Correction serie n°2

Ex 1

$$f(x) = x_1^2 + 4x_2^2 - 4x_1 - 8x_2$$

$$\nabla f(x) = \begin{pmatrix} 2x_1 - 4 \\ 8x_2 - 8 \end{pmatrix}$$

1) Méthode de gradient à pas fixe $t = 0{,}1$

→ $k = 0$ $\quad x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\nabla f(x^{(0)}) = \begin{pmatrix} -4 \\ -8 \end{pmatrix} \;;\quad d_0 = -\nabla f(x^{(0)}) = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$x^{(1)} = x^{(0)} + t\,d_0$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0{,}1 \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0{,}4 \\ 0{,}8 \end{pmatrix}$$

→ $k = 1$ $\quad x^{(1)} = \begin{pmatrix} 0{,}4 \\ 0{,}8 \end{pmatrix}$

$$\nabla f(x^{(1)}) = \begin{pmatrix} 2 \times 0{,}4 - 4 \\ 8 \times 0{,}8 - 8 \end{pmatrix} = \begin{pmatrix} -3{,}2 \\ -1{,}6 \end{pmatrix}$$

$$d^{(1)} = -\nabla f(x^{(1)}) = \begin{pmatrix} 3{,}2 \\ 1{,}6 \end{pmatrix}$$

$$x^{(2)} = x^{(1)} + t\,d_1$$

$$= \begin{pmatrix} 0{,}4 \\ 0{,}8 \end{pmatrix} + 0{,}1 \begin{pmatrix} 3{,}2 \\ 1{,}6 \end{pmatrix} = \begin{pmatrix} 0{,}72 \\ 0{,}96 \end{pmatrix}$$

k	$x^{(k)}$	$\nabla f(x^k)$	d_k	$x^{(k+1)} = x^{(k)} + t\,d_k$
0	$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\nabla f(x^{(0)}) = \begin{pmatrix} -4 \\ -8 \end{pmatrix}$	$d_0 = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$	$x^{(1)} = \begin{pmatrix} 0{,}4 \\ 0{,}8 \end{pmatrix}$
1	$x^{(1)} = \begin{pmatrix} 0{,}4 \\ 0{,}8 \end{pmatrix}$	$\nabla f(x^{(1)}) = \begin{pmatrix} -3{,}2 \\ -1{,}6 \end{pmatrix}$	$d_1 = \begin{pmatrix} 3{,}2 \\ 1{,}6 \end{pmatrix}$	$x^{(2)} = \begin{pmatrix} 0{,}72 \\ 0{,}96 \end{pmatrix}$

2) méthode à pas optimal à partir $(0,0)$

Rappel

- $d_k = -\nabla f(x^{(k)})$
- trouver t minimisant $f(x^{(k)} + t\, d_k)$
- $x^{(k+1)} = x^{(k)} + t\, d_k$

→ $k=0$

- $x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- $d_0 = -\nabla f(x^0) = +\begin{pmatrix} 4 \\ 8 \end{pmatrix}$
- soit $\phi(t) = f(x^{(0)} + t\, d_0)$

$$= f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 8 \end{pmatrix}\right)$$

$$= f\begin{pmatrix} 4t \\ 8t \end{pmatrix}$$

$$= (4t)^2 + 4(8t)^2 - 4(4t) - 8(8t)$$

$$\phi'(t) = 2\times 4\times 4.t + 4\times 2\times 8\times 8t - 4\times 4 - 8\times 8$$

$$= 544t - 80$$

$\phi'(t) = 0 \Rightarrow t = \dfrac{80}{544} = \dfrac{5}{34}.$

t	0		$\frac{5}{34}$		$+\infty$
ϕ'		$-$	0	$+$	
ϕ		$\searrow$		$\nearrow$	

d'après le T.V $\boxed{t_0 = \frac{5}{34}}$ est bien un minimum.

$$t_0 = \min_{t>0} f(x^{(0)} + t\, d_0)$$

- $x^{(1)} = x^{(0)} + t_0\, d_0$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{5}{34}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{10}{17} \\ \frac{20}{17} \end{pmatrix} = \begin{pmatrix} 0,588 \\ 1,176 \end{pmatrix}$$

→ $k=1$

- $x^{(1)} = \begin{pmatrix} 10/17 \\ 20/17 \end{pmatrix}$
- $d_1 = -\nabla f(x^{(1)}) = -\begin{pmatrix} 2\times\frac{10}{17} - 4 \\ 8\times\frac{20}{17} - 8 \end{pmatrix} = \begin{pmatrix} \frac{48}{17} \\ -\frac{24}{17} \end{pmatrix}$

- $\varphi(t) = f\left(x^{(1)} + t\, d_1\right)$

$$= f\left(\begin{pmatrix} \frac{10}{17} \\ \frac{20}{17} \end{pmatrix} + t \begin{pmatrix} \frac{48}{17} \\ -\frac{24}{17} \end{pmatrix}\right) = f\begin{pmatrix} \frac{10}{17} + \frac{48}{17}t \\ \frac{20}{17} - \frac{24}{17}t \end{pmatrix}$$

$$= \left(\frac{10}{17} + \frac{48}{17}t\right)^2 + 4\left(\frac{20}{17} - \frac{24}{17}t\right)^2 - 4\left(\frac{10}{17} + \frac{48}{17}t\right) - 8\left(\frac{20}{17} - \frac{24}{17}t\right)$$

$$\varphi'(t) = 2 \times \frac{48}{17} \times \left(\frac{10}{17} + \frac{48}{17}t\right) + 2 \times 4 \times \frac{-24}{17}\left(\frac{20}{17} - \frac{24}{17}t\right) - 4\,\frac{48}{17} + 8 \times \frac{24}{17}.$$

$$= t\left(2 \times \left(\frac{48}{17}\right)^2 + 8 \times \left(\frac{24}{17}\right)^2\right) + 2 \times \frac{48}{17} \times \frac{10}{17} - 8 \times \frac{24}{17}\,\frac{20}{17} - 4 \times \frac{48}{17} + 8 \times \frac{24}{17}$$

$$\varphi'(t) = 0 \Rightarrow t = \frac{30}{96} = \frac{15}{48}$$

t	0		$\frac{15}{48}$		$+\infty$
φ'		$-$	0	$+$	
		$\searrow$		$\nearrow$	

$$t_1 = \min .\ f\left(x^{(1)} + t\, d_1\right)$$

- $x^{(2)} = x^{(1)} + t_1 d^1$

$$= \begin{pmatrix} \frac{10}{17} + \frac{48}{17} \times \frac{15}{48} \\ \frac{20}{17} - \frac{24}{17} \times \frac{15}{48} \end{pmatrix} = \begin{pmatrix} \frac{25}{17} \\ \frac{25}{34} \end{pmatrix} = \Big($$

k	$x^{(k)}$	$\nabla f(x^{(k)})$	d_k	t_k	$x^{(k+1)}$
$k=0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$-\begin{pmatrix} 4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 8 \end{pmatrix}$	$\frac{5}{34}$	$\begin{pmatrix} 10/17 \\ 20/17 \end{pmatrix}$
$k=1$	$\begin{pmatrix} 10/17 \\ 20/17 \end{pmatrix}$	$\begin{pmatrix} -48/17 \\ 24/17 \end{pmatrix}$	$\begin{pmatrix} 48/17 \\ -24/17 \end{pmatrix}$	$15/48$	$\begin{pmatrix} 25/17 \\ 25/34 \end{pmatrix}$

Ex 1 $\quad f(x) = x_1^2 + 4x_2^2 - 4x_1 - 8x_2$

4°) Méthode de gradient conjugué à partir de $(0,0)$

→ $k=0$ • $x^{(1)} = x^{(0)} + t\, d_0$ avec $d_0 = -\nabla f(x^0)$

d'après 2°) $\quad x^{(1)} = \begin{pmatrix} \frac{10}{17} \\ \frac{20}{17} \end{pmatrix}$

• $\beta_0 = \dfrac{\|\nabla f(x^1)\|^2}{\|\nabla f(x^0)\|^2}$ $\quad$ d'après 2°) $\nabla f(x^0) = \begin{pmatrix} -4 \\ -8 \end{pmatrix}$

$\|u\|^2 = \langle u, u \rangle = u^t u.$

$\|\nabla f(x^0)\|^2 = (-4)^2 + (-8)^2$

et $\nabla f(x^1) = \begin{pmatrix} 2x_1 - 4 \\ 8x_2 - 8 \end{pmatrix} = \begin{pmatrix} 2 \times \frac{10}{17} - 4 \\ 8 \times \frac{20}{17} - 8 \end{pmatrix} = \begin{pmatrix} -\frac{48}{17} \\ \frac{24}{17} \end{pmatrix}$

$\Rightarrow \|\nabla f(x^1)\|^2 = \left(-\frac{48}{17}\right)^2 + \left(\frac{24}{17}\right)^2 = \dfrac{2880}{(17)^2}$

$\Rightarrow \beta_0 = \dfrac{2880}{23120} = 0{,}125 = \dfrac{36}{(17)^2}$

$d_1 = -\nabla f(x^1) + \beta_0 d_0 \quad ; \quad d_0 = -\nabla f(x^0) = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$= -\begin{pmatrix} -\frac{48}{17} \\ \frac{24}{17} \end{pmatrix} + \dfrac{36}{(17)^2} \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$d_1 = \begin{pmatrix} 3{,}322 \\ -0{,}415 \end{pmatrix}$

→ $k=1$ • $t_1 = \min\limits_{t>0} \phi(t)$ avec $\phi(t) = f(x^1 + t\, d_1)$

$\phi(t) = f\left(\begin{pmatrix} \frac{10}{17} \\ \frac{20}{17} \end{pmatrix} + t \begin{pmatrix} 3{,}22 \\ -0{,}415 \end{pmatrix}\right) = f\begin{pmatrix} \frac{10}{17} + 3{,}322\, t \\ \frac{20}{17} - 0{,}415\, t \end{pmatrix}$

$$= \left(\frac{10}{17} + 3{,}322\,t\right)^2 + 4\left(\frac{20}{17} - 0{,}415\,t\right)^2$$

$$-4\left(\frac{10}{17} + 3{,}322t\right) - 8\left(\frac{20}{17} - 0{,}415t\right)$$

$$\phi'(t) = 2 \times 3{,}322\,(0{,}588 + 3{,}322\,t) - 4 \times 2 \times 0{,}415\,(1{,}176 - 0{,}415\,t)$$

$$- 4 \times 3{,}322 + 8 \times 0{,}415$$

$$= (2 \times 3{,}322 \times 0{,}588) - (4 \times 2 \times 0{,}415 \times 1{,}176) - (4 \times 3{,}322 + 8 \times 0{,}415)$$

$$+ 2 \times (3{,}322)^2\, t + 4 \times 2 \times (0{,}415)^2\, t \cancel{- 4 \times 3{,}322 + 8 \times 0{,}415}$$

$$= -9{,}954 + 23{,}45\, t$$

$$\phi'(t) = 0 \quad \Rightarrow \quad t = \frac{+9{,}954}{23{,}45} = 0{,}424 =$$

$$t_1 = \min_{t>0} \phi(t)$$

$$\boxed{t_1 = 0{,}424}$$

t	0		$0{,}424$		$+\infty$
$\phi'(t)$		$-$	0	$+$	
$\phi(t)$		$\searrow$		$\nearrow$	

- $x^2 = x^1 + t_1 d_1$

$$= \begin{pmatrix} \frac{10}{17} \\ \frac{20}{17} \end{pmatrix} + 0{,}424 \begin{pmatrix} 3{,}322 \\ -0{,}415 \end{pmatrix} = \begin{pmatrix} 0{,}588 + 0{,}424 \times 3{,}322 \\ 1{,}176 - 0{,}424 \times 0{,}415 \end{pmatrix}$$

$$= \begin{pmatrix} 1{,}997 \\ 1{,}0 \end{pmatrix} \approx \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

à 2 chiffres.

- $\beta_1 = \dfrac{\|\nabla f(x^2)\|^2}{\|\nabla f(x^1)\|^2}$; $\nabla f(x^2) = \begin{pmatrix} 2 \times 2 - 4 \\ 8 \times 1 - 8 \end{pmatrix} = 0$

$\Rightarrow$ $x^{(2)}$ est le minimum local de f.

5) Méthode de Newton en partant de $(0,0)$ 4

$\rightarrow k = 0$ $\quad \cdot\ x^0 = (0,0)$

$\cdot\ d_0 ?$ $\quad \nabla^2 f(x^0)\, d_0 = -\nabla f(x^0) \quad (*)$

sachant que $\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$ $\quad \nabla f(x) = \begin{pmatrix} 2x_1 - 4 \\ 8x_2 - 8 \end{pmatrix}$

$(*) \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} d_0' \\ d_0'' \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$\Rightarrow \begin{cases} 2 d_0' = 4 \\ 8 d_0'' = 8 \end{cases} \Rightarrow \begin{cases} d_0' = 2 \\ d_0'' = 1 \end{cases}$

$\cdot\ x^{(1)} = x^0 + d_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\| \nabla f(x^1) \|^2 = 0 \longrightarrow x^{(1)}$ est un minimum local de f