2.2 复微分与解析函数

定义 2.3

复变量 z 的函数 $f(z)$ 称为在某个区域 R 内解析，如果它 1) 在此域内单值；2) 有限；3) 在域 R 内任一点处存在极限

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\equiv f'(z),$$

且其值与 Δz 趋向于零的方式无关 (即该极限存在且唯一)。 ♣

注意：我们把 $f(z)$ 在给定一个点 z 上满足上述定义叫 $f(z)$ 在 z 处可导或者可微。若 $f(z)$ 在区域 R 内处处可导，则称 $f(z)$ 是区域 R 内的解析函数。更严格的名词定义将在后续学了泰勒展开后再回头讨论[6]

例题 2.9 根据定义证明

$$f(z)=z^2$$

是复平面上的解析函数

z^* 在复平面上不解析

根据所给的定义容易证明，略。

注意：复变量函数的导数的定义形式上和大一学过的实数几乎一样，所以各种求导规则原则上可以拿来直接套用。只是要注意复变函数是两个实数变量的组合，所以一个复变量函数根据上面的导数定义，满足可导和解析的条件可能非常苛刻。

现寻求函数 $f(z)=u(x,y)+iv(x,y)$ 解析的条件。我们先讨论复微分/解析的必要条件。可以简单利用 Δz 从实轴和虚轴上趋于 0 给出的极限必须一样，得到一个必要条件。

$$f'(z)\xlongequal[\Delta x\to 0]{\Delta y=0}\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)+iv(x+\Delta x,y)-u(x,y)-iv(x,y)}{\Delta x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\xlongequal[\Delta x=0]{\Delta y\to 0}\lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)+iv(x,y+\Delta y)-u(x,y)-iv(x,y)}{i\Delta y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

[6]Consider the function $f(z)$ on the domain $G\subset \mathrm{Domain}(f)$.
f is called *differentiable* at the point z_0 if the differential quotient

重要公式 2.6

$$\left.\frac{df}{dz}\right|_{z_0}=f'(z)|_{z_0}=\left.\frac{\partial f}{\partial x}\right|_{z_0}=\left.\frac{1}{i}\frac{\partial f}{\partial y}\right|_{z_0} \tag{2.19}$$

♠

exists.

If f is (arbitrarily often) differentiable in the domain G it is called *holomorphic*. We shall state without proof that, if a holomorphic function is differentiable, it is also arbitrarily often differentiable.

If a function can be expanded as a convergent power series, like $f(z)=\sum_{n=0}^{\infty}a_nz^n$, in the domain G then it is called *analytic* in the domain G. We state without proof that holomorphic functions are analytic, and *vice versa*; that is the terms "holomorphic" and "analytic" will be used synonymously.

概念上说清楚：在某一点可导则可微，反之亦然，不区分

在区域内处处可微，则全纯 (holomorphic) 或者不区分叫解析 (analytic) 函数.

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}. \tag{2.20}$$

比较实部和虚部得到

重要公式 2.7

Cauchy-Riemann 条件

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y},\\ \frac{\partial v}{\partial x}&=-\frac{\partial u}{\partial y}.\end{aligned} \tag{2.21}$$

例题 2.10 证明:

$$\begin{aligned}|f'(z)|^2&=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\\ &=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\end{aligned}$$

证明

$$\frac{df}{dz}=f'(z)=\frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y} \tag{2.22}$$

再求模方，则得到第一行的两个表达式；第一行分别带入柯西-黎曼条件

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y},\\ \frac{\partial v}{\partial x}&=-\frac{\partial u}{\partial y}.\end{aligned} \tag{2.23}$$

得到第二行的公式。

可见复变函数可导或者在区域内解析需要满足很强的条件。但是正是因为复解析的条件苛刻，导致复解析函数有绝妙的性质。

我们先讨论利用复解析的 Cauchy-Riemann 条件示意性的推导复变函数最核心的定理——柯西定理 (也可以看成 Cauchy-Riemann 条件最直接的一个应用的例题)

定理 2.1

(柯西定理) 如果函数 $f(z)$ 在区域 R 内解析, 则此函数沿着复平面中区域 R 内的任一简单闭曲线 C 的积分都等于零。

$$\oint_c f(z)dz=0$$

证明 假定

$$f(z)=u+iv \text{ 和} dz=dx+idy,$$

$$
\begin{aligned}
&I=\oint_c f(z)dz=\oint_c [u(x,y)+iv(x,y)][dx+idy]\\
&=\oint_c [udx-vdy]+i[vdx+udy]\\
&\text{利用}\oint_c Pdx+Qdy=\iint dxdy\,(Q_x-P_y)\,(\text{Green Theorem，}\;Q_x=\partial Q/\partial x, P_x=\partial P/\partial y)\\
&\Rightarrow I=\int\int dxdy\,\{[-v_x-u_y]+i\,[u_x-v_y]\}\,.\\
&\;\text{Cauchy-Riemann 条件}\quad v_x=-u_y\quad u_x=v_y\\
&\Rightarrow I=0=\oint_c f(z)dz
\end{aligned}
\tag{2.24}
$$

数学系的复变函数书上有更严格的证明 (条件可以进一步放松)，然后我们这里只需要掌握这种简单但体现复变函数核心思想的证明方法。

例题 2.11 计算

$$\oint_{|z|=1} zdz$$

$$\oint_{|z|=1} z^2dz$$

$$\oint_{|z|=1} z^{100}dz$$

$$\oint_{|z-3|+|z+3|=10} z^{100}dz$$

$$\oint_{\text{扇形}} z^2dz$$

$$\oint_{\text{扇形}} e^{iz^2}dz$$

2.2.1 简要复习实变函数微积分基本定理

微积分基本定理: 广义 Stokes 公式

1. Newton-Leibniz $\int_a^x f(t)dt=\Phi(x)-\Phi(a),\quad a\le x\le b$

2. Green

$$\oint_L P(x,y)dx+Q(x,y)dy=\iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

3. Stokes

$$
\begin{aligned}
\oint_L Pdx+Qdy+Rdz=\iint_\Sigma\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx\\
+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy
\end{aligned}
$$

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4. Gauss

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy = \iiint_{V} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

5. 广义 Stokes

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega$$

这些公式是微积分基本定理 (从一维到任意高维微积分)。一般的证明, 特别是广义 Stokes 定理, 有兴趣的同学可以学习外微分的知识。

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2.3 Cauchy' s Theorem and Its Applications

The solution of a large number of problems can be reduced, in the last analysis, to the evaluation of definite integrals; thus mathematicians have been much occupied with this task... However, among many results obtained, a number were initially discovered by the aid of a type of induction based on the passage from real to imaginary. Often passage of this kind led directly to remarkable results. Nevertheless this part of the theory, as has been observed by Laplace, is subject to various difficulties... After having reflected on this subject and brought together various results mentioned above, I hope to establish the passage from the real to the imaginary based on a direct and rigorous analysis; my researches have thus led me to the method which is the object of this memoir... A. L. Cauchy, 1827

2.3.1 菲涅尔积分 The Fresnel Integrals

$S(R) = \int_0^R \sin x^2 dx \quad C(R) = \int_0^R \cos x^2 dx$
$R \to \infty$ 可解析积分

$$S \equiv \int_0^\infty \sin x^2 dx \quad C \equiv \int_0^\infty \cos x^2 dx \tag{2.25}$$

同样的方法，不同的积分路径，可以见后面讲傅立叶变换时证明 Gaussian 函数的傅立叶变换还是 Gaussian。

2.3.2 柯西定理构造积分围道严格积分

Solution:

1. 选定积分函数：观察 S 和 C 分别对应 $\int_0^\infty e^{ix^2} dx$ 的 real 和 imaginary，猜测可以构造被积分函数

$$f(z) = e^{iz^2} \tag{2.26}$$

2. 构造积分围道：构造半径为 R, 弧度为 $\pi/4$ 的扇形积分围道，如图2.5所示
3. 应用柯西定理：

$$\oint_{C_1+C_2+C_3} e^{iz^2} dz = 0 \equiv I_1 + I_2 + I_3 \tag{2.27}$$

$$I_1 = \int_{C_1} e^{iz^2} dz \quad I_2 = \int_{C_2} e^{iz^2} dz \quad I_3 = \int_{C_3} e^{iz^2} dz \tag{2.28}$$

$$C_1 : z = x \quad dz = dx \quad z^2 = x^2 \quad x \in [0, R] \tag{2.29}$$

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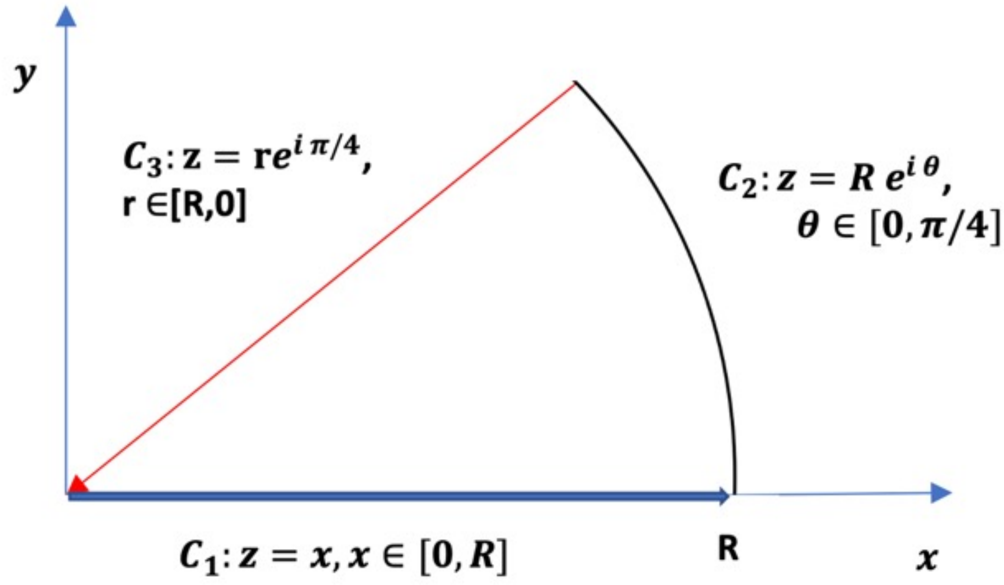

图 2.5: 积分围道

$$I_1 = \int_0^R dx e^{ix^2} \tag{2.30}$$

$R \to \infty$ 时，real 和 imaginary 分别对应 S 和 C

$$C_3 : z = re^{i\frac{\pi}{4}}, r \in [R, 0] \tag{2.31}$$

$$dz = e^{i\frac{\pi}{4}} dr \tag{2.32}$$

$$z^2 = r^2 e^{i\frac{\pi}{2}} = ir^2 \tag{2.33}$$

$$e^{iz^2} = e^{-r^2} \tag{2.34}$$

$$I_3 = \int_R^0 e^{-r^2} e^{i\frac{\pi}{4}} dr = -\int_0^R e^{-r^2} e^{i\pi/4} dr = -e^{i\frac{\pi}{4}} \int_0^R e^{-r^2} dr \tag{2.35}$$

$R \to \infty$ 时, 积分部分对应 Gaussian 积分（$G = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$）

$$C_2 : z = Re^{i\theta}, \theta \in \left[0, \frac{\pi}{4}\right] \tag{2.36}$$

$$dz = d\theta i Re^{i\theta}, z^2 = R^2 e^{i2\theta} \tag{2.37}$$

$$I_2 = \int_0^{\frac{\pi}{4}} d\theta e^{iR^2 e^{i2\theta}} i Re^{i\theta} \tag{2.38}$$

$R \to \infty$ 时, $I_2 \to 0$, 先得到主要结果，再证明这个。

由柯西定理，

$$R \to \infty \Rightarrow I_1 + I_2 + I_3 = 0, \quad I_2 = 0 \tag{2.39}$$

即

$$\int_0^\infty dx e^{ix^2} - e^{i\frac{\pi}{4}} \int_0^\infty e^{-r^2} dr = 0 \tag{2.40}$$

$$\boxed{\begin{aligned} \int_0^\infty dx e^{ix^2} = C + iS = e^{i\frac{\pi}{4}} \int_0^\infty e^{-r^2} dr = e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2} \qquad (2.41) \\ \int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4} \end{aligned}}$$

Joke: 柯西叫菲涅尔遛了一个圈，结果变成了高斯

补充知识高斯积分：

I will now show you how to do the integral $G \equiv \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}x^2}$. The trick is to square the integral, call the dummy integration variable in one of the integrals y, and then pass to polar coordinates

$$dxdy = rd\theta dr \tag{2.42}$$

$$\begin{aligned} G^2 &= \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}x^2} \int_{-\infty}^{+\infty} dy e^{-\frac{1}{2}y^2} = 2\pi \int_0^{+\infty} dr r e^{-\frac{1}{2}r^2} \\ &= 2\pi \int_0^{+\infty} dw e^{-w} = 2\pi \end{aligned}$$

Thus, we obtain

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

Gassian 类型的积分是物理中最核心的积分，特别是理论物理中，应该是最重要的积分。学有余力的同学可以阅读课外材料[7]

补充证明：

[7] Believe it or not, a significant fraction of the theoretical physics literature consists of varying and elaborating this basic Gaussian integral. The simplest extension is almost immediate:

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}}$$

as can be seen by scaling $x \to x/\sqrt{a}$.

An important variant is the integral

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2 + Jx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{J^2/2a}$$

To see this, take the expression in the exponent and "complete the square": $-ax^2/2 + Jx = -(a/2)\left(x^2 - 2Jx/a\right) = -(a/2)(x - J/a)^2 + J^2/2a$. The x integral can now be done by shifting $x \to x + J/a$, giving the factor of $(2\pi/a)^{\frac{1}{2}}$. Another important variant is obtained by replacing J by iJ :

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2 + iJx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{-J^2/2a}$$

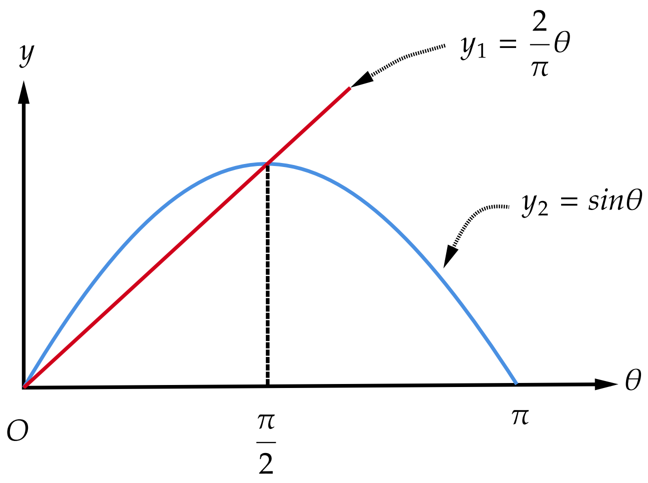

图 2.6: Jordan 不等式的证明

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{4}} d\theta e^{iR^2 e^{i2\theta}} iRe^{i\theta} \\ &= \int_0^{4} d\theta e^{iR^2(\cos 2\theta + i \sin 2\theta)} iRe^{i\theta} \\ &= \int_0^{\frac{\pi}{4}} d\theta e^{iR^2 \cos 2\theta} e^{-R^2 \sin 2\theta} iRe^{i\theta} \end{aligned} \tag{2.43}$$

$$\begin{aligned} | \ e^{i\theta} \ | &= 1 \\ | \ e^{iR^2 \cos\theta} \ | &= 1 \end{aligned} \tag{2.44}$$

$$|I_2| \leq \int_0^{\frac{\pi}{4}} d\theta \left| e^{iR^2 \cos 2\theta} \right| \left| e^{-R^2 \sin 2\theta} \right| |iR| \left| e^{i\theta} \right| = \int_0^{\frac{\pi}{4}} d\theta e^{-R^2 \sin 2\theta} R = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{2} e^{-R^2 \sin\alpha} R \tag{2.45}$$

To get yet another variant, replace a by $-ia$:

$$\int_{-\infty}^{+\infty} dx e^{\frac{1}{2}iax^2 + iJx} = \left(\frac{2\pi i}{a}\right)^{\frac{1}{2}} e^{-iJ^2/2a}$$

Let us promote a to a real symmetric N by N matrix A_{ij} and x to a vector $x_i (i, j = 1, \cdots, N)$. Then

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 dx_2 \cdots dx_N e^{-\frac{1}{2}x \cdot A \cdot x + J \cdot x} = \left(\frac{(2\pi)^N}{\det[A]}\right)^{\frac{1}{2}} e^{\frac{1}{2}J \cdot A^{-1} \cdot J}$$

where $x \cdot A \cdot x = x_i A_{ij} x_j$ and $J \cdot x = J_i x_i$ (with repeated indices summed.) To derive this important relation, diagonalize A by an orthogonal transformation O so that $A = O^{-1} \cdot D \cdot O$, where D is a diagonal matrix. Call $y_i = O_{ij} x_j$. In other words, we rotate the coordinates in the N-dimensional Euclidean space we are integrating over. The expression in the exponential in the integrand then becomes $-\frac{1}{2} y \cdot D \cdot y + (OJ) \cdot y$. Using $\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 \cdots dx_N = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dy_1 \cdots dy_N$, we factorize the left-hand side of (22) into a product of N integrals, each of the form $\int_{-\infty}^{+\infty} dy_i e^{-\frac{1}{2} D_{ii} y_i^2 + (OJ)_i y_i}$. Putting in some i 's $(A \to -iA, J \to iJ)$, we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 dx_2 \cdots dx_N e^{(i/2) x \cdot A \cdot x + iJ \cdot x} \\ &= \left(\frac{(2\pi i)^N}{\det[A]}\right)^{\frac{1}{2}} e^{-(i/2) J \cdot A^{-1} \cdot J} \end{aligned}$$

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重要公式 2.8

若当 (Jordan) 不等式

$$\frac{2\theta}{\pi} < \sin\theta, \qquad \text{当} 0 < \theta < \frac{\pi}{2}$$

证明: 分别作出 $y_1 = \frac{2}{\pi}\theta$ 及 $y_2 = \sin\theta$ 的函数曲线图 (图 2.6). 易见在开区间 $\left(0, \frac{\pi}{2}\right)$ 中, 有 $\sin\theta > \frac{2}{\pi}\theta$; 而在闭区间 $\left[0, \frac{\pi}{2}\right]$ 的端点, 有 $\sin\theta = \frac{2}{\pi}\theta$.

比较图中直线和曲线的函数值得到

$\sin\alpha > \frac{2\alpha}{\pi}$

$$\Rightarrow e^{-R^2\sin\alpha} < e^{-R^2\frac{2\alpha}{\pi}}$$

$$\begin{aligned} |I_2| &< \int_0^{\frac{\pi}{2}} \frac{d\alpha}{2} e^{-R^2\frac{2\alpha}{\pi}} R \\ &= \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \overset{R\to\infty}{=} 0 \end{aligned} \tag{2.46}$$

故得证明 $R \to \infty$ 时，I_2 为 0，该证明方法很有用，将来会用来证明 jordan 引理，大圆弧引理等。

2.3.3 物理直觉方法

Fresnel Integrals Using Wick Rotation

不严格的，半物理直觉，从 Gaussian 型积分作 Wick rotation

$$G = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \tag{2.47}$$

$$x = e^{-i\frac{\pi}{4}} t \tag{2.48}$$

$$dx = dt e^{-i\frac{\pi}{4}} \tag{2.49}$$

$$x^2 = e^{-i\frac{\pi}{2}} t^2 = -it^2 \tag{2.50}$$

$$\int_0^\infty e^{-x^2} dx = G = \int_0^{\infty e^{\frac{-i\pi}{4}}} e^{-i\frac{\pi}{4}} dt e^{it^2} = \int_0^\infty e^{-i\frac{\pi}{4}} dt e^{it^2} \tag{2.51}$$

$$\Rightarrow \int_0^\infty dt e^{+it^2} = e^{i\frac{\pi}{4}} \int_0^\infty e^{-x^2} dx = e^{i\frac{\pi}{4}} \left(\frac{\sqrt{\pi}}{2}\right) = C + iS \tag{2.52}$$

2.3.4 柯西定理进阶

上节课讨论了最简单的柯西定理及其简单应用。可以不加证明的推广到复连通区域的柯西定理 (证明思路大致和下面的柯西公式的推导的前半部分类似)

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定理 2.2

若 $f(z)$ 在闭复通区域 $\bar{D}$ 解析，则 $f(z)$ 沿所有内、外边界线 $(L = L_0 + \sum_k L_k)$ 正方向积分之和为零

$$\oint_L f(z)\mathrm{d}z = \oint_{L_0} f(z)\mathrm{d}z + \sum_k \oint_{L_k} f(z)\mathrm{d}z = 0$$

“正方向”是指，当沿内、外边界线环行时，$\bar{D}$ 保持在左边．换句话说，外边界线取逆时针方向，内边界线取顺时针方向．作为约定，今后积分号中没标明方向的积分均沿正方向．

思考题

- ◇ 逐步总结复变函数求导的方法？定义法？实变函数求导规则？Cauchy-Riemann？其它方法？
- ◇

$$f(z) = z^2$$

的实部和虚部在电磁学中对应的物理意义？
- ◇ 实变函数和复变函数求导以及沿着曲线的积分有什么异同？

学而时习之

1. 若 $f(z)$ 为解析函数，请问 $f^*(z)$ 是否为解析函数？$f^*(z^*)$ 是否为解析函数？
2. $$\oint_{|z-3|+|z+3|=10} (z^2 + z^5 + z^7)dz$$
3. 在极坐标下，推导出复变函数解析需要满足的 Cauchy-Riemann 条件
4. 已经解析函数 $f(z)$ 的虚部为 $v(x, y) = \sqrt{-x + \sqrt{x^2 + y^2}}$，求该解析函数的实部以及 $f(z)$ 的完整表达式

课程预告

- □ 课前十分钟四位同学到黑板上不看讲义计算，如果有同学觉得例题简单的话，可以做一道上面的习题代替我指定的例题——
陈伟亮：假设已经知道圆弧上积分为 0，计算 $\int_0^\infty \sin x^2 dx$ $\int_0^\infty \cos x^2 dx$;
陈泽华：根据解析函数的定义，推导 Cauchy-Riemann 条件
丁思宇：根据 Cauchy-Riemann 条件，推导柯西定理：如果函数 $f(z)$ 在区域 R 内解析，则此函数沿着复平面中区域 R 内的任一简单闭曲线 C 的积分都等于零。
$$\oint_c f(z)dz = 0$$
方嘉源：假设已经知道圆弧上积分为 0，计算 $\int_0^\infty \sin x^2 dx$ $\int_0^\infty \cos x^2 dx$;
- □ 通过更多的例子，更加深入的讲解复变函数可导和解析的条件(柯西-黎曼条件）及其最重要的应用——得到复变函数的核心，即柯西定理。然后是柯西定理的应用，例如柯西积分公式。对应教材上 2.3 3.1 3.2.3.3 节