1. Considere el modelo linealizado

$$\log V_i = \log\alpha + \beta_1 \log K_i + \beta_2 \log L_i + \log\eta_i, \qquad i = 1,\dots,n.$$

Sea $Y_i = \log V_i$, $x_i = \log K_i$, $z_i = \log L_i$, $\epsilon_i = \log\eta_i$ y $\theta = \log\alpha$. Entonces, podemos escribir el modelo en la forma lineal, $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ con

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \qquad \boldsymbol{X} = \begin{pmatrix} 1 & x_1 & z_1 \\ \vdots & \vdots & \vdots \\ 1 & x_n & z_n \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \theta \\ \beta_1 \\ \beta_2 \end{pmatrix}.$$

La restricción $\beta_1 + \beta_2 = 1$ puede ser escrita como $\boldsymbol{G}\boldsymbol{\beta} = \boldsymbol{g}$ con

$$\boldsymbol{G} = (0,1,1), \qquad g = 1.$$

En este caso, $\boldsymbol{G} = (\boldsymbol{G}_r, \boldsymbol{G}_q)$, $\boldsymbol{\beta} = (\boldsymbol{\beta}_r^\top, \boldsymbol{\beta}_q^\top)^\top$ con

$$\boldsymbol{G}_r = (0,1), \qquad \boldsymbol{G}_q = (1), \qquad \boldsymbol{\beta}_r = (\theta, \beta_1)^\top, \qquad \boldsymbol{\beta}_q = \beta_2.$$

Es decir,

$$\boldsymbol{X}_r = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \qquad \boldsymbol{X}_q = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

De este modo, obtenemos

$$\boldsymbol{Y}_R = \boldsymbol{Y} - \boldsymbol{X}_q \boldsymbol{G}_q^{-1} \boldsymbol{g} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} - \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} Y_1 - z_1 \\ \vdots \\ y_n - z_n \end{pmatrix},$$

$$\boldsymbol{X}_R = \boldsymbol{X}_r - \boldsymbol{X}_q \boldsymbol{G}_q^{-1} \boldsymbol{G}_r = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} - \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} (0,1) = \begin{pmatrix} 1 & x_1 - z_1 \\ \vdots & \vdots \\ 1 & x_n - z_n \end{pmatrix} = \begin{pmatrix} 1 & D_1 \\ \vdots & \vdots \\ 1 & D_n \end{pmatrix},$$

donde $D_i = x_i - z_i$, para $i = 1,\dots,n$. Ahora

$$\boldsymbol{X}_R^\top \boldsymbol{X}_R = \begin{pmatrix} 1 & \dots & 1 \\ D_1 & \dots & D_n \end{pmatrix} \begin{pmatrix} 1 & D_1 \\ \vdots & \vdots \\ 1 & D_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n D_i \\ \sum_{i=1}^n D_i & \sum_{i=1}^n D_i^2 \end{pmatrix} = n \begin{pmatrix} 1 & \overline{D} \\ \overline{D} & \overline{D}^2 + S_D/n \end{pmatrix},$$

donde $S_D = \sum_{i=1}^n (D_i - \overline{D})^2$. De este modo

$$|\boldsymbol{X}_R^\top \boldsymbol{X}_R| = n^2(\overline{D}^2 + S_D/n - \overline{D}^2) = nS_D.$$

Por tanto, obtenemos

$$(\boldsymbol{X}_R^\top \boldsymbol{X}_R)^{-1} = \frac{1}{S_D} \begin{pmatrix} \overline{D}^2 + S_D/n & -\overline{D} \\ -\overline{D} & 1 \end{pmatrix}.$$

Mientras que,

$$\boldsymbol{X}_R^\top \boldsymbol{Y}_R = \begin{pmatrix} 1 & \dots & 1 \\ D_1 & \dots & D_n \end{pmatrix} \begin{pmatrix} Y_1 - z_1 \\ \vdots \\ Y_n - z_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n (Y_i - z_i) \\ \sum_{i=1}^n D_i(Y_i - z_i) \end{pmatrix}.$$

Luego, obtenemos que $\widetilde{\beta}_1$ es dado por

$$\begin{aligned} \widetilde{\beta}_1 &= \frac{1}{S_D}\Big\{ \sum_{i=1}^n D_i(Y_i - z_i) - \overline{D} \sum_{i=1}^n (Y_i - z_i) \Big\} \\ &= \frac{\sum_{i=1}^n (D_i - \overline{D})(Y_i - \overline{Y} - (z_i - \overline{z}))}{\sum_{i=1}^n (D_i - \overline{D})^2}. \end{aligned}$$

Finalmente, tenemos que:

$$\widetilde{\beta}_2 = \boldsymbol{G}_q^{-1}(\boldsymbol{g} - \boldsymbol{G}_r \widetilde{\boldsymbol{\beta}}_r) = 1 - (0,1) \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\beta}_1 \end{pmatrix} = 1 - \widetilde{\beta}_1.$$

2. Note que $b^\star$ es un estimador lineal. En efecto, podemos escribir

$$b^\star = \frac{1}{n-1} \boldsymbol{a}^\top \boldsymbol{Y},$$

donde $\boldsymbol{a} = (a_1, a_2, \dots, a_n)^\top$, con

$$\begin{aligned} a_1 &= \frac{1}{\triangle x_2}, \\ a_i &= \frac{1}{\triangle x_i} - \frac{1}{\triangle x_{i-1}}, \qquad i = 2, \dots, n-1, \\ a_n &= \frac{1}{\triangle x_n}. \end{aligned}$$

Podemos escribir,

$$\begin{aligned} b^\star &= \frac{1}{n-1} \sum_{i=2}^n \frac{\alpha + \theta z_i + \epsilon_i - \alpha - \theta z_{i-1} - \epsilon_{i-1}}{\triangle x_i} \\ &= \frac{1}{n-1} \sum_{i=2}^n \frac{\theta(z_i - z_{i-1}) + (\epsilon_i - \epsilon_{i-1})}{\triangle x_i}. \end{aligned}$$

Además, $z_i - z_{i-1} = x_i - \overline{x} - x_{i-1} + \overline{x} = \triangle x_i$, luego

$$b^\star = \frac{1}{n-1}\Big\{ (n-2+1)\theta + \sum_{i=2}^n \frac{\epsilon_i - \epsilon_{i-1}}{\triangle x_i} \Big\} = \theta + \frac{1}{n-1} \sum_{i=2}^n \frac{\epsilon_i - \epsilon_{i-1}}{\triangle x_i}.$$

De esta manera,

$$\mathsf{E}(b^\star) = \theta + \frac{1}{n-1} \sum_{i=2}^n \frac{\mathsf{E}(\epsilon_i) - \mathsf{E}(\epsilon_{i-1})}{\triangle x_i} = \theta.$$

Por tanto, $b^\star$ es un estimador lineal e insesgado. Sin embargo, por el Teorema de Gauss-Markov, tenemos que $b^\star$ **no** es BLUE.

3. Tenemos
$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \phi\boldsymbol{Z} + \boldsymbol{\epsilon}.$$
De este modo, pre-multiplicando por $(\boldsymbol{I} - \boldsymbol{H})$ sigue que:
$$(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{X}\boldsymbol{\beta} + \phi(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z} + (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{\epsilon},$$
como $(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{X} = \boldsymbol{0}$ y $\boldsymbol{e} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$ obtenemos
$$\boldsymbol{e} = \phi(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z} + \boldsymbol{u}, \qquad \boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z}.$$
Note que
$$\boldsymbol{u} \sim \mathsf{N}_n(\boldsymbol{0}, \sigma^2(\boldsymbol{I} - \boldsymbol{H})).$$
De este modo,
$$\widehat{\phi} = (\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{W})^{-1}\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{e},$$
con $\boldsymbol{W} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z}$. Usando que $(\boldsymbol{I} - \boldsymbol{H})^+ = \boldsymbol{I} - \boldsymbol{H}$, tenemos
$$\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{W} = \boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{W} = \boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})^\top(\boldsymbol{I} - \boldsymbol{H})^2\boldsymbol{Z} = \boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z}.$$
Análogamente,
$$\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{e} = \boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})^\top(\boldsymbol{I} - \boldsymbol{H})^2\boldsymbol{Y} = \boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}.$$
Lo que lleva al estimador de ϕ,
$$\widehat{\phi} = \frac{\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{e}}{\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{W}} = \frac{\boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}}{\boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z}},$$
con varianza
$$\mathsf{var}(\widehat{\phi}) = \frac{\sigma^2}{\boldsymbol{W}^\top(\boldsymbol{I} - \boldsymbol{H})^+\boldsymbol{W}} = \frac{\sigma^2}{\boldsymbol{Z}^\top(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Z}}.$$

4.a. El modelo $\mathcal{M}_k$ puede ser escrito como
$$\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_1 \\ \boldsymbol{\epsilon}_2 \end{pmatrix},$$
con $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1^\top, \boldsymbol{\epsilon}_2^\top)^\top \sim \mathsf{N}_n(\boldsymbol{0}, \sigma^2\boldsymbol{I}_n)$. Luego, el estimador ML para $\boldsymbol{\beta}$ es dado por
$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\boldsymbol{\beta}}_2 \end{pmatrix} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{Y},$$
donde
$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix}.$$
Ahora, tenemos que
$$\boldsymbol{X}^\top\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1^\top & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1^\top\boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2^\top\boldsymbol{X}_2 \end{pmatrix},$$

y

$$\boldsymbol{X}^\top \boldsymbol{Y} = \begin{pmatrix} \boldsymbol{X}_1^\top & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}_2^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1^\top \boldsymbol{Y}_1 \\ \boldsymbol{X}_2^\top \boldsymbol{Y}_2 \end{pmatrix}.$$

De este modo,

$$\widehat{\boldsymbol{\beta}}_1 = (\boldsymbol{X}_1^\top \boldsymbol{X}_1)^{-1} \boldsymbol{X}_1^\top \boldsymbol{Y}_1, \qquad \widehat{\boldsymbol{\beta}}_2 = (\boldsymbol{X}_2^\top \boldsymbol{X}_2)^{-1} \boldsymbol{X}_2^\top \boldsymbol{Y}_2.$$

Mientras que el estimador ML para σ^2 adopta la forma

$$\begin{aligned} \widehat{\sigma}^2 &= \frac{1}{n}(\boldsymbol{Y}^\top \boldsymbol{Y} - \widehat{\boldsymbol{\beta}}^\top \boldsymbol{X}^\top \boldsymbol{Y}) = \frac{1}{n}(\boldsymbol{Y}_1^\top \boldsymbol{Y}_1 + \boldsymbol{Y}_2^\top \boldsymbol{Y}_2 - \widehat{\boldsymbol{\beta}}_1^\top \boldsymbol{X}_1^\top \boldsymbol{Y}_1 - \widehat{\boldsymbol{\beta}}_2^\top \boldsymbol{X}_2^\top \boldsymbol{Y}_2) \\ &= \frac{1}{n}(\boldsymbol{Y}_1^\top \boldsymbol{Y}_1 - \widehat{\boldsymbol{\beta}}_1^\top \boldsymbol{X}_1^\top \boldsymbol{Y}_1 + \boldsymbol{Y}_2^\top \boldsymbol{Y}_2 - \widehat{\boldsymbol{\beta}}_2^\top \boldsymbol{X}_2^\top \boldsymbol{Y}_2) = \frac{1}{n}\{Q_1(\widehat{\boldsymbol{\beta}}_1) + Q_2(\widehat{\boldsymbol{\beta}}_2)\}, \end{aligned}$$

donde $Q_i(\widehat{\boldsymbol{\beta}}_i) = \boldsymbol{Y}_i^\top \boldsymbol{Y}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_i^\top \boldsymbol{Y}_i = \|\boldsymbol{Y}_i - \boldsymbol{X}_i \widehat{\boldsymbol{\beta}}_i\|^2$, para $i = 1, 2$.

4.b. Note que la función de log-verosimilitud para el modelo $\mathcal{M}_k$, es dada por

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\|\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}\|^2 \\ &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2} Q(\boldsymbol{\beta}). \end{aligned}$$

De este modo, evaluando $\ell(\boldsymbol{\theta})$ en $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\beta}}_1^\top, \widehat{\boldsymbol{\beta}}_2^\top, \widehat{\sigma}^2)^\top$, sigue que

$$\begin{aligned} \ell(\widehat{\boldsymbol{\theta}}) &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \widehat{\sigma}^2 - \frac{1}{2\widehat{\sigma}^2} Q(\widehat{\boldsymbol{\beta}}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log Q(\widehat{\boldsymbol{\beta}})/n - \frac{n}{2} \\ &= -\frac{n}{2}(1 + \log 2\pi) - \frac{n}{2}\log Q(\widehat{\boldsymbol{\beta}}) + \frac{n}{2}\log n, \end{aligned}$$

sabemos que

$$Q(\widehat{\boldsymbol{\beta}}) = Q_1(\widehat{\boldsymbol{\beta}}_1) + Q_2(\widehat{\boldsymbol{\beta}}_2).$$

Luego, obtenemos que el criterio de información de Schwarz es dado por:

$$\begin{aligned} SIC &= -2\,\ell(\widehat{\boldsymbol{\theta}}) + (2p+1)\log n \\ &= n\log(1 + 2\pi) + n\log(Q_1(\widehat{\boldsymbol{\beta}}_1) + Q_2(\widehat{\boldsymbol{\beta}}_2)) - n\log n + (2p+1)\log n \\ &= n\log(1 + 2\pi) + n\log(Q_1(\widehat{\boldsymbol{\beta}}_1) + Q_2(\widehat{\boldsymbol{\beta}}_2)) + (2p + 1 - n)\log n \end{aligned}$$