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Posterior properties of the generalized gamma family of distributions and its higher moments

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Abstract: The generalized gamma distribution is an important model that has proven to be very flexible in practice for modeling data from several areas. In this paper, considering a Bayesian analysis we have provided necessary and sufficient conditions to check whether or not improper priors lead proper posterior distributions. Further, we also have discussed sufficient conditions to verify if the obtained posterior moments are finite. An interesting aspect of our findings was that one can check if the posterior is proper or improper and also if its posterior moments are finite by looking directly at the behavior of the proposed improper prior. The proposed methodology was applied in different objective priors.

Key words and phrases: Bayesian Inference; Generalized Gamma Distribution; Objective Prior; Reference Prior.

1 Introduction

The generalized gamma (GG) distribution plays an important role in statistics and has proven to be very flexible in practice for modeling data from several areas, such as climatology, meteorology medicine, reliability and image processing data, among others. Introduced by Stacy (1962) the GG distribution unify many important models such as the exponential, Weibull, gamma, lognormal, generalized normal, Nakagami-m, half-normal, Rayleigh, Maxwell-Boltzmann and chi distribution, to list a few. A random variable X follows a GG distribution if its probability density function (PDF) is given by

$$f(x|\boldsymbol{\theta}) = \frac{\alpha}{\Gamma(\phi)}\mu^{\alpha\phi}x^{\alpha\phi-1}\exp\left(-(\mu x)^{\alpha}\right), \quad x>0 \tag{1.1}$$

where $\Gamma(\phi) = \int_0^{\infty} e^{-x}x^{\phi-1}dx$ is the gamma function, $\boldsymbol{\theta} = (\phi, \mu, \alpha)$, $\alpha > 0$ and $\phi > 0$ are the shape parameters and $\mu > 0$ is a scale parameter.

The parameter estimators for the GG distribution have been discussed earlier considering the maximum likelihood (ML) method (Stacy and Mihram 1965). However, the ML estimators are not well-behaved (Hager and Bain 1970) and its asymptotic properties may not be achieved even for samples greater than 400 (Prentice 1974). From a Bayesian point of view, a subjective analysis can be considered where the prior distribution supplies information from an expert (see O'Hagan et al. (2006)). On the other hand, in many situations, we are interested in obtaining a prior distribution which guarantees that the information provided by the data will not be overshadowed by subjective information. In this case, an objective analysis is recommended by considering non-informative priors that are derived by formal rules (Bernardo 2005). Although several studies have considered weakly informative priors (flat priors) as presumed non-informative priors, Bernardo (2005) argued that using simple proper priors presumed to be non-informative, often hides important unwarranted assumptions which may easily dominate, or even invalidate the statistical analysis and should be strongly discouraged.

Objective priors have been discussed for the generalized gamma distribution (see, Van Noortwijk (2001) and Ramos et al. (2017)). The obtained priors are constructed by formal rules (Kass and Wasserman 1996) and are usually improper, i.e., do not correspond to proper probability distribution and could lead to improper posteriors, which is undesirable. According to Northrop and Attalides (2016), there are no simple conditions that can be used to prove that an improper prior yields a proper posterior for a particular distribution, therefore a case-by-case investigation is needed to check the propriety of posterior distribution. This study overcomes this problem by providing in a simple way necessary and sufficient conditions to check whether or not objective priors lead proper posterior distributions for the generalized gamma distribution. As a result, one can easily check if the obtained posterior is proper or improper directly looking at the behavior of the improper prior.

The proposed methodology is fully illustrated in twelve improper priors such as independent uniform priors, Jeffreys' rule (Kass and Wasserman 1996), Jeffreys' prior (Jeffreys 1946), maximal data information (MDI) prior (Zellner 1977, 1984), reference priors (Bernardo 1979, 2005; Berger et al. 2015a), to list a few. We proved that among the priors considered only one reference prior returned a proper posterior distribution. The proper reference posterior has excellent theoretical properties such as invariance property under one-to-one transformations of the parameters, consistency under marginalization and consistent sampling properties. Despite the fact that the posterior distribution may be proper the posterior moments can be infinite. Therefore, we also provided sufficient conditions to verify if the posterior moments are finite.

The remainder of this paper is organized as follows. Section 2 presents a theorem that provides necessary and sufficient conditions for the posterior distributions to be proper and also sufficient conditions to check if the posterior moments of the parameters are finite. Section 3 presents the applications of our main theorem in different objective priors. Finally, Section 4 summarizes the study.

2 Bayesian Analysis

The joint posterior distribution for $\boldsymbol{\theta}$ is given by the product of the likelihood function and the prior distribution $\pi(\boldsymbol{\theta})$ divided by a normalizing constant $d(\boldsymbol{x})$, resulting in

$$p(\boldsymbol{\theta}|\boldsymbol{x}) = \frac{\pi(\boldsymbol{\theta})}{d(\boldsymbol{x})}\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}\right\}, \tag{2.2}$$

where

$$d(\boldsymbol{x}) = \int\limits_{\mathcal{A}} \pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}\right\}d\boldsymbol{\theta} \tag{2.3}$$

and $\mathcal{A} = \{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space of $\boldsymbol{\theta}$. Consider any prior in the form $\pi\left(\boldsymbol{\theta}\right) \propto \pi(\mu)\pi(\alpha)\pi(\phi)$, the main aim is to find necessary and sufficient conditions for this class of posterior to be proper, i.e., $d(\boldsymbol{x}) < \infty$.

The following propositions are useful to prove the results related to the posterior distribution. Let $\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\}$ denote the *extended real number line* with the usual order $(\geq)$, let $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{R}_0^+$ denote the positive real numbers including 0.

Definition 1. Let $\mathrm{g} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$ and $\mathrm{h} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$, where $\mathcal{U} \subset \mathbb{R}$. We say that $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if there exist $M \in \mathbb{R}^+$ such that $\mathrm{g}(x) \leq M\,\mathrm{h}(x)$ for every $x \in \mathcal{U}$. If $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ and $\mathrm{h}(x) \lesssim \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \propto \mathrm{h}(x)$.

Definition 2. Let $a \in \overline{\mathbb{R}}$, $\mathrm{g} : \mathcal{U} \to \mathbb{R}^+$ and $\mathrm{h} : \mathcal{U} \to \mathbb{R}^+$, where $\mathcal{U} \subset \mathbb{R}$. We say that $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$ if $\limsup_{x \to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} < \infty$. If $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{h}(x) \underset{x \to a}{\lesssim} \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \underset{x \to a}{\propto} \mathrm{h}(x)$.

The meaning of the relations $\mathrm{g}(x) \underset{x \to a^+}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x \to a^-}{\lesssim} \mathrm{h}(x)$ for $a \in \mathbb{R}$ are defined analogously.

Note that, if for some $c \in \mathbb{R}^+$ we have $\lim_{x \to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = c$, then $\mathrm{g}(x) \underset{x \to a}{\propto} \mathrm{h}(x)$. The following proposition is a direct consequence of the above definition.

Proposition 1. *For $a \in \overline{\mathbb{R}}$ and $r \in \mathbb{R}^+$, let $f_1(x) \underset{x \to a}{\lesssim} f_2(x)$ and $g_1(x) \underset{x \to a}{\lesssim} g_2(x)$ then the following hold*

$$f_1(x) g_1(x) \underset{x \to a}{\lesssim} f_2(x) g_2(x) \quad \text{and} \quad f_1(x)^r \underset{x \to a}{\lesssim} f_2(x)^r.$$

The following proposition relates Definition 1 and Definition 2.

Proposition 2. *Let $\mathrm{g} : (a,b) \to \mathbb{R}^+$ and $\mathrm{h} : (a,b) \to \mathbb{R}^+$ be continuous functions on $(a,b) \subset \mathbb{R}$, where $a \in \overline{\mathbb{R}}$ and $b \in \overline{\mathbb{R}}$. Then $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if and only if $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x \to b}{\lesssim} \mathrm{h}(x)$.*

Proof. Suppose that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$. Then, by Definition 2, $\limsup_{x\to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = w$ for some $w \in \mathbb{R}^+$. Therefore, from the definition of lim sup there exist some $a' \in (a, b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{3w}{2}$ for every $x \in (a, a']$. Proceeding analogously, there must exist some $v \in \mathbb{R}^+$ and $b' \in (a', b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{3v}{2}$ for every $x \in [b', b)$. On the other hand, since $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}$ is continuous in $[a', b']$, the Weierstrass Extreme Value Theorem states that there exist some $x_1 \in [a', b']$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}$ for every $x \in [a', b']$. Finally, choosing $M = \max\left(\dfrac{3w}{2}, \dfrac{3v}{2}, \dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}\right) < \infty$, it follows that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$, which by Definition 1 means that $g(x) \lesssim h(x)$.

Now suppose $g(x) \lesssim h(x)$. By Definition 1, there exist some $M < 0$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$. This implies that $\limsup_{x\to a} \frac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M < \infty$ which by Definition 2 means that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$. The proof that $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$ must also be satisfied is analogous to the previous case. Therefore the theorem is proved. □

Note that if $\mathrm{g} : (a, b) \to \mathbb{R}^+$ and $\mathrm{h} : (a, b) \to \mathbb{R}^+$ are continuous functions on $(a, b) \subset \mathbb{R}$, then by continuity it follows directly that $\lim_{x\to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = \dfrac{\mathrm{g}(c)}{\mathrm{h}(c)} > 0$ and therefore $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$ for every $c \in (a, b)$. This fact and the Proposition 2 imply directly the following.

Proposition 3. *Let* $\mathrm{g} : (a,b) \to \mathbb{R}^+$ *and* $\mathrm{h} : (a,b) \to \mathbb{R}^+$ *be continuous functions in* $(a,b) \subset \mathbb{R}$*, where* $a \in \overline{\mathbb{R}}$ *and* $b \in \overline{\mathbb{R}}$*, and let* $c \in (a,b)$*. Then if* $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ *(or* $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$*) we have that* $\int_a^c g(t)\ dt \lesssim \int_a^c h(t)\ dt$ *(respectively* $\int_c^b g(t)\ dt \lesssim \int_c^b h(t)\ dt$ *).*

Theorem 1. *Suppose that* $\pi(\alpha,\beta,\mu) < \infty$ *for all* $(\alpha,\beta,\mu) \in \mathbb{R}^3_+$*, that* $n \in \mathbb{N}^+$*, and suppose that* $\pi(\mu,\alpha,\phi) = \pi(\mu)\pi(\alpha)\pi(\mu)$ *where*

$$\pi(\mu) \underset{\mu\to 0^+}{\lesssim} \mu^{k_0}, \quad \pi(\mu) \underset{\alpha\to\infty}{\lesssim} \mu^{k_\infty}, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\lesssim} \alpha^{q_0},$$

$$\pi(\alpha) \underset{\alpha\to\infty}{\lesssim} \alpha^{q_\infty}, \quad \pi(\phi) \underset{\phi\to 0^+}{\lesssim} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\lesssim} \phi^{r_\infty},$$

such that $k_0 \geq -1$*,* $k_\infty \leq -1$*,* $q_\infty < r_0$*,* $2r_\infty + 1 < q_0$ *and* $n > -q_0$*, then* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is proper.*

Proof. See Appendix A.2 □

Theorem 2. *Suppose that* $\pi(\alpha,\beta,\mu) > 0\ \forall(\alpha,\beta,\mu) \in \mathbb{R}^3_+$ *and that* $n \in \mathbb{N}^+$*, then the following items are valid*

i) $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\beta)$ *for all* $\beta \in [b_0,b_1]$ *where* $0 \leq b_0 < b_1$ *and one of the following hold*

- $\pi(\mu) \underset{\mu\to 0^+}{\gtrsim} \mu^{k_0}$ *where* $k_0 < -1$*, or*
- $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^{k_\infty}$ *and* $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$*, where* $k_\infty > -1$ *and* $q_0 \in \mathbb{R}$*,*

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then $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

ii) $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\beta)$ *in which*

$$\pi(\mu) \underset{\mu\to 0^+}{\gtrsim} \mu^{k_0} \quad \text{and} \quad \pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^{k_\infty},$$

where $k_0 \geq -1$ *and* $k_\infty \leq -1$, *and one of the following occur*

- $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *and* $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ *where* $q_\infty \geq r_0$, *or*
- $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ *and* $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$ *where* $2r_\infty + 1 \geq q_0$, *or* $n \leq -q_0$,

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

Proof. See Appendix A.3 □

Theorem 3. *Suppose that* $0 < \pi(\alpha,\beta,\mu) < \infty$ *for all* $(\alpha,\beta,\mu) \in \mathbb{R}^3_+$, *and suppose that* $\pi(\mu,\alpha,\phi) = \pi(\mu)\pi(\alpha)\pi(\phi)$ *where*

$$\pi(\mu) \underset{\mu\to 0^+}{\propto} \mu^{k_0}, \quad \pi(\mu) \underset{\alpha\to\infty}{\propto} \mu^{k_\infty}, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^{q_0},$$

$$\pi(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty}, \quad \pi(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty},$$

then $\alpha^q\phi^r\mu^k\pi(\alpha,\phi,\mu)$ *leads to a proper posterior if and only if* $-1-k_0 \leq k \leq -1-k_\infty$, $2r + (2r_\infty + 1 - q_0) < q < r + (r_0 - q_\infty)$ *and* $n \geq -q_0$.

Proof. By Theorems 2 and 3 we have that $\alpha^q\beta^r\mu^k\pi(\alpha,\beta,\mu)$ leads to a proper posterior if and only if $k + k_0 \leq -1$, $k + k_\infty \geq -1$, $q + q_\infty < r + r_0$,

$2(r + r_\infty) \le q + q_0$ and $n > -q_0$. Combining theses inequalities the proof is completed. $\square$

3 Some common objective priors

A naive approach to obtain objective priors is to consider uniform priors contained in the interval $(0, \infty)$. However, uniform priors are usually not attractive due to its lack of invariance over reparametrizations. The uniform prior for GG distribution is given by $\pi_1(\phi, \mu, \alpha) \propto 1$.

Corollary 1. *The posterior distribution obtained using a joint uniform prior is improper for all $n \in \mathbb{N}^+$.*

Proof. Since $\pi_1(\phi, \mu, \alpha) = \mu^0\alpha^0\phi^0$ we apply Theorem 2 ii) with $k_0 = k_\infty = q_\infty = r_0 = 0$ and since $q_\infty \ge r_0$ we have that $\pi(\alpha, \beta, \mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

Another common approach was suggested by Jeffreys' that considered different procedures for constructing objective priors. For $\theta \in (0, \infty)$ (see, Kass and Wasserman (1996)), Jeffreys suggested to use the prior $\pi(\theta) = \theta^{-1}$. The main justification for this choice is its invariance under power transformations of the parameters. As the parameters of the GG distribution are contained in the interval $(0,\infty)$, the prior using Jeffreys' first rule is $\pi_2\left(\phi,\mu,\alpha\right)\propto(\phi\mu\alpha)^{-1}$.

Corollary 2. *The posterior distribution obtained using Jeffreys' first rule is improper for all $n\in\mathbb{N}^+$.*

Proof. Since $\pi_1(\phi,\mu,\alpha)=\mu^{-1}\alpha^{-1}\phi^{-1}$ we can apply Theorem 2 ii) with $k_0=k_\infty=q_\infty=r_0=-1$, where $q_\infty\geq r_0$, and therefore we have that $\pi(\alpha,\beta,\mu)$ leads to an improper posterior for all $n\in\mathbb{N}^+$. □

Zellner (1977, 1984) discussed another procedure to obtain an objective prior that is based on the information measure known as Shannon entropy. Such prior is known as MDI prior and can be obtained by solving

$$\pi_3\left(\boldsymbol{\theta}\right)\propto\exp\left(\int f(t\,|\,\phi,\mu,\alpha)\log f(t\,|\,\phi,\mu,\alpha)dt\right). \tag{3.4}$$

Ramos et al. (2017) showed that the MDI prior (3.4) for the GG distribution is given by

$$\pi_3\left(\boldsymbol{\theta}\right)\propto\frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\}. \tag{3.5}$$

Corollary 3. *The joint posterior density using the MDI prior (3.5) is improper for any $n\in\mathbb{N}^+$.*

Proof. Since $\psi(\phi)<0$ for all $\phi\in(0,1]$ (see Abramowitz and Stegun (1972)), we have that $\exp\left(-\psi(\phi)\dfrac{1}{\alpha}\right) \geq 1$ for all $\phi \in [0.5, 1]$ and therefore

$$\pi_3(\boldsymbol{\theta}) \gtrsim \alpha\mu \frac{\exp\left(\psi(\phi)\phi - \phi\right)}{\Gamma(\phi)}.$$

in the interval $[0.5, 1]$. It follows that the hypothesis in Theorem 2 i) is satisfied with $b_0 = 0.5$, $b_1 = 1$, $k_\infty = 1 > -1$ and $q_0 = 1$, and therefore we have that $\pi_3(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

4 Priors based on the Fisher information matrix

The priors discussed in this section belong to the class of improper priors given by

$$\pi_j\left(\boldsymbol{\theta}\right) \propto \frac{\pi_j(\phi)}{\mu}, \tag{4.6}$$

where j is the index related to a particular prior. Therefore, our main focus in this section will be to study the behavior of the priors $\pi_j(\phi)$.

One important objective prior is based on Jeffreys' general rule (Jeffreys 1946) and known as Jeffreys' prior. This prior is obtained through the square root of the determinant of the Fisher information matrix and has been widely used due to its invariance property under one-to-one transformations. The Fisher information matrix for the GG distribution was derived by Hager and Bain (1970) and its elements are given by

$$I_{\alpha,\alpha}(\boldsymbol{\theta}) = \frac{1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2}{\alpha^2},\ I_{\alpha,\mu}(\boldsymbol{\theta}) = -\frac{\psi(\phi)}{\alpha},\ I_{\mu,\phi}(\boldsymbol{\theta}) = \frac{\alpha}{\mu},$$

$$I_{\alpha,\phi}(\boldsymbol{\theta}) = -\frac{1+\phi\psi(\phi)}{\mu},\quad I_{\mu,\mu}(\boldsymbol{\theta}) = \frac{\phi\alpha^2}{\mu^2} \quad \text{and} \quad I_{\phi,\phi}(\boldsymbol{\theta}) = \psi'(\phi).$$

Van Noortwijk (2001) provided the Jeffreys prior for the GG distribution, which can be expressed by (4.6) with

$$\pi_4(\phi) \propto \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}. \tag{4.7}$$

Corollary 4. *The posterior distribution using the Jeffreys prior (4.7) is improper for all $n \in \mathbb{N}^+$.*

Proof. Ramos et al. (2017) proved that

$$\sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to 0^+}{\propto} 1 \ \text{ and } \ \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to\infty}{\propto} \frac{1}{\phi}. \tag{4.8}$$

Since $\pi_4(\phi) \underset{\phi\to 0^+}{\propto} 1$, the hypotheses of Theorem 2 ii) hold with $k_0 = k_\infty = -1$ and $r_0 = q_\infty = 0$, where $q_\infty \geq r_0$, and therefore $\pi_4(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. □

Fonseca et al. (2008) considered the scenario where the Jeffreys prior has an independent structure, i.e., the prior has the form $\pi_{J2}(\boldsymbol{\theta}) \propto \sqrt{|\operatorname{diag} I(\boldsymbol{\theta})|}$, where $\operatorname{diag} I(\cdot)$ is the diagonal matrix of $I(\cdot)$. For the GG distribution the prior is given by (4.6) with

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)}. \tag{4.9}$$

Corollary 5. *The posterior distribution using the independent Jeffreys' prior (4.9) is improper for all $n \in \mathbb{N}^+$.*

Proof. By Abramowitz and Stegun (1972), we have the recurrence relations

$$\psi(\phi) = -\frac{1}{\phi} + \psi(\phi+1) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi^2} + \psi'(\phi+1). \tag{4.10}$$

It follows that

$$2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1 =$$
$$2\left(-\frac{1}{\phi} + \psi(\phi+1)\right) + \phi\left(\frac{1}{\phi^2} + \psi'(\phi+1)\right) + \phi\left(\frac{1}{\phi^2} - \frac{2}{\phi}\psi(\phi+1) + \psi(\phi+1)^2\right) + 1 =$$
$$1 + \phi\left(\psi(\phi+1)^2 + \psi'(\phi+1)\right).$$

Hence, $2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1 \underset{\phi\to 0^+}{\propto} 1$, which implies that

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}}, \tag{4.11}$$

then, Theorem 2 ii) can be applied with $k_0 = k_\infty = -1$, $r_0 = -\frac{1}{2}$ and $q_\infty = 0$ where $q_\infty \geq r_0$ and therefore $\pi_5(\boldsymbol{\theta})$ leads to an improper posterior. $\square$

This approach can be further extended considering that only one parameter is independent. For instance, let (θ_1, θ_2) be dependent parameters and θ_3 be independent then under the partition the $((\theta_1, \theta_2), \theta_3)$-Jeffreys prior is given by

$$\pi(\boldsymbol{\theta}) \propto \sqrt{\left(I_{11}(\boldsymbol{\theta})I_{22}(\boldsymbol{\theta}) - I_{12}^2(\boldsymbol{\theta})\right) I_{33}(\boldsymbol{\theta})}. \tag{4.12}$$

For the GG distribution the partition $((\phi,\mu),\alpha)$-Jeffreys' prior is of the form (4.6) with

$$\pi_6\left(\phi\right) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)}. \tag{4.13}$$

Corollary 6. *The posterior distribution using the $((\phi,\mu),\alpha)$-Jeffreys' prior is improper for all $n \in \mathbb{N}^+$.*

Proof. From the recurrence relations (4.10) we have that

$$\phi\psi^{'}(\phi)-1=\frac{1}{\phi}+\phi\psi^{'}(\phi+1)-1 \Rightarrow \phi\psi^{'}(\phi)-1 \underset{\phi\to 0^+}{\propto} \frac{1}{\phi}. \tag{4.14}$$

Together with the relation (4.11) this implies that

$$\pi_6\left(\phi\right) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}}. \tag{4.15}$$

Therefore, Theorem 2 ii) can be applied with $k_0=k_\infty=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$ and therefore $\pi_6(\boldsymbol{\theta})$ leads to an improper posterior. □

On the other hand, the partition $((\phi,\alpha),\mu)$-Jeffreys prior is given by (4.6) where

$$\pi_7\left(\phi\right) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2}. \tag{4.16}$$

Corollary 7. *The posterior distribution using the independent Jeffreys' prior (4.16) is improper for all $n \in \mathbb{N}^+$.*

Proof. From (4.10) we have that

$$
\begin{aligned}
\pi_7^{\frac{1}{2}}(\phi) \propto\ & \phi\psi^{'}(\phi)\left(1+2\psi(\phi)+\phi\psi^{'}(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2 \\
&= \left(\phi^{-1}+\phi\psi'(\phi+1)\right)\left(1+\phi\left(\psi(\phi+1)^2+\psi'(\phi+1)\right)\right)-\phi\left(-\phi^{-1}+\psi(\phi+1)\right)^2 \\
&= \phi\left(\psi'(\phi+1)-\psi(\phi+1)^2+\phi\psi'(\phi+1)(\psi(\phi+1)^2+\psi'(\phi+1))\right)+\psi(\phi+1)^2 \\
&\quad +2\psi(\phi+1)+\psi'(\phi+1) \\
&\propto\ \phi\psi^{'}(\phi)\left(1+2\psi(\phi)+\phi\psi^{'}(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2 \\
&\underset{\phi\to 0^+}{\propto}\ \psi(1)^2+2\psi(1)+\psi'(1)=\gamma^2-2\gamma+\frac{\pi}{6}>0,
\end{aligned}
$$

then, Theorem 2 ii) can be applied with $k_0 = k_\infty = -1$, $r_0 = 0$ and $q_\infty = 0$ where $q_\infty \geq r_0$ and therefore $\pi_7(\boldsymbol{\theta})$ leads to an improper posterior. $\square$

Finally, the $((\alpha,\mu),\phi)$-Jeffreys prior is given by (4.6) where

$$
\pi_8(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)}\,. \tag{4.17}
$$

Corollary 8. *The posterior distribution using the independent Jeffreys' prior (4.17) is improper for all $n \in \mathbb{N}^+$.*

Proof. From the recurrence relations (4.10) we have that

$$
\phi^2\psi^{'}(\phi)+\phi-1 = \phi\left(1+\phi\psi^{'}(\phi+1)\right) \Rightarrow \phi^2\psi^{'}(\phi)+\phi-1 \underset{\phi\to 0^+}{\propto} \phi \tag{4.18}
$$

as $\psi'(\phi) \propto \frac{1}{\phi^2}$ it follows that

$$
\pi_8(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}},
$$

and Theorem 2 ii) can be applied with $k_0 = k_\infty = -1$, $r_0 = -\frac{1}{2}$ and $q_\infty = 0$ where $q_\infty \geq r_0$. Therefore $\pi_8(\boldsymbol{\theta})$ leads to an improper posterior. $\square$

5 Reference priors

Introduced by Bernardo (1979) with further developments by Berger and Bernardo (1989, 1992); Berger et al. (1992) reference priors play an important role in objective Bayesian analysis. The reference priors have desirable properties, such as invariance, consistent marginalization and consistent sampling properties. Bernardo (2005) reviewed different procedures to derive reference priors considering ordered parameters of interest. The following proposition will be applied to obtain reference priors for the Generalized Gamma distribution.

Proposition 4. *[Bernardo (1979), pg 40, Theorem 14] Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ be a vector with the ordered parameters of interest and $p(\boldsymbol{\theta}|\boldsymbol{x})$ be the posterior distribution that has an asymptotically normal distribution with dispersion matrix $V(\hat{\boldsymbol{\theta}}_n)/n$, where $\hat{\boldsymbol{\theta}}_n$ is a consistent estimator of $\boldsymbol{\theta}$ and $H(\boldsymbol{\theta}) = V^{-1}(\boldsymbol{\theta})$. In addition, V_j is the upper $j \times j$ submatrix of V, $H_j = V_j$ and $h_{j,j}(\boldsymbol{\theta})$ is the lower right element of H_j. If the parameter space of θ_j is independent of $\boldsymbol{\theta}_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_m)$, for $j = 1, \ldots, m$, and $h_{j,j}(\boldsymbol{\theta})$*

are factorized in the form $h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j)g_j(\boldsymbol{\theta}_{-j}), \quad j = 1, \ldots, m,$ *then the reference prior for the ordered parameters* $\boldsymbol{\theta}$ *is given by*

$$\pi(\boldsymbol{\theta}) = \pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1),$$

where $\pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) = f_j(\theta_j)$, *for* $j = 1, \ldots, m$, *and there is no need for compact approximations, even if the conditional priors are not proper.*

The reference priors obtained from Proposition 4 belong to the class of improper priors given by

$$\pi(\boldsymbol{\theta}) \propto \frac{\pi(\phi)}{\alpha\mu}, \tag{5.19}$$

therefore, our focus will be in the behavior of $\pi(\phi)$. Let (α, ϕ, μ) be the ordered parameters of interest, then conditional priors of the (α, ϕ, μ)-reference prior are given by

$$\pi(\alpha) \propto \frac{1}{\alpha}, \qquad \pi(\phi|\alpha) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}, \qquad \pi(\mu|\alpha, \phi) \propto \frac{1}{\mu}\cdot$$

Therefore, (α, ϕ, μ)-reference prior is of the form (5.19) with

$$\pi_9(\phi) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}\cdot$$

Corollary 9. *The posterior density using the* (α, ϕ, μ)*-reference prior is improper for all* $n \in \mathbb{N}^+$.

Proof. By equation (4.14) we have that

$$\pi_9(\phi) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}} \underset{\phi \to 0^+}{\propto} \frac{1}{\phi},$$

therefore, item (ii) of Theorem 2 can be applied with $k_0 = k_\infty = r_0 = q_\infty = -1$ where $q_\infty \geq r_0$ which implies that $\pi_9(\alpha, \phi, \mu)$ leads to an improper posterior.

$\square$

On the other hand, if (α, μ, ϕ) are the ordered parameters, then the conditional reference priors are

$$\pi(\alpha) \propto \frac{1}{\alpha}, \qquad \pi(\mu|\alpha) \propto \frac{1}{\mu}, \qquad \pi(\phi|\alpha, \mu) \propto \sqrt{\psi'(\phi)},$$

and the (α, μ, ϕ)-reference prior is of the form (5.19) with

$$\pi_{10}(\phi) \propto \sqrt{\psi'(\phi)}.$$

Corollary 10. *The posterior density using the (α, μ, ϕ)-reference prior is improper for all $n \in \mathbb{N}^+$.*

Proof. By the equation (4.10) we have that $\sqrt{\psi'(\phi)} \underset{\phi \to 0^+}{\propto} \frac{1}{\phi}$. Thus, as in Corollary 9 we have that $\pi_{10}(\alpha, \mu, \phi)$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

In the case of (μ, ϕ, α) be the vector of ordered parameters, we have that the conditional priors are

$$\pi(\mu) \propto \frac{1}{\mu}, \;\; \pi(\phi|\mu) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi^2) + 1}}, \;\; \pi(\alpha|\phi, \mu) \propto \frac{1}{\alpha}.$$

and the (μ, ϕ, α)-reference prior is of the form (5.19) with

$$\pi_{11}(\phi) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi^2) + 1}}.$$

Corollary 11. *The posterior density using the (μ, ϕ, α)-reference prior is improper for all $n \in \mathbb{N}^+$.*

Proof. From Abramowitz and Stegun (1972), we have

$$\psi(\phi) = \log(\phi) - \frac{1}{2\phi} - \frac{1}{12\phi^2} + o\left(\frac{1}{\phi^2}\right) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi} + \frac{1}{2\phi^2} + o\left(\frac{1}{\phi^2}\right), \tag{5.20}$$

where it follows directly that

$$\psi(\phi)^2 = \log(\phi)^2 - \frac{\log(\phi)}{\phi} + o\left(\frac{1}{\phi}\right).$$

Therefore $2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1 = \phi\log(\phi)^2 + \log(\phi) + 2 + o(1)$ and

$$\begin{aligned}
\pi_{11}(\phi) &\propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1}} \\
&= \sqrt{\frac{\left(\frac{1}{\phi} + \frac{1}{2\phi^2} + o\left(\frac{1}{\phi^2}\right)\right)\left(\phi\log(\phi)^2 + \log(\phi) + 2 + o(1)\right) - \log(\phi)^2 + \frac{\log(\phi)}{\phi} + o\left(\frac{1}{\phi}\right)}{\phi\log(\phi)^2 + \log(\phi) + 2 + o(1)}} \\
&= \sqrt{\frac{\frac{1}{\phi}\left(\log(\phi)^2 + o(\log(\phi)^2)\right)}{\phi\left(\log(\phi)^2 + o(\log(\phi)^2)\right)}} = \frac{1}{\phi}\sqrt{\frac{1 + o(1)}{1 + o(1)}}.
\end{aligned}$$

Thus

$$\pi_{11}(\phi) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2 + 1}} \underset{\phi \to 0^+}{\propto} \frac{1}{\phi},$$

and therefore Theorem 2 ii) can be applied with $k_0 = k_\infty = q_0 = r_\infty = -1$ where $2r_\infty + 1 \geq q_0$. Therefore $\pi_{11}(\boldsymbol{\theta})$ leads to an improper posterior.

$\square$

If (μ, α, ϕ) are the ordered parameters then the conditional priors are given by

$$\pi(\mu) \propto \frac{1}{\mu}, \quad \pi(\alpha|\mu) \propto \frac{1}{\alpha}, \quad \pi(\phi|\alpha, \mu) \propto \sqrt{\psi'(\phi)}$$

and the joint (μ, α, ϕ)-reference prior has the same form of $\pi_{10}(\boldsymbol{\theta})$ and its posterior is improper from Corollary 10.

Finally, let (ϕ, α, μ) be the ordered parameters, then the conditional priors are

$$\pi(\phi) \propto \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}}, \quad \pi(\alpha|\phi) \propto \frac{1}{\alpha}, \quad \pi(\mu|\alpha, \phi) \propto \frac{1}{\mu}$$

and the (ϕ, α, μ)-reference prior is of the form (5.19) with

$$\pi_{12}(\phi) \propto \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}}. \tag{5.21}$$

It is important to point out that (ϕ, μ, α)-reference prior is the same as the (ϕ, α, μ)-reference prior, which completes all possible reference priors obtained from Proposition 4.

Corollary 12. *The posterior distribution using the (ϕ, α, μ)-reference prior (5.21) is proper for $n \geq 2$ and its higher moments are improper for all $n \in \mathbb{N}^+$.*

Proof. From (4.8) and by the asymptotic relations (5.20) we have that

$$\phi^2\psi^{'}(\phi)+\phi-1 = 2\phi-\frac{1}{2}+o\left(1\right) \underset{\phi\to\infty}{\propto} \phi$$

which together with equation (4.18) implies that

$$\sqrt{\phi^2\psi'(\phi)+\phi-1} \underset{\phi\to 0^+}{\propto} \sqrt{\phi} \ \text{ and } \ \sqrt{\phi^2\psi'(\phi)+\phi-1} \underset{\phi\to\infty}{\propto} \sqrt{\phi}.$$

Hence, from the above proportionalities we have that

$$\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}} \ \text{ and } \ \sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}} \underset{\phi\to\infty}{\propto} \frac{1}{\sqrt{\phi^3}}.$$

Therefore, Theorem 1 can be applied with $k_0 = k_\infty = q_0 = q_\infty = -1$, $r_0 = -\frac{1}{2}$ and $r_\infty = -\frac{3}{2}$ where $k_0 \geq -1$, $k_\infty \leq -1$, $q_\infty < r_0$ and $2r_\infty + 1 < q_0$, and therefore $\pi_{12}(\alpha, \mu, \phi)$ leads to an proper posterior for every $n > -q_0 = 1$.

In order to prove that the higher moments are improper suppose $\alpha^q\phi^r\mu^k\pi(\boldsymbol{\theta})$ leads to a proper posterior for $r \in \mathbb{N}$, $q \in \mathbb{N}$ and $r \in \mathbb{N}$. By Theorem 3 we have $k + k_0 \leq -1$, $k + k_\infty \geq -1$, $q + q_\infty < r + r_0$, $2(r + r_\infty) \leq q + q_0$ and $n \geq -q_0$, i.e., $k = 0$ and $2r - 1 < q < r + \frac{1}{2}$. The inequality $2r - 1 < r + \frac{1}{2}$ leads to $r < \frac{3}{2}$, i.e., $r = 0$ or $r = 1$. By the previous inequality, the case where $r = 0$ leads to $-1 < q < \frac{1}{2}$, that is, $q = 0$. Now, for $r = 1$ we have the inequality $1 < q < \frac{3}{2}$ which do not have integer solution. Therefore, the only possible solution is $q = r = k = 0$ which implies that the higher moments are improper. $\square$

6 Discussion

We have provided in a simple way necessary and sufficient conditions to check whether or not improper priors lead to proper posterior distributions for the GG distribution. In this case, one can easily check if the obtained posterior is proper or improper directly looking at the behavior of the improper prior. From the main theorem, we proved that the uniform prior, the prior obtained from Jeffreys' first rule and the MDI prior lead to improper posteriors.

The impropriety of the posterior using the Jeffreys' priors (Van Noortwijk 2001) led us to consider the scenario where the Jeffreys prior has an independent structure (Fonseca et al. 2008). However, the four possible objective priors also returned improper posteriors. An alternative was to consider the reference priors discussed in Bernardo (1979) with further developments by Berger and Bernardo (1989, 1992); Berger et al. (1992). Since these priors are sensitive to the ordering of the unknown parameters, from Proposition 4 we obtained six reference priors, two of them were similar to other reference priors. Among the four distinct reference priors, we proved that only one leads to a proper posterior distribution without the need of compact approximations or truncate possible values of the parameters. The proper reference posterior has excellent theoretical properties such as invariance property under one-to-one transformations of the parameters, consistency under marginalization and consistent sampling properties and should be used to make inference in the parameters of the GG distribution.

Due to the consistent marginalization property of the reference prior the reference marginal posterior distribution of ϕ and α is

$$p_{12}(\phi,\alpha|\boldsymbol{x}) \propto \alpha^{n-2}\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}}\left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{\alpha}}}{\sum_{i=1}^{n} t_i^{\alpha}}\right)^{n\phi},$$

while the conditional posterior distributions for μ given ϕ and α is given by

$$p_{12}(\mu|\phi,\alpha,\boldsymbol{x}) \sim \mathrm{GG}\left(n\phi, \left(\sum_{i=1}^{n} t_i^{\alpha}\right)^{\frac{1}{\alpha}}, \alpha\right).$$

These results are useful to obtain posterior estimates using Markov chain Monte Carlo methods. Since we proved that the posterior mean for the parameter does not return finite values, the posterior median can be an alternative as a posterior estimate.

There are a large number of possible extensions of this current work. Roy and Dey (2014) considered an objective Bayesian analysis for the generalized extreme value regression distribution. The proposed results can be further extended for the generalized gamma regression distribution. Another extension that can be explored is to consider other objective priors that are also constructed by formal rules such as other types of reference priors (Berger et al. 2015b) and probability matching priors (Mukerjee and

Dey 1993; Datta and Mukerjee 2012).

Disclosure statement

No potential conflict of interest was reported by the author(s)

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A Appendix

A.1 Useful Proportionalities

The following proportionalities are useful to prove results related to the posterior distribution and its proofs can be seen in Ramos et al. (2017).

Proposition 5. *Let* $\mathrm{p}(\alpha) = \log\left(\dfrac{\frac{1}{n}\sum_{i=1}^{n} t_i^{\alpha}}{\sqrt[n]{\prod_{i=1}^{n} t_i^{\alpha}}}\right)$, $\mathrm{q}(\alpha) = \mathrm{p}(\alpha) + \log n$, *for* $t_1, t_2, \ldots, t_n$ *positive and not all equal,* $k \in \mathbb{R}^+$, $r \in \mathbb{R}^+$ *and* $t_m = \max\{t_1, \ldots, t_n\}$, *then* $\mathrm{p}(\alpha) > 0$, $\mathrm{q}(\alpha) > 0$ *and the following results hold*

$$\mathrm{p}(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^2 \quad \textit{and} \quad \mathrm{p}(\alpha) \underset{\alpha\to\infty}{\propto} \alpha;$$

$$\mathrm{q}(\alpha) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \mathrm{q}(\alpha) \underset{\alpha\to\infty}{\propto} \alpha;$$

$$\frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \underset{\phi\to 0^+}{\propto} \phi^{n-1} \quad \textit{and} \quad \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \underset{\phi\to\infty}{\propto} \phi^{\frac{n-1}{2}} n^{n\phi};$$

$$\gamma\left(k, r\,\mathrm{q}(\alpha)\right) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \gamma\left(k, r\,\mathrm{q}(\alpha)\right) \underset{\alpha\to\infty}{\propto} 1; \tag{A.22}$$

$$\Gamma\left(k, r\,\mathrm{p}(\alpha)\right) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \Gamma\left(k, r\,\mathrm{p}(\alpha)\right) \underset{\alpha\to\infty}{\propto} \alpha^{k-1} e^{-rk(\boldsymbol{x})\alpha}; \tag{A.23}$$

where $k = \log\left(\dfrac{t_m}{\sqrt[n]{\prod_{i=1}^{n} t_i}}\right) > 0$; $\gamma(y,x) = 1 - \Gamma(y,x)$ *and* $\Gamma(y,x) = \int_x^{\infty} w^{y-1} e^{-w}\, dw$ *is the upper incomplete gamma function.*

A.2 Proof of Theorem 1

Since $\pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\} \pi(\mu)\mu^{n\alpha\phi-1} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$
\begin{aligned}
d(\boldsymbol{x}) &= \int\limits_{\mathcal{A}} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\boldsymbol{\theta} \\
&= \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha.
\end{aligned}
$$

Now, since $k_0 \geq -1$ and $k_\infty \leq -1$ by hypothesis we have that $\pi(\mu) \underset{\mu\to 0^+}{\lesssim} \mu^{-1}$, $\pi(\mu) \underset{\mu\to\infty}{\lesssim} \mu^{-1}$ and therefore $\pi(\mu) \lesssim \mu^{-1}$ and therefore

$$
\begin{aligned}
d(\boldsymbol{x}) &\lesssim \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^{\alpha}\right)^\phi \mu^{n\alpha\phi-1} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha \\
&= \int\limits_0^\infty \int\limits_0^\infty \pi(\alpha)\alpha^{n-1} \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^{\alpha}\right)^\phi \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}\, d\phi\, d\alpha \\
&= \int\limits_0^\infty \int\limits_0^\infty \pi(\alpha)\alpha^{n-1} \pi(\phi) \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha
\end{aligned}
$$

where $\mathrm{q}(\alpha)$ is given in Proposition 5. Therefore, from the proportionalities in Proposition 5 it follows that

$$\begin{aligned}
d(\boldsymbol{x}) &\lesssim \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^{n-1}\pi(\phi)\frac{\Gamma(n\phi)}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&\propto \int_0^1 \int_0^1 f(\alpha,\phi)\, d\phi\, d\alpha + \int_1^\infty \int_0^1 f(\alpha,\phi)\, d\phi\, d\alpha + \int_0^1 \int_1^\infty g(\alpha,\phi)\, d\phi\, d\alpha + \int_1^\infty \int_1^\infty g(\alpha,\phi)\, d\phi\, d\alpha \\
&= s_1 + s_2 + s_3 + s_4,
\end{aligned} \tag{A.24}$$

where $f(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}$, $g(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{\frac{n-1}{2}}e^{-n\,\mathrm{p}(\alpha)\phi}$ and s_1, s_2, s_3 and s_4 denote the respective four real numbers in the sum that precedes it. It follows that $d(\boldsymbol{x}) < \infty$, if and only if $s_1 < \infty$, $s_2 < \infty$, $s_3 < \infty$ and $s_4 < \infty$. Now, using the proportionalities in Proposition 5 it follows that

$$\begin{aligned}
s_1 &\lesssim \int_0^1 \alpha^{q_0+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&= \int_0^1 \alpha^{q_0+n-1} \frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\, d\alpha \propto \int_0^1 \alpha^{q_0+n-1}\, d\alpha < \infty,
\end{aligned}$$

$$\begin{aligned}
s_2 &\lesssim \int_1^\infty \alpha^{q_\infty+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty-r_0-1}\, d\alpha < \infty,
\end{aligned}$$

$$s_3 \lesssim \int_0^1 \alpha^{q_0+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$

$$= \int_0^1 \alpha^{q_0+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{(q_0-2r_\infty-1)-1}\, d\alpha < \infty, \text{ and}$$

$$s_4 \lesssim \int_1^\infty \alpha^{q_\infty+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$

$$= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty+n-2} e^{-nk\alpha}\, d\alpha < \infty,$$

where in the last line $k \in \mathbb{R}^+$ is given in Proposition 5. Therefore, from $s_i < \infty, i = 1, \ldots, 4$, we have that $d = s_1 + s_2 + s_3 + s_4 < \infty$.

A.3 Proof of Theorem 2

Suppose that hypothesis of item $i)$ hold.

First suppose that $\pi(\mu) \underset{\mu\to 0^+}{\gtrsim} \mu^{k_\infty}$ with $k_0 < -1$. For $h = \sqrt{\frac{-k_0-1}{2n}} > 0$, fixing $0 < \alpha \leq h$ and $0 < \phi \leq h$ we have that $n\alpha\phi + (k_0+1) - 1 \leq nh^2 + (k_0+1) - 1 = \frac{(k_0+1)}{2} - 1 < -1$. Moreover, for every $\alpha > 0$ fixed we have that $\exp\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\} \underset{\mu\to 0^+}{\propto} 1$, hence, from Proposition 3 we have that

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \gtrsim \int_0^1 \mu^{n\alpha\phi+(k_0+1)-1} = \infty,$$

for all $\alpha \in (0,h]$ and $\phi \in (0,h]$. Therefore

$$
\begin{aligned}
d(\boldsymbol{x}) &\gtrsim \int_{h/2}^{h}\int_{h/2}^{h} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{\phi} \int_0^{\infty} \mu^{n\alpha\phi+(k+1)-1}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\phi\, d\alpha \\
&\propto \int_{h/2}^{h}\int_{h/2}^{h} \infty\, d\phi\, d\alpha = \infty,
\end{aligned}
$$

that is, $d(\boldsymbol{x}) = \infty$. Now suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^{k_\infty}$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k_\infty > -1$ and $q_0 \in \mathbb{R}$. Then, from the proportionalities in Proposition 5 we have that

$$
\begin{aligned}
d(\boldsymbol{x}) &\propto \int_{\mathcal{A}} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\boldsymbol{\theta} \\
&= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\phi\, d\alpha \\
&\gtrsim \int_0^{\infty}\int_0^{\infty}\int_1^{\infty} \alpha^{n+q_0} \frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{\phi} \mu^{n\alpha\phi+(k_\infty+1)-1}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\phi\, d\alpha \\
&= \int_0^{\infty}\int_0^{\infty}\int_{\sum_{i=1}^{n} x_i^{\alpha}}^{\infty} \alpha^{n+q_0} \frac{\pi(\phi)}{\Gamma(\phi)^n}\frac{\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{\phi}}{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{n\phi+\frac{k_\infty+1}{\alpha}}} u^{n\phi+\frac{k_\infty+1}{\alpha}-1} e^{-u}\, du\, d\phi\, d\alpha \\
&\geq \int_0^{\infty}\int_1^{\infty} \frac{\pi(\phi)}{\Gamma(\phi)^n} n^{-n\phi} u^{n\phi-1} e^{-u} \int_0^{\infty} \alpha^{n+q_0} e^{-\,\mathrm{p}(\alpha)\left(n\phi+\frac{k_\infty+1}{\alpha}\right)+(\log u-\log n)\frac{k+1}{\alpha}}\, d\alpha\, du\, d\phi
\end{aligned}
$$

where in the above we used the change of variables $u = \mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}$ in the integral, in the last inequality we used the fact that $\sum_{i=1}^{n} x_i^{\alpha} \geq 1$ for $\alpha \geq 0$, and $\mathrm{p}(\alpha)$ is given as in Proposition 5. Now, since $\mathrm{p}(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^2$ from Proposition 5 it follows due to the Proposition 2 that $\mathrm{p}(\alpha) \propto \alpha^2$ for $\alpha \in [0,1]$

and therefore $\lim_{\alpha\to 0^+} e^{-\,\mathrm{p}(\alpha)\left(n\phi+\frac{k_\infty+1}{\alpha}\right)} = \lim_{\alpha\to 0^+} e^{-\frac{p(\alpha)}{\alpha^2}(n\phi\alpha+k_\infty+1)\alpha} = e^0 = 1$. Thus, since $n \geq 1$ and $\log u - \log n > 0$ for $u \geq 3n > e\cdot n$, and since $\int_0^1 \alpha^H e^{\frac{L}{\alpha}} = \infty$ for every $H \in \mathbb{R}$ and $L \in R^+$ (which can be easily checked via the change of variable $\beta = \frac{1}{\alpha}$ in the integral), it follows that

$$d(x) \gtrsim \int_0^\infty \int_{3n}^\infty \pi(\phi)\frac{1}{\Gamma(\phi)^n} n^{-n\phi}u^{n\phi-1}e^{-u}\int_0^1 \alpha^{n+q_0}e^{(\log u-\log n)\frac{k+1}{\alpha}}\,d\alpha\,du\,d\phi$$
$$= \int_0^\infty \int_{3n}^\infty \infty\,du\,d\phi = \infty,$$

and therefore $d(x) = \infty$.

Now suppose the hypothesys of ii) hold. First suppose that

$$\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0} \qquad \text{and } \pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$$

where $q_\infty \geq r_0$. Then, following the same steps that resulted in (A.24) and the same expressions for s_i, where $i = 1, \cdots, 4$, we have that $d(\boldsymbol{x}) \gtrsim s_1 + s_2 + s_3 + s_4$ where

$$s_2 \gtrsim \int_1^\infty \alpha^{q_\infty+n-1}\int_0^1 \phi^{n+r_0-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha$$
$$= \int_1^\infty \alpha^{q_\infty+n-1}\frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\,d\alpha \propto \int_1^\infty \alpha^{q_\infty-r_0-1}\,d\alpha = \infty$$

and therefore $d(\boldsymbol{x}) = \infty$.

Now suppose that

$$\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0} \qquad \text{and} \qquad \pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$$

where $2r_\infty + 1 \geq q_0$ or $n \leq -q_0$. Then, following the same steps that resulted in (A.24) and the same expressions for s_i, where $i = 1, \cdots, 4$, we have that $d(\boldsymbol{x}) \gtrsim s_1 + s_2 + s_3 + s_4$ where

$$
\begin{aligned}
s_3 &\gtrsim \int_0^1 \alpha^{q_0+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha \\
&= \int_0^1 \alpha^{q_0+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{(q_0-2r_\infty-1)-1}\, d\alpha = \infty
\end{aligned}
$$

which implies $d(\boldsymbol{x}) = \infty$.

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