1. Tenemos que
$$Y_{ij} = \mu_{ij} + \epsilon_{ij}, \qquad i = 1,2,3; j = 1,\dots,n.$$

a) Aquí,
$$\boldsymbol{\mu} = \boldsymbol{X\beta} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{1}_m & \mathbf{1}_m \\ \mathbf{1}_m & -\mathbf{1}_m \end{pmatrix} \begin{pmatrix} \tau \\ \theta \end{pmatrix},$$
luego
$$\boldsymbol{X} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{1}_m & \mathbf{1}_m \\ \mathbf{1}_m & -\mathbf{1}_m \end{pmatrix}.$$
Además, $\boldsymbol{Y} = (\boldsymbol{Y}_1^\top, \boldsymbol{Y}_2^\top, \boldsymbol{Y}_3^\top)^\top$, con $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2}, \dots, Y_{im})^\top$.

b) Note que
$$\boldsymbol{X}^\top\boldsymbol{X} = \begin{pmatrix} \mathbf{1}_m^\top & \mathbf{1}_m^\top & \mathbf{1}_m^\top \\ \mathbf{0} & \mathbf{1}_m^\top & -\mathbf{1}_m \end{pmatrix} \begin{pmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{1}_m & \mathbf{1}_m \\ \mathbf{1}_m & -\mathbf{1}_m \end{pmatrix} = \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix} = m\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$
y
$$\boldsymbol{X}^\top\boldsymbol{Y} = \begin{pmatrix} \mathbf{1}_m^\top & \mathbf{1}_m^\top & \mathbf{1}_m^\top \\ \mathbf{0} & \mathbf{1}_m^\top & -\mathbf{1}_m \end{pmatrix} \begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \\ \boldsymbol{Y}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m^\top\boldsymbol{Y}_1 + \mathbf{1}_m^\top\boldsymbol{Y}_2 + \mathbf{1}_m^\top\boldsymbol{Y}_3 \\ \mathbf{1}_m^\top\boldsymbol{Y}_2 - \mathbf{1}_m^\top\boldsymbol{Y}_3 \end{pmatrix} = m\begin{pmatrix} \overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3 \\ \overline{Y}_2 - \overline{Y}_3 \end{pmatrix}.$$
Es decir, $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y} = (\widehat{\tau}, \widehat{\theta})^\top$, con
$$\widehat{\tau} = \frac{1}{3}(\overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3), \qquad \widehat{\theta} = \frac{1}{2}(\overline{Y}_2 - \overline{Y}_3).$$
Es fácil notar que $\widehat{\tau}$ y $\widehat{\theta}$ son independientes con
$$\mathsf{var}(\widehat{\theta}) = \frac{\sigma^2}{2m}.$$

c) La hipótesis $H_0 : \theta = 0$, puede ser escrita como $H_0 : \boldsymbol{G\beta} = \boldsymbol{g}$ con $\boldsymbol{G} = (0,1)$ y $g = 0$. En cuyo caso, $q = 1$. Así,
$$\boldsymbol{G}\widehat{\boldsymbol{\beta}} - \boldsymbol{g} = (0,1)\begin{pmatrix} \widehat{\tau} \\ \widehat{\theta} \end{pmatrix} = \widehat{\theta},$$
y
$$\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top = \frac{1}{m}(0,1)\begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2m}.$$
Por otro lado, note que
$$\widehat{\boldsymbol{\beta}}^\top\boldsymbol{X}^\top\boldsymbol{X}\widehat{\boldsymbol{\beta}} = m(\widehat{\tau}, \widehat{\theta})\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \widehat{\tau} \\ \widehat{\theta} \end{pmatrix} = 3m\widehat{\tau}^2 + 2m\widehat{\theta}^2.$$

De este modo,

$$\begin{aligned}
Q(\widehat{\boldsymbol{\beta}}) &= \boldsymbol{Y}^\top\boldsymbol{Y} - \widehat{\boldsymbol{\beta}}^\top\boldsymbol{X}^\top\boldsymbol{X}\widehat{\boldsymbol{\beta}} \\
&= \boldsymbol{Y}_1^\top\boldsymbol{Y}_1 + \boldsymbol{Y}_2^\top\boldsymbol{Y}_2 + \boldsymbol{Y}_3^\top\boldsymbol{Y}_3 - 3m\overline{Y}^2 - \frac{m}{2}(\overline{Y}_2 - \overline{Y}_3)^2 \\
&= \sum_{i=1}^3\sum_{j=1}^m Y_{ij}^2 - 3m\overline{Y}^2 - \frac{m}{2}(\overline{Y}_2 - \overline{Y}_3)^2 \\
&= \sum_{i=1}^3\sum_{j=1}^m (Y_{ij} - \overline{Y})^2 - \frac{m}{2}(\overline{Y}_2 - \overline{Y}_3)^2.
\end{aligned}$$

Entonces, el estadístico F adopta la forma:

$$F = \frac{2m\widehat{\theta}^2}{s^2} \sim F(1, 3m-2),$$

con

$$s^2 = \frac{1}{3m-2}\Big\{\sum_{i=1}^3\sum_{j=1}^m (Y_{ij} - \overline{Y})^2 - \frac{m}{2}(\overline{Y}_2 - \overline{Y}_3)^2\Big\},$$

y se rechaza $H_0 : \theta = 0$, si

$$F > F_{1-\alpha}(1, 3m-2).$$

2. Para mostrar que $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{B}\boldsymbol{G}$ es idempotente, basta mostrar que $\boldsymbol{B}\boldsymbol{G}$ es idempotente. En efecto,

$$\boldsymbol{G}\boldsymbol{B}\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}\boldsymbol{G} = \boldsymbol{G},$$

de ahí que $\boldsymbol{B}\boldsymbol{G}$ es idempotente. Además,

$$\operatorname{rg}(\boldsymbol{B}\boldsymbol{G}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{G}) = \operatorname{tr}(\boldsymbol{G}\boldsymbol{B}) = q,$$

así $\operatorname{rg}(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{G}) = \operatorname{tr}(\boldsymbol{A}) = p - q = r$, lo que muestra la parte *(i)*.

Para notar la parte *(ii)*, sea $\boldsymbol{C} = \boldsymbol{X}\boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top$, luego

$$\boldsymbol{C}^2 = \boldsymbol{X}\boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{X}\boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top = \boldsymbol{X}\boldsymbol{A}^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top,$$

como $\boldsymbol{A}$ es idempotente, sigue que $\boldsymbol{C}^2 = \boldsymbol{C}$. Esto permite escribir

$$\operatorname{rg}(\boldsymbol{C}) = \operatorname{tr}(\boldsymbol{C}) = \operatorname{tr}(\boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{X}) = \operatorname{tr}(\boldsymbol{A}) = r.$$

Por otro lado,

$$\boldsymbol{C}^\top = (\boldsymbol{X}\boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top)^\top = \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{A}^\top\boldsymbol{X}^\top.$$

Tenemos que,

$$\boldsymbol{A}^\top = \boldsymbol{I} - \boldsymbol{G}^\top\boldsymbol{B}^\top = \boldsymbol{I} - \boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1},$$

luego premultiplicando por $(\boldsymbol{X}^\top\boldsymbol{X})^{-1}$ y factorizando lleva a,

$$\begin{aligned}
(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{A}^\top &= (\boldsymbol{X}^\top\boldsymbol{X})^{-1} - (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top(\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{G}^\top)^{-1}\boldsymbol{G}(\boldsymbol{X}^\top\boldsymbol{X})^{-1} \\
&= (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{G})(\boldsymbol{X}^\top\boldsymbol{X})^{-1} = \boldsymbol{A}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}, \qquad \textbf{(1)}
\end{aligned}$$

así

$$\boldsymbol{C}^\top = \boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{A}^\top \boldsymbol{X} = \boldsymbol{X}\boldsymbol{A}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X} = \boldsymbol{C}.$$

Finalmente, Ecuación **(1)** permite notar la primera igualdad de la parte *(iii)*. Ahora, por **(1)**, sigue que

$$\boldsymbol{A}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{A}^\top = \boldsymbol{A}^2(\boldsymbol{X}^\top \boldsymbol{X})^{-1} = \boldsymbol{A}(\boldsymbol{X}^\top \boldsymbol{X})^{-1},$$

pues $\boldsymbol{A}$ es idempotente y esto termina la prueba.