1.3.5 Exponential Fourier series

From the above Fourier series and Euler equation under certain "mild" conditions – that is, f must be piecewise continuous, periodic with period L, and (Riemann) integrable – f can be decomposed into an *exponential Fourier series*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}e^{in\phi}\mathrm{d}\phi = \delta_{mn}$$

More generally,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi inx}{L}\right) . \tag{1.34}$$

Using the orthogonal identity

$$\int_{x_0}^{x_0+L} \exp\left(-\frac{2\pi ikx}{L}\right)\exp\left(\frac{2\pi ipx}{L}\right)dx = \begin{cases} L & \text{for } k=p \\ 0 & \text{for } k\neq p \end{cases} \tag{1.35}$$

we can obtain

$$c_n = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\exp\left(-\frac{2\pi inx}{L}\right)dx \tag{1.36}$$

例题 1.6

- Find a complex Fourier series for $f(x)=x$ in the range $-2<x<2$

For $k=0$, c_0=0.

$$\begin{aligned} c_k &= \frac{1}{4}\int_{-2}^{2} x\exp\left(-\frac{\pi ikx}{2}\right)dx \\ &= \left[-\frac{x}{2\pi ik}\exp\left(-\frac{\pi ikx}{2}\right)\right]_{-2}^{2} + \int_{-2}^{2}\frac{1}{2\pi ik}\exp\left(-\frac{\pi ikx}{2}\right)dx \\ &= -\frac{1}{\pi ik}[\exp(-\pi ik)+\exp(\pi ik)] + \left[\frac{1}{k^2\pi^2}\exp\left(-\frac{\pi ikx}{2}\right)\right]_{-2}^{2} \\ &= \frac{2i}{\pi k}\cos\pi k - \frac{2i}{k^2\pi^2}\sin\pi k = \frac{2i}{\pi k}(-1)^k \end{aligned} \tag{1.37}$$

$$x = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2i(-1)^k}{k\pi}\exp\left(\frac{\pi ikx}{2}\right) \tag{1.38}$$

1.3.6 Revisit Parseval's theorem

$$\begin{aligned} \frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2dx &= \sum_{k=-\infty}^{\infty} |c_k|^2 \\ &= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2+b_k^2) \end{aligned} \tag{1.39}$$

It is worth noticing that a_k, b_k and c_k are the Fourier coefficients of $f(x)$. $f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2\pi}{L}kx\right)+b_k\sin\left(\frac{2\pi}{L}kx\right)\right]$[6].

$f(x)=\sum_{k=-\infty}^{\infty}c_k\exp\left(\frac{2\pi ikx}{L}\right)$.

1.4 Fourier transformation 傅立叶变换

$$f(x)=\sum_{n=-\infty}^{\infty}c_n\exp\left(\frac{2\pi inx}{L}\right) . \tag{1.40}$$

$$c_n=\frac{1}{L}\int_{-L/2}^{L/2}f(x)\exp\left(-\frac{2\pi inx}{L}\right)dx \tag{1.41}$$

重新定义傅立叶级数展开的变量

$$\begin{aligned} k &\equiv n\frac{2\pi}{L} \\ \Delta k &= \frac{2\pi}{L} \end{aligned} \tag{1.42}$$

$$f(x)=\sum_{k=-\infty}^{\infty}c_k e^{ikx}, \text{ with}$$

$$c_k=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x')e^{-ikx'}dx'.$$

or

$$f(x)=\frac{1}{L}\sum_{k=-\infty}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x')e^{-ikx'}e^{ikx}dx'.$$

上面的定义有 $\Delta k=2\pi/L$, or $1/L=\Delta k/2\pi$. Then

$$f(x)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x')e^{-ikx'}e^{ikx}dx'\Delta k. \tag{1.43}$$

Now, in the "aperiodic" limit $L\to\infty$, $\sum_{k=-\infty}^{\infty}\to\int_{-\infty}^{\infty}$, $\Delta k\to dk$, .

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty}f(x')e^{-ikx'}dx'}_{\tilde{f}(k)}\,e^{ikx}dk \tag{1.44}$$

namely,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}dk \tag{1.45}$$

[6] I emphasize this because some students felt confused after last class.

with

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x')e^{-ikx'}dx' \tag{1.46}$$

Or,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk \tag{1.47}$$

with

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')e^{-ikx'}dx' \tag{1.48}$$

做傅立叶变换时请务必同时写出正拟变换，这样才能保证系数和符号的自洽性 It is better to write down the equations for Fourier and inverse Fourier transformation simultaneously to avoid the inconsistent of the formulae.

例题 1.7 著名的"top-hat" 函数的傅立叶变换 Find the Fourier transformation and inverse transformation of the following function

$$f(x) = \begin{cases} 1 & (|x| < a) \\ 0 & (|x| > a) \end{cases} \tag{1.49}$$

解 Solution:

Firstly, to avoid confusion of the constant coefficients, it is better to write down the Fourier transformation and inverse transformation in the beginning.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$$

with

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x')e^{-ikx'}dx'$$

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= -\frac{1}{\mathrm{i}\omega}\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{-a}^{a} = \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega a} - \mathrm{e}^{-\mathrm{i}\omega a}\right) \\ &= 2\frac{\sin a\omega}{\omega} \end{aligned} \tag{1.50}$$

副产品:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \tag{1.51}$$

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega}\mathrm{d}\omega \tag{1.52}$$

From Eq.(1.49), we have $f(0) = 1$. Therefore, one could obtain

$$\int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega}\mathrm{d}\omega = \pi \quad (a > 0) \tag{1.53}$$

(这样我们集齐了第三课龙珠，而且结论更加一般)

一般性讨论见脚注[7]

1.4.1 重要应用：不确定原理与 Parseval 公式

Find the Fourier transform of the normalized Gaussian distribution[8]

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right), \quad -\infty < t < \infty \tag{1.59}$$

This Gaussian distribution is centered on $t = 0$ and has a root mean square deviation $\Delta t = \sigma$.

[7] We obtain the *Fourier transformation* and the *Fourier inversion* $\mathcal{F}^{-1}[\mathcal{F}[f(x)]] = \mathcal{F}[\mathcal{F}^{-1}[f(x)]] = f(x)$ by

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')e^{-ik(x'-x)}dx'dk, \text{ whereby} \\ \mathcal{F}^{-1}[\tilde{f}](x) &= f(x) = \alpha\int_{-\infty}^{\infty} \tilde{f}(k)e^{\pm ikx}dk, \text{ and} \\ \mathcal{F}[f](k) &= \tilde{f}(k) = \beta\int_{-\infty}^{\infty} f(x')e^{\mp ikx'}dx'. \end{aligned} \tag{1.54}$$

$\mathcal{F}[f(x)] = \tilde{f}(k)$ is called the *Fourier transform* of $f(x)$. *Per* convention, either one of the two sign pairs $+-$ or $-+$ must be chosen. The factors α and β must be chosen such that

$$\alpha\beta = \frac{1}{2\pi}; \tag{1.55}$$

that is, the factorization can be "spread evenly among α and β," such that $\alpha = \beta = 1/\sqrt{2\pi}$, or "unevenly," such as, for instance, $\alpha = 1$ and $\beta = 1/2\pi$, or $\alpha = 1/2\pi$ and $\beta = 1$.

Most generally, the Fourier transformations can be rewritten (change of integration constant), with arbitrary $A, B \in \mathbb{R}$, as

$$\begin{aligned} \mathcal{F}^{-1}[\tilde{f}](x) &= f(x) = B\int_{-\infty}^{\infty} \tilde{f}(k)e^{iAkx}dk, \text{ and} \\ \mathcal{F}[f](k) &= \tilde{f}(k) = \frac{A}{2\pi B}\int_{-\infty}^{\infty} f(x')e^{-iAkx'}dx'. \end{aligned} \tag{1.56}$$

The choice $A = 2\pi$ and $B = 1$ renders a very symmetric form of (1.56); more precisely,

$$\begin{aligned} \mathcal{F}^{-1}[\tilde{f}](x) &= f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)e^{2\pi ikx}dk, \text{ and} \\ \mathcal{F}[f](k) &= \tilde{f}(k) = \int_{-\infty}^{\infty} f(x')e^{-2\pi ikx'}dx'. \end{aligned} \tag{1.57}$$

[8] The probability density function for a Gaussian distribution of a random variable X, with mean $E[X] = \mu$ and variance $V[X] = \sigma^2$, takes the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{1.58}$$

The factor $1/\sqrt{2\pi}$ arises from the normalisation of the distribution,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Write down the transformation formulae

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$$

with

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x')e^{-ikx'}dx'$$

The Fourier transform of $f(t)$ is given by

$$\begin{aligned}
\widetilde{f}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right)\exp(-i\omega t)dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\frac{t}{\sigma\sqrt{2}} + i\frac{\omega\sigma}{\sqrt{2}}\right)^2\right)\exp(-\frac{\omega^2\sigma^2}{2})dt \qquad (1.60)\\
&= \exp(-\frac{\omega^2\sigma^2}{2})\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\left(\frac{t}{\sigma\sqrt{2}} + i\frac{\omega\sigma}{\sqrt{2}}\right)^2\right)dt
\end{aligned}$$

The variable transformation $x = \frac{t}{\sigma\sqrt{2}} + i\frac{\omega\sigma}{\sqrt{2}}$ yields $dt = dx\sqrt{2}\sigma$;

$$\widetilde{f}(\omega) = \exp(-\frac{\omega^2\sigma^2}{2})\frac{1}{\sigma\sqrt{2\pi}}\sqrt{2}\sigma \int_{-\infty+i\frac{\omega\sigma}{\sqrt{2}}}^{+\infty+i\frac{\omega\sigma}{\sqrt{2}}} e^{-x^2}dx \qquad (1.61)$$

The *Gaussian integral* is

$$\Gamma(1/2) = \int_0^{\infty} x^{-1/2}e^{-x}dx = \int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi} \quad . \qquad (1.62)$$

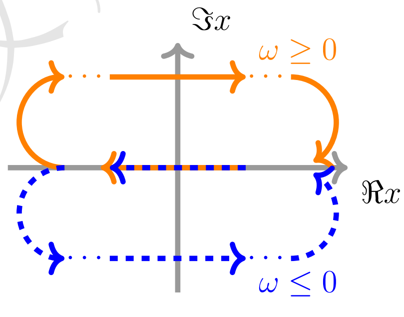

图 1.1: Integration paths to compute the Fourier transform of the Gaussian.

Let us rewrite the integration (1.61) into the Gaussian integral by considering the closed paths (depending on whether ω is positive or negative) depicted in Fig. 1.1. whose "left and right pieces vanish" strongly as the real part goes to (minus) infinity. Moreover, by the Cauchy's integral theorem,

$$\oint_{\mathcal{C}} dx e^{-x^2} = \int_{+\infty}^{-\infty} e^{-x^2}dx + \int_{-\infty+i\frac{\omega\sigma}{\sqrt{2}}}^{+\infty+i\frac{\omega\sigma}{\sqrt{2}}} e^{-x^2}dx = 0, \qquad (1.63)$$

because e^{-x^2} is analytic in the region $0 \leq |\mathrm{Im}\, x| \leq \sigma|\omega|/\sqrt{2}$. Thus, by substituting

$$\int_{-\infty+i\frac{\omega\sigma}{\sqrt{2}}}^{+\infty+i\frac{\omega\sigma}{\sqrt{2}}} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}, \tag{1.64}$$

by insertion of the value $\sqrt{\pi}$ for the *Gaussian integral*, we finally obtain

$$\widetilde{f}(\omega) = \exp(-\frac{\omega^2\sigma^2}{2})\frac{1}{\sigma\sqrt{2\pi}}\sqrt{2}\sigma \underbrace{\int_{-\infty+i\frac{\omega\sigma}{\sqrt{2}}}^{+\infty+i\frac{\omega\sigma}{\sqrt{2}}} e^{-x^2}\ dx}_{=\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}} = \exp\left(\frac{-\sigma^2\omega^2}{2}\right) \tag{1.65}$$

Finally,

$$\widetilde{f}(\omega) = \exp\left(\frac{-\sigma^2\omega^2}{2}\right) \tag{1.66}$$

which is another Gaussian distribution, centered on zero and with a root mean square deviation $\Delta\omega = 1/\sigma$.

It is interesting to note, and an important property, that the Fourier transform of a Gaussian is another Gaussian.

In the above example the root mean square deviation in t was τ, and so it is seen that the deviations or ‘spreads’ in t and in ω are inversely related:

$$\Delta\omega\Delta t = 1 \tag{1.67}$$

Similarly, we could also obtain $\Delta k \Delta x = 1$ for a Gaussian wave packet. [9]

The uncertainty relations as usually expressed in quantum mechanics can be related to this if the de Broglie and Einstein relationships for momentum and energy are introduced; they are

$$p = \frac{h}{2\pi}k \quad \text{and} \quad E = \frac{h}{2\pi}\omega$$

Here $\frac{h}{2\pi}$ is Planck’s constant h divided by 2π.

对于量子力学中的微观粒子 $\Delta x \cdot \Delta p \gtrsim \hbar/2$

Recap:

高斯型函数的傅里叶变换还是高斯型函数

The Fourier transformation of Gaussian function is still Gaussian function.

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right), \quad \widetilde{f}(\omega) = \exp\left(\frac{-\sigma^2\omega^2}{2}\right) \tag{1.69}$$

$\Delta t = \sigma$, $\Delta\omega = \frac{1}{\sigma}$, $\Delta\omega\Delta t = 1$.

[9] In quantum mechanics, we have the Heisenberg uncertainty principle

$$\Delta E \Delta t \geq \frac{h}{4\pi} \quad \text{and} \quad \Delta p \Delta x \geq \frac{h}{4\pi} \tag{1.68}$$

This is the basic principle of microscopic physics, which is widely used in various advanced physics courses.

1.4.2 傅立叶变换解微分方程简介

暂时不需要掌握，下半学期专门学微分方程求解的时候会再讲一遍. 这里只是作为傅立叶变换微分性质的应用:

不用证明，会用就行

重要公式 1.1

(i) 微分性质 Differentiation:

$$\mathcal{F}\left[f'(t)\right] = i\omega \tilde{f}(\omega)$$

This may be extended to higher derivatives, so that

$$\mathcal{F}\left[f''(t)\right] = i\omega\mathcal{F}\left[f'(t)\right] = -\omega^2\tilde{f}(\omega)$$

and so on. 下一小节，用傅立叶变换解微分方程需要用到这个性质

设入射波的电场强度 $\vec{E} = \vec{E}_0 e^{-i\omega t}$, 振子的固有频率为 ω_0, 而 γ 表征唯象阻尼力, 则电子的运动方程为

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = \frac{e}{m}\vec{E}.$$

该方程的解为故意写这个公式，是因为后面接着讨论傅立叶变换和拉普拉斯变换可以提供很简单的求解这类二阶微分方程的方法。

$$\vec{r} = \frac{e\vec{E}}{m\left[\omega_0^2 - \omega^2 - i\omega\gamma\right]},$$

例题 1.8

$$y''(x) + py'(x) + qy(x) = f(x).$$

Because of linearity,

$$(ik)^2\tilde{y} + (ikp)\tilde{y} + q\tilde{y} = \tilde{f},$$

and so

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{f}(k)e^{ikx}}{q + ikp - k^2}dk$$

学而时习之

1. 计算下面函数的傅立叶变换（物理中非常常用）$f(t) = e^{-\alpha|t|}$, with $\alpha > 0$
2.
$$\int_{3+4i}^{4-3i}\left(4z^2 - 3iz\right)dz$$
has the same value on the two paths: (a) the straight line connecting the integration limits, and (b) an arc on the circle $|z| = 5$

课程预告

- □ 傅立叶变换拓展和 delta 函数（需要预习的同学请看教程 10.1 节）
- □ 课前十分钟四位同学到黑板上不看讲义计算，如果有同学觉得例题简单的话，可以做一道上面的习题代替我指定的例题——

唐翼，唐誉珊：对

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) \tag{1.70}$$

做傅立叶变换

王灿坚．王千慧：用傅立叶变换的方法证明

$$\int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega} \mathrm{d}\omega = \pi \quad (a > 0) \tag{1.71}$$