1. Sabemos que $\widehat{\alpha} = \overline{y} - \widehat{\beta}\,\overline{x}$, de este modo

$$e_i = y_i - \widehat{\alpha} - \widehat{\beta}\,x_i = y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x}).$$

En efecto,

$$\sum_{i=1}^n e_i = \sum_{i=1}^n \big(y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x})\big) = \sum_{i=1}^n (y_i - \overline{y}) - \widehat{\beta} \sum_{i=1}^n (x_i - \overline{x}) = 0.$$

Para notar que $\sum_{i=1}^n e_i x_i = 0$, considere

$$\sum_{i=1}^n e_i x_i = \sum_{i=1}^n (y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x}))x_i = \sum_{i=1}^n (y_i - \overline{y})x_i - \widehat{\beta} \sum_{i=1}^n (x_i - \overline{x})x_i.$$

Es fácil notar que

$$\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y}) = \sum_{i=1}^n x_i(y_i - \overline{y}) - \overline{x} \sum_{i=1}^n (y_i - \overline{y}) = \sum_{i=1}^n x_i(y_i - \overline{y}),$$

y análogamente para $\sum_{i=1}^n (x_i - \overline{x})^2 = \sum_{i=1}^n x_i(x_i - \overline{x})$, lo que permite escribir:

$$\sum_{i=1}^n e_i x_i = \sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y}) - \widehat{\beta} \sum_{i=1}^n (x_i - \overline{x})^2.$$

Como $\widehat{\beta} = S_{XY}/S_{XX}$, sigue que el resultado.

Por otro lado,

$$\begin{aligned}\sum_{i=1}^n e_i\,\widehat{y}_i &= \sum_{i=1}^n (y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x}))(\overline{y} + \widehat{\beta}(x_i - \overline{x})) \\ &= \overline{y} \sum_{i=1}^n (y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x})) + \widehat{\beta} \sum_{i=1}^n (y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x}))(x_i - \overline{x}).\end{aligned}$$

Podemos reconocer que $\sum_{i=1}^n (y_i - \overline{y} - \widehat{\beta}(x_i - \overline{x})) = \sum_{i=1}^n e_i = 0$. Es decir, tenemos que:

$$\sum_{i=1}^n e_i\,\widehat{y}_i = \widehat{\beta}\Big[\sum_{i=1}^n (y_i - \overline{y})(x_i - \overline{x}) - \widehat{\beta} \sum_{i=1}^n (x_i - \overline{x})^2\Big] = 0,$$

pues $\widehat{\beta} = S_{XY}/S_{XX}$.

Para otra manera de verificar estas ecuaciones, considere

$$\widehat{\boldsymbol{Y}} = \boldsymbol{H}\boldsymbol{Y}, \qquad \boldsymbol{e} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y},$$

además, $\boldsymbol{HX} = \boldsymbol{X}$, con $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X}^\top$. Para el modelo bajo consideración tenemos que la matriz de diseño es dada por:

$$\boldsymbol{X} = (\boldsymbol{1}, \boldsymbol{x}),$$

donde $\boldsymbol{1} = (1, \dots, 1)^n$, $\boldsymbol{x} = (x_1, \dots, x_n)^\top$. Así desde,

$$\boldsymbol{HX} = \boldsymbol{X}, \quad \Rightarrow \quad \boldsymbol{H}(\boldsymbol{1}, \boldsymbol{x}) = (\boldsymbol{1}, \boldsymbol{x}),$$

es decir, $\boldsymbol{H1} = \boldsymbol{1}$ y $\boldsymbol{Hx} = \boldsymbol{x}$. Luego,

$$\sum_{i=1}^n e_i = \boldsymbol{1}^\top \boldsymbol{e} = \boldsymbol{1}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}.$$

Ahora,

$$(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{1} = \boldsymbol{1} - \boldsymbol{H1} = \boldsymbol{1} - \boldsymbol{1} = \boldsymbol{0},$$

de ahí que $\sum_{i=1}^n e_i = 0$. Por otro lado,

$$\sum_{i=1}^n e_i x_i = \boldsymbol{e}^\top \boldsymbol{x} = \boldsymbol{Y}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{x} = \boldsymbol{Y}^\top(\boldsymbol{x} - \boldsymbol{Hx}),$$

como $\boldsymbol{Hx} = \boldsymbol{x}$, sigue que $\sum_{i=1}^m e_i x_i = 0$.

Finalmente,

$$\sum_{i=1}^n e_i \widehat{Y}_i = \boldsymbol{e}^\top \widehat{\boldsymbol{Y}} = \boldsymbol{Y}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{HY} = \boldsymbol{Y}^\top(\boldsymbol{H} - \boldsymbol{H}^2)\boldsymbol{Y} = 0,$$

pues $\boldsymbol{H}^2 = \boldsymbol{H}$.

2. Tenemos que $\boldsymbol{Y} \sim \mathsf{N}_n(\mu \boldsymbol{1}_n, \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma} = (1-\rho)\boldsymbol{I}_n + \rho \boldsymbol{1}_n \boldsymbol{1}_n^\top$ donde $\rho > -1/(n-1)$. Además,

$$\overline{Y} = \frac{1}{n}\boldsymbol{1}^\top \boldsymbol{Y}, \qquad \sum_{i=1}^n (Y_i - \overline{Y})^2 = \boldsymbol{Y}^\top \boldsymbol{CY}, \qquad \boldsymbol{C} = \boldsymbol{I}_n - \frac{1}{n}\boldsymbol{1}_n \boldsymbol{1}_n^\top.$$

Para mostrar la independencia entre $\overline{Y}$ y $\sum_{i=1}^n (Y_i - \overline{Y})^2$, debemos verificar que $\boldsymbol{C\Sigma 1} = \boldsymbol{0}$. En efecto,

$$\boldsymbol{C\Sigma 1} = \boldsymbol{C}\{(1-\rho)\boldsymbol{I} + \rho \boldsymbol{1}\boldsymbol{1}^\top\}\boldsymbol{1} = (1-\rho)\boldsymbol{C1} + \rho \boldsymbol{C}\boldsymbol{1}\boldsymbol{1}^\top \boldsymbol{1} = \boldsymbol{0}$$

pues $\boldsymbol{C1} = \boldsymbol{0}$.

3.a. La matriz de diseño para este caso es dada por:

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ x_{15,1} & x_{15,2} \end{pmatrix}.$$

3.b. Tenemos que

$$\boldsymbol{X}^\top \boldsymbol{X} = \begin{pmatrix} 15.00 & 374.50 \\ 374.50 & 9492.75 \end{pmatrix}, \qquad \boldsymbol{X}^\top \boldsymbol{Y} = \begin{pmatrix} 6.03 \\ 158.25 \end{pmatrix},$$

luego,

$$\det(\boldsymbol{X}^\top \boldsymbol{X}) = 15 \cdot 9482.75 - 374.5^2 = 1991.$$

De este modo,

$$(\boldsymbol{X}^\top \boldsymbol{X})^{-1} = \frac{1}{1991} \begin{pmatrix} 9492.75 & -374.50 \\ -374.50 & 15.00 \end{pmatrix}.$$

Luego,

$$\begin{aligned} \widehat{\boldsymbol{\beta}} &= \frac{1}{1991} \begin{pmatrix} 9492.75 & -374.50 \\ -374.50 & 15.00 \end{pmatrix} \begin{pmatrix} 6.03 \\ 158.25 \end{pmatrix} = \frac{1}{1991} \begin{pmatrix} 9492.75 \cdot 6.03 - 374.5 \cdot 158.25 \\ -374.5 \cdot 6.03 + 15 \cdot 158.25 \end{pmatrix} \\ &= \frac{1}{1991} \begin{pmatrix} -2083.642 \\ 115.515 \end{pmatrix} = \begin{pmatrix} -1.046531 \\ 0.058019 \end{pmatrix}. \end{aligned}$$

Note que

$$\begin{aligned} Q(\widehat{\boldsymbol{\beta}}) &= \boldsymbol{Y}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{Y} = \boldsymbol{Y}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \boldsymbol{H} \boldsymbol{Y} = \boldsymbol{Y}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \widehat{\boldsymbol{Y}} \\ &= \boldsymbol{Y}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \boldsymbol{X} \widehat{\boldsymbol{\beta}}. \end{aligned}$$

Ahora,

$$\begin{aligned} \boldsymbol{Y}^\top \boldsymbol{X} \widehat{\boldsymbol{\beta}} &= (6.03, 158.25) \begin{pmatrix} -1.046531 \\ 0.058019 \end{pmatrix} = -6.03 \cdot 1.046531 + 158.25 \cdot 0.058019 \\ &= 2.870925, \end{aligned}$$

luego,

$$Q(\widehat{\boldsymbol{\beta}}) = 3.03 - 2.870925 = 0.159075.$$

Finalmente el MLE de σ^2 resulta:

$$\widehat{\sigma}^2 = \frac{0.159075}{15} = 0.010605.$$