{X}\widetilde{\boldsymbol{\beta}})^\top\boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top(\boldsymbol{Y}-\boldsymbol{X}\widetilde{\boldsymbol{\beta}})\\ &\qquad= (\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})^\top\boldsymbol{B}^\top\boldsymbol{X}^\top\boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})\\ &\qquad= (\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})^\top\boldsymbol{B}^\top\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})\\ &\qquad= (\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g}).\end{aligned}$$

Finalmente,

$$R = \frac{(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}}-\boldsymbol{g})}{n\widetilde{\sigma}^2}$$

Por otro lado, el estadístico gradiente es dado por

$$T = \boldsymbol{U}^\top(\widetilde{\boldsymbol{\beta}})(\widehat{\boldsymbol{\beta}} - \widetilde{\boldsymbol{\beta}}) = (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top \boldsymbol{B}^\top \boldsymbol{X}^\top \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \widetilde{\boldsymbol{\beta}})$$

Sabemos que $\widetilde{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}} - \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})$, esto lleva a

$$\begin{aligned} T &= (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top \boldsymbol{B}^\top \boldsymbol{X}^\top \boldsymbol{X}(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}} + \boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})) \\ &= (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top \boldsymbol{B}^\top \boldsymbol{X}^\top \boldsymbol{X}\boldsymbol{B}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}) \\ &= (\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g})^\top (\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g}) \end{aligned}$$

Observación:

Es decir, podemos escribir:

$$
\begin{aligned}
W &= \frac{qn}{n-p} F \\
R &= \left\{ 1 + \left(\frac{n-p}{q} \right) F^{-1} \right\}^{-1} \\
T &= \frac{qs^2}{n-p} F
\end{aligned}
$$

Lo que permite notar que basta usar el estadístico F para probar hipótesis lineales en el modelo de regresión lineal.

Regiones de confianza

Sea $\boldsymbol{\gamma} = \boldsymbol{G}\boldsymbol{\beta}$ un vector q-dimensional. Sabemos que

$$(\boldsymbol{\gamma} - \widehat{\boldsymbol{\gamma}})^\top (\boldsymbol{G}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}(\boldsymbol{\gamma} - \widehat{\boldsymbol{\gamma}}) \leq q s^2 F_{1-\alpha}(q, n-p),$$

luego, puede ser usada como una región de confianza.

En particular, para $\boldsymbol{G} = \boldsymbol{I}$ tenemos

$$R(\boldsymbol{\beta}) = \{\boldsymbol{\beta} : (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})^\top \boldsymbol{X}^\top \boldsymbol{X}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \leq p s^2 F_{1-\alpha}(p, n-p)\}.$$

Además, para el caso en que $\operatorname{rg}(\boldsymbol{G}) = 1$, tenemos un intervalo de confianza para $\gamma = \boldsymbol{a}^\top \boldsymbol{\beta}$, digamos

$$CI(\gamma) = \big[\boldsymbol{a}^\top \widehat{\boldsymbol{\beta}} \mp t_{1-\alpha/2}(n-p)\, s\sqrt{\boldsymbol{a}^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{a}}\big],$$

o bien,

$$R(\boldsymbol{a}^\top \boldsymbol{\beta}) = \{\gamma : (\gamma - \boldsymbol{a}^\top \widehat{\boldsymbol{\beta}})^2 \leq s^2 \boldsymbol{a}^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{a} F_{1-\alpha}(1, n-p)\}.$$