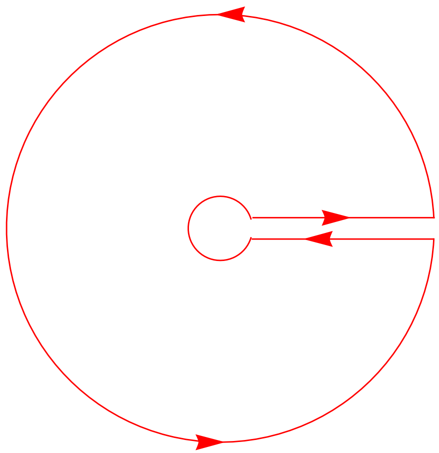
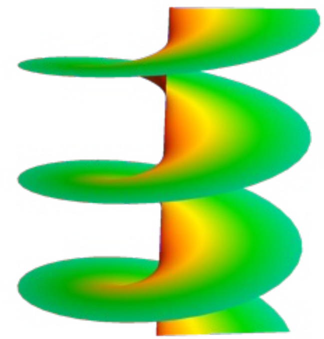
数学物理方法

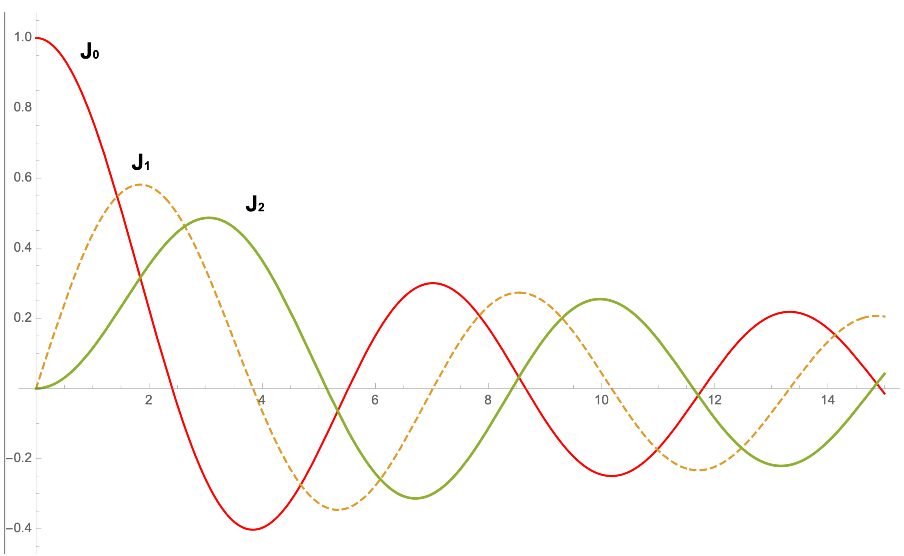


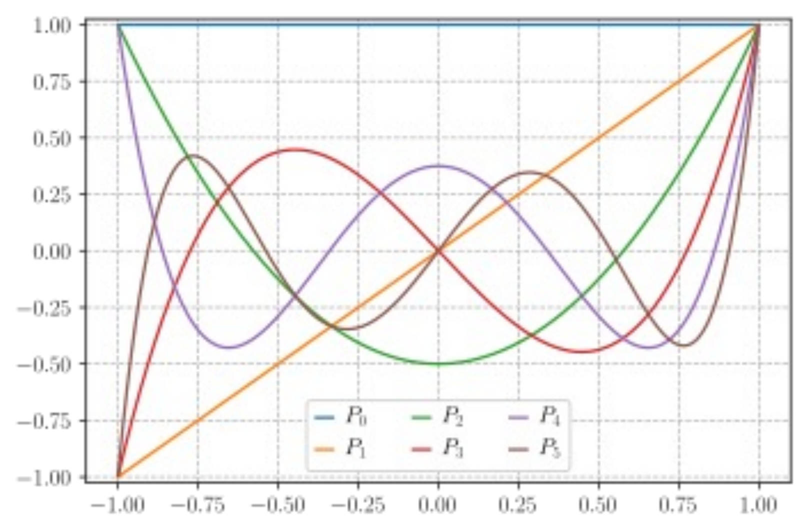


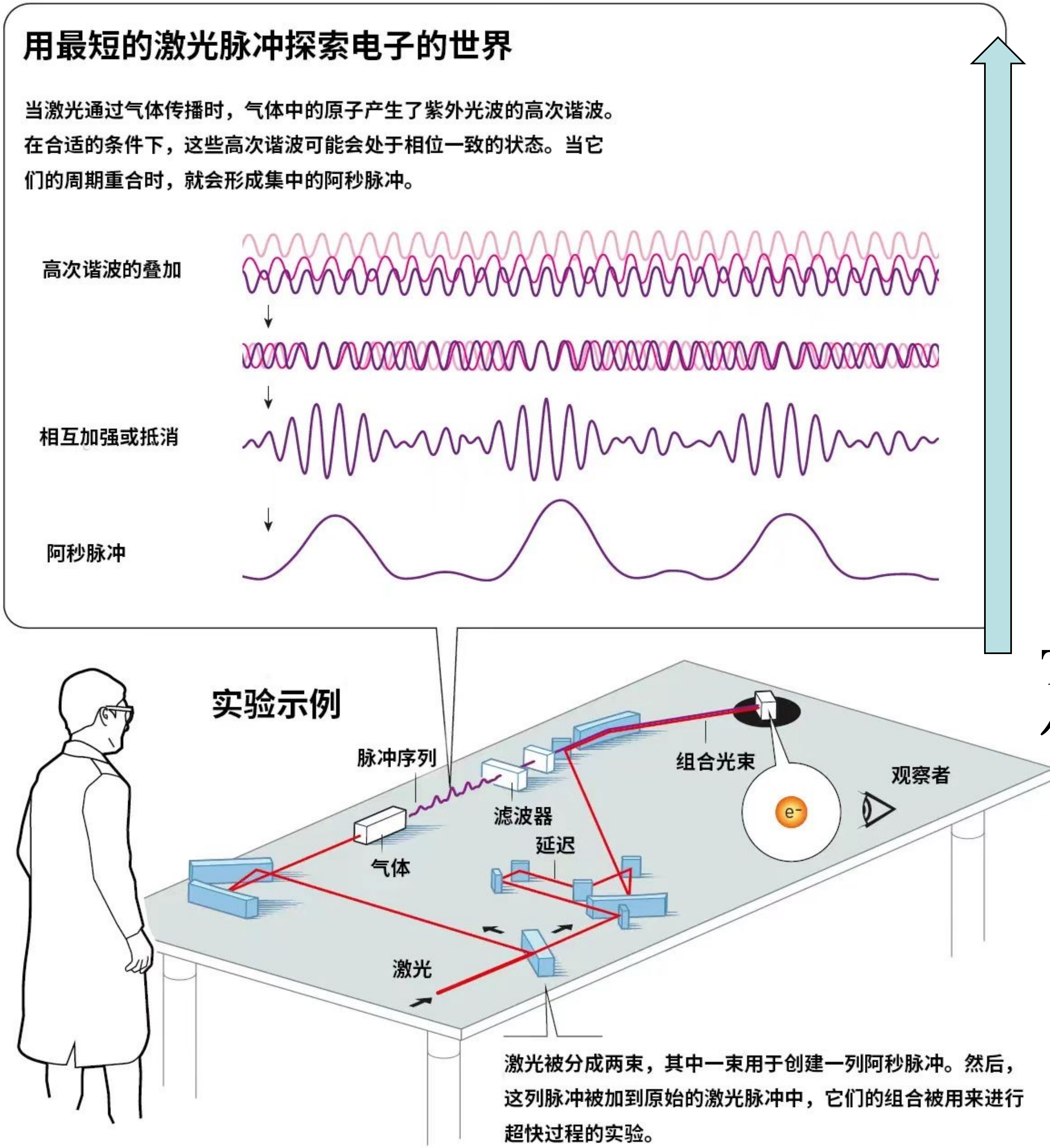
复平面上的傅立叶分析

天琴中心 黄发朋

https://fapenghuang.github.io/teaching/







10^{-18}
seconds

Today:

傅立叶级数展开

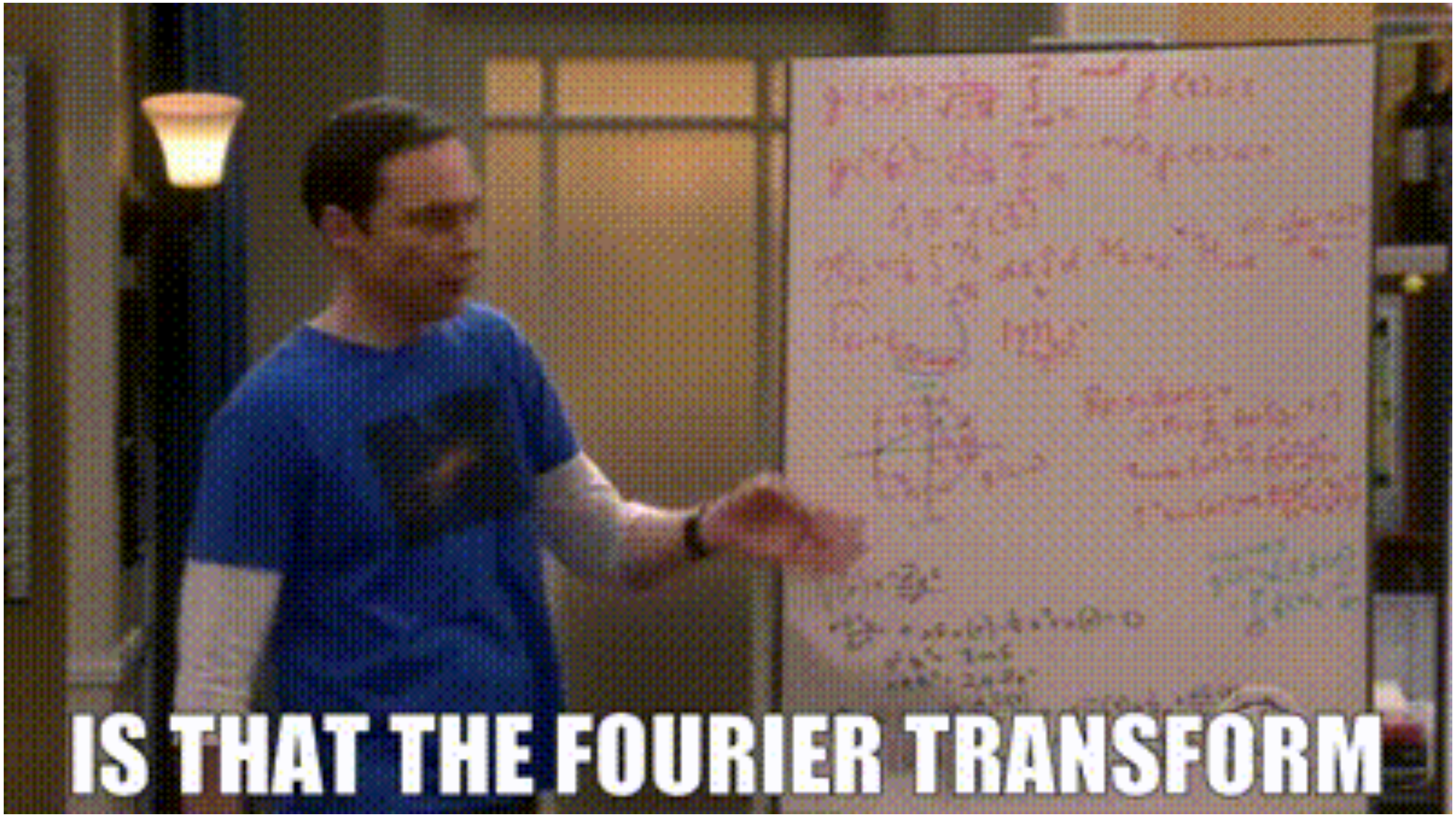
无处不在的傅立叶分析

PYTHAGOREAN THEOREM
(GEOMETRY)
$a^2 + b^2 = c^2$
This theorem, proposed by ancient Greek mathematician Pythagoras, relates the lengths of the sides of a right-angled triangle. It is a fundamental principle in Euclidean geometry and has been used extensively in various fields, including physics, computer graphics, and engineering design.
FOURIER TRANSFORM
(SIGNAL PROCESSING)
$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
Developed by Joseph Fourier in the early 19th century, the Fourier Transform allows a mathematical function to be decomposed into its constituent frequencies. This technique is fundamental in the field of signal processing and has found applications in areas like image processing, audio signal processing, and solving partial differential equations.
EINSTEIN'S EQUATION
(SPECIAL RELATIVITY)
$E = mc^2$
This equation, E=mc², was derived by Albert Einstein as a part of his theory of special relativity in 1905. It states that mass and energy are interchangeable, altering forever the way we understand the universe. It also led to the development of nuclear energy and nuclear weapons.
HEISENBERG'S UNCERTAINTY PRINCIPLE
(QUANTUM MECHANICS)
$\Delta x \Delta p \geq \frac{\hbar}{2}$
Formulated by Werner Heisenberg in 1927, the uncertainty principle is one of the core tenets of quantum mechanics. It states that the more precisely the position of a particle is determined, the less precisely its momentum can be known, and vice versa. It underpins the inherently probabilistic nature of quantum mechanical phenomena.
DIRAC EQUATION
(QUANTUM FIELD THEORY)
$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$
Proposed by Paul Dirac in 1928, this equation describes fermions—particles with half-integer spin—and led to the prediction of antimatter. It's critical in quantum field theory and has shaped our understanding of the fundamental nature of particles and the universe.
EULER'S IDENTITY
(COMPLEX ANALYSIS)
$e^{i\pi} + 1 = 0$
Named after the Swiss mathematician Leonhard Euler, this identity is a special case of Euler's formula from complex analysis, and it beautifully combines five of the most important numbers in mathematics: 0, 1, π, e, and i. It's considered by many to be the most beautiful equation in mathematics due to its elegant linking of these fundamental numbers.
MAXWELL'S EQUATIONS
(ELECTRODYNAMICS)
$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
Formulated by James Clerk Maxwell and Oliver Heaviside in 1862, these four equations describe how electric and magnetic fields interact. They underpin all of classical electrodynamics, optics, and electric circuits and paved the way for the advent of technologies such as radio, television, and radar.
EINSTEIN'S FIELD EQUATIONS
(GENERAL RELATIVITY)
$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$
Published by Albert Einstein in 1915, these equations describe how matter and energy in the Universe distort spacetime to create the phenomenon we experience as gravity. They are fundamental to cosmology and astrophysics, predicting phenomena like gravitational waves and black holes.
SCHRODINGER EQUATION
(QUANTUM MECHANICS)
$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$
Proposed by Erwin Schrödinger in 1926, this equation describes how the quantum state of a quantum system changes with time. It forms the basis of quantum mechanics and has been crucial in the development of many modern technologies, from semiconductors to MRI scanners.

复平面上的傅立叶分析

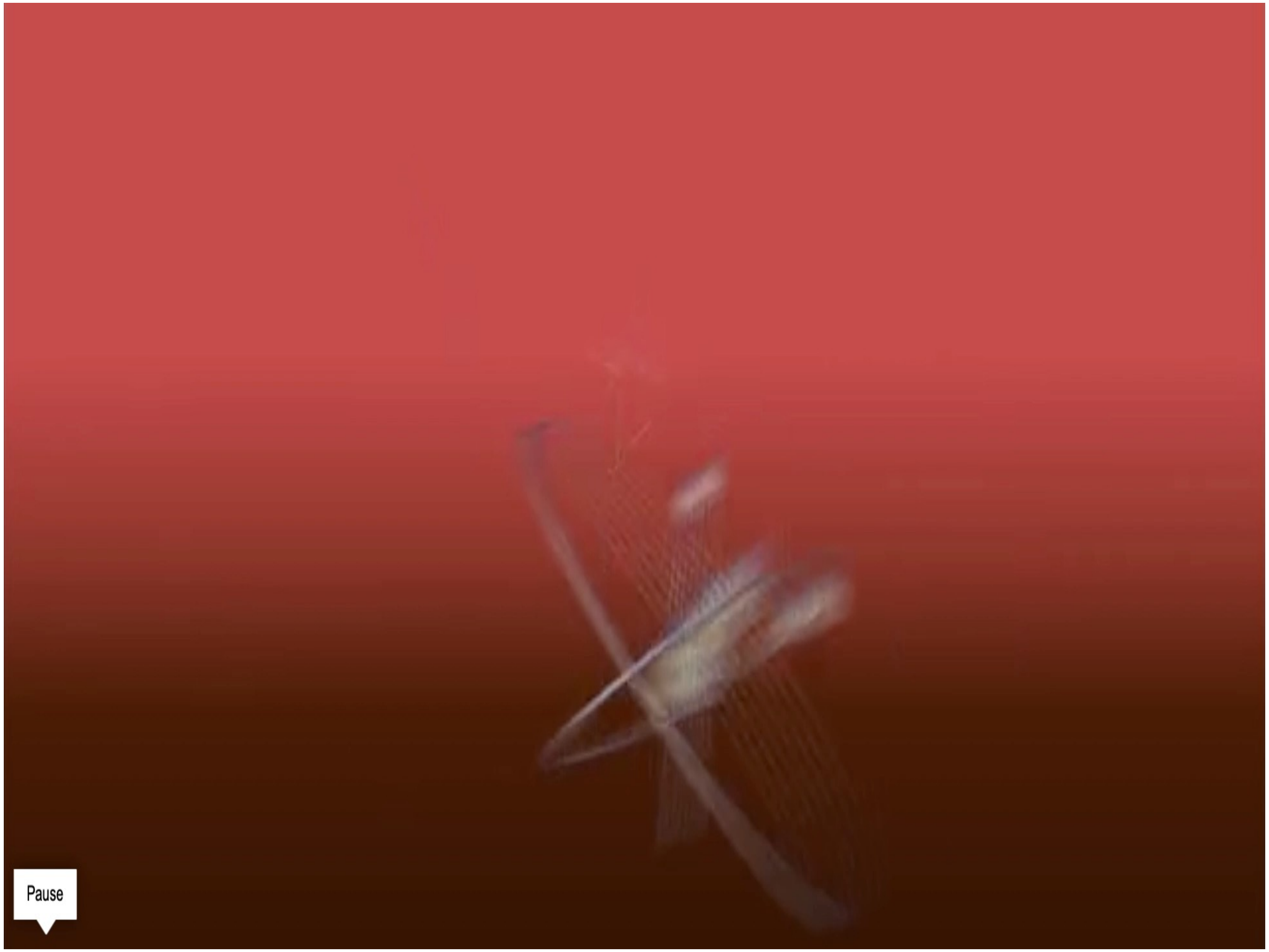
无处不在的傅立叶分析

所有方向受用终身的知识——没有之一



物理即傅立叶分析!

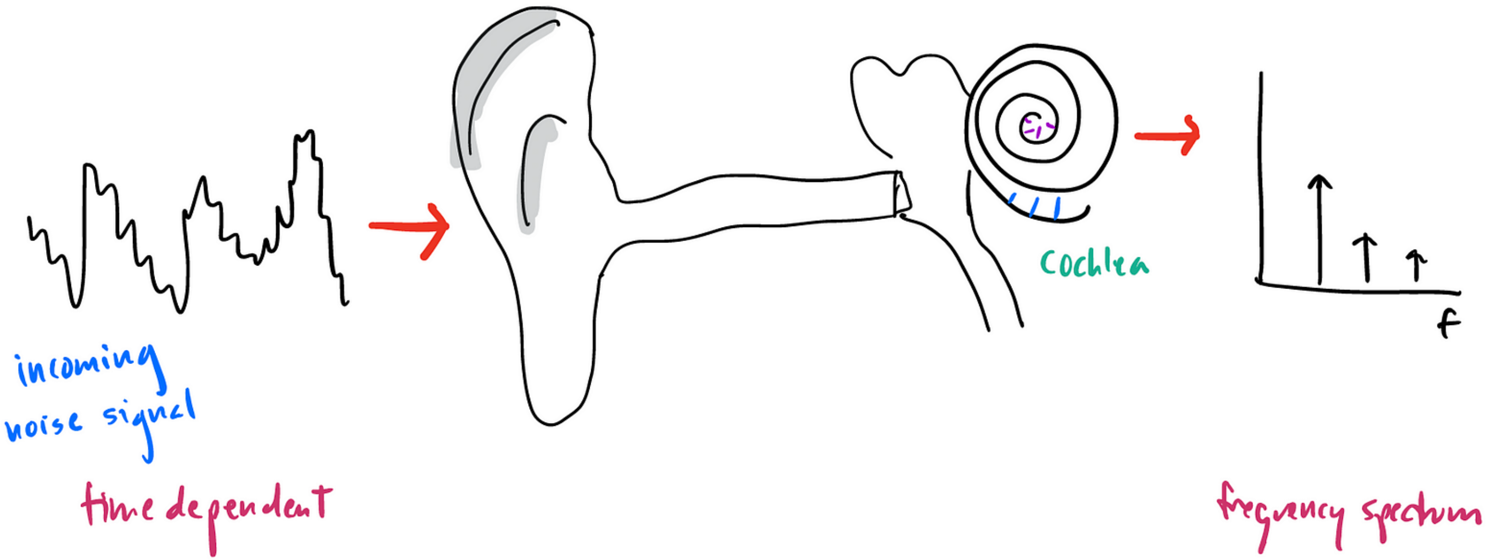
The fundamental problem with elementary particle is Fourier transformation…



量子力学、光学、量子场论、电动力学、计算机、热力学与统计物理、数理统计、理论力学、拓扑学微分几何等很多现代数学分支

无处不在的傅立叶分析

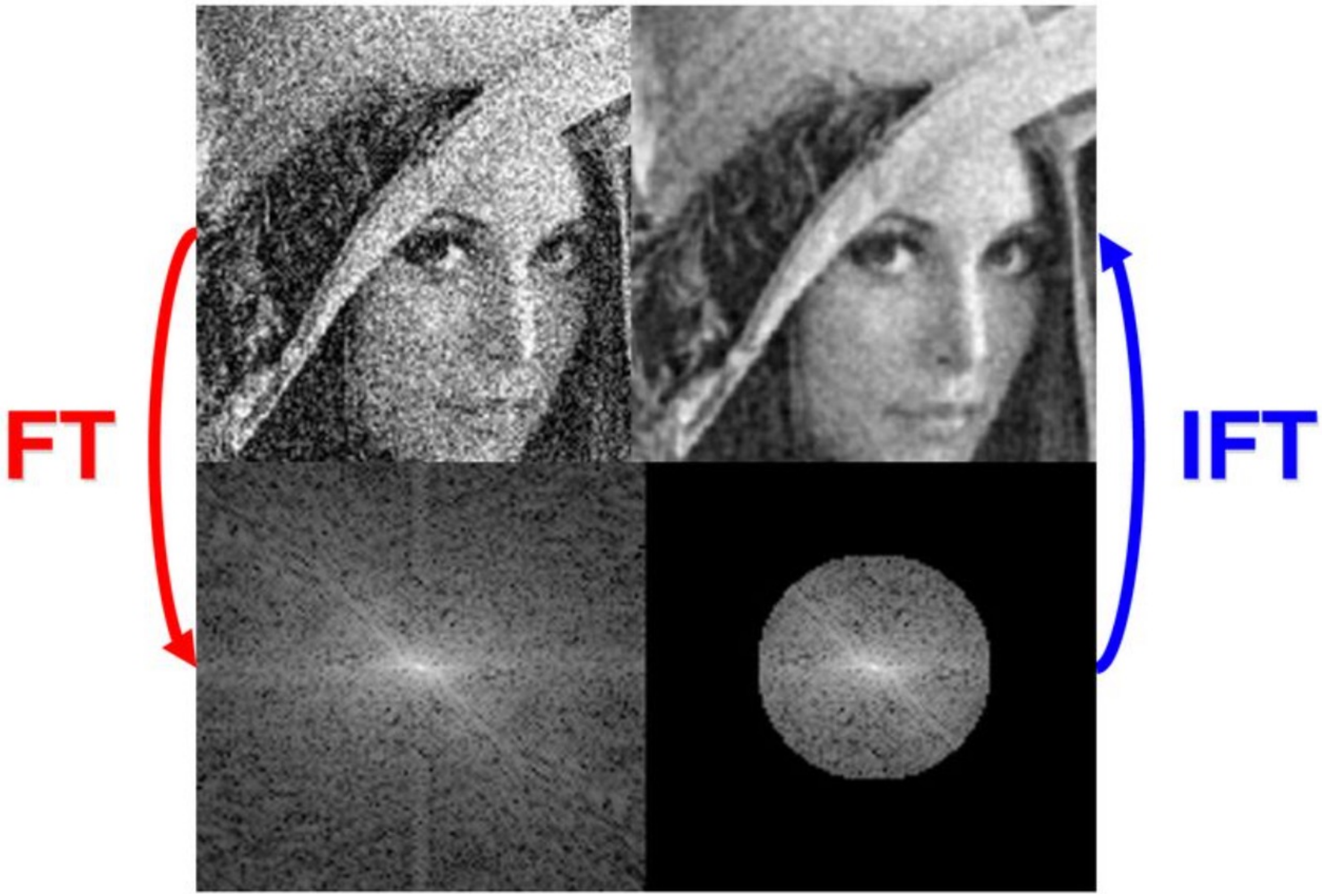
Hearing, a Biological Application of the Fourier Transform



我们耳听目视的过程本质上也是傅立叶变换/分析

无处不在的傅立叶分析

Noise Suppression



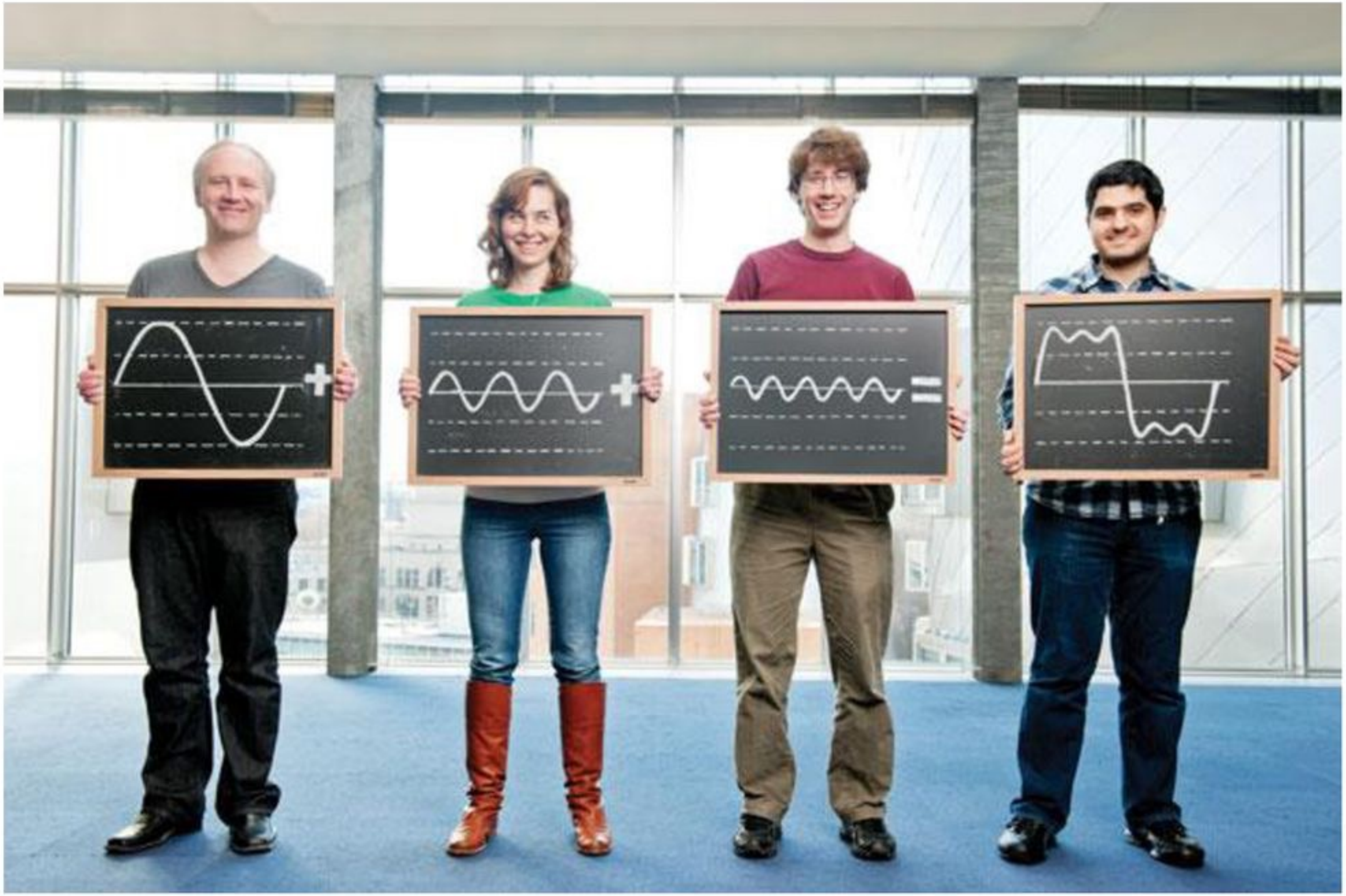
例子：信号处理、图片压缩

无处不在的傅立叶分析

例子：音频压缩

无处不在的傅立叶分析

As Sum of Waves



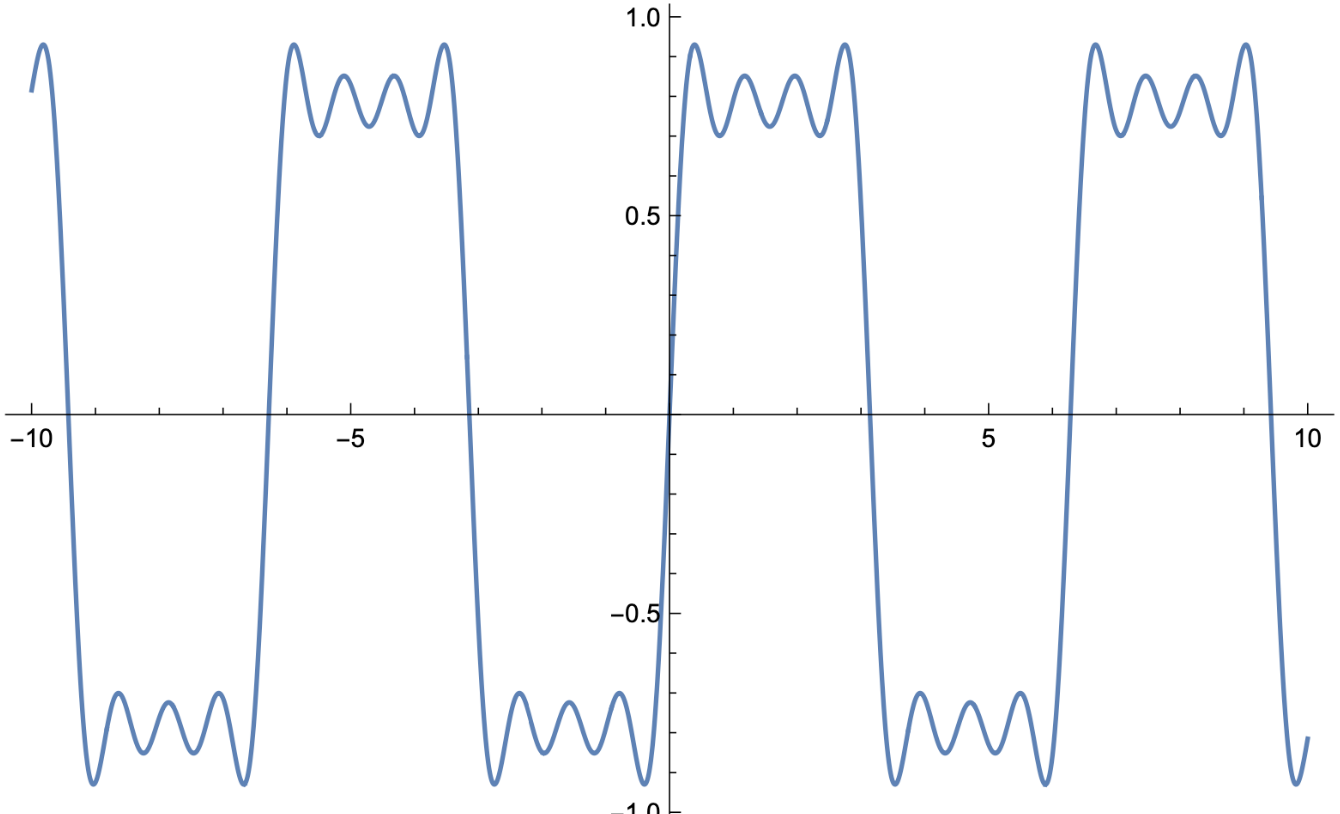
无处不在的傅立叶分析

In[34]:=

(*傅立叶级数4项的部分和：逐步逼近锯齿波*)

```
Plot[Sum[Sin[(2 n - 1) t] / (2 n - 1), {n, 1, 4}], {t, -10, 10}]
```

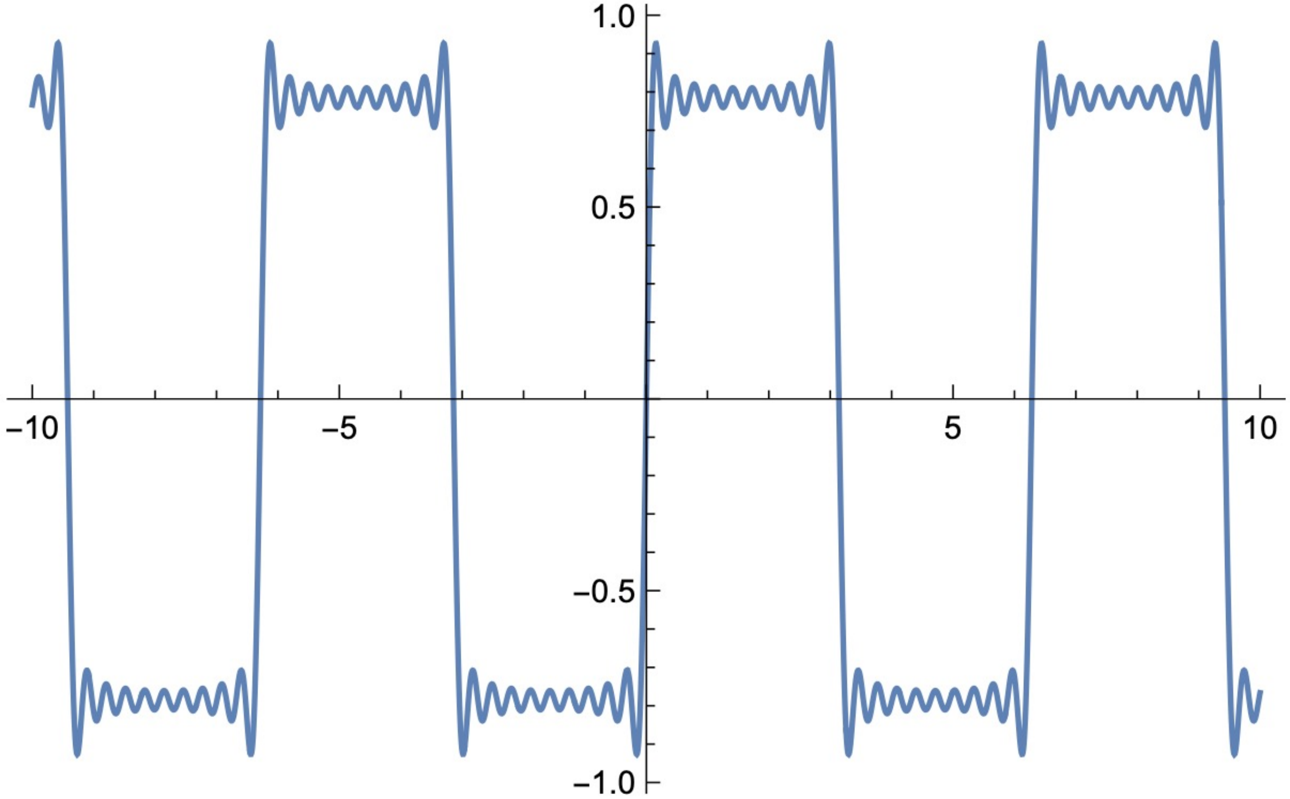
Out[34]=



无处不在的傅立叶分析

```
Plot[Sum[Sin[(2 n - 1) t] / (2 n - 1), {n, 1, 10}], {t, -10, 10}]
```

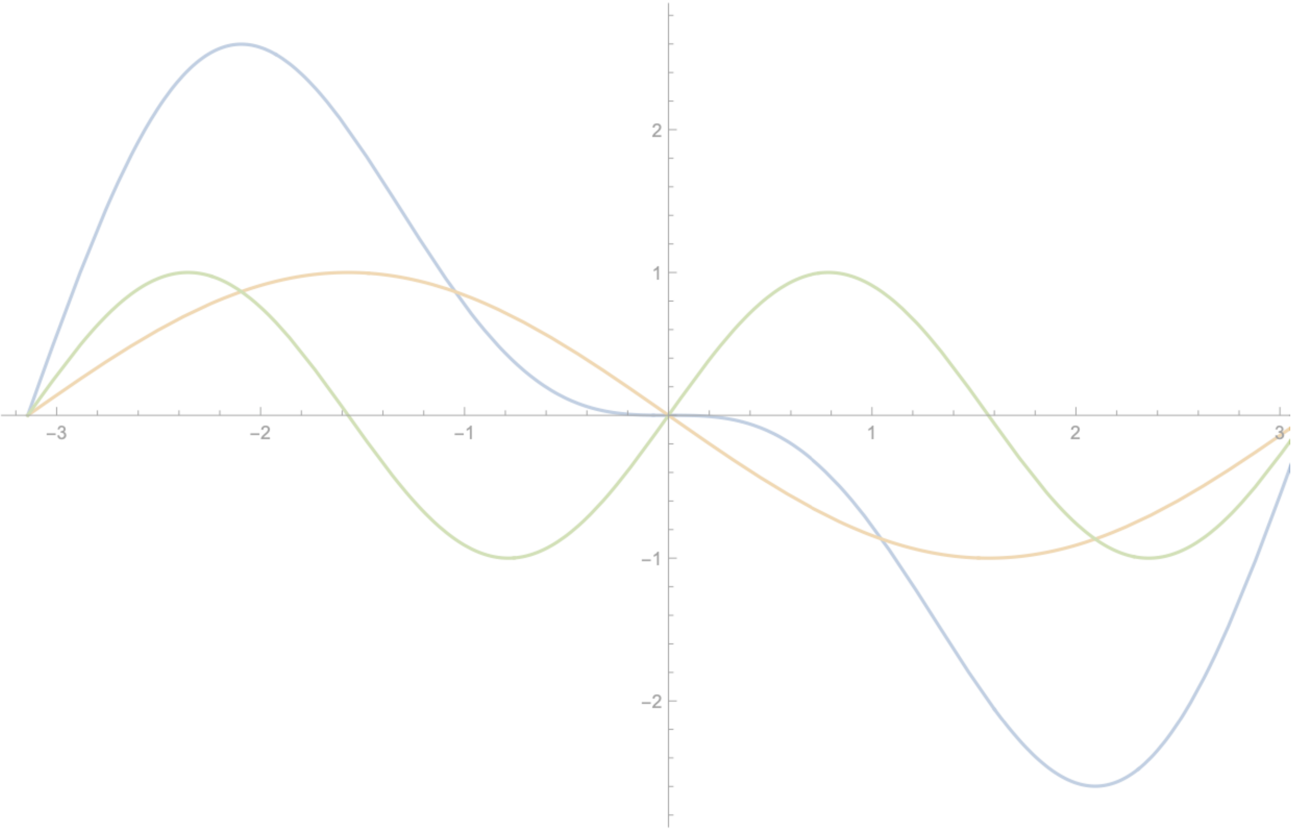
Out[44]=



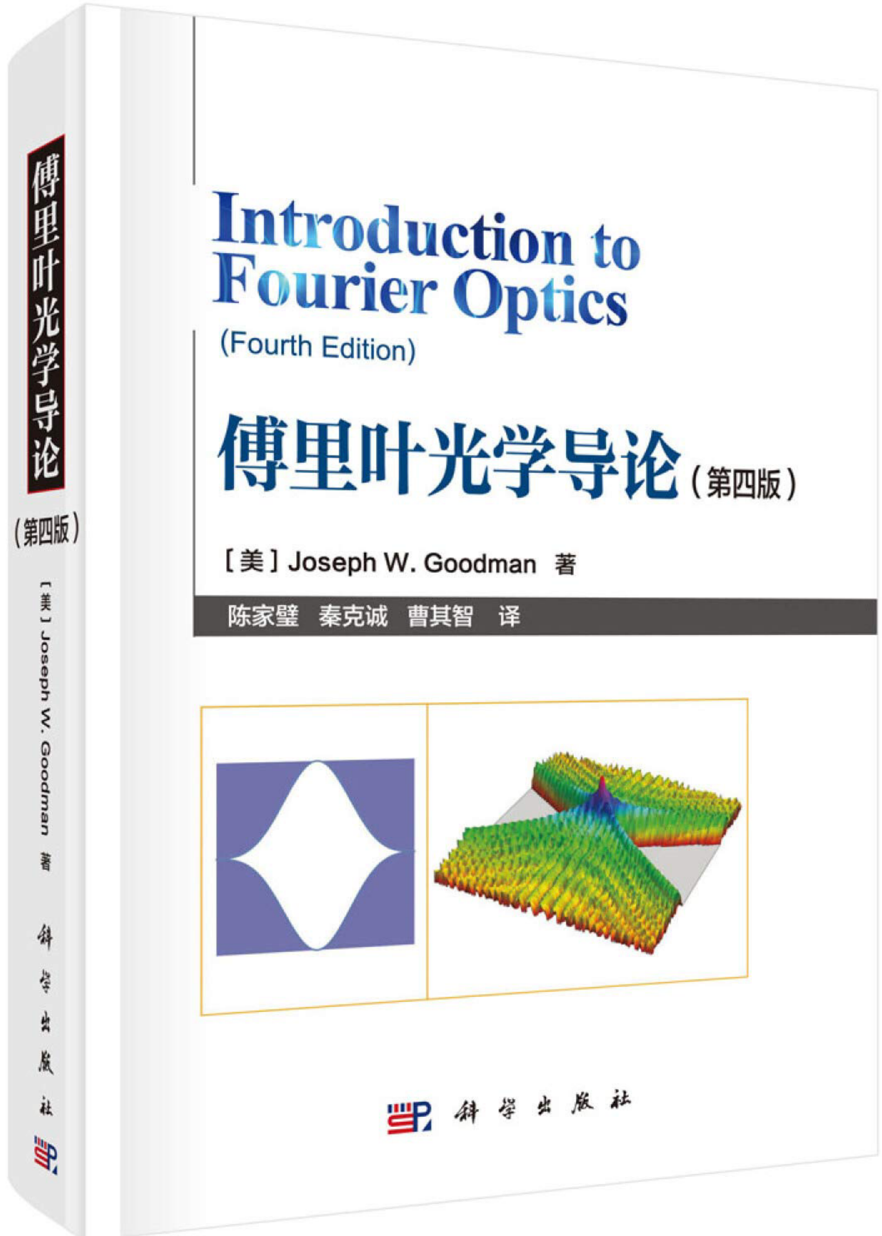
无处不在的傅立叶分析

(*
傅立叶级数展开逼近一个函数示意图
*)

Out[38]=



无处不在的傅立叶分析



傅里叶光学导论
(第四版)
[美] Joseph W. Goodman 著
科学出版社
Introduction to Fourier Optics
(Fourth Edition)
傅里叶光学导论(第四版)
[美] Joseph W. Goodman 著
陈家璧 秦克诚 曹其智 译
科学出版社

无处不在的傅立叶分析

夫琅和费衍射就是傅立叶变换

Fraunhofer diffraction is a Fourier transform

$$\boxed{E\left(x_0, y_0\right) \propto \iint \exp\left\{-\frac{jk}{z}\left(x_0 x_1 + y_0 y_1\right)\right\} Aperture(x_1, y_1)\, dx_1 dy_1}$$

This is just a Fourier Transform! (actually, two of them, in two variables)

Interestingly, it's a Fourier Transform from position, x_1, to another position variable, x_0 (in another plane, i.e., a different z position).

Usually, the Fourier "conjugate variables" have reciprocal units (e.g., t and ω, or x and k).

The conjugate variables here are really x_1 and kx_0/z, which do have reciprocal units.

无处不在的傅立叶分析

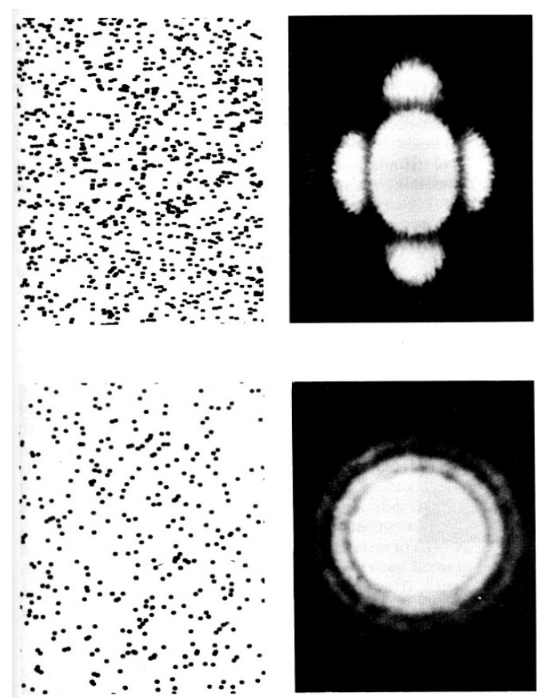
Fraunhofer Diffraction: an interesting example

Randomly placed identical holes yield a diffraction pattern whose gross features reveal the shape of the holes.

Hole pattern | Diffraction pattern

Square holes

Round holes



无处不在的傅立叶分析

第一次导论课:

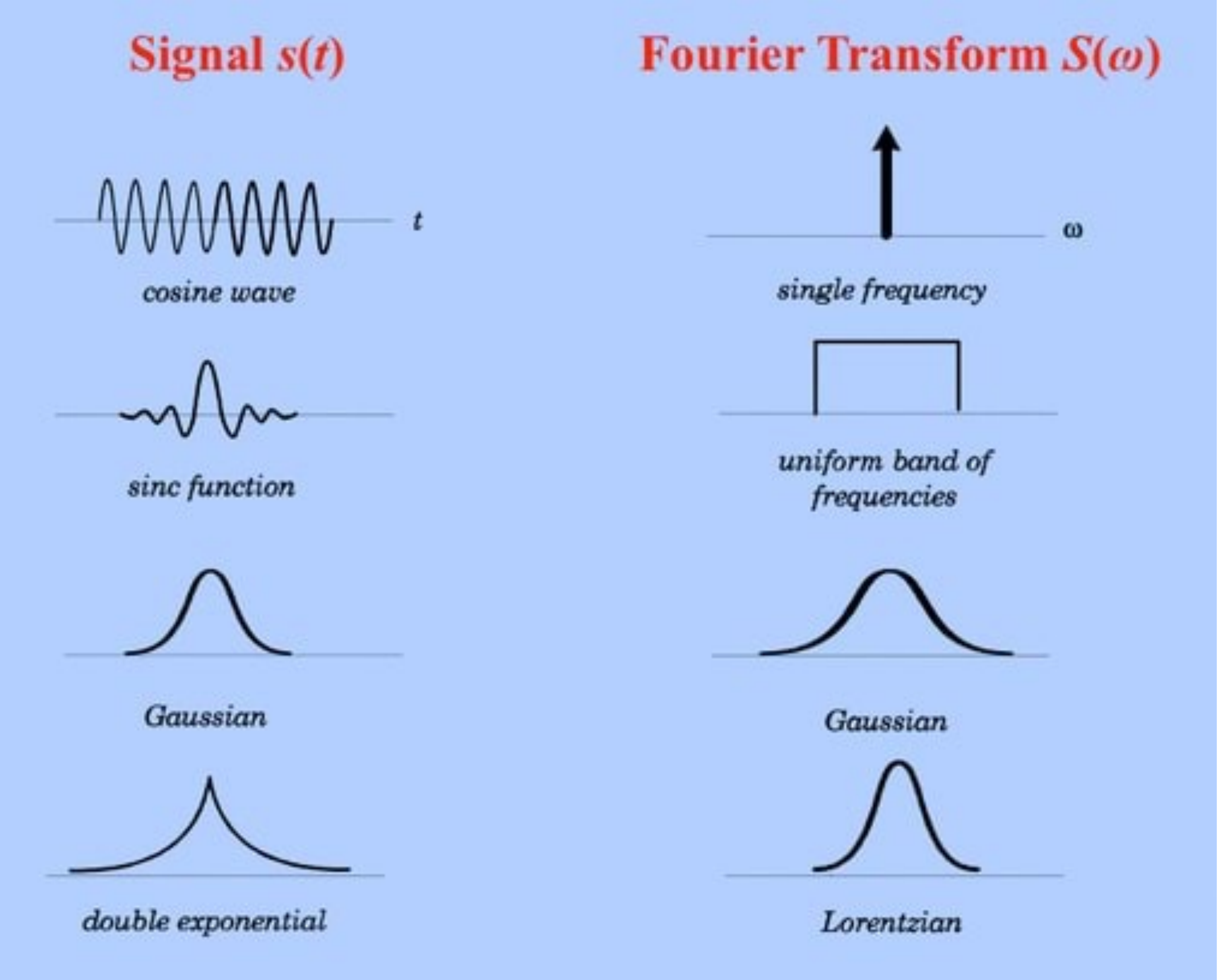
必须会证明：Gaussian(高显老师?)函数的傅立叶变换(在动量空间) 还是Gaussian(还是高显老师)函数

量子力学(海森堡测不准原理)

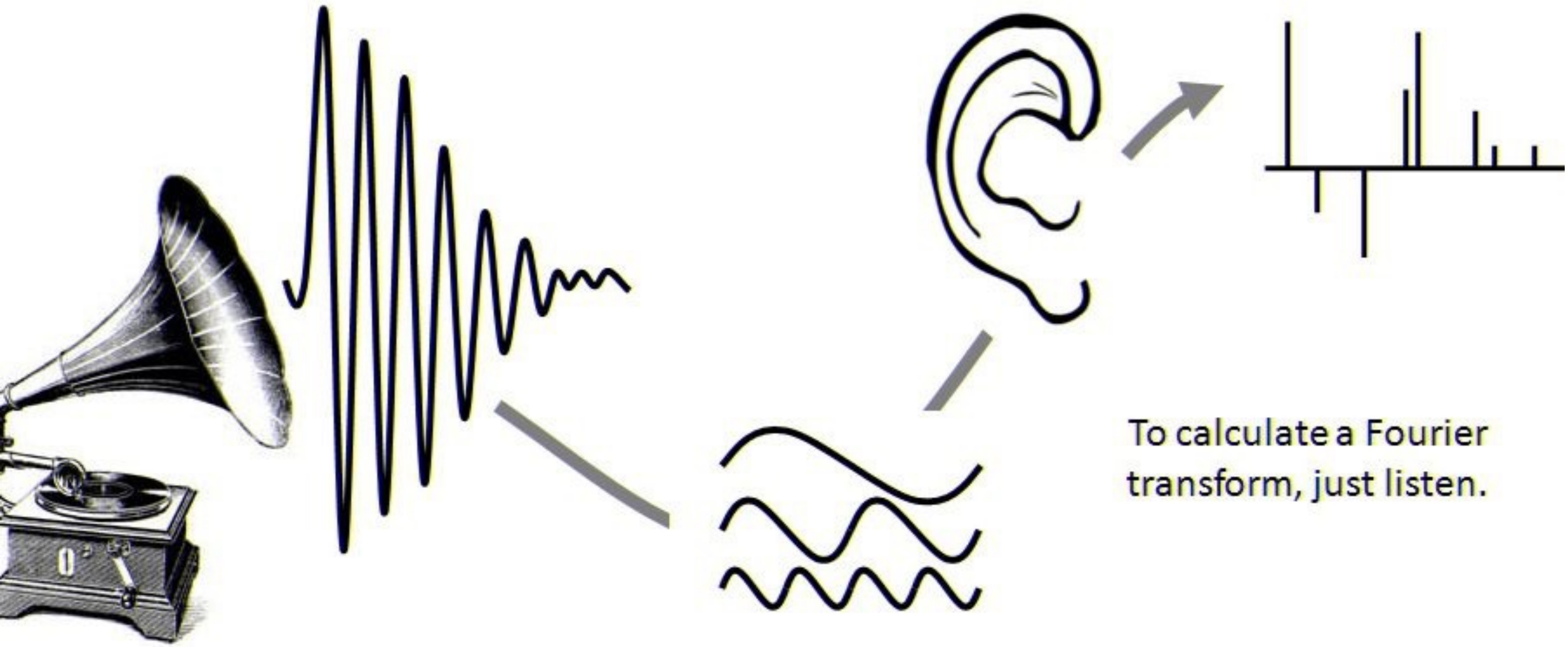
除了物理和各种实际应用上傅立叶变换无处不在，在数学问题中也威力巨大：暴力求解微分方程，级数求和.....

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = \frac{e}{m}\vec{E} \qquad \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

常用的傅立叶变换


Signal s(t)
Fourier Transform S(ω)
t
ω
cosine wave
single frequency
sinc function
uniform band of frequencies
Gaussian
Gaussian
double exponential
Lorentzian

如何计算傅立叶变换



To calculate a Fourier transform, just listen!欲知如何计算傅立叶变换，请耐心听课....能听会算，归根结底是复平面上的积分