1.3 傅立叶级数展开

傅立叶级数展开的广泛应用和基本思想：见 ppt 和视频，
我们不做严谨的数学讨论，直接给出一组最常见的正交完备基

$$1, \sin 2\pi kx/L, \cos 2\pi px/L$$

Based on the following orthogonal functions (please proof the following orthogonal equations by yourself[4].), we can obtain Fourier series from Fourier conjecture [5].

$$\begin{aligned}
&\int_{x_0}^{x_0+L} \sin\left(\tfrac{2\pi kx}{L}\right)\cos\left(\tfrac{2\pi px}{L}\right)dx = 0 \quad \text{for all } k \text{ and } p,\\
&\int_{x_0}^{x_0+L} \cos\left(\tfrac{2\pi kx}{L}\right)\cos\left(\tfrac{2\pi px}{L}\right)dx = \begin{cases} L & \text{for } k=p=0,\\ \frac{1}{2}L & \text{for } k=p>0,\\ 0 & \text{for } k\neq p,\end{cases}\\
&\int_{x_0}^{x_0+L} \sin\left(\tfrac{2\pi kx}{L}\right)\sin\left(\tfrac{2\pi px}{L}\right)dx = \begin{cases} 0 & \text{for } k=p=0,\\ \frac{1}{2}L & \text{for } k=p>0,\\ 0 & \text{for } k\neq p\end{cases}
\end{aligned} \tag{1.8}$$

x_0 is arbitrary but is often taken as 0 or $-L/2$. We choose $x_0 = -L/2$ in most cases.

Suppose that a function $f(x)$ is periodic in the interval $[-\frac{L}{2}, \frac{L}{2}]$ – with period L. (Alternatively, the function may be only defined in this interval.), namely,

$$f(L+x) = f(x). \tag{1.9}$$

With certain “mild” conditions – that is, f must be piecewise continuous, periodic with period

[4]证明正交性的提示：

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \ (1)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \ (2)$$

(1)+(2) 除以 2:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \ (3)$$
$$\int_{-\pi}^{\pi}(\cos nx)(\cos kx)dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos((n-k)x)+\cos((n+k)x)]dx$$
$$= \frac{1}{2}\left[\frac{1}{n-k}\sin((n-k)x)+\frac{1}{n+k}\sin((n+k)x)\right]\Bigg|_{-\pi}^{\pi} = 0$$

[5]We only explicitly show the conditions of orthogonal functional basis. The completeness condition is not discussed here.

L, and (Riemann) integrable – f can be decomposed into a *Fourier series*

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2\pi}{L}kx\right)+b_k\sin\left(\frac{2\pi}{L}kx\right)\right],\text{ with}$$

$$\begin{aligned} a_k&=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi}{L}kx\right)dx \text{ for } k\geq 0\\ b_k&=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi}{L}kx\right)dx \text{ for } k>0. \end{aligned} \tag{1.10}$$

a_0 can be derived by just considering the functional scalar product of $f(x)$ with the constant identity function $g(x)=1$; that is,

$$\begin{aligned} \langle g\mid f\rangle&=\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx\\ =\int_{-\frac{L}{2}}^{\frac{L}{2}}\left\{\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi}{L}nx\right)+b_n\sin\left(\frac{2\pi}{L}nx\right)\right]\right\}dx&=a_0\frac{L}{2}, \end{aligned} \tag{1.11}$$

and hence

$$a_0=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \tag{1.12}$$

In a very similar manner, the other coefficients can be computed by considering $\langle\cos\left(\frac{2\pi}{L}kx\right)\mid f(x)\rangle$ $\langle\sin\left(\frac{2\pi}{L}kx\right)\mid f(x)\rangle$ and exploiting the *orthogonality relations for sines and cosines*)

$$\int_{-\frac{L}{2}}^{\frac{L}{2}}\cos\left(\frac{2\pi}{L}kx\right)f(x)dx=a_k\int_{-\frac{L}{2}}^{\frac{L}{2}}\cos\left(\frac{2\pi}{L}kx\right)\cos\left(\frac{2\pi}{L}kx\right)dx=\frac{L}{2}a_k \tag{1.13}$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}}\sin\left(\frac{2\pi}{L}kx\right)f(x)dx=a_k\int_{-\frac{L}{2}}^{\frac{L}{2}}\sin\left(\frac{2\pi}{L}kx\right)\sin\left(\frac{2\pi}{L}kx\right)dx=\frac{L}{2}b_k \tag{1.14}$$

例题 1.1 奇函数的傅立叶级数展开 (Compute the Fourier series of odd function example)

$$f(t)=\begin{cases}-1 & \text{for } -\frac{1}{2}T\leq t<0\\ +1 & \text{for } 0\leq t<\frac{1}{2}T\end{cases} \tag{1.15}$$

解 傅立叶级数展开技巧：首先观察奇偶性 This function is an odd function and so the series will contain only sine terms

$$\begin{aligned} b_k&=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi kt}{T}\right)dt\\ &=\frac{4}{T}\int_{0}^{T/2}\sin\left(\frac{2\pi kt}{T}\right)dt\\ &=\frac{2}{\pi k}\left[1-(-1)^k\right] \end{aligned} \tag{1.16}$$

$$f(t)=\frac{4}{\pi}\left(\sin\omega t+\frac{\sin 3\omega t}{3}+\frac{\sin 5\omega t}{5}+\cdots\right) \tag{1.17}$$

Here, $\omega\equiv 2\pi/T$ is the so-called angular frequency.

1.3.1 从勾股定理推广到 Parseval’s theorem

Parseval’s theorem:

$$\frac{1}{L}\int_{x_0}^{x_0+L}|f(x)|^2dx \;=\left(\frac{1}{2}a_0\right)^2+\frac{1}{2}\sum_{k=1}^{\infty}(a_k^2+b_k^2) \tag{1.18}$$

例题 1.2 级数求和

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

解 直接利用上面的级数展开表达式和推广的勾股定理，选择 $f(x)$ 为 (1.15)，利用其傅立叶级数展开的结果，

$$\frac{1}{L}\int_{t_0}^{t_0+L}|f(t)|^2dt=\frac{1}{2}\sum_{k=1}^{\infty}(b_k^2) \tag{1.19}$$

$$L=T$$

$$|f(t)|^2=1$$

$$b_k=\frac{4}{(2k-1)\pi}$$

带入得到

$$1=\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{4}{(2k-1)\pi}\right)^2$$

得到级数求和

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$$

例题 1.3 偶函数的傅立叶级数展开 (Compute the Fourier series of even function example)

$$f(x)=|x|=\begin{cases}-x, & \text{for } -\pi\le x<0,\\ +x, & \text{for } 0\le x\le\pi.\end{cases}$$

解 傅立叶级数展开技巧：首先观察奇偶性 First observe that $L=2\pi$, and that $f(x)=f(-x)$; that is, f is an *even* function of x; hence $b_n=0$, and the coefficients a_n can be obtained by considering only the integration between 0 and π.

For $n=0$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}dx f(x)=\frac{2}{\pi}\int_{0}^{\pi}x dx=\pi.$$

For $n>0$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx=\frac{2}{\pi}\int_{0}^{\pi}x\cos(nx)dx= \\
&= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}x\Big|_0^{\pi}-\int_0^{\pi}\frac{\sin(nx)}{n}dx\right]=\frac{2}{\pi}\frac{\cos(nx)}{n^2}\Big|_0^{\pi}= \\
&= \frac{2}{\pi}\frac{\cos(n\pi)-1}{n^2}=-\frac{4}{\pi n^2}\sin^2\frac{n\pi}{2}=\begin{cases}0 & \text{for even } n\\ -\dfrac{4}{\pi n^2} & \text{for odd } n\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x) &= \frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{9}+\frac{\cos 5x}{25}+\cdots\right)= \\
&= \frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2}.
\end{aligned}$$

思考题

1. 类似奇函数展开得到级数求和的方法，利用上面偶函数的傅立叶级数展开求级数之和

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

2. 从勾股的角度类比思考傅立叶级数展开的思想

学而时习之

1. 将 $f(x)=x$ 在 $-\pi<x\le\pi$ 做傅立叶级数展开. 并证明

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

2. 证明

$$\int_{x_0}^{x_0+L}\cos\left(\frac{2\pi kx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx=\begin{cases}L & \text{for } k=p=0\\ \frac{1}{2}L & \text{for } k=p>0\\ 0 & \text{for } k\neq p\end{cases}$$

3. 证明

$$\frac{1}{2\pi}\int_0^{2\pi}e^{-im\phi}e^{in\phi}\mathrm{d}\phi=\delta_{mn}$$

4. 计算

$$\oint_{|z|=2}\frac{dz}{z^2+z}$$

课程预告

- 傅立叶级数展开和傅立叶变换
- 课前十分钟四位同学到黑板上不看讲义计算，如果有同学觉得例题简单的话，可以做一道上面的习题代替我指定的例题——
  苏润梓，孙中一：对

$$f(t)=\begin{cases}-1 & \text{for } -\frac{1}{2}T\le t<0\\ +1 & \text{for } 0\le t<\frac{1}{2}T\end{cases}\tag{1.20}$$

做傅立叶级数展开

谭雅凝，汤霆锋：做傅立叶级数展开

$$f(x)=|x|=\begin{cases}-x, & \text{for } -\pi\le x<0,\\ +x, & \text{for } 0\le x\le\pi.\end{cases}$$