1. Note que

$$\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{Y} = \tfrac{1}{2}\boldsymbol{Z}^\top \boldsymbol{B}\boldsymbol{Z}, \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{A}^\top & \boldsymbol{0} \end{pmatrix}$$

De ahí que,

$$\mathsf{E}(\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{Y}) = \tfrac{1}{2}\,\mathsf{E}(\boldsymbol{Z}^\top \boldsymbol{B}\boldsymbol{Z}) = \tfrac{1}{2}\big(\operatorname{tr}\boldsymbol{B}\boldsymbol{\Sigma} + \boldsymbol{\mu}^\top \boldsymbol{B}\boldsymbol{\mu}\big),$$

como

$$\boldsymbol{B}\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{A}^\top & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{\Sigma}_{21} & \boldsymbol{A}\boldsymbol{\Sigma}_{22} \\ \boldsymbol{A}^\top\boldsymbol{\Sigma}_{11} & \boldsymbol{A}^\top\boldsymbol{\Sigma}_{12} \end{pmatrix},$$

$$\boldsymbol{\mu}^\top \boldsymbol{B}\boldsymbol{\mu} = (\boldsymbol{\mu}_1^\top, \boldsymbol{\mu}_2^\top)\begin{pmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{A}^\top & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} = \boldsymbol{\mu}_1^\top \boldsymbol{A}\boldsymbol{\mu}_2 + \boldsymbol{\mu}_2^\top \boldsymbol{A}^\top \boldsymbol{\mu}_1.$$

Luego,

$$\begin{aligned} \mathsf{E}(\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{Y}) &= \tfrac{1}{2}(\operatorname{tr}\boldsymbol{A}^\top\boldsymbol{\Sigma}_{12} + \operatorname{tr}\boldsymbol{A}\boldsymbol{\Sigma}_{21}) + \tfrac{1}{2}(\boldsymbol{\mu}_1^\top \boldsymbol{A}\boldsymbol{\mu}_2 + \boldsymbol{\mu}_2^\top \boldsymbol{A}^\top \boldsymbol{\mu}_1) \\ &= \operatorname{tr}\boldsymbol{A}^\top\boldsymbol{\Sigma}_{12} + \boldsymbol{\mu}_1^\top \boldsymbol{A}\boldsymbol{\mu}_2. \end{aligned}$$

2.a. Tenemos que

$$U_i = (1,-1)\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \boldsymbol{a}^\top \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathsf{N}_1(\boldsymbol{a}^\top\boldsymbol{\mu}, \boldsymbol{a}^\top\boldsymbol{\Sigma}\boldsymbol{a}),$$

donde $\boldsymbol{a} = (1,-1)^\top$. En este caso,

$$\boldsymbol{a}^\top\boldsymbol{\mu} = (1,-1)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mu_1 - \mu_2,$$

$$\boldsymbol{a}^\top\boldsymbol{\Sigma}\boldsymbol{a} = (1,-1)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma_{11} + \sigma_{22} - 2\sigma_{12}.$$

Es decir,

$$U_i \sim \mathsf{N}_1(\mu_1 - \mu_2, \sigma_{11} + \sigma_{22} - 2\sigma_{12}).$$

2.b. Sea $\delta = \mu_1 - \mu_2$, $\omega^2 = \sigma_{11} + \sigma_{22} - 2\sigma_{12}$. Evidentemente,

$$\begin{aligned} \mathsf{P}(|U_i| \le c) &= \mathsf{P}(-c \le U_i \le c) = \mathsf{P}(U_i \le c) - \mathsf{P}(U_i \le -c) \\ &= \mathsf{P}\left(\frac{U_i - \delta}{\omega} \le \frac{c-\delta}{\omega}\right) - \mathsf{P}\left(\frac{U_i - \delta}{\omega} \le \frac{-c-\delta}{\omega}\right) \end{aligned}$$

Como $Z = (U_i - \delta)/\omega \sim \mathsf{N}(0,1)$, sigue que

$$\mathsf{P}(|U_i| \le c) = \Phi\Big(\frac{c-\delta}{\omega}\Big) - \Phi\Big(\frac{-c-\delta}{\omega}\Big).$$

2.c. Sabemos que

$$\mathsf{E}(U_i^2) = \mathsf{var}(U_i) + \mathsf{E}^2(U_i) = \sigma_{11} + \sigma_{22} - 2\sigma_{12} + (\mu_1 - \mu_2)^2,$$

Cuando $\sigma_{12} = 0$, tenemos

$$\mathsf{E}(U_i^2 | \sigma_{12} = 0) = \sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2.$$

Luego,

$$\rho_c = 1 - \frac{\sigma_{11} + \sigma_{22} - 2\sigma_{12} + (\mu_1 - \mu_2)^2}{\sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2} = \frac{2\sigma_{12}}{\sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2}.$$

2.d. Note que,

$$\begin{aligned}
\rho_c &= \frac{\sqrt{\sigma_{11}\sigma_{22}}}{\sqrt{\sigma_{11}\sigma_{22}}} \frac{2\sigma_{12}}{\sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \frac{2\sqrt{\sigma_{11}\sigma_{22}}}{\sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2} \\
&= \rho_{12}\Big\{\frac{\sigma_{11} + \sigma_{22} + (\mu_1 - \mu_2)^2}{2\sqrt{\sigma_{11}\sigma_{22}}}\Big\}^{-1} = \rho_{12}\Big\{\Big[\frac{\mu_1 - \mu_2}{(\sigma_{11}\sigma_{22})^{1/4}}\Big]^2 + \frac{1}{2}\Big(\sqrt{\frac{\sigma_{11}}{\sigma_{22}}} + \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}\Big)\Big\}^{-1} \\
&= \rho_{12}\, C,
\end{aligned}$$

con $C = [a^2 + \frac{1}{2}(b + b^{-1})]^{-1}$, donde $a = (\mu_1 - \mu_2)/(\sigma_{11}\sigma_{22})^{1/4}$ y $b = \sqrt{\sigma_{11}/\sigma_{22}}$.

3. Tenemos,

$$\mathsf{Cov}(\boldsymbol{AX}, \boldsymbol{X}^\top\boldsymbol{BX}) = \mathsf{E}\{[\boldsymbol{AX} - \mathsf{E}(\boldsymbol{AX})][\boldsymbol{X}^\top\boldsymbol{BX} - \mathsf{E}(\boldsymbol{X}^\top\boldsymbol{BX})]^\top\}.$$

Sabemos que

$$\mathsf{E}(\boldsymbol{AX}) = \boldsymbol{A\mu}, \qquad \mathsf{E}(\boldsymbol{X}^\top\boldsymbol{BX}) = \operatorname{tr}(\boldsymbol{B\Sigma}) + \boldsymbol{\mu}^\top\boldsymbol{B\mu}.$$

Luego,

$$\mathsf{Cov}(\boldsymbol{AX}, \boldsymbol{X}^\top\boldsymbol{BX}) = \mathsf{E}\{\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{\mu})[\boldsymbol{X}^\top\boldsymbol{BX} - \boldsymbol{\mu}^\top\boldsymbol{B\mu} - \operatorname{tr}(\boldsymbol{B\Sigma})]^\top\}.$$

Es fácil notar que,

$$(\boldsymbol{X} - \boldsymbol{\mu})^\top\boldsymbol{B}(\boldsymbol{X} - \boldsymbol{\mu}) + 2(\boldsymbol{X} - \boldsymbol{\mu})^\top\boldsymbol{B\mu} = \boldsymbol{X}^\top\boldsymbol{BX} - \boldsymbol{\mu}^\top\boldsymbol{B\mu}$$

Es decir,

$$\mathsf{Cov}(\boldsymbol{AX}, \boldsymbol{X}^\top\boldsymbol{BX}) = \mathsf{E}\{\boldsymbol{AZ}[\boldsymbol{Z}^\top\boldsymbol{BZ} + 2\boldsymbol{Z}^\top\boldsymbol{B\mu} - \operatorname{tr}(\boldsymbol{B\Sigma})]^\top\},$$

con $\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{\mu} \sim \mathsf{N}_p(\boldsymbol{0}, \boldsymbol{\Sigma})$. De este modo,

$$\mathsf{Cov}(\boldsymbol{AX}, \boldsymbol{X}^\top\boldsymbol{BX}) = \boldsymbol{A}\,\mathsf{E}(\boldsymbol{ZZ}^\top\boldsymbol{BZ}) + 2\boldsymbol{A}\,\mathsf{E}(\boldsymbol{ZZ}^\top)\boldsymbol{B\mu} - \boldsymbol{A}\,\mathsf{E}(\boldsymbol{Z})\operatorname{tr}(\boldsymbol{B\Sigma}).$$

Es fácil mostrar que

$$\mathsf{E}(\boldsymbol{ZZ}^\top\boldsymbol{BZ}) = \Big(\sum_{i=1}^p\sum_{j=1}^p b_{ij}\,\mathsf{E}(Z_iZ_jZ_k)\Big) = \boldsymbol{0}.$$

Además, $\mathsf{E}(\boldsymbol{Z}) = \boldsymbol{0}$ y $\mathsf{E}(\boldsymbol{ZZ}^\top) = \boldsymbol{\Sigma}$, de donde sigue que

$$\mathsf{Cov}(\boldsymbol{AX}, \boldsymbol{X}^\top\boldsymbol{BX}) = 2\boldsymbol{A\Sigma B\mu},$$

como deseado.