ARTICLE TEMPLATE

# Objective Bayesian analysis for censored generalized gamma model

Eduardo Ramos[a,c], Pedro Luiz Ramos[b*], and Jeremias Leão[c]

[a] Departamento de Estatística, Universidade Federal do Amazonas, Manaus, Brazil
[b] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile

**ARTICLE HISTORY**
Compiled May 4, 2023

**Abstract**
The are currently many important lifetime models that have proven to be very flexible in practice for modeling data from several areas such as gamma, Weibull, Nakagami, generalized half-normal, generalized gamma distribution and some of its special cases . In this paper, considering a Bayesian analysis we have provided necessary and sufficient conditions to check whether or not improper priors lead proper posterior distributions for the cited distributions. Further, we also have discussed sufficient conditions to verify if the obtained posterior moments are finite. An interesting aspect of our findings was that one can check if the posterior is proper or improper and also if its posterior moments are finite by looking directly at the behavior of the proposed improper prior. The proposed methodology was applied in different objective priors.

**KEYWORDS**
Bayesian Inference; Generalized Gamma Distribution; Objective Prior; Reference Prior.

Lifetime models plays an important role in statistics and has proven to be very flexible in practice for modeling data from several other areas, such as climatology, meteorology medicine, reliability and image processing data, among others. Introduced by Stacy [21] the generalized gamma distribution unify many important models (see, for instance, Table 1). Let X be a random variable follows a GG distribution if its probability density function (PDF) is given by

$$f(x|\boldsymbol{\theta}) = \frac{\alpha}{\Gamma(\phi)}\mu^{\alpha\phi}x^{\alpha\phi-1}\exp\left(-(\mu x)^{\alpha}\right), \quad x > 0 \tag{1}$$

where $\Gamma(\phi) = \int_0^{\infty} e^{-x}x^{\phi-1}dx$ is the gamma function, $\boldsymbol{\theta} = (\phi, \mu, \alpha)$, $\alpha > 0$ and $\phi > 0$ are the shape parameters and $\mu > 0$ is a scale parameter.

The parameter estimators for the GG distribution and its particular cases have been discussed earlier considering the maximum likelihood (ML) method [22]. However, the ML estimators are not well-behaved [11] for the GG distribution and its asymptotic properties may not be achieved even for samples larger than 400 [18]. From a Bayesian point of view, a subjective analysis can be considered where the prior distribution supplies information from an expert (see [17]). On the other hand, in many situations,

*Corresponding author. Email: pedro.ramos@mat.uc.cl

**Table 1.** The Generalized Gamma family of distributions

| Generalized Gamma | $\mu$ | $\phi$ | $\alpha$ |
|---|---|---|---|
| Exponential | $\cdot$ | 1 | 1 |
| Rayleigh | $\cdot$ | 1 | 2 |
| Haf-Normal | $\cdot$ | 0.5 | 2 |
| Maxwell Boltzmann | $\cdot$ | $\frac{3}{2}$ | 2 |
| scaled chi-square | $\cdot$ | 0.5n | 1 |
| chi-square | 2 | 0.5n | 1 |
| Weibull | $\cdot$ | 1 | $\cdot$ |
| Generalized Haf-Normal | $\cdot$ | 2 | $\cdot$ |
| Gamma | $\cdot$ | $\cdot$ | 1 |
| Erlang | $\cdot$ | $n$ | $\cdot$ |
| Nakagami | $\cdot$ | $\cdot$ | 2 |
| Wilson-Hilferty | $\cdot$ | $\cdot$ | 3 |
| Lognormal | $\cdot$ | $\phi \to \infty$ | $\cdot$ |

$n \in \mathbb{N}$

we are interested in obtaining a prior distribution which guarantees that the information provided by the data will not be overshadowed by subjective information. In this case, an objective analysis is recommended by considering non-informative priors that are derived by formal rules [8]. Although several studies have considered weakly informative priors (flat priors) as presumed non-informative priors, [8] argued that using simple proper priors presumed to be non-informative, often hides significant unwarranted assumptions which may easily dominate, or even invalidate the statistical analysis and should be strongly discouraged.

Objective priors have been discussed for the gamma, Weibull, Nakagami or the generalized gamma distribution (see for instance, [14], [**?** ], [**?** ] [23]; [19]). The obtained priors are constructed by formal rules [13] and are usually improper, i.e., do not correspond to proper probability distribution and could lead to improper posteriors, which is undesirable. According to Northrop and Attalides [16], there are no simple conditions that can be used to prove that improper prior yields a proper posterior for a particular distribution, therefore a case-by-case investigation is needed to check the propriety of the posterior distribution. This study overcomes this problem by providing in a simple way necessary and sufficient conditions to check whether or not objective priors lead proper posterior distributions for the generalized gamma distribution and its submodels. As a result, one can easily check if the obtained posterior is proper or improper directly looking at the behavior of the improper prior.

The proposed methodology is fully illustrated in many improper priors such as independent uniform priors, Jeffreys' rule ([13]), Jeffreys' prior [12], maximal data information (MDI) prior [24], [25], reference priors [7], [8], [5], to list a few. We proved that among the priors considered for the GG distribution, only one reference prior returned a proper posterior distribution. The proper reference posterior has excellent theoretical properties such as invariance property under one-to-one transformations of the parameters, consistency under marginalization and consistent sampling properties. Although the posterior distribution may be proper, the posterior moments can be infinite. Therefore, we also provided sufficient conditions to verify if the posterior moments are finite. These results have also been discussed for its submodels under different objective priors. Finally, we revisited a realistic data analysis case which considered the generalized gamma as a component model. Our theoretical results are applied to show that the Bayesian inference was misused to analyze this data since the posterior distribution using the Jeffreys prior returned an improper posterior.

The remainder of this paper is organized as follows. Section 2 presents theorems that provide necessary and sufficient conditions for the posterior distributions to be proper and also sufficient conditions to check if the posterior moments of the parameters are finite. Sections 3-5 present the applications of our theorems in different objective priors. Section 6 illustrates the relevance of our theoretical results in a real data set. Finally, Section 7 summarizes the study with concluding remarks.

## 1. Bayesian Analysis

The joint posterior distribution for $\boldsymbol{\theta}$ is given by the product of the likelihood function and the prior distribution $\pi(\boldsymbol{\theta})$ divided by a normalizing constant $d(\boldsymbol{x})$, resulting in

$$p(\boldsymbol{\theta}|\boldsymbol{x}) = \frac{\pi(\boldsymbol{\theta})}{d(\boldsymbol{x})}\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}, \tag{2}$$

where

$$d(\boldsymbol{x}) = \int\limits_{\mathcal{A}} \pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}d\boldsymbol{\theta} \tag{3}$$

and $\mathcal{A} = \{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space of $\boldsymbol{\theta}$. Consider any prior in the form $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\alpha)\pi(\phi)$, our main aim is to find necessary and sufficient conditions for this class of posterior to be proper, i.e., $d(\boldsymbol{x}) < \infty$.

The following definitions and propositions are useful to prove the results related to the posterior distribution. Let $\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\}$ denote the *extended real number line* with the usual order $(\geq)$, let $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{R}_0^+$ denote the positive real numbers including 0, and denote $\overline{\mathbb{R}}^+$ and $\overline{\mathbb{R}}_0^+$ analogously. Moreover, if $M \in \mathbb{R}^+$ and $a \in \overline{\mathbb{R}}^+$, we define $M\cdot a$ as the usual product if $a\in\mathbb{R}$, and $M\cdot a = \infty$ if $a = \infty$.

**Definition 1.1.** Let $a \in \overline{\mathbb{R}}_0^+$ and $b \in \overline{\mathbb{R}}_0^+$. We say that $a \lesssim b$ if there exist $M \in \mathbb{R}^+$ such that $a \leq M\cdot b$. If $a \lesssim b$ and $b \lesssim a$ then we say that $a \propto b$.

In other words, by the Definition 1.1 we have that $a \lesssim b$ if either $a < \infty$ or $b = \infty$, and we have that $a \propto b$ if either $a < \infty$ and $b < \infty$, or $a = b = \infty$.

**Definition 1.2.** Let $\mathrm{g} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$ and $\mathrm{h} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$, where $\mathcal{U} \subset \mathbb{R}$. We say that $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if there exist $M \in \mathbb{R}^+$ such that $\mathrm{g}(x) \leq M\,\mathrm{h}(x)$ for every $x \in \mathcal{U}$. If $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ and $\mathrm{h}(x) \lesssim \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \propto \mathrm{h}(x)$.

**Definition 1.3.** Let $\mathcal{U} \subset \mathbb{R}$, $a \in \overline{\mathcal{U}}\cup\{\infty\}$, $\mathrm{g} : \mathcal{U} \to \mathbb{R}^+$ and $\mathrm{h} : \mathcal{U} \to \mathbb{R}^+$. We say that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ if $\limsup_{x\to a}\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} < \infty$. If $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{h}(x) \underset{x\to a}{\lesssim} \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \underset{x\to a}{\propto} \mathrm{h}(x)$.

The meaning of the relations $\mathrm{g}(x) \underset{x\to a^+}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to a^-}{\lesssim} \mathrm{h}(x)$ for $a \in \mathbb{R}$ are defined analogously. Note that, if for some $d \in \mathbb{R}^+$ we have $\lim_{x\to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = d$, then it follows directly that $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$. The following proposition is a direct consequence of the above definition.

**Proposition 1.4.** *Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $c \in [a,b]$, $r \in \mathbb{R}^+$, and let $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ be continuous functions with domain $(a,b)$ such that $f_1(x) \underset{x\to c}{\lesssim} f_2(x)$ and $g_1(x) \underset{x\to c}{\lesssim} g_2(x)$. Then the following hold*

$$f_1(x)g_1(x) \underset{x\to c}{\lesssim} f_2(x)g_2(x) \quad \textit{and} \quad f_1(x)^r \underset{x\to c}{\lesssim} f_2(x)^r.$$

The following proposition relates Definition 1.2 and Definition 1.3.

**Proposition 1.5.** *Let $\mathrm{g} : (a,b) \to \mathbb{R}_0^+$ and $\mathrm{h} : (a,b) \to \mathbb{R}^+$ be continuous functions on $(a,b) \subset \mathbb{R}$, where $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$. Then $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if and only if $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$.*

***Proof.*** See Appendix B.2. □

Note that if $\mathrm{g} : (a,b) \to \mathbb{R}^+$ and $\mathrm{h} : (a,b) \to \mathbb{R}^+$ are continuous functions on $(a,b) \subset \mathbb{R}$, then by continuity it follows directly that $\lim_{x\to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = \dfrac{\mathrm{g}(c)}{\mathrm{h}(c)} > 0$ and therefore $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$ for every $c \in (a,b)$. This fact and the Proposition 1.5 imply directly the following.

**Proposition 1.6.** *Let $\mathrm{g} : (a,b) \to \mathbb{R}^+$ and $\mathrm{h} : (a,b) \to \mathbb{R}^+$ be continuous functions in $(a,b) \subset \mathbb{R}$, where $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$, and let $c \in (a,b)$. Then if $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ (or $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$) we have that $\int_a^c g(t)\,dt \lesssim \int_a^c h(t)\,dt$ (respectively $\int_c^b g(t)\,dt \lesssim \int_c^b h(t)\,dt$ ).*

### 1.1. *Case when $\alpha$ is known*

Let $p(\boldsymbol{\theta}|\boldsymbol{x},\alpha)$ be of the form (2) but considering $\alpha$ fixed and $\boldsymbol{\theta} = (\phi,\mu)$, the normalizing constant is given by

$$d(\boldsymbol{x};\alpha) \propto \int\limits_{\mathcal{A}} \frac{\pi(\boldsymbol{\theta})}{\Gamma(\phi)^n} \left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\} \mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^{n} x_i^\alpha\right\} d\boldsymbol{\theta}, \tag{4}$$

where $\mathcal{A} = \{(0,\infty)\times(0,\infty)\}$ is the parameter space. For any prior distribution in the form: $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\phi)$, our purpose is to find necessary and sufficient conditions for $d(\boldsymbol{x};\alpha) < \infty$.

**Theorem 1.7.** *Suppose that $\pi(\mu,\phi) < \infty$ for all $(\mu,\phi) \in \mathbb{R}_+^2$, that $n \in \mathbb{N}^+$, and* *suppose that* $\pi(\mu,\phi)=\pi(\mu)\pi(\phi)$ *where*

$$\pi(\mu)\lesssim\mu^k,\quad \pi(\phi)\underset{\phi\to 0^+}{\lesssim}\phi^{r_0}\qquad\text{and}\qquad \pi(\phi)\underset{\phi\to\infty}{\lesssim}\phi^{r_\infty},$$

*such that* $k=-1$ *with* $n>-r_0$*, or* $k>-1$ *with* $n>-r_0-1$*, then* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is proper.*

***Proof.*** See Appendix B.3. □

**Theorem 1.8.** *Suppose that* $\pi(\mu,\phi)>0\ \forall(\mu,\phi)\in\mathbb{R}^2_+$*,* $n\in\mathbb{N}^+$*,* $\pi(\mu,\phi)\gtrsim\pi(\mu)\pi(\phi)$*, where* $\pi(\mu)\gtrsim\mu^k$ *and one of the following hold:*

*i)* $k<-1$*; or*
*ii)* $k>-1$ *where* $\pi(\phi)\underset{\phi\to 0^+}{\gtrsim}\phi^{r_0}$ *with* $n\leq -r_0-1$*; or*
*iii)* $k=-1$ *where* $\pi(\phi)\underset{\phi\to 0^+}{\gtrsim}\phi^{r_0}$ *with* $n\leq -r_0$*,*

*then* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

***Proof.*** See Appendix B.4 □

**Theorem 1.9.** *Let* $\pi(\phi,\mu)=\pi(\phi)\pi(\mu)$ *and the behavior of* $\pi(\mu)$*,* $\pi(\phi)$ *be given by*

$$\pi(\mu)\propto\mu^k,\quad \pi(\phi)\underset{\mu\to 0^+}{\propto}\phi^{r_0}\qquad\text{and}\quad \pi(\phi)\underset{\phi\to\infty}{\propto}\phi^{r_\infty},$$

*for* $k\in\mathbb{R}$*,* $r_0\in\mathbb{R}$ *and* $r_\infty\in\mathbb{R}$*. The posterior related to* $\pi(\phi,\mu)$ *is proper if and only if* $k=-1$ *with* $n>-r_0$*, or* $k>-1$ *with* $n>-r_0-1$*, and in this case the posterior mean of* $\phi$ *and* $\mu$ *are finite, as well as all moments.*

***Proof.*** Since the posterior is proper, by Theorem 1.7 we have that $k=-1$ with $n>-r_0$ or $k>-1$ with $n>-r_0-1$.

Let $\pi^*(\phi,\mu)=\phi\pi(\phi,\mu)$. Then $\pi^*(\phi,\mu)=\pi^*(\phi)\pi^*(\mu)$, where $\pi^*(\phi)=\phi\pi(\phi)$ and $\pi^*(\mu)=\pi(\mu)$, and we have

$$\pi^*(\mu)\propto\mu^k,\quad \pi^*(\phi)\underset{\phi\to 0^+}{\propto}\phi^{r_0+1}\qquad\text{and}\quad \pi^*(\phi)\underset{\phi\to\infty}{\propto}\phi^{r_\infty+1}.$$

Since $k=-1$ with $n>-r_0>-(r_0+1)$ or $k>-1$ with $n>-(r_0+1)-1$, it follows from Theorem 1.7 that the posterior

$$\pi^*(\phi,\mu)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

related to the prior $\pi^*(\phi,\mu)$ is proper. Therefore

$$E[\phi|\boldsymbol{x}]=\int_0^\infty\int_0^\infty\phi\pi(\phi,\mu)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi<\infty.$$

Analogously one can prove that

$$E[\mu|\boldsymbol{x}] = \int_0^\infty \int_0^\infty \mu\pi(\phi,\mu)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi < \infty.$$

Therefore we have proved that if a prior $\pi(\phi,\mu)$ satisfying the assumptions of the theorem leads to a proper posterior, then the priors $\phi\pi(\phi,\mu)$ and $\mu\pi(\phi,\mu)$ also leads to proper posteriors, and it follows by induction that $\phi^r\mu^s\pi(\phi,\mu)$ also leads to proper posteriors for any $r$ and $s \in \mathbb{N}$, which concludes the proof. □

### 1.2. *Case when $\phi$ is known*

Let $p(\boldsymbol{\theta}|\boldsymbol{x},\phi)$ be of the form (2) but considering fixed $\phi$ and $\boldsymbol{\theta}=(\mu,\alpha)$, the normalizing constant is given by

$$d(\boldsymbol{x};\phi) = \int_{\mathcal{A}} \pi\left(\boldsymbol{\theta}\right)\alpha^n\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\boldsymbol{\theta}, \tag{5}$$

where $\mathcal{A}=\{(0,\infty)\times(0,\infty)\}$ is the parameter space. For any prior distribution in the form: $\pi\left(\boldsymbol{\theta}\right)\propto\pi(\mu)\pi(\alpha)$, our purpose is to find necessary and sufficient conditions for these class of posterior be proper, i.e., $d(\boldsymbol{x};\phi)<\infty$. The following propositions are useful.

**Theorem 1.10.** *Suppose that $\pi(\mu,\alpha)<\infty$ for all $(\mu,\alpha)\in\mathbb{R}_+^2$, that $n\in\mathbb{N}^+$, and suppose that $\pi(\mu,\alpha)=\pi(\alpha)\pi(\mu)$ where*

$$\pi(\mu)\lesssim\mu^k, \quad \pi(\alpha)\underset{\alpha\to0^+}{\lesssim}\alpha^{q_0}, \quad \pi(\alpha)\underset{\alpha\to\infty}{\lesssim}\alpha^{q_\infty},$$

*such that $k=-1$, $n>-q_0$ and $q_\infty\in\mathbb{R}$. then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.*

***Proof.*** See Appendix B.5. □

**Theorem 1.11.** *Suppose that $\pi(\alpha,\mu)>0$ $\forall(\alpha,\mu)\in\mathbb{R}_+^2$ and that $n\in\mathbb{N}^+$, and suppose that $\pi(\mu,\alpha)\gtrsim\pi(\mu)\pi(\alpha)$ where $\pi(\mu)\gtrsim\mu^k$ and one of the following hold*

*i)* $k<-1$;
*ii)* $k>-1$ *such that* $\pi(\alpha)\underset{\alpha\to0^+}{\gtrsim}\alpha^{q_0}$ *with* $q_0\in\mathbb{R}$; *or*
*iii)* $k=-1$ *such that* $\pi(\alpha)\underset{\alpha\to0^+}{\gtrsim}\alpha^{q_0}$ *with* $n\leq-q_0$

*then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.*

***Proof.*** See Appendix B.6. □

**Theorem 1.12.** *Let $\pi(\mu,\alpha)=\pi(\mu)\pi(\alpha)$ and the behavior of $\pi(\mu)$, $\pi(\alpha)$ be given by*

$$\pi(\mu)\propto\mu^k, \quad \pi(\alpha)\underset{\mu\to0^+}{\propto}\alpha^{q_0} \quad \text{and} \quad \pi(\alpha)\underset{\alpha\to\infty}{\propto}\alpha^{q_\infty},$$

*for $k\in\mathbb{R}$, $q_0\in\mathbb{R}$ and $q_\infty\in\mathbb{R}$. The posterior related to $\pi(\mu,\alpha)$ is proper if and only if $k=-1$ with $n>-q_0$, and in this case the posterior mean of $\alpha$ is finite for this prior,* *as well as all moments relative to $\alpha$, and the posterior mean of $\mu$ is not finite.*

***Proof.*** Since the posterior is proper, by Theorem 1.11 we have that $k = -1$ and $n > -q_0$.

Let $\pi^*(\mu, \alpha) = \alpha\pi(\mu, \alpha)$. Then $\pi^*(\mu, \alpha) = \pi^*(\mu)\pi^*(\alpha)$, where $\pi^*(\alpha) = \alpha\pi(\alpha)$ and $\pi^*(\mu) = \pi(\mu)$, and we have

$$\pi^*(\mu) \propto \mu^{-1}, \quad \pi^*(\alpha) \underset{\mu\to 0^+}{\propto} \alpha^{q_0+1} \quad \text{and} \quad \pi^*(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty+1}.$$

But since $n > -q_0 > -(q_0+1)$ it follows from Theorem 1.10 that the posterior

$$\pi^*(\mu, \alpha)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

relative to the prior $\pi^*(\mu, \alpha)$ is proper. Therefore

$$E[\alpha|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \alpha\pi(\mu, \alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha < \infty.$$

Analogously one can prove using the item ii) of the Theorem 1.11 that

$$E[\mu|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \mu\pi(\mu, \alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha = \infty$$

since in this case $\mu\pi(\mu) \propto \mu^0$.

Therefore we have proved that if a prior $\pi(\mu, \alpha)$ satisfying the assumptions of the theorem leads to a proper posterior, then the prior $\alpha\pi(\mu, \alpha)$ also leads to proper posteriors, and it follows by induction that $\alpha^r\pi(\mu, \alpha)$ also leads to proper posteriors for any $r$ in $\mathbb{N}$, which concludes the proof. $\square$

### 1.3. *General case when $\phi$, $\alpha$ and $\mu$ are unknown*

**Theorem 1.13.** *Suppose that $\pi(\alpha, \beta, \mu) < \infty$ for all $(\alpha, \beta, \mu) \in \mathbb{R}^3_+$, that $n \in \mathbb{N}^+$, and suppose that $\pi(\mu, \alpha, \phi) = \pi(\mu)\pi(\alpha)\pi(\mu)$ where*

$$\pi(\mu) \lesssim \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\lesssim} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\lesssim} \alpha^{q_\infty},$$

$$\pi(\phi) \underset{\phi\to 0^+}{\lesssim} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\lesssim} \phi^{r_\infty},$$

*such that $k = -1$, $q_\infty < r_0$, $2r_\infty + 1 < q_0$, $n > -q_0$ and $n > -r_0$, then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.*

***Proof.*** See Appendix B.7 $\square$

**Theorem 1.14.** *Suppose that $\pi(\alpha, \phi, \mu) > 0$ $\forall(\alpha, \phi, \mu) \in \mathbb{R}^3_+$ and that $n \in \mathbb{N}^+$, then the following items are valid*

*i)* $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\phi)$ *for all* $\phi \in [b_0, b_1]$ *where* $0 \leq b_0 < b_1$*, such that* $\pi(\mu) \gtrsim \mu^k$ *and one of the following hold*
- $k < -1$*;*
- $k > -1$*; where* $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ *with* $q_0 \in \mathbb{R}$*; or*
- $k > -1$*; where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n < -r_0 - 1$ *and* $b_0 = 0$.

*then* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

*ii)* $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\beta)$ *such that* $\pi(\mu) \gtrsim \mu^{-1}$ *and one of the following occur*
- $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *and* $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ *where either* $q_\infty \geq r_0$ *or* $n \leq -r_0$*;*
- $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ *and* $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$ *where either* $2r_\infty + 1 \geq q_0$ *or* $n \leq -q_0$*;*

*then* $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

***Proof.*** See Appendix B.8 □

**Theorem 1.15.** *Suppose that* $0 < \pi(\alpha,\beta,\mu) < \infty$ *for all* $(\alpha,\beta,\mu) \in \mathbb{R}^3_+$*, and suppose that* $\pi(\mu,\alpha,\phi) = \pi(\mu)\pi(\alpha)\pi(\phi)$ *where*

$$\pi(\mu) \propto \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty},$$

$$\pi(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty},$$

*then the posterior is proper if and only if* $k = -1$*,* $q_\infty < r_0$*,* $2r_\infty + 1 < q_0$*,* $n > -q_0$ *and* $n > -r_0$*. Moreover, if the posterior is proper then* $\alpha^q\phi^r\mu^j\pi(\alpha,\phi,\mu)$ *leads to a proper posterior if and only if* $j = 0$*, and* $2(r + r_\infty) + 1 - q_0 < q < r + r_0 - q_\infty$*.*

***Proof.*** Notice that under our hypothesis, Theorems 1.14 and 1.15 are complementary, and thus the first part of the theorem is proved. Analogously, by the Theorems 1.14 and 1.15 the prior $\alpha^q\beta^r\mu^l\pi(\alpha,\beta,\mu)$ leads to a proper posterior if and only if $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) + 1 < q + q_0$, $n > -q_0 - q$ and $n > -r_0 - r$. The last two proportionalities are already satisfied since $n > -q_0$ and $n > -r_0$. Combining the other inequalities the proof is completed. □

## 2. Some common objective priors

A naive approach to obtain objective priors is to consider uniform priors contained in the interval $(0,\infty)$. However, uniform priors are usually not attractive due to its lack of invariance over reparametrizations. The uniform prior is given by $\pi_1(\boldsymbol{\theta}) \propto 1$.

**Corollary 2.1.** *The posterior distribution obtained using a joint uniform prior when* $\phi$ *is known is improper for all* $n \in \mathbb{N}^+$*.*

***Proof.*** Since $\pi(\theta) \propto 1$ we can apply the item ii) of Theorem 1.11 with $k = q_0 = -1$ and it follows that the posterior is improper for all $n \in \mathbb{N}^+$. □

**Corollary 2.2.** *The posterior distribution (2) using a joint uniform prior is improper for all* $n \in \mathbb{N}^+$*.*

***Proof.*** Since $\pi_1(\phi,\mu,\alpha) = \mu^0\alpha^0\phi^0$ we apply Theorem 1.14 ii) with $k = q_\infty = r_0 = 0$ and since $q_\infty \geq r_0$ we have that $\pi(\alpha,\beta,\mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$. □

Another common approach was suggested by Jeffreys' that considered different procedures for constructing objective priors. For $\theta \in (0,\infty)$ (see, Kass and Wasserman [13]), Jeffreys suggested to use the prior $\pi(\theta) = \theta^{-1}$. The main justification for this choice is its invariance under power transformations of the parameters. As the parameters of the GG distribution are contained in the interval $(0,\infty)$, the prior using Jeffreys' first rule is $\pi_2\left(\phi,\mu,\alpha\right) \propto (\phi\mu\alpha)^{-1}$.

**Corollary 2.3.** *The posterior distribution obtained using Jeffreys' first rule when $\alpha$ is known is proper for all $n > 1$ as well as its higher moments.*

***Proof.*** Since $\pi_2(\phi,\mu,\alpha) \propto \phi^{-1}\alpha^{-1}\mu^{-1}$ we can apply Theorem 1.12 with $k = r_0 = r_\infty = -1$ and it follows that the posterior is proper for $n > -r_0 = 1$ as well as its moments. □

**Corollary 2.4.** *The posterior distribution (2) obtained using Jeffreys' first rule is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** Since $\pi_1(\phi,\mu,\alpha) = \phi^{-1}\alpha^{-1}\mu^{-1}$ we can apply Theorem 1.14 ii) with $k = q_\infty = r_0 = -1$, where $q_\infty \geq r_0$, and therefore we have that $\pi(\alpha,\beta,\mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$. □

Zellner [24]-[25], discussed another procedure to obtain an objective prior that is based on the information measure known as Shannon entropy. Such prior is known as MDI prior and can be obtained by solving

$$\pi_3\left(\boldsymbol{\theta}\right) \propto \exp\left(\int f(t\,|\,\phi,\mu,\alpha)\log f(t\,|\,\phi,\mu,\alpha)dt\right). \tag{6}$$

Ramos et al. [19] showed that the MDI prior (6) for the GG distribution is given by

$$\pi_3\left(\boldsymbol{\theta}\right) \propto \frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\}, \tag{7}$$

where $\psi(k) = \frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function. It is important to point out that either when $\phi$ or $\alpha$ are known the MDI prior has the same form presented above.

**Corollary 2.5.** *The joint posterior densities using the MDI prior when $\phi$ is known is improper for any $n \in \mathbb{N}^+$.*

***Proof.*** Let $\phi$ be fixed. Notice that $\pi_3(\theta) \propto \pi(\alpha)\pi(\mu)$, where $\pi(\alpha) = \alpha e^{-\frac{\psi(\phi)}{\alpha}}$ and $\pi(\mu) = \mu$. Since $\psi(\phi) < 0$ for all $\phi > 0$, one can prove easily via the change of variable $u = \frac{1}{\alpha}$ that $e^{-\frac{\psi(\phi)}{\alpha}} \underset{\alpha\to 0^+}{\gtrsim} 1$ and thus $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha$. Thus we can apply item ii) of Theorem 1.11 with $k = 1$ and $q_0 = 1$ to conclude the posterior is improper for all $n \in \mathbb{N}^+$. □

**Corollary 2.6.** *The joint posterior density (2) using the MDI prior (7) is improper for any $n \in \mathbb{N}^+$.*

***Proof.*** Since $\psi(\phi) < 0$ for all $\phi \in (0,1]$ (see Abramowitz and Stegun [1]), we have that $\exp\left(-\psi(\phi)\frac{1}{\alpha}\right) \geq 1$ for all $\phi \in [0.5,1]$ and therefore

$$\pi_3(\boldsymbol{\theta}) \gtrsim \alpha\mu\frac{\exp\left(\psi(\phi)\phi - \phi\right)}{\Gamma(\phi)},$$

in the interval $[0.5,1]$. It follows that the hypothesis in Theorem 1.14, i) is satisfied with $b_0 = 0.5$, $b_1 = 1$, $k = 1 > -1$ and $q_0 = 1$, and therefore we have that $\pi_3(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

## 3. Priors based on the Fisher information matrix

The priors discussed in this section belong to the class of improper priors given by

$$\pi_j\left(\boldsymbol{\theta}\right) \propto \frac{\pi_j(\phi)}{\mu}, \tag{8}$$

where $j$ is the index related to a particular prior. Therefore, our main focus in this section will be to study the behavior of the priors $\pi_j(\phi)$.

One important objective prior is based on Jeffreys' general rule [12] and known as Jeffreys' prior. This prior is obtained through the square root of the determinant of the Fisher information matrix and has been widely used due to its invariance property under one-to-one transformations. The Fisher information matrix for the GG distribution was derived by Hager and Bain [11] and its elements are given by

$$I_{\alpha,\alpha}(\boldsymbol{\theta}) = \frac{1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2}{\alpha^2},\ I_{\alpha,\mu}(\boldsymbol{\theta}) = -\frac{\psi(\phi)}{\alpha},\ I_{\mu,\phi}(\boldsymbol{\theta}) = \frac{\alpha}{\mu},$$

$$I_{\alpha,\phi}(\boldsymbol{\theta}) = -\frac{1+\phi\psi(\phi)}{\mu},\quad I_{\mu,\mu}(\boldsymbol{\theta}) = \frac{\phi\alpha^2}{\mu^2} \quad \text{and} \quad I_{\phi,\phi}(\boldsymbol{\theta}) = \psi'(\phi),$$

where $\psi'(k) = \frac{\partial}{\partial k}\psi(k)$ is the trigamma function.

Van Noortwijk [23] provided the Jeffreys' prior for the GG distribution, which can be expressed by (8) with

$$\pi_4\left(\phi\right) \propto \sqrt{\phi^2\psi'\left(\phi\right)^2 - \psi'\left(\phi\right) - 1}. \tag{9}$$

**Corollary 3.1.** *The posterior distribution (2) using the Jeffreys' prior (9) is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** Ramos et al. [19] proved that

$$\sqrt{\phi^2\psi'\left(\phi\right)^2 - \psi'\left(\phi\right) - 1} \underset{\phi\to 0^+}{\propto} 1 \ \text{ and } \ \sqrt{\phi^2\psi'\left(\phi\right)^2 - \psi'\left(\phi\right) - 1} \underset{\phi\to\infty}{\propto} \frac{1}{\phi}. \tag{10}$$

Since $\pi_4\left(\phi\right) \underset{\phi\to 0^+}{\propto} 1$, the hypotheses of Theorem 1.14, ii) hold with $k = -1$ and $r_0 = q_\infty = 0$, where $q_\infty \geq r_0$, and therefore $\pi_4(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

Let $\alpha$ be known, then the Jeffreys' prior has the form (8) where $\pi(\phi)$ is given by

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi) - 1}. \tag{11}$$

**Corollary 3.2.** *The joint posterior densities when $\alpha$ is known using the prior (11) is proper for $n \geq 1$ as well as its higher moments.*

***Proof.*** Here, we have $\pi(\beta) = \beta^{-1}$. Following Abramowitz and Stegun [1] we have that $\lim_{z\to 0^+} \dfrac{\psi'(z)}{z^{-2}} = 1$, then $\lim_{\phi\to 0^+} \dfrac{\phi\psi'(\phi) - 1}{\phi^{-1}} = \lim_{\phi\to 0^+} \dfrac{\psi'(\phi)}{\phi^{-2}} - \phi = 1$, and thus

$$\phi\psi'(\phi) - 1 \underset{\phi\to 0^+}{\propto} \phi^{-1}, \tag{12}$$

which implies $\sqrt{\phi\psi'(\phi) - 1} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$. Moreover, from Abramowitz and Stegun [1], we have that $\psi'(z) = \dfrac{1}{z} + \dfrac{1}{2z^2} + o\left(\dfrac{1}{z^3}\right)$ and thus

$$\frac{\phi\psi'(\phi) - 1}{\phi^{-1}} = \frac{1}{2} + o\left(\frac{1}{\phi}\right) \Rightarrow \lim_{\phi\to\infty} \frac{\sqrt{\phi\psi'(\phi) - 1}}{\phi^{-\frac{1}{2}}} = \frac{1}{\sqrt{2}},$$

which implies $\sqrt{\phi\psi'(\phi) - 1} \underset{\phi\to\infty}{\propto} \phi^{-\frac{1}{2}}$.

Therefore we can apply Theorem 1.9 with $k = -1$ and $r_0 = r_\infty = -\frac{1}{2}$ and therefore the posterior is proper and the posterior moments are finite for all $n > -r_0 = \frac{1}{2}$. $\square$

The proper posterior distribution using the Jeffreys' prior cited above include as particular cases the posterior distributions for the parameters of the Gamma, Nakagami and Wilson-Hilferty distributions.

Fonseca et al. [10] considered the scenario where the Jeffreys' prior has an independent structure, i.e., the prior has the form $\pi_{J2}(\boldsymbol{\theta}) \propto \sqrt{|\operatorname{diag} I(\boldsymbol{\theta})|}$, where $\operatorname{diag} I(\cdot)$ is the diagonal matrix of $I(\cdot)$. For the GG distribution the prior is given by (8) with

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2\right)}. \tag{13}$$

**Corollary 3.3.** *The posterior distribution using the independent Jeffreys' prior (13) is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** By Abramowitz and Stegun [1], we have the recurrence relations

$$\psi(\phi) = -\frac{1}{\phi} + \psi(\phi + 1) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi^2} + \psi'(\phi + 1). \tag{14}$$

It follows that

$$
\begin{aligned}
&2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1=\\
&2\left(-\frac{1}{\phi}+\psi(\phi+1)\right)+\phi\left(\frac{1}{\phi^2}+\psi'(\phi+1)\right)+\phi\left(\frac{1}{\phi^2}-\frac{2}{\phi}\psi(\phi+1)+\psi(\phi+1)^2\right)+1=\\
&1+\phi\left(\psi(\phi+1)^2+\psi'(\phi+1)\right).
\end{aligned}
$$

Hence, $2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1 \underset{\phi\to 0^+}{\propto} 1$, which implies that

$$
\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}}, \tag{15}
$$

then, Theorem 1.14 ii) can be applied with $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$ and therefore $\pi_5(\boldsymbol{\theta})$ leads to an improper posterior. □

This approach can be further extended considering that only one parameter is independent. For instance, let $(\theta_1,\theta_2)$ be dependent parameters and $\theta_3$ be independent then under the partition the $((\theta_1,\theta_2),\theta_3)$-Jeffreys' prior is given by

$$
\pi(\boldsymbol{\theta}) \propto \sqrt{\left(I_{11}(\boldsymbol{\theta})I_{22}(\boldsymbol{\theta})-I_{12}^2(\boldsymbol{\theta})\right)I_{33}(\boldsymbol{\theta})}. \tag{16}
$$

For the GG distribution the partition $((\phi,\mu),\alpha)$-Jeffreys' prior is of the form (8) with

$$
\pi_6(\phi) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)}. \tag{17}
$$

**Corollary 3.4.** *The posterior distribution using the $((\phi,\mu),\alpha)$-Jeffreys' prior is improper for all $n\in\mathbb{N}^+$.*

***Proof.*** From equation (12) we have that $\phi\psi'(\phi)-1 \underset{\phi\to 0^+}{\propto} \frac{1}{\phi}$ which combined with the relation (15) implies that

$$
\pi_6(\phi) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}}. \tag{18}
$$

Therefore, Theorem 1.14, ii) can be applied with $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty\geq r_0$ and therefore $\pi_6(\boldsymbol{\theta})$ leads to an improper posterior. □

On the other hand, the partition $((\phi,\alpha),\mu)$-Jeffreys' prior is given by (8) where

$$
\pi_7(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2}. \tag{19}
$$

**Corollary 3.5.** *The posterior distribution using the independent Jeffreys' prior (19) is improper for all $n\in\mathbb{N}^+$.*

***Proof.*** From (14) we have that

$$\begin{aligned}
\pi_7^{\frac{1}{2}}(\phi) \propto\ & \phi\psi^{'}(\phi)\left(1+2\psi(\phi)+\phi\psi^{'}(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2 \\
&= \left(\phi^{-1}+\phi\psi'(\phi+1)\right)\left(1+\phi\left(\psi(\phi+1)^2+\psi'(\phi+1)\right)\right)-\phi\left(-\phi^{-1}+\psi(\phi+1)\right)^2 \\
&= \phi\left(\psi'(\phi+1)-\psi(\phi+1)^2+\phi\psi'(\phi+1)(\psi(\phi+1)^2+\psi'(\phi+1))\right)+\psi(\phi+1)^2 \\
&\quad +2\psi(\phi+1)+\psi'(\phi+1) \\
&\propto \phi\psi^{'}(\phi)\left(1+2\psi(\phi)+\phi\psi^{'}(\phi)+\phi\psi(\phi)^2\right)-\phi\psi(\phi)^2 \\
&\underset{\phi\to 0^+}{\propto} \psi(1)^2+2\psi(1)+\psi'(1)=\gamma^2-2\gamma+\frac{\pi}{6}>0,
\end{aligned}$$

then, Theorem 1.14, ii) can be applied with $k=-1$, $r_0=0$ and $q_\infty=0$ where $q_\infty \geq r_0$ and therefore $\pi_7(\boldsymbol{\theta})$ leads to an improper posterior. □

Finally, the $((\alpha,\mu),\phi)$-Jeffreys' prior is given by (8) where

$$\pi_8(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)}\,. \tag{20}$$

**Corollary 3.6.** *The posterior distribution using the independent Jeffreys' prior (20) is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** From the recurrence relations (14), we have that

$$\phi^2\psi^{'}(\phi)+\phi-1=\phi\left(1+\phi\psi^{'}(\phi+1)\right) \Rightarrow \phi^2\psi^{'}(\phi)+\phi-1 \underset{\phi\to 0^+}{\propto} \phi \tag{21}$$

as $\psi'(\phi) \propto \frac{1}{\phi^2}$ it follows that

$$\pi_8(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}},$$

and Theorem 1.14 ii) can be applied with $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$. Therefore $\pi_8(\boldsymbol{\theta})$ leads to an improper posterior. □

## 4. Reference priors

Introduced by Bernardo [7] with further developments ([2], [3],[4]) reference priors play an important role in objective Bayesian analysis. The reference priors have desirable properties, such as invariance, consistent marginalization, and consistent sampling properties. Bernardo [8] reviewed different procedures to derive reference priors considering ordered parameters of interest. The following proposition will be applied to obtain reference priors for the Generalized Gamma distribution.

**Proposition 4.1.** *[Bernardo [7], pg 40, Theorem 14] Let $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_m)$ be a vector with the ordered parameters of interest and $p(\boldsymbol{\theta}|\boldsymbol{x})$ be the posterior distribution that has an asymptotically normal distribution with dispersion matrix $V(\hat{\boldsymbol{\theta}}_n)/n$, where $\hat{\boldsymbol{\theta}}_n$ is a consistent estimator of $\boldsymbol{\theta}$ and $H(\boldsymbol{\theta})=V^{-1}(\boldsymbol{\theta})$. In addition, $V_j$ is the upper $j\times j$ submatrix of $V$, $H_j=V_j$ and $h_{j,j}(\boldsymbol{\theta})$ is the lower right element of $H_j$. If the parameter* *space of $\theta_j$ is independent of $\boldsymbol{\theta}_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_m)$, for $j = 1, \ldots, m$, and $h_{j,j}(\boldsymbol{\theta})$ are factorized in the form $h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j) g_j(\boldsymbol{\theta}_{-j})$, $\quad j = 1, \ldots, m$, then the reference prior for the ordered parameters $\boldsymbol{\theta}$ is given by*

$$\pi(\boldsymbol{\theta}) = \pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1),$$

*where $\pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) = f_j(\theta_j)$, for $j = 1, \ldots, m$, and there is no need for compact approximations, even if the conditional priors are not proper.*

The reference priors obtained from Proposition 4.1 belong to the class of improper priors given by

$$\pi(\boldsymbol{\theta}) \propto \frac{\pi(\phi)}{\alpha\mu}, \tag{22}$$

therefore, our focus will be in the behavior of $\pi(\phi)$. Let $(\alpha, \phi, \mu)$ be the ordered parameters of interest, then conditional priors of the $(\alpha, \phi, \mu)$-reference prior are given by

$$\pi(\alpha) \propto \frac{1}{\alpha}, \quad \pi(\phi|\alpha) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}, \quad \pi(\mu|\alpha, \phi) \propto \frac{1}{\mu}\cdot$$

Therefore, $(\alpha, \phi, \mu)$-reference prior is of the form (22) with

$$\pi_9(\phi) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}\cdot$$

**Corollary 4.2.** *The posterior density using the $(\alpha, \phi, \mu)$-reference prior is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** By equation (12) we have that $\pi_9(\phi) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}} \underset{\phi \to 0^+}{\propto} \frac{1}{\phi}$. Therefore, item ii) of Theorem 1.14 can be applied with $k = r_0 = q_\infty = -1$ where $q_\infty \geq r_0$ which implies that $\pi_9(\alpha, \phi, \mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$.

□

On the other hand, if $(\alpha, \mu, \phi)$ are the ordered parameters, then the conditional reference priors are

$$\pi(\alpha) \propto \frac{1}{\alpha}, \quad \pi(\mu|\alpha) \propto \frac{1}{\mu}, \quad \pi(\phi|\alpha, \mu) \propto \sqrt{\psi'(\phi)},$$

and the $(\alpha, \mu, \phi)$-reference prior is of the form (22) with

$$\pi_{10}(\phi) \propto \sqrt{\psi'(\phi)}.$$

**Corollary 4.3.** *The posterior density using the $(\alpha, \mu, \phi)$-reference prior is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** Since $\psi'(\phi) \underset{\phi\to 0^+}{\propto} \frac{1}{\phi^2}$ we have that $\sqrt{\psi'(\phi)} \underset{\phi\to 0^+}{\propto} \frac{1}{\phi}$. Thus, as in Corollary 4.2 we have that $\pi_{10}(\alpha,\mu,\phi)$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

**Corollary 4.4.** *Consider that $\alpha$ is known and assumed to be $\alpha = 1$, then $\pi(\phi,\mu) \propto \sqrt{\psi'(\phi)}/\mu$ is the $(\mu,\phi)$-reference prior for the Gamma distribution. The joint posterior densities when $\alpha = 1$ using the $(\mu,\phi)$-reference is proper for $n \geq 2$ as well as its higher moments.*

***Proof.*** Following Abramowitz and Stegun [1] we have that $\psi'(\phi) \underset{\phi\to 0^+}{\propto} \frac{1}{\phi^2}$ and $\psi'(\phi) \underset{\phi\to\infty^+}{\propto} \frac{1}{\phi}$ and thus $\pi_{10}(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-1}$ and $\pi_{10}(\phi) \underset{\phi\to\infty^+}{\propto} \phi^{-\frac{1}{2}}$, therefore we can apply Theorem 1.9 with $k = -1$, $r_0 = -1$ and $r_\infty = -\dfrac{1}{2}$ and it follows that the posterior as well as all its moments are proper for all $n > -r_0 = 1$. $\square$

**Corollary 4.5.** *Let $\phi$ be known and assumed to be $\phi = 1$, then $\pi(\mu,\alpha) \propto \alpha^{-1}\mu^{-1}$ is the $(\alpha,\mu)$-reference prior for the Weibull distribution. The joint posterior densities when $\phi = 1$ using the $(\alpha,\mu)$-reference is proper for $n \geq 2$ although its higher moments relative to $\mu$ are improper.*

***Proof.*** In Theorem 1.12 consider that $k = -1$ and $q_0 = q_\infty = -1$ and the proof is completed. $\square$

In the case of $(\mu,\phi,\alpha)$ be the vector of ordered parameters, we have that the conditional priors are

$$\pi(\mu) \propto \frac{1}{\mu},\quad \pi(\phi|\mu) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi^2) + 1}},\quad \pi(\alpha|\phi,\mu) \propto \frac{1}{\alpha}$$

and the $(\mu,\phi,\alpha)$-reference prior is of the form (22) with

$$\pi_{11}(\phi) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi^2) + 1}}.$$

**Corollary 4.6.** *The posterior density using the $(\mu,\phi,\alpha)$-reference prior is improper for all $n \in \mathbb{N}^+$.*

***Proof.*** From Abramowitz and Stegun [1], we have

$$\psi(\phi) = \log(\phi) - \frac{1}{2\phi} - \frac{1}{12\phi^2} + o\left(\frac{1}{\phi^2}\right) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi} + \frac{1}{2\phi^2} + o\left(\frac{1}{\phi^2}\right), \tag{23}$$

where it follows directly that

$$\psi(\phi)^2 = \log(\phi)^2 - \frac{\log(\phi)}{\phi} + o\left(\frac{1}{\phi}\right).$$

Therefore $2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1=\phi\log(\phi)^2+\log(\phi)+2+o(1)$ and

$$\begin{aligned}
\pi_{11}(\phi) &\propto \sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\\
&= \sqrt{\frac{\left(\frac{1}{\phi}+\frac{1}{2\phi^2}+o\left(\frac{1}{\phi^2}\right)\right)(\phi\log(\phi)^2+\log(\phi)+2+o(1))-\log(\phi)^2+\frac{\log(\phi)}{\phi}+o\left(\frac{1}{\phi}\right)}{\phi\log(\phi)^2+\log(\phi)+2+o(1)}}\\
&= \sqrt{\frac{\frac{1}{\phi}\left(\log(\phi)^2+o(\log(\phi)^2)\right)}{\phi\left(\log(\phi)^2+o(\log(\phi)^2)\right)}}=\frac{1}{\phi}\sqrt{\frac{1+o(1)}{1+o(1)}}.
\end{aligned}$$

Thus

$$\pi_{11}(\phi)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\underset{\phi\to0^+}{\propto}\frac{1}{\phi},$$

and therefore Theorem 1.14 ii) can be applied with $k=q_0=r_\infty=-1$ where $2r_\infty+1\geq q_0$. Thus, $\pi_{11}(\boldsymbol{\theta})$ leads to an improper posterior.

□

If $(\mu,\alpha,\phi)$ are the ordered parameters then the conditional priors are given by

$$\pi(\mu)\propto\frac{1}{\mu},\qquad \pi(\alpha|\mu)\propto\frac{1}{\alpha},\qquad \pi(\phi|\alpha,\mu)\propto\sqrt{\psi'(\phi)}$$

and the joint $(\mu,\alpha,\phi)$-reference prior has the same form of $\pi_{10}(\boldsymbol{\theta})$ and from Corollary 4.3 its posterior is improper.

Finally, let $(\phi,\alpha,\mu)$ be the ordered parameters, then the conditional priors are

$$\pi(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}},\qquad \pi(\alpha|\phi)\propto\frac{1}{\alpha},\qquad \pi(\mu|\alpha,\phi)\propto\frac{1}{\mu}$$

and the $(\phi,\alpha,\mu)$-reference prior is of the form (22) with

$$\pi_{12}(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}}. \tag{24}$$

It is necessary to point out that $(\phi,\mu,\alpha)$-reference prior is the same as the $(\phi,\alpha,\mu)$-reference prior, which completes all possible reference priors obtained from Proposition 4.1.

**Corollary 4.7.** *The posterior distribution using the $(\phi,\alpha,\mu)$-reference prior (24) is proper for $n\geq2$ and its higher moments are improper for all $n\in\mathbb{N}^+$.*

***Proof.*** See Appendix B.9. □

## 5. Conclusion

We have provided in a simple way necessary and sufficient conditions to check whether or not improper priors lead to proper posterior distributions for the GG distribution and its sub-models. In this case, one can quickly check if the obtained posterior is proper or improper directly looking at the behavior of the improper prior. If the posterior is proper, sufficient conditions are also presented to check if the related posterior moments are finite.

From the conditions presented in Theorem 1.14 and 1.15, we proved that for the GG distribution, the uniform prior, the prior obtained from Jeffreys' first rule and the MDI prior lead to improper posteriors. The impropriety of the posterior using the Jeffreys' priors [23] led us to consider the scenario where the Jeffreys' prior has an independent structure [10]. However, the four possible objective priors also returned improper posteriors. An alternative was to consider the reference priors discussed in Bernardo [7] with further developments [2]. Since these priors are sensitive to the ordering of the unknown parameters, from Proposition 4.1, we obtained six reference priors, two of them were similar to other reference priors. Among the four distinct reference priors, we proved that only one leads to a proper posterior distribution without the need for compact approximations or truncate possible values of the parameters. The proper reference posterior has excellent theoretical properties such as invariance property under one-to-one transformations of the parameters, consistency under marginalization and consistent sampling properties and should be used to make inference in the parameters of the GG distribution.

In addition, we also presented the application of some objective priors in the Theorems discussed in subsections 1.1 and 1.2. For instance, when $\phi$ is known both uniform and MDI priors lead to improper posteriors. Moreover, if $\phi = 1$ the $(\alpha, \mu)$-reference prior for the Weibull distribution returned a proper posterior, but the posterior mean of $\mu$ and its higher moments are not finite. By considering $\alpha$ fixed, we showed that both the Jeffreys' first rule and the Jeffreys' prior returned proper posterior distributions as well as finite higher moments. Finally, our theoretical results were applied to show that the Bayesian approach was misused to analyze the data set related to the annual maximum discharges of the river Rhine at Lobith, Netherlands, hence, using a proper posterior distribution, the correct posterior estimates were computed.

There are a large number of possible extensions of this current work. Roy and Dey [20] considered an objective Bayesian analysis for the generalized extreme value regression distribution. The proposed results can be further extended for the generalized gamma regression distribution. Another extension that can be explored is to consider other objective priors that are also constructed by formal rules such as other types of reference priors [6] and probability matching priors [15], [9].

## Disclosure statement

No potential conflict of interest was reported by the author(s)

## References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* NBS, Washington, D.C., 10 edition, 1972.

[2] J. O. Berger and J. M. Bernardo. Estimating a product of means: Bayesian analysis with reference priors. *Journal of the American Statistical Association*, 84(405):200–207, 1989.
[3] J. O. Berger and J. M. Bernardo. Ordered group reference priors with application to the multinomial problem. *Biometrika*, 79(1):25–37, 1992.
[4] J. O. Berger, J. M. Bernardo, et al. On the development of reference priors. *Bayesian statistics*, 4(4):35–60, 1992.
[5] J. O. Berger, J. M. Bernardo, D. Sun, et al. Overall objective priors. *Bayesian Analysis*, 10(1):189–221, 2015.
[6] J. O. Berger, J. M. Bernardo, D. Sun, et al. Overall objective priors. *Bayesian Analysis*, 10(1):189–221, 2015.
[7] J. M. Bernardo. Reference posterior distributions for bayesian inference. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 113–147, 1979.
[8] J. M. Bernardo. Reference analysis. *Handbook of statistics*, 25:17–90, 2005.
[9] G. S. Datta and R. Mukerjee. *Probability matching priors: higher order asymptotics*, volume 178. Springer Science & Business Media, 2012.
[10] T. C. Fonseca, M. A. Ferreira, and H. S. Migon. Objective bayesian analysis for the student-t regression model. *Biometrika*, 95(2):325–333, 2008.
[11] H. W. Hager and L. J. Bain. Inferential procedures for the generalized gamma distribution. *Journal of the American Statistical Association*, 65(332):1601–1609, 1970.
[12] H. Jeffreys. An invariant form for the prior probability in estimation problems. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 186, pages 453–461. The Royal Society, 1946.
[13] R. E. Kass and L. Wasserman. The selection of prior distributions by formal rules. *Journal of the American Statistical Association*, 91(435):1343–1370, 1996.
[14] R. B. Miller. Bayesian analysis of the two-parameter gamma distribution. *Technometrics*, 22(1):65–69, 1980.
[15] R. Mukerjee and D. K. Dey. Frequentist validity of posterior quantiles in the presence of a nuisance parameter: higher order asymptotics. *Biometrika*, 80(3):499–505, 1993.
[16] P. Northrop and N. Attalides. Posterior propriety in bayesian extreme value analyses using reference priors. *Statistica Sinica*, 26(2), 2016.
[17] A. O'Hagan, C. E. Buck, A. Daneshkhah, J. R. Eiser, P. H. Garthwaite, D. J. Jenkinson, J. E. Oakley, and T. Rakow. *Uncertain judgements: eliciting experts' probabilities*. John Wiley & Sons, 2006.
[18] R. L. Prentice. A log gamma model and its maximum likelihood estimation. *Biometrika*, 61(3):539–544, 1974.
[19] P. L. Ramos, J. A. Achcar, F. A. Moala, E. Ramos, and F. Louzada. Bayesian analysis of the generalized gamma distribution using non-informative priors. *Statistics*, 51(4): 824–843, 2017.
[20] V. Roy and D. K. Dey. Propriety of posterior distributions arising in categorical and survival models under generalized extreme value distribution. *Statistica Sinica*, pages 699–722, 2014.
[21] E. W. Stacy. A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, pages 1187–1192, 1962.
[22] E. W. Stacy and G. A. Mihram. Parameter estimation for a generalized gamma distribution. *Technometrics*, 7(3):349–358, 1965.
[23] J. M. Van Noortwijk. Bayes estimates of flood quantiles using the generalised gamma distribution. *System and Bayesian Reliability*, pages 351–374, 2001.
[24] A. Zellner. Maximal data information prior distributions. *In New Methods in the applications of Bayesian Methods*, 1977.
[25] A. Zellner. Maximal data information prior distributions. *Basic Issues in Econometrics*, 1984.

## Appendix A.

## Appendix B. Appendix A:

### B.1. *Useful Proportionalities*

The following proportionalities are useful to prove results related to the posterior distribution and its proofs can be seen in [19].

**Proposition B.1.** *Let* $\mathrm{p}(\alpha)=\log\left(\dfrac{\frac{1}{n}\sum_{i=1}^{n}t_i^{\alpha}}{\sqrt[n]{\prod_{i=1}^{n}t_i^{\alpha}}}\right)$, $\mathrm{q}(\alpha)=\mathrm{p}(\alpha)+\log n$, *for* $t_1,t_2,\ldots,t_n$ *positive and not all equal,* $h\in\mathbb{R}^+$*,* $r\in\mathbb{R}^+$ *and* $t_m=\max\{t_1,\ldots,t_n\}$*, then* $\mathrm{p}(\alpha)>0$, $\mathrm{q}(\alpha)>0$ *and the following results hold*

$$\mathrm{p}(\alpha)\underset{\alpha\to0^+}{\propto}\alpha^2\quad\textit{and}\quad\mathrm{p}(\alpha)\underset{\alpha\to\infty}{\propto}\alpha;$$

$$\mathrm{q}(\alpha)\underset{\alpha\to0^+}{\propto}1\quad\textit{and}\quad\mathrm{q}(\alpha)\underset{\alpha\to\infty}{\propto}\alpha;$$

$$\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}\underset{\phi\to0^+}{\propto}\phi^{n-1}\quad\textit{and}\quad\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}\underset{\phi\to\infty}{\propto}\phi^{\frac{n-1}{2}}n^{n\phi};$$

$$\gamma\left(h,r\,\mathrm{q}(\alpha)\right)\underset{\alpha\to0^+}{\propto}1\quad\textit{and}\quad\gamma\left(h,r\,\mathrm{q}(\alpha)\right)\underset{\alpha\to\infty}{\propto}1;\tag{B1}$$

$$\Gamma\left(h,r\,\mathrm{p}(\alpha)\right)\underset{\alpha\to0^+}{\propto}1\quad\textit{and}\quad\Gamma\left(h,r\,\mathrm{p}(\alpha)\right)\underset{\alpha\to\infty}{\propto}\alpha^{k-1}e^{-rk(\boldsymbol{x})\alpha};\tag{B2}$$

*where* $k(\boldsymbol{x})=\log\left(\frac{t_m}{\sqrt[n]{\prod_{i=1}^{n}t_i}}\right)>0$*;* $\gamma(y,x)=1-\Gamma(y,x)$ *and* $\Gamma(y,x)=\int_x^{\infty}w^{y-1}e^{-w}\,dw$ *is the upper incomplete gamma function.*

### B.2. *Proof of Proposition 1.5*

Suppose that $\mathrm{g}(x)\underset{x\to a}{\lesssim}\mathrm{h}(x)$ and $\mathrm{g}(x)\underset{x\to b}{\lesssim}\mathrm{h}(x)$. Then, by Definition 1.3 we have that $\limsup_{x\to a}\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}=w$ for some $w\in\mathbb{R}^+$. Therefore, from the definition of lim sup there exist some $a'\in(a,b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}\leq\dfrac{3w}{2}$ for every $x\in(a,a']$. Proceeding analogously, there must exist some $v\in\mathbb{R}^+$ and $b'\in(a',b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}\leq\dfrac{3v}{2}$ for every $x\in[b',b)$. On the other hand, since $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}$ is continuous in $[a',b']$, the Weierstrass Extreme Value Theorem states that there exist some $x_1\in[a',b']$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}\leq$

$\dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}$ for every $x \in [a', b']$. Finally, choosing $M = \max\left(\dfrac{3w}{2}, \dfrac{3v}{2}, \dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}\right) < \infty$, it follows that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$, which by Definition 1.2 means that $g(x) \lesssim h(x)$.

Now suppose $g(x) \lesssim h(x)$. By Definition 1.2, there exist some $M < 0$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$. This implies that $\limsup_{x \to a} \frac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M < \infty$ which by Definition 1.3 means that $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$. The proof that $\mathrm{g}(x) \underset{x \to b}{\lesssim} \mathrm{h}(x)$ must also be satisfied is analogous to the previous case. Therefore the theorem is proved.

### B.3. *Proof of Theorem 1.7*

Let $\alpha \in \mathbb{R}^+$ be fixed. Since $\frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\} \pi(\mu)\mu^{n\alpha\phi-1} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$
\begin{aligned}
d(\boldsymbol{x}; \alpha) &= \int_{\mathcal{A}} \frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\boldsymbol{\theta} \\
&= \int_0^\infty \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \, d\phi.
\end{aligned}
$$

Since $\pi(\mu) \lesssim \mu^k$ and $k \geq -1$ by hypothesis it follows that

$$
\begin{aligned}
d(\boldsymbol{x}; \alpha) &\lesssim \int_0^\infty \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi+k} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \, d\phi \\
&= \int_0^\infty \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \frac{\alpha\Gamma\left(n\phi + \frac{k+1}{\alpha}\right)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi+\frac{k+1}{\alpha}}} d\mu \, d\phi.
\end{aligned}
$$

Now suppose that $k > -1$. Then, since $k + 1 > 0$, $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi \to 0^+}{\propto} 1$ and $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi \to \infty}{\propto} \Gamma(n\phi)(n\phi)^{\frac{k+1}{\alpha}}$ (see Abramowitz and Stegun [1]). Therefore, from the proportionalities in Proposition B.1 it follows that

$$
\begin{aligned}
d(\boldsymbol{x}; \alpha) &\lesssim \int_0^1 \pi(\phi) \frac{1}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi} d\phi + \int_1^\infty \pi(\phi) \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \phi^{\frac{k+1}{\alpha}} e^{-n\,\mathrm{q}(\alpha)\phi} \, d\phi \\
&\propto \int_0^1 \pi(\phi)\phi^n e^{-n\,\mathrm{q}(\alpha)\phi} d\phi + \int_1^\infty \pi(\phi)\phi^{\frac{n-1}{2}+\frac{k+1}{\alpha}} e^{-n\,\mathrm{p}(\alpha)\phi} \, d\phi = s_1(\boldsymbol{x}; \alpha) + s_2(\boldsymbol{x}; \alpha)
\end{aligned}
\tag{B3}
$$

where $\mathrm{q}(\alpha)$ and $\mathrm{p}(\alpha)$ are given in Proposition B.1 and $s_1(\boldsymbol{x}; \alpha)$ and $s_2(\boldsymbol{x}; \alpha)$ denote the respective two integrals in the sum that precedes it. It follows that $d(\boldsymbol{x}; \alpha) < \infty$ if $s_1(\boldsymbol{x}; \alpha) < \infty$ and $s_2(\boldsymbol{x}; \alpha) < \infty$. Now, using the proportionalities in Proposition B.1

it follows that, since $n + r_0 > -1$, $\mathrm{q}(\alpha) > 0$ and $\mathrm{p}(\alpha) > 0$, then

$$s_1(\boldsymbol{x};\alpha) \lesssim \int_0^1 \phi^{n+r_0} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi \;=\; \frac{\gamma(n + r_0 + 1, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}} \;< \infty,$$

and

$$s_2(\boldsymbol{x};\alpha) \lesssim \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}+\frac{k+1}{\alpha}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi \;=\; \frac{\Gamma(\frac{n+1+2r_\infty}{2} + \frac{k+1}{\alpha}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}+\frac{k+1}{\alpha}}} \;< \infty,$$

therefore, we have that $d(\boldsymbol{x};\alpha) < \infty$.

The case where $k = -1$ and $n > -r_0$ is completely analogous to the previous case, with the only difference in the proof being that $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi\to0^+}{\propto} \phi^{-1}$ in this case, instead of $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi\to0^+}{\propto} 1$.

### B.4. *Proof of Theorem 1.8*

Let $\alpha \in \mathbb{R}^+$ be fixed. Suppose that hypothesis of item *i*) hold, that is, $\pi(\mu) \gtrsim \mu^k$ with $k < -1$. Notice that, for $0 < \phi \leq -\frac{(k+1)}{n\alpha}$ we have that $n\alpha\phi + k \leq -1$. Moreover, for every $\alpha > 0$ fixed we have that $\exp\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\} \underset{\mu\to0^+}{\propto} 1$. Hence, from Proposition 1.6 we have that

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \gtrsim \int_0^1 \mu^{n\alpha\phi+k} d\mu = \infty,$$

for all $\phi \in (0, -\frac{(k+1)}{n\alpha}]$. Therefore

$$\begin{aligned} d(\boldsymbol{x};\alpha) &\gtrsim \int_0^{-\frac{(k+1)}{n\alpha}} \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \int_0^\infty \mu^{n\alpha\phi+k} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi \\ &\gtrsim \int_0^{-\frac{(k+1)}{n\alpha}} \infty\; d\phi \;= \infty, \end{aligned}$$

that is, $d(\boldsymbol{x};\alpha) = \infty$.

Now suppose that hypothesis of *ii*) hold. First suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\phi) \underset{\phi\to0^+}{\gtrsim} \phi^{r_0}$, where $k > -1$ and $n < -r_0 - 1$. Then, following the same steps that resulted in (B3) we have that

$$d(\boldsymbol{x};\alpha) \gtrsim \int_0^1 \phi^{n+r_0} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi \propto \int_0^1 \phi^{n+r_0}\, d\phi = \infty$$

and therefore $d(\boldsymbol{x};\alpha)=\infty$.

The case where $k=-1$, and $n<-r_0$ follows analogously

## B.5. *Proof of Theorem 1.10*

Let $\phi\in\mathbb{R}^+$ be fixed. Since $\pi(\alpha)\alpha^n\frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\}\pi(\mu)\mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}\geq 0$ always, by Tonelli's theorem we have:

$$\begin{aligned} d(\boldsymbol{x};\phi) &= \int_{\mathcal{A}} \pi(\alpha)\alpha^n\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\boldsymbol{\theta} \\ &= \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\,d\alpha. \end{aligned} \tag{B4}$$

Now, since $\pi(\mu)\lesssim\mu^{-1}$ by hypothesis it follows that

$$\begin{aligned} d(\boldsymbol{x};\phi) &\lesssim \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\,d\alpha \\ &= \int_0^\infty \pi(\alpha)\alpha^{n-1}\frac{\left(\prod_{i=1}^n x_i^\alpha\right)^\phi}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}\,d\alpha = \int_0^\infty \pi(\alpha)\alpha^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha \end{aligned}$$

where $\mathrm{q}(\alpha)$ is given in Proposition B.1. Therefore, from the proportionalities in Proposition B.1 it follows that

$$\begin{aligned} d(\boldsymbol{x};\phi) &\lesssim \int_0^\infty \pi(\alpha)\alpha^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha \\ &\propto \int_0^1 \alpha^{q_0+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha+\int_1^\infty \alpha^{q_\infty+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha = s_1(\boldsymbol{x};\phi)+s_2(\boldsymbol{x};\phi). \end{aligned} \tag{B5}$$

where $s_1(\boldsymbol{x};\phi)$ and $s_2(\boldsymbol{x};\phi)$ denote the respective two real numbers in the sum that precedes it. It follows that $d(\boldsymbol{x};\phi)<\infty$ if $s_1(\boldsymbol{x};\phi)<\infty$ and $s_2(\boldsymbol{x};\phi)<\infty$.

By Proposition B.1, $\mathrm{q}(\alpha)>0$, which implies that $e^{-n\,\mathrm{q}(\alpha)\phi}\leq 1$. Moreover, since $q_0+n>0$ we have that

$$s_1(\boldsymbol{x};\phi)=\int_0^1 \alpha^{q_0+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha \quad \leq \int_0^1 \alpha^{q_0+n-1}\,d\alpha<\infty$$

Additionally, by Proposition B.1, $\mathrm{q}(\alpha)\underset{\alpha\to\infty}{\propto}\alpha$ and therefore by Proposition 1.5 there exists $c>0$ such that $\mathrm{q}(\alpha)\leq c\alpha$ for all $\alpha\in[1,\infty)$. Therefore

$$s_2(\boldsymbol{x};\phi)=\int_1^\infty \alpha^{q_\infty+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha \leq \int_1^\infty \alpha^{q_\infty+n-1}e^{-n\phi c\alpha}\,d\alpha=\frac{\Gamma(q_\infty+n,n\phi c)}{(n\phi c)^{q_\infty+n}}<\infty,$$

hence, $d(\boldsymbol{x};\phi)<\infty$.

### B.6. *Proof of Theorem 1.11*

Let $\phi\in\mathbb{R}^+$ be fixed. Suppose that $\pi(\mu)\gtrsim\mu^k$ where $k<-1$. Notice that, for $0<\alpha\leq\frac{k+1}{n\phi}$ it follows that $n\phi+\frac{k+1}{\alpha}\leq 0$ and since $\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}\underset{\mu\to 0^+}{\propto}1$, we have that

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\gtrsim\int_0^1\mu^{n\alpha\phi+k}d\mu=\infty,$$

for all $\alpha\in(0,\frac{k+1}{n\phi}]$. Therefore

$$d(\boldsymbol{x};\phi)\gtrsim\int_0^{\frac{k+1}{n\phi}}\pi(\alpha)\alpha^{n-1}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi\int_0^1\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\,d\alpha=\int_0^{\frac{k+1}{n\phi}}\infty\;d\alpha=\infty$$

hence $d(\boldsymbol{x};\phi)=\infty$.

Now suppose that $\pi(\mu)\gtrsim\mu^k$ and $\pi(\alpha)\underset{\alpha\to 0^+}{\gtrsim}\alpha^{q_0}$, where $k>-1$ and $q_0\in\mathbb{R}$. Then

$$\begin{aligned}
d(\boldsymbol{x};\phi)&\gtrsim\int_0^1\int_0^\infty\alpha^{n+q_0}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi\mu^{n\alpha\phi+k}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\,d\alpha\\
&=\int_0^1\int_0^\infty\alpha^{n+q_0}\frac{\left(\prod_{i=1}^n x_i^\alpha\right)^\phi}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi+\frac{k+1}{\alpha}}}u^{n\phi+\frac{k+1}{\alpha}-1}e^{-u}\,du\,d\alpha=\\
&=\int_0^1\int_0^\infty\alpha^{n+q_0}\left(\prod_{i=1}^n x_i^\alpha\right)^{-\frac{k+1}{\alpha}}n^{-n\phi-\frac{k+1}{\alpha}}e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)}u^{n\phi+\frac{k+1}{\alpha}-1}e^{-u}\,du\,d\alpha\\
&=\int_0^\infty\left(\prod_{i=1}^n x_i\right)^{-(k+1)}n^{-n\phi}u^{n\phi-1}e^{-u}\int_0^1\alpha^{n+q_0}e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)}e^{(\log u-\log n)\frac{k+1}{\alpha}}\,d\alpha\,du
\end{aligned}$$

where in the above we used the change of variables $u=\mu^\alpha\sum_{i=1}^n x_i^\alpha$ in the integral and $\mathrm{p}(\alpha)$ is given as in Proposition B.1.

Now, since $\mathrm{p}(\alpha)\underset{\alpha\to 0^+}{\propto}\alpha^2$ from Proposition B.1 it follows that $\lim_{\alpha\to 0^+}e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)}=\lim_{\alpha\to 0^+}e^{-\frac{p(\alpha)}{\alpha^2}(n\phi\alpha+k+1)\alpha}=e^0=1$. These two facts together applied to the above inequality leads to

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^\infty n^{-n\phi}\left(\prod_{i=1}^n x_i\right)^{-(k+1)} u^{n\phi-1}e^{-u}\int_0^1 \alpha^{n+q_0}e^{(\log u-\log n)\frac{k+1}{\alpha}}\,d\alpha\,du$$

Thus, since $n \geq 1$ and $\log u - \log n > 0$ for $u \geq 3n > e\cdot n$, and since $\int_0^1 \alpha^H e^{\frac{L}{\alpha}} = \infty$ for every $H \in \mathbb{R}$ and $L \in R^+$ (which can be easily checked via the change of variable $\beta = \frac{1}{\alpha}$ in the integral), it follows that

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^\infty n^{-n\phi}\left(\prod_{i=1}^n x_i\right)^{-(k+1)} u^{n\phi-1}e^{-u}\cdot\infty\,du\;=\infty, \tag{B6}$$

and therefore $d(\boldsymbol{x};\phi) = \infty$.

Now suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k \leq -1$ and $n \leq -q_0$. Then, following the same steps that resulted in (B5) we have that

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^1 \alpha^{q_0+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\alpha.$$

but since by Proposition B.1 we have that $q(\alpha) \underset{\alpha\to 0^+}{\propto} 0$ it follows that $e^{-nq(\alpha)\phi} \underset{\alpha\to 0^+}{\propto} 1$ and therefore

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^1 \alpha^{q_0+n-1}\,d\alpha = \infty.$$

## B.7. *Proof of Theorem 1.13*

Since $\pi(\alpha)\alpha^n\frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\}\pi(\mu)\mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$\begin{aligned}d(\boldsymbol{x}) &= \int_{\mathcal{A}} \pi(\alpha)\alpha^n\frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\boldsymbol{\theta}\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n\frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\,d\phi\,d\alpha.\end{aligned}$$

Now, since $\pi(\mu) \lesssim \mu^{-1}$ we have that

$$
\begin{aligned}
d(\boldsymbol{x}) &\lesssim \int_0^\infty \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi-1} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha \\
&= \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^{n-1} \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}\, d\phi\, d\alpha \\
&= \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^{n-1}\pi(\phi) \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha
\end{aligned}
$$

where $\mathrm{q}(\alpha)$ is given in Proposition B.1. Therefore, from the proportionalities in Proposition B.1 it follows that

$$
\begin{aligned}
d(\boldsymbol{x}) &\lesssim \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^{n-1}\pi(\phi) \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&\propto \int_0^1 \int_0^1 f(\alpha,\phi)\, d\phi\, d\alpha + \int_1^\infty \int_0^1 f(\alpha,\phi)\, d\phi\, d\alpha + \int_0^1 \int_1^\infty g(\alpha,\phi)\, d\phi\, d\alpha + \int_1^\infty \int_1^\infty g(\alpha,\phi)\, d\phi\, d\alpha \\
&= s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x}),
\end{aligned} \tag{B7}
$$

where $f(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}$, $g(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{\frac{n-1}{2}}e^{-n\,\mathrm{p}(\alpha)\phi}$ and $s_1(\boldsymbol{x})$, $s_2(\boldsymbol{x})$, $s_3(\boldsymbol{x})$ and $s_4(\boldsymbol{x})$ denote the respective four real numbers in the sum that precedes it. It follows that $d(\boldsymbol{x}) < \infty$, if and only if $s_1(\boldsymbol{x}) < \infty$, $s_2(\boldsymbol{x}) < \infty$, $s_3(\boldsymbol{x}) < \infty$ and $s_4(\boldsymbol{x}) < \infty$. Now, using the proportionalities in Proposition B.1 it follows that

$$
\begin{aligned}
s_1(\boldsymbol{x}) &\lesssim \int_0^1 \alpha^{q_0+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&= \int_0^1 \alpha^{q_0+n-1} \frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\, d\alpha \propto \int_0^1 \alpha^{q_0+n-1}\, d\alpha < \infty,
\end{aligned}
$$

where in the last inequality the condition $n > -q_0$ was used, and in the equality that precedes it the condition $n > -r_0$ was used to ensure that $\gamma(n+r_0, n\,\mathrm{q}(\alpha))$ is well defined and that the equality holds,

$$
\begin{aligned}
s_2(\boldsymbol{x}) &\lesssim \int_1^\infty \alpha^{q_\infty+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\
&= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty-r_0-1}\, d\alpha < \infty,
\end{aligned}
$$

where just as in the $s_1(\boldsymbol{x})$ case, the condition $n > -r_0$ was used in order for the above equality to hold,

$$s_3(\boldsymbol{x}) \lesssim \int_0^1 \alpha^{q_0+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_0^1 \alpha^{q_0+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{q_0-2r_\infty-2}\, d\alpha < \infty,$$

where in the last inequality the condition $q_0 > 2r_\infty + 1$ was used, and finally

$$s_4(\boldsymbol{x}) \lesssim \int_1^\infty \alpha^{q_\infty+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty+n-2} e^{-nk\alpha}\, d\alpha < \infty,$$

where in the above $k \in \mathbb{R}^+$ is given in Proposition B.1. Therefore, from $s_i(\boldsymbol{x}) < \infty, i = 1, \ldots, 4$, we have that $d = s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x}) < \infty$.

## B.8. *Proof of Theorem 1.14*

Suppose that hypothesis of item $i$) hold.

First suppose that $\pi(\mu) \gtrsim \mu^k$ with $k < -1$. Denoting $h = \sqrt{\frac{-k-1}{2n}} > 0$, it follows that for $0 < \alpha \leq h$ and $0 < \phi \leq h$ we have that $n\alpha\phi + k \leq nh^2 + k = \frac{(k-1)}{2} < -1$. Moreover, for every $\alpha > 0$ fixed we have that $\exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \underset{\mu\to 0^+}{\propto} 1$, hence, from Proposition 1.6 we have that

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \gtrsim \int_0^\infty \mu^{n\alpha\phi+k} = \infty,$$

for all fixed $\alpha \in (0, h]$ and $\phi \in (0, h]$. Therefore

$$d(\boldsymbol{x}) \gtrsim \int_{h/2}^h \int_{h/2}^h \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \int_0^\infty \mu^{n\alpha\phi+k} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha$$
$$\propto \int_{h/2}^h \int_{h/2}^h \infty\; d\phi\; d\alpha = \infty,$$

that is, $d(\boldsymbol{x}) = \infty$.

Now suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k > -1$ and $q_0 \in \mathbb{R}$.

Under these hypothesis, in equation (B6) it was proved that

$$d(\boldsymbol{x};\phi) \propto \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^n \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\alpha = \infty$$

for every $\phi > 0$, and therefore

$$\begin{aligned} d(\boldsymbol{x}) &\propto \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^n \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\alpha\, d\phi \\ &= \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \cdot \infty\, d\phi = \infty \end{aligned}$$

and thus $d(\boldsymbol{x}) = \infty$.

Suppose on the other hand that the hypotheses of ii) hold. Since $\pi(\mu) \gtrsim \mu^{-1}$, following the same steps that resulted in (B7) and the same expressions for $s_i(\boldsymbol{x})$, where $i = 1, \cdots, 4$, we have that $d(\boldsymbol{x}) \gtrsim s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x})$. We now divide the proof that $d(\boldsymbol{x}) = \infty$ in four cases:

- Suppose that $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ and $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ with $n \leq -r_0$. Then

$$\begin{aligned} s_2(\boldsymbol{x}) &\gtrsim \int_1^\infty \alpha^{q_\infty+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\ &= \int_1^\infty \alpha^{q_\infty+n-1} \cdot \infty\, d\alpha = \infty \end{aligned}$$

  which implies $d(\boldsymbol{x}) = \infty$.
- Suppose that $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ and $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ with $q_\infty \geq r_0$ and $n > -r_0$. Then

$$\begin{aligned} s_2(\boldsymbol{x}) &\gtrsim \int_1^\infty \alpha^{q_\infty+n-1} \int_0^1 \phi^{n+r_0-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \\ &= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty-r_0-1}\, d\alpha = \infty \end{aligned}$$

  which implies $d(\boldsymbol{x}) = \infty$.
- Suppose that $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ and $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$ with $n \leq -q_0$. Then, by Proposition B.1 we have that $q(\alpha) \underset{\alpha\to 0^+}{\propto} 0$ from where it follows that $e^{-nq(\alpha)\phi} \underset{\alpha\to 0^+}{\propto} 1$

and therefore

$$s_1(\boldsymbol{x}) \gtrsim \int_0^1 \pi(\phi)\phi^{n-1} \int_0^1 \alpha^{q_0+n-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha\, d\phi$$
$$\propto \int_0^1 \pi(\phi)\phi^{n-1} \int_0^1 \alpha^{q_0+n-1}\, d\alpha\, d\phi = \int_0^1 \pi(\phi)\phi^{n-1}\cdot\infty\, d\phi = \infty,$$

which implies $d(\boldsymbol{x}) = \infty$.

- Suppose that $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ and $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$ with $2r_\infty + 1 \geq q_0$. Then

$$s_3(\boldsymbol{x}) \gtrsim \int_0^1 \alpha^{q_0+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_0^1 \alpha^{q_0+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{q_0-2r_\infty-2}\, d\alpha = \infty$$

which implies $d(\boldsymbol{x}) = \infty$.

Therefore the proof is completed.

## B.9. *Proof of Corollary 4.7*

From (10) and by the asymptotic relations (23) we have that

$$\phi^2\psi'(\phi) + \phi - 1 = 2\phi - \frac{1}{2} + o(1) \underset{\phi\to\infty}{\propto} \phi$$

which together with equation (21) implies that

$$\sqrt{\phi^2\psi'(\phi) + \phi - 1} \underset{\phi\to 0^+}{\propto} \sqrt{\phi} \quad\text{and}\quad \sqrt{\phi^2\psi'(\phi) + \phi - 1} \underset{\phi\to\infty}{\propto} \sqrt{\phi}.$$

Hence, from the above proportionalities we have that

$$\sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}} \underset{\phi\to 0^+}{\propto} \frac{1}{\sqrt{\phi}} \quad\text{and}\quad \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}} \underset{\phi\to\infty}{\propto} \frac{1}{\sqrt{\phi^3}}.$$

Therefore, Theorem 1.13 can be applied with $k = q_0 = q_\infty = -1$, $r_0 = -\frac{1}{2}$ and $r_\infty = -\frac{3}{2}$ where $k = -1$, $q_\infty < r_0$ and $2r_\infty + 1 < q_0$, and therefore $\pi_{12}(\alpha, \mu, \phi)$ leads to a proper posterior for every $n > -q_0 = 1$.

In order to prove that the higher moments are improper suppose $\alpha^q\phi^r\mu^j\pi(\boldsymbol{\theta})$ leads to a proper posterior for $r \in \mathbb{N}$, $q \in \mathbb{N}$ and $k \in \mathbb{N}$. By Theorem 1.15 we have $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) \leq q + q_0$ and $n \geq -q_0$, i.e., $k = 0$ and $2r - 1 < q < r + \frac{1}{2}$. The inequality $2r - 1 < r + \frac{1}{2}$ leads to $r < \frac{3}{2}$, i.e., $r = 0$ or $r = 1$. By the previous inequality, the case where $r = 0$ leads to $-1 < q < \frac{1}{2}$, that is, $q = 0$. Now, for $r = 1$ we have the inequality $1 < q < \frac{3}{2}$ which do not have integer solution. Therefore, the only possible values for which $\alpha^q \phi^r \mu^j \pi(\boldsymbol{\theta})$ is proper is $q = r = j = 0$, that is, the higher moments are improper.