Extended Essay-Mathematics

The number of infinite loops in third degree polynomials when finding roots using Newton-Raphson's iterative method

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Abstract

The Newton-Raphson method is an iterative method for solving equations and known for its fast convergence. However, it has its limitations and one of them is the occurrence of an infinite loop. The infinite loop is when the Newton-Raphson method fails to converge due to the fact that its third estimate equals its first. The research question of this essay is: When finding real roots of third degree polynomial using Newton-Raphson iterative method, to what extent can the number of possible infinite loops be determined? The algebraic and geometric conditions for the existence of the infinite loop were derived. Algebraically, solving a ninth degree polynomial derived from Newton-Raphson method gives a maximum of five real solutions. These solutions represent either real roots of the third degree polynomial or values leading to infinite loop(s). Geometrically, a pair of similar triangles has to exist for an infinite loop to occur. These conditions were then investigated on different cases of third degree polynomials. A method for dividing the function into different parts, and analysing the algebraic and geometric conditions for the existence of infinite loops was presented and observations were made. Analysing these observations combined, enabled a justified generalization that the number of possible infinite loops in third degree polynomials ranges from zero to a maximum of two. This essay could be considered the ground for further studies of infinite loops for higher degree polynomials or other functions.

Word count: 235

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Introduction

The "Abel Ruffini" theorem states that there is no general algebraic solution to fifth degree polynomial equations or higher with arbitrary coefficients (Bajnok, 2013). Mathematicians have developed different iterative methods in order to calculate approximate values of the zeros of polynomials or other functions. Iterative methods are important because they are often the only way to solve certain equations. Moreover, technology used nowadays for solving equations is in many cases based on these iterative methods. The Newton-Raphson method is an iterative method with fast convergence. In fact, according to Zeggeren and Storey (1970): "*The Newton-Raphson method is probably the most popular (and is certainly the best known) method of finding numerically the roots of a set of non-linear equations.*" (p.137). However, the Newton-Raphson iterative method has a certain limitation concerning the first estimate leading to an infinite loop. The study of this particular limitation was not extensive maybe because of the rareness of its occurrence with integers. Nevertheless, studying when it occurs algebraically and geometrically leads to more understanding of the mathematics behind it; which is the main objective of this extended essay. The research question is: When finding real roots of third degree polynomial using Newton-Raphson iterative method, to what extent can the number of possible infinite loops be determined? The plan is to derive the algebraic and geometric conditions for an infinite loop to occur. Then, the infinite loop for different cases of third degree polynomials will be investigated graphically in order to reach a generalization. The third degree polynomials could be considered the ground for further studies of infinite loops for higher degree polynomials or other functions.

Brief history of the Newton-Raphson method

Named after Isaac Newton and Joseph Raphson, the Newton-Raphson method was first published in 1690 by Joseph Raphson. However, it is not evident whether Joseph Raphson had previous knowledge about Newton's work on solving by iterative methods. In fact, Kapadia, Chan and Moyé (2005) suggested that: *"It was actually Joseph Raphson who first published in 1690. He did this without previous knowledge about Newton's work on this problem."* (p.573), while on the other hand, Dahlquist and Björck (2008) suggested that: *"Joseph Raphson was allowed to see Newton's work and Newton's method was first published in a book by Raphson 1690."* (p.685). A view proposed in Sosmath.com (2016) suggested that the history behind the Newton-Raphson method lies on an ancient Babylonian approach of finding an approximation to $\sqrt{2}$. However, the link between the ancient Babylonian approach and the Newton-Raphson method is not very clear. According to Sebah (2001), around 1669 Isaac Newton introduced a new algorithm to solving polynomial equations. It was illustrated on the example: $y^3 - 2y - 5 = 0$. In order to find an accurate root of this equation, one must choose a first estimate; in this example, consider $y = 2$. Writing $y = 2 + p$ and substituting in the main equation gives: $p^3 + 6p^2 + 10p - 1 = 0$. Supposedly, the value of p is a very small decimal, and hence the first part of the equation can be neglected as powering small decimals tend to be 0. Therefore, $10p - 1 = 0$; which gives $p = 0.1$. Repeating the same process again; the value of y should be coming closer to the real zero. Note that the first chosen estimate must be very close to the real root or else the value of p will not be a small value, and thus, $p^3 + 6p^2$ will not be neglected; and the algorithm will not work. According to Newton, the algorithm proposed had no relation with calculus. In 1690, Joseph Raphson somehow made a further step to the algorithm proposed by Newton, and so he proposed a new method. He illustrated his algorithm on the example: $x^3 - bx + c = 0$, starting with the estimate $x \approx g$, then a better approximation in the form $x \approx g + \frac{c + g^3 - bg}{b - 3g^2}$ which reduces to $x \approx g - \frac{c + g^3 - bg}{3g^2 - b}$. It can be noticed that this algorithm is strongly related to Newton-Raphson method since the denominator of the fraction is the derivative of its numerator. Furthermore, Sebah (2016) presented this observation as a start for the further studies taken into account of the Newton-Raphson method. For the years to come, the Newton-Raphson method was studied and generalized by other mathematicians such as Simpson (18^{th} century), Cauchy (19^{th} century) and Kantorovich (20^{th} century).

Newton-Raphson method

The Newton-Raphson method can be applied to find real roots of $f(x) = 0$ when they are in an interval where *f(x)* is continuous and differentiable. According to Harcet et al. (2014) (p.52):

- A function *f(x)* is continuous at the point *a* where $a \in D_f$ (Domain of *f*) if: $\lim_{x \to a} f(x) = f(a)$ And if *f(x)* is continuous at each point on an interval *I*, then the function is continuous on *I*.

- A continuous function *f(x)* is differentiable at $x = a$ if and only if the $\lim_{h \to 0} \frac{f(x+h) - f(a)}{h}$ exists and is finite. And if *f(x)* is differentiable at all points on an interval *I*, then the function is differentiable on *I*, and the derivative of the function is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, $x \in I$.

The Newton-Raphson method is based on the concept that the *x*-intercept of the tangent to *f(x)* at a certain point (called first estimate) is closer to the real root than the first estimate. Figure 1 represents a function $f(x)$ that is continuous and differentiable on an interval including a root R_1.

After an estimate x_1 is picked, the estimate x_2 is the x-intercept of the tangent of *f(x)* at point having $x = x_1$. Indeed, x_2 is closer to the real root R_1 than x_1 and hence, the estimation of the real root improves for every new estimation.

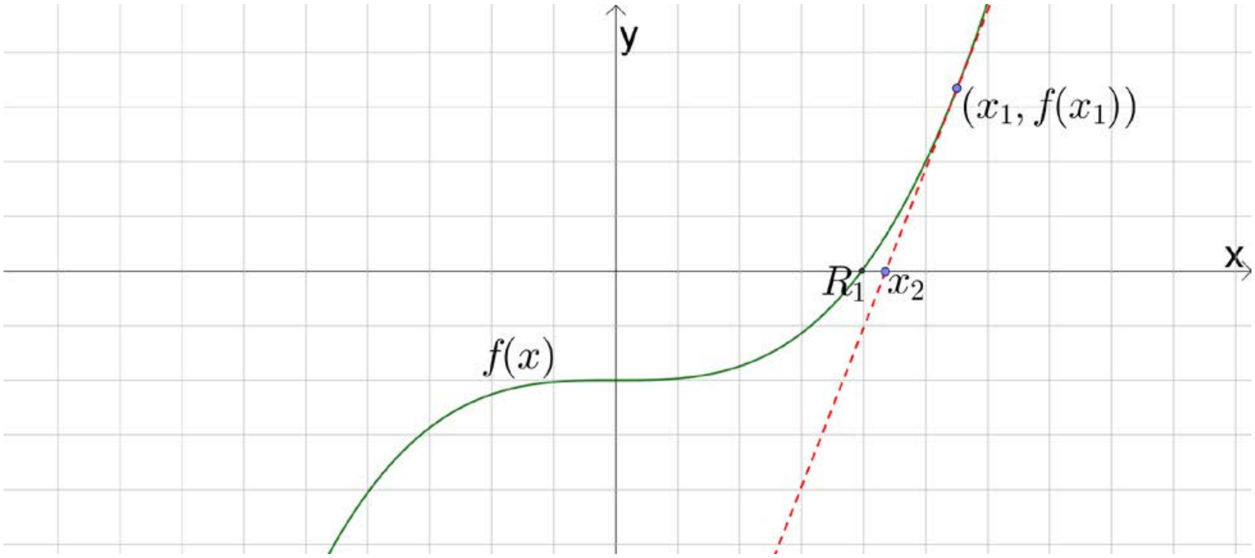


Figure 1

Consider function *f(x)* and first estimate x_1. The equation of the tangent at point having $x = x_1$ is $y = f'(x_1)x + c$. Knowing that the tangent passes by the point $(x_1, f(x_1))$ and substituting, $f(x_1) = f'(x_1)x_1 + c$, hence $c = f(x_1) - f'(x_1)x_1$

Therefore, the equation of the tangent is $y = f'(x_1)x + f(x_1) - f'(x_1)x_1$

Its x-intercept (when $y = 0$) would be: $x = \dfrac{f'(x_1)x_1 - f(x_1)}{f'(x_1)} = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ which equals the second estimate x_2. Hence, the general formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ is obtained, where x_n is the estimate and x_{n+1} is the next estimate.

It can be understood from the general formula that:

1- The derivative $f'(x_n)$ must not equal to 0. Hence, the estimate x_n must not be a stationary point.

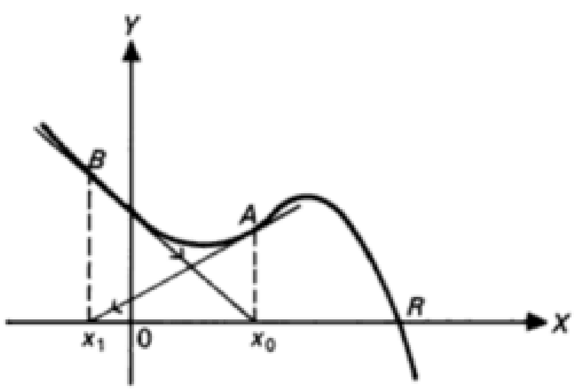
2- The function *f(x)* must be continuous at this estimate point, or else *f(x)* will not exist.

3- The function *f(x)* must be differentiable at this estimate point, or else $f'(x_n)$ will not exist.

The infinite loop

An interesting limitation mentioned by Balachandra and Shantha (2004):

"*The initial value* x_0 *leads to the next iterate* x_1*. But, the tangent to the curve* $y = f(x)$ *at the point B corresponding to* $x = x_1$ *cuts the x-axis again at* x_0*. The tangent at* $A(x = x_0)$ *hits the x-axis again at the earlier point* x_1*. Thus, there results an infinite loop between* x_0 *and* x_1 *without giving the solution*." (p.50) And they clarified on the graph below:



Therefore, in some cases, using Newton-Raphson method fails in converging to the real root due to the consecutive estimates entering an infinite loop. Furthermore, a similar example was stated on Wikipedia (2016) where $f(x) = x^3 - 2x + 2$ had an infinite loop between 0 and 1.

$x_1 = 0$

$$x_2 = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1$$

$x_3 = 1 - \frac{f(1)}{f'(1)} = 0$, which is equal to x_1 and hence this leads to an infinite loop; where the value of x is alternating between 0 and 1.

This example can be illustrated on figure 2 below:

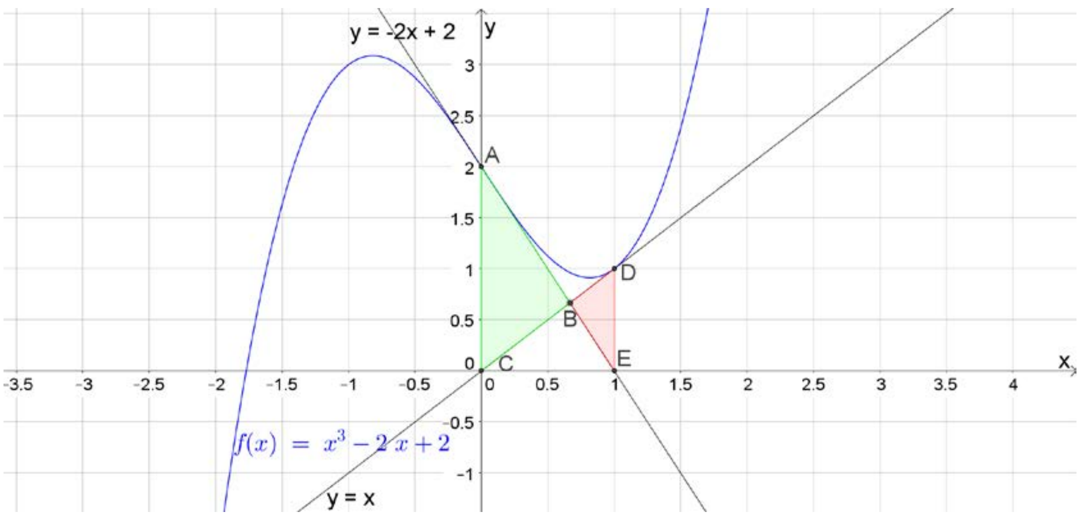


Figure 2

From figure 2, it is clear that the *x*-intercept of each tangent is on the same vertical line as the point of intersection of the other tangent with the function. This means that AC and DE are vertical. Since $\angle ABC = \angle EBD$ (vertically opposite) and $\angle ACB = \angle EDB$ (alternate interior since AC parallel to ED), hence, the two triangles ABC and EBD are similar. To generalize, geometrically, in order for an infinite loop to occur, the triangle formed between: the point of intersection of the first tangent and curve, *x*-intercept of the second tangent, and point of intersection of the two tangents, must be similar to the triangle formed between: the point of intersection of the second tangent and curve, *x*-intercept of the first tangent and point of intersection of the two tangents.

Algebraically, the infinite loop occurs when $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_1$

It should be noticed that if x_1 is equal to the real root, then: $x_2 = x_1 - \frac{0}{f'(x_1)} = x_1$; which will lead to $x_1 = x_2 = x_3$.

In order to generalize the infinite loop algebraic condition, consider function *f(x)* with first estimate $x_1 = a$. According to Newton-Raphson method: $x_2 = a - \frac{f(a)}{f'(a)}$ and $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

Since there exist an infinite loop: $x_3 = x_1 = a$, therefore: $a - \frac{f(a)}{f'(a)} - \frac{f(a - \frac{f(a)}{f'(a)})}{f'(a - \frac{f(a)}{f'(a)})} = a$

Hence, the algebraic condition for an infinite loop to take place is such that:

$$\frac{f(a)}{f'(a)} + \frac{f(a - \frac{f(a)}{f'(a)})}{f'(a - \frac{f(a)}{f'(a)})} = 0 \quad \text{…. Equation 1}$$

Applying equation 1 on $f(x) = x^3 - 2x + 2$ and substituting *x* in terms of *a*:

$$f(a) = a^3 - 2a + 2$$
$$f'(a) = 3a^2 - 2$$
$$a - \frac{f(a)}{f'(a)} = a - \frac{a^3 - 2a + 2}{3a^2 - 2} = \frac{2a^3 - 2}{3a^2 - 2}$$

Hence,

$$\frac{a^3-2a+2}{3a^2-2}+\frac{f(\frac{2a^3-2}{3a^2-2})}{f'(\frac{2a^3-2}{3a^2-2})}=0$$

$$\frac{a^3-2a+2}{3a^2-2}+\frac{(\frac{2a^3-2}{3a^2-2})^3-2(\frac{2a^3-2}{3a^2-2})+2}{3(\frac{2a^3-2}{3a^2-2})^2-2}=0$$

Expanding the equation using technology (Geogebra 5.0):

((a^3-2a+2)/(3a^2-2))+(((2a^3-2)/(3a^2-2))^3-2((2a^3-2)/(3a^2-2))+2)/(3*((2a^3-2)/(3a^2-2))^2-2)=0

Expand: $$\frac{10\,a^9-39\,a^7+15\,a^6+54\,a^5-30\,a^4-42\,a^3+36\,a^2-4\,a}{18\,a^8-39\,a^6-36\,a^5+54\,a^4+24\,a^3-18\,a^2-4}=0$$

It can be noticed that solving the equation above results in ninth degree polynomial. Which makes sense because the numerator of the fraction includes $f(a-\frac{f(a)}{f'(a)})$; where $a-\frac{f(a)}{f'(a)}$ is cubic, as well as the function *f(x)*. This will always be the case when applying on any further polynomials.

Using technology to solve for *a:*

((a^3-2a+2)/(3a^2-2))+(((2a^3-2)/(3a^2-2))^3-2((2a^3-2)/(3a^2-2))+2)/(3*((2a^3-2)/(3a^2-2))^2-2)=0

Solve: $\{a=-1.7692923542, a=0, a=0.1333458576, a=1, a=1.0249665376\}$

Solving for *a* resulted in five real solutions. One solution is the only real root of the function $R_1 \approx -1.77$ (3 sig. fig.), which confirms what was mentioned earlier that equation 1 gives solutions either when $x_1 = x_2 = x_3$ (which is the real root) or $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_1$ (which is an infinite loop).

Two solutions are the infinite loop between 0 and 1. Then the two other solutions suggesting another infinite loop for $x \approx 0.133$ and $x \approx 1.03$.

It must be noticed that the latter two values that cause the infinite loop for $f(x) = x^3 - 2x + 2$ are irrational numbers. Which means practically they will not be selected as first estimates. Even if selected to three significant figures, the Newton-Raphson formula will not show an infinite loop:

$$x_1 = 0.133$$

$$x_2 = 0.133 - \frac{f(0.133)}{f'(0.133)} = 0.133 - \frac{1.736}{-1.947} \approx 1.03$$

$x_3 = 1.03 - \dfrac{f(1.03)}{f'(1.03)} = 1.03 - \dfrac{1.03}{1.18} \approx 0.157$, which is $\neq 0.133$; however approximately close.

Even approximating all values to three decimal places will not give answers where $x_1 = x_3$

Nevertheless, the objective is to improve the understanding of the infinite loop.

Studying infinite loop in different third degree polynomials

Cirrito and Tobin (2004) classified third degree polynomials according to the four cases below (p.94):

$$f(x) = a(x-k)^3$$

$$f(x) = a(x-k)^2(x-m)$$

$$f(x) = a(x-k)(x^2 + px + q)$$

$$f(x) = a(x-k)(x-m)(x-n)$$

For functions in the form $f(x)=a(x-k)^3$, to even generalize more, taking the vertical translation *d* into consideration; it will be in the form: $f(x)=a(x-k)^3+d$. This third degree function has one real root, since the function cuts the *x*-axis only at:

$$a(x-k)^3+d=0$$
$$(x-k)^3=-\frac{d}{a}$$
$$x=\sqrt[3]{-\frac{d}{a}}+k$$

And it has only one stationary point of inflection since:

$f'(x)=3a(x-k)^2$ and $f''(x)=6a(x-k)$

$f'(x)=0$ and $f''(x)=0$ only when $x=k$

Applying equation 1 on two examples: $f_1(x)=x^3-1$ and $f_2(x)=-2x^3-3$, and using technology to solve for *a*, the following table is obtained.

$f_1(x)=x^3-1$	$f_2(x)=-2x^3+1$
$\{\mathbf{a=1}\}$	$\left\{\mathbf{a=\frac{\sqrt[3]{4}}{2}}\right\}$

It can be seen that there is only one real solution for *a*, which is the real root of the function. Hence, it suggests that no infinite loop exists for this case. Geometrically, it can be noticed that third degree polynomials of this case cannot have estimates that would lead to infinite loops since the sign of the gradient never changes. They would always be either increasing or decreasing for $x\in R$ In fact, for: $f(x)=a(x-k)^3+d$, $f'(x)=3a(x-k)^2$ which always have the same sign as *a* since $(x-k)^2$ is always positive.

Investigations in this essay concerning different third degree cases for each function will be by dividing the function into different parts; parts in which the boundaries are either: the real roots, points of inflexion or stationary points. Studying this case in more depth, consider the following graph of *f(x)* with R_1 representing the single real root. In this case, the division of parts results in three parts: $\mathrm{A}_1, \mathrm{A}_2,$ and A_3 represented in figure 3 below.

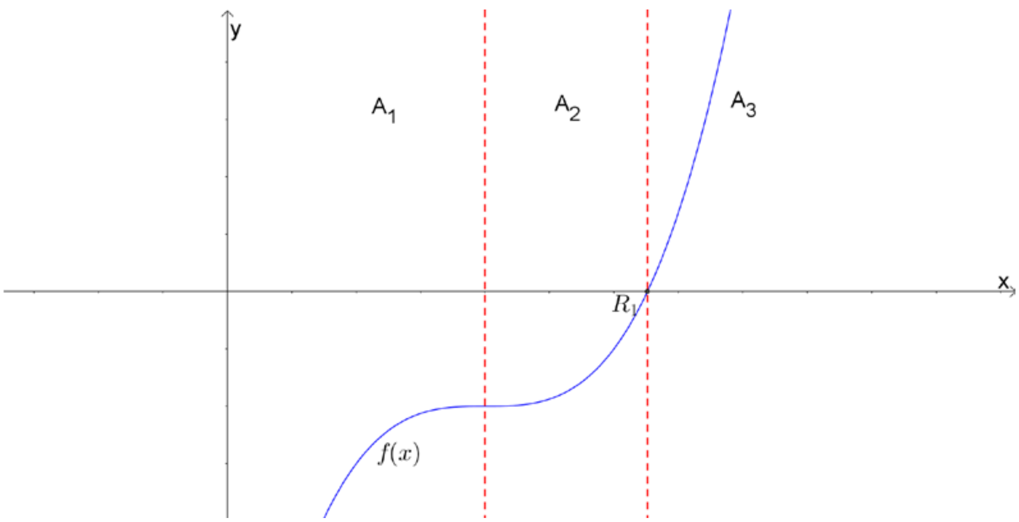


Figure 3

Possible tangents for each part with increment $x = 0.1$ will be illustrated in figures 3.1, 3.2 and 3.3.

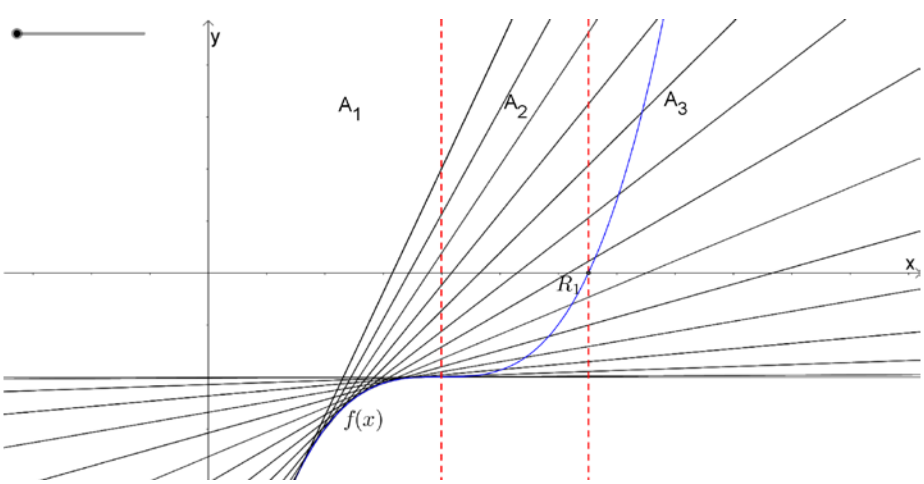


Figure 3.1

From observation of figure 3.1, first estimate in A_1 (tangent in A_1) would give second estimate (x-intercept of first tangent) in any of all three parts.

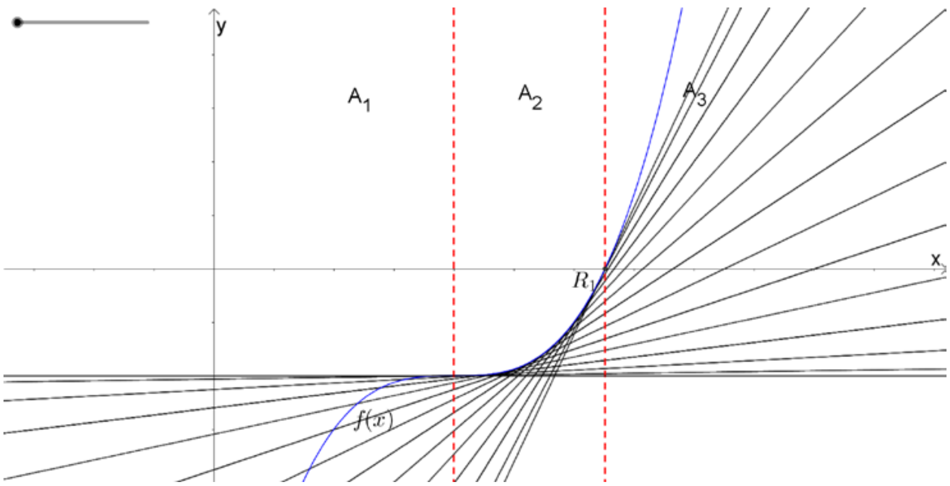


Figure 3.2

From observation of figure 3.2, first estimate in A_2 would give second estimate only in A_3.

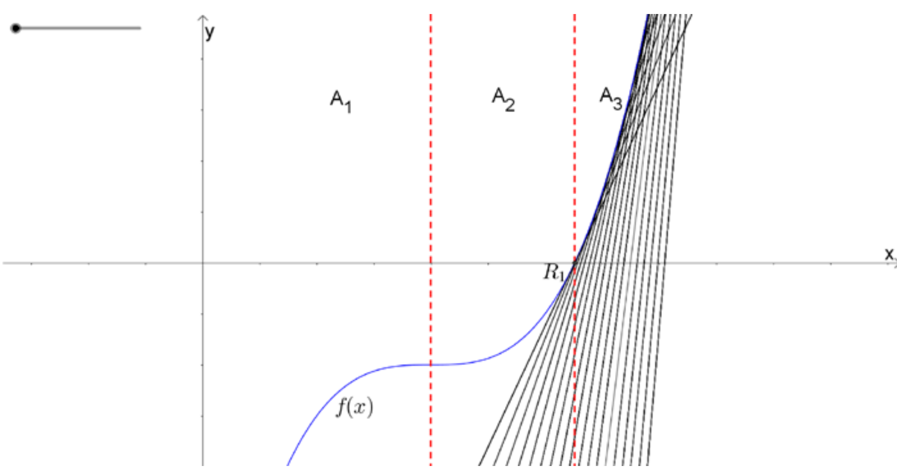


Figure 3.3

From observation of figure 3.3, first estimate in A_3 would only give second estimate in A_3 again.

Graphing tangents in different parts showed that estimates in A_3 will approach the real root R_1.

The observations could be summarized in the following table.

First estimate	Second estimate
In A_1	In A_1
	In A_2
	In A_3
In A_2	In A_3
In A_3	In A_3

The tangent in one part does not fall in another part then return. Hence, it is not possible for a similar triangle to exist, and thus no possibility for an infinite loop for this case. It should be noted that the point of inflexion of the function being above or below the *x*-axis will not affect the results in any way. Even for functions in the form $f(x) = a(x-k)^3 + d$ when $d = 0$, which means the function having three identical real roots (or repeated roots) instead of one single real root since the point of inflexion lies on the *x*-axis, the results will be unaffected; since the possibility of at least one pair of similar triangles to be formulated would not exist. Also reflection of the function along either the *x*-axis or the *y*-axis would not affect the similar triangles that may have been formulated if infinite loops were to exist. To conclude, there is no possibility for any infinite loop to exist for third degree polynomials in the form: $f(x) = a(x-k)^3 + d$.

For functions in the form: $f(x) = a(x-k)^2(x-m)$. Third degree polynomials having one real root and two identical real roots, solving $a(x-k)^2(x-m) = 0$ gives $x - m = 0$ or $(x-k)^2 = 0$. Hence, the single real root $x = m$ and the repeated root $x = k$.

Applying equation 1 on two examples: $f_3(x) = x^3 - x^2 - x + 1$ and $f_4(x) = x^3 - 3x + 2$, and using technology to solve for *a*, the following table is obtained.

$f_3(x) = x^3 - x^2 - x + 1$	$f_4(x) = x^3 - 3x + 2$
$\{a = -1\}$	$\{a = -2\}$

It can be seen that there is only one real solution for *a*, which is the single real root of the function. Hence, it suggests that no infinite loop exists for these cases.

Studying this case in more depth, consider the following graph of *f(x)* with R_1 representing the single real root, R_2 the repeated one and A_1, A_2, A_3 and A_4 the four different parts of *f(x)*.

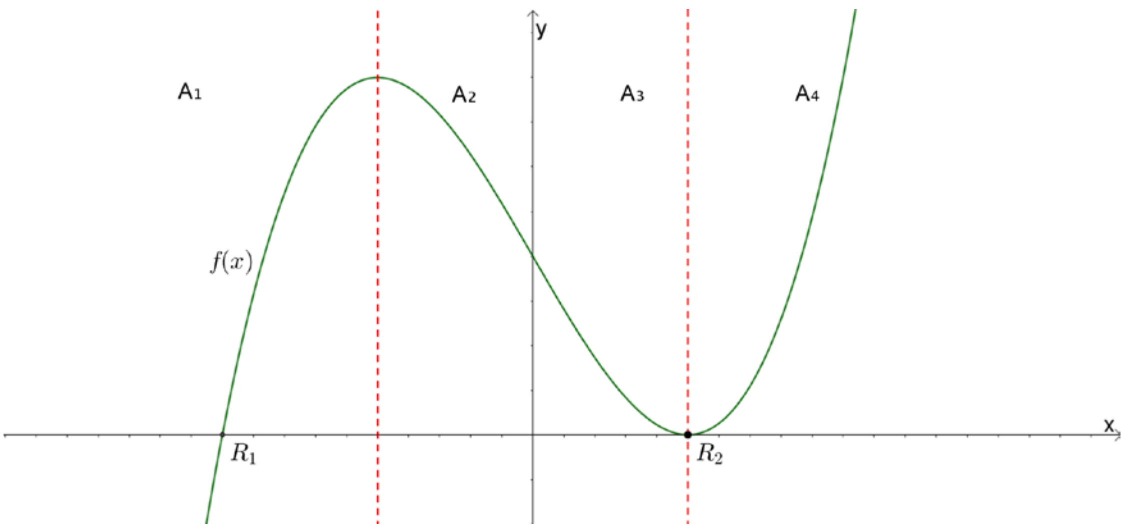


Figure 4

Possible tangents for each part with increment $x = 0.1$ will be illustrated in figures 4.1, 4.2, 4.3 and 4.4

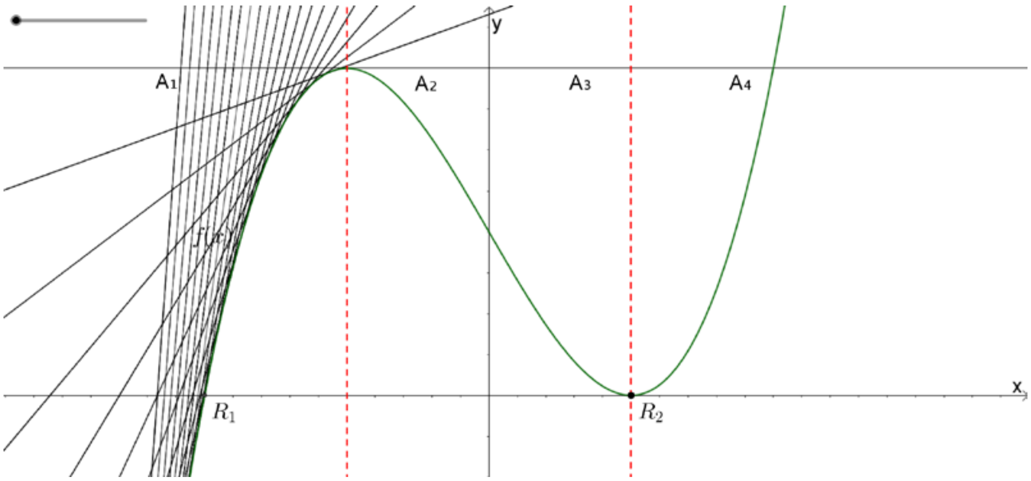


Figure 4.1

From observation of figure 4.1, first estimate in A_1 would give second estimate only in A_1 again.

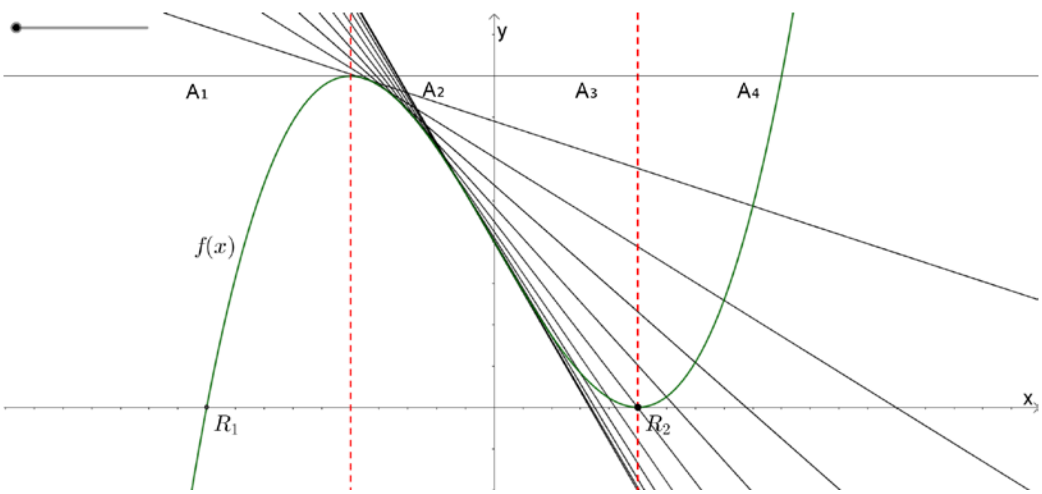


Figure 4.2

From observation of figure 4.2, first estimate in A_2 would give second estimate in any of A_3 or A_4

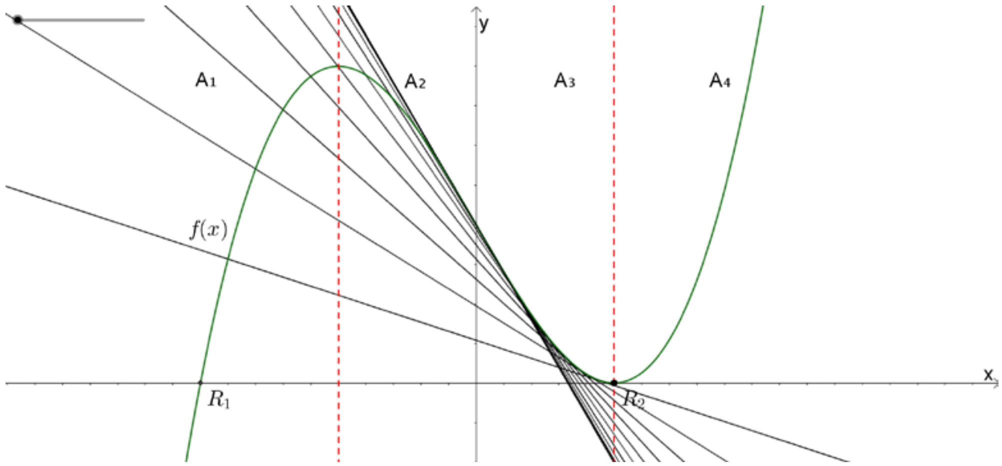


Figure 4.3

From observation of figure 4.3, first estimate in A_3 would give second estimate only in A_3 again.

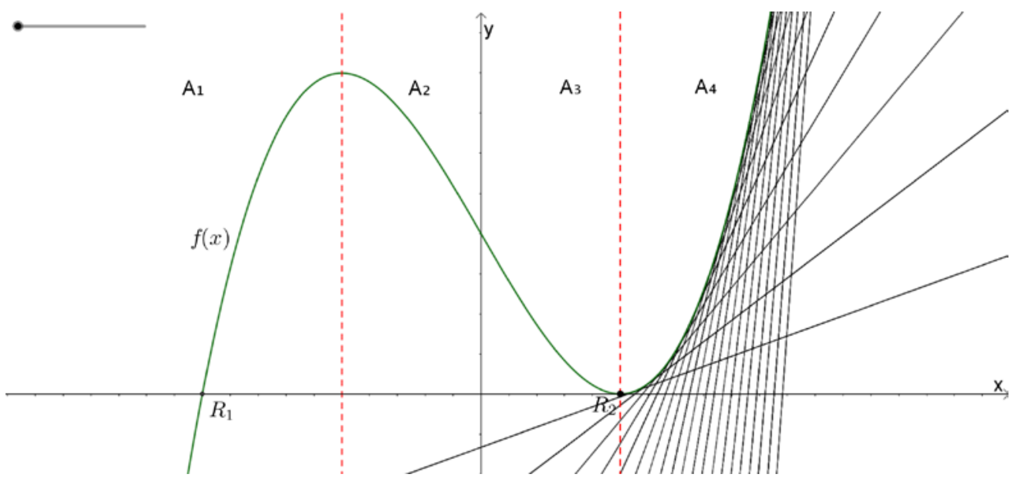


Figure 4.4

From observation of figure 4.4, first estimate in A_4 would give second estimate only in A_4 again.

Graphing tangents in different parts showed that: estimates in A_1 will approach the real root R_1, estimates in A_3 will approach the real root R_2 and estimates in A_4 will approach the real root R_2.

The observations could be summarized in the following table.

First estimate	Second estimate
In A_1	In A_1
In A_2	In A_3
	In A_4
In A_3	In A_3
In A_4	In A_4

The tangent in one part does not fall into another part and return. Hence, it is not possible for a similar triangle to exist. It should be noted that that any stretch or reflection of the function along the *x*-axis or the *y*-axis would not affect the results in any way. In fact, it will not affect the non-existence of similar triangles that may have been formed for infinite loops to exist. To conclude, there is no possibility for an infinite loop to exist for third degree polynomials in the form: $f(x) = a(x-k)^2(x-m)$.

For functions in the form: $f(x) = a(x-k)(x^2 + px + q)$. Third degree polynomials having one real root and an irreducible real quadratic; the curve cuts the *x*-axis at only one point. The single real root is represented by $x = k$ in the original form, while the irreducible quadratic is: $x^2 + px + q$. Applying equation 1 on two examples: $f_5(x) = x^3 - 4x + 4$ and $f_6(x) = x^3 - 3x + 3$, and using technology to solve for *a*, the following table is obtained.

$f_5(x) = x^3 - 4x + 4$
$\{a = -2.382975767906, a = -0.4500670218406, a = 0.7727510241192, a = 1.232882700505, a = 1.393261655339\}$
$f_6(x) = x^3 - 3x + 3$
$\{a = -2.103803402736, a = -0.297254939404, a = 0.5235401267294, a = 1.116132413527, a = 1.245800556385\}$

It can be seen that there are five real solutions for *a,* which are: the real root of the function, two estimates leading to an infinite loop and two other estimates leading to another infinite loop. Hence, it suggests that there exists only two infinite loops for this case. Geometrically, in order for an infinite loop to exist, one pair of similar triangles must exist. Using technology to draw the triangles of the following function $f_5(x) = x^3 - 4x + 4$: two pairs of similar triangles exist, and that is due to the two existing infinite loops. First pair between tangent at point having $x \approx 0.773$ and tangent at point having $x \approx 1.39$ in figure 5.

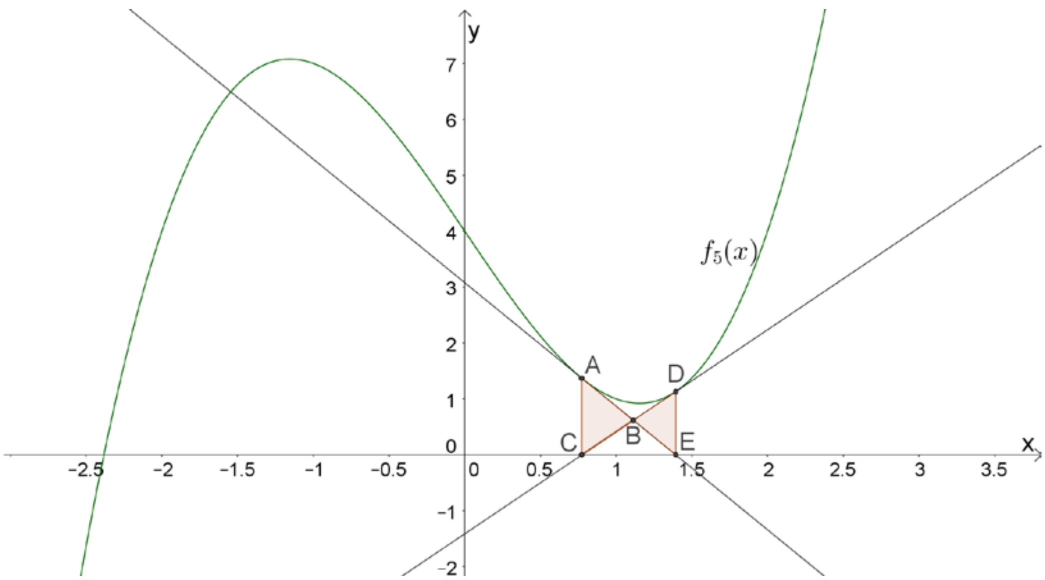


Figure 5

Zooming in to show the angles of each triangle on the following figure 5.1.

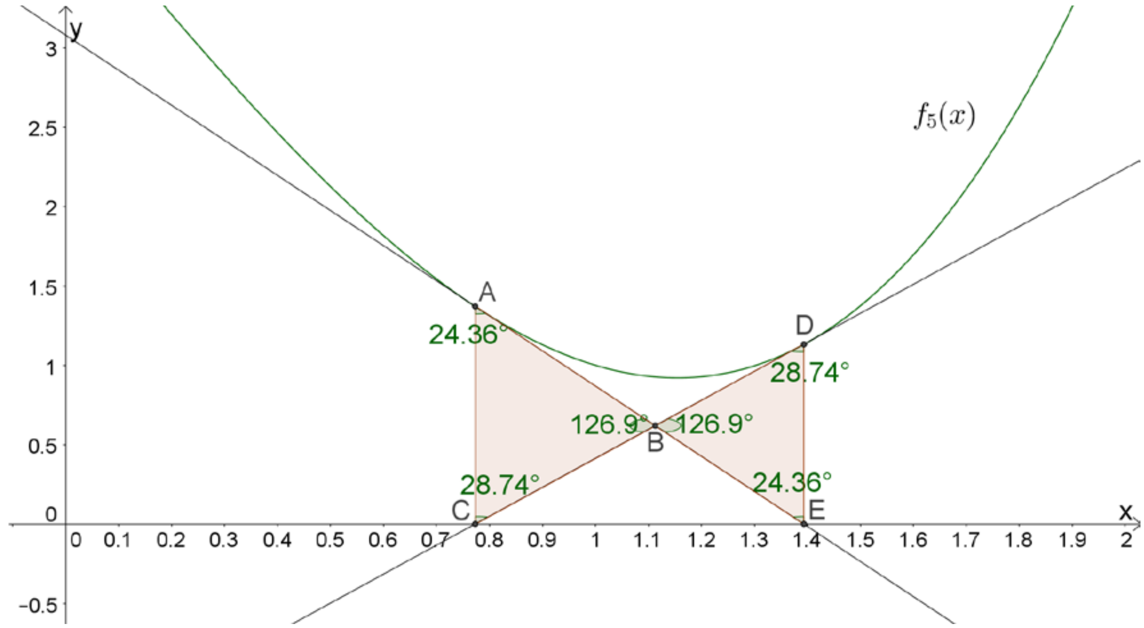


Figure 5.1

Since AC and DE are vertical, there will exist equal alternate interior angles: $\angle ACB = \angle EDB$ and $\angle CAB = \angle DEB$. As seen on figure 5.1, the triangles are indeed similar.

Second pair between tangent at point having $x \approx 0.450$ and tangent at point having $x \approx 1.23$ on figure 6 below.

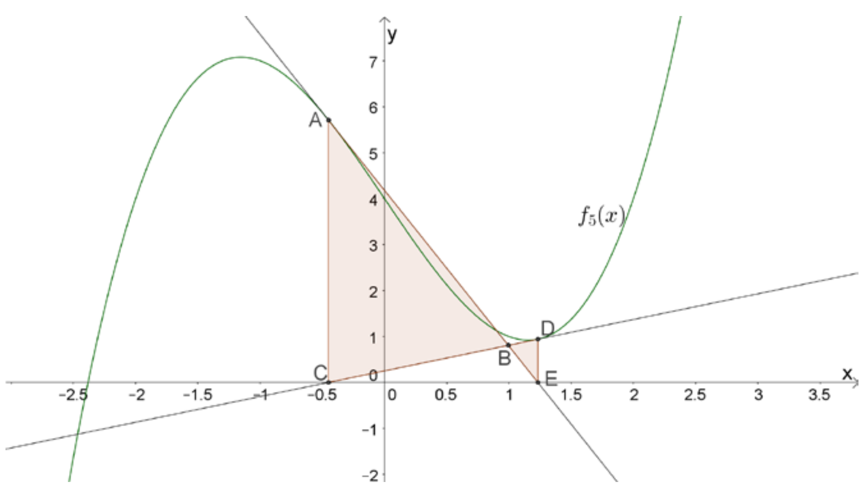


Figure 6

Zooming in to show the angles of each triangle on figure 6.1 below.

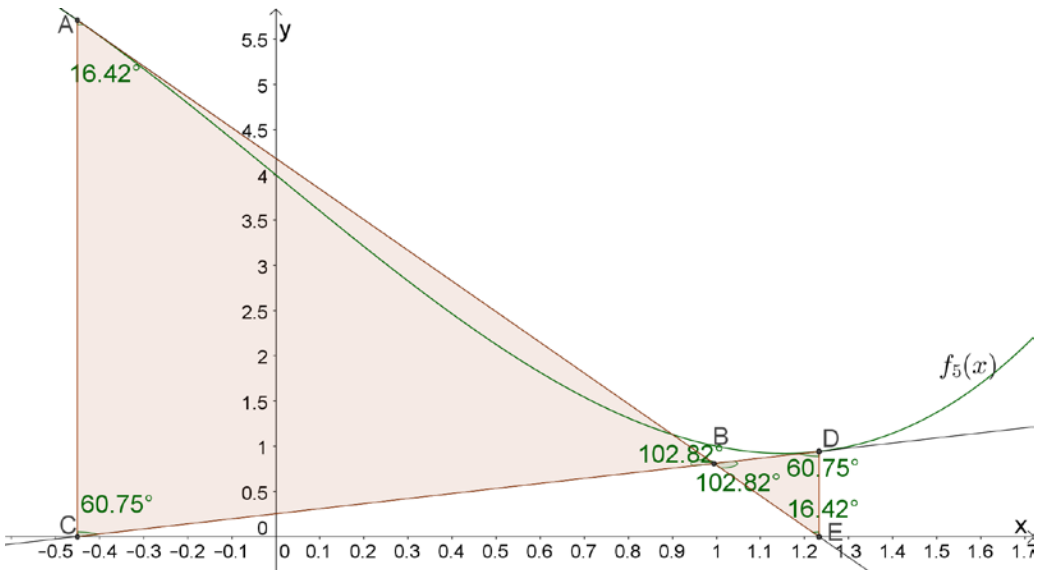


Figure 6.1

Since AC and DE are vertical, there will exist equal alternate interior angles: $\angle \text{ACB} = \angle \text{EDB}$ and $\angle \text{CAB} = \angle \text{DEB}$. As seen on figure 6.1, the triangles are indeed similar. The two pair of triangles have equal angles, which proves their similarity, and hence, two infinite loops would exist as it was explained earlier about the geometric condition. Studying this case in more depth, consider the following graph of *f(x)* in figure 7 with R_1 representing the single real root and $\text{A}_1, \text{A}_2, \text{A}_3$ and A_4 the four different parts of *f(x)*.

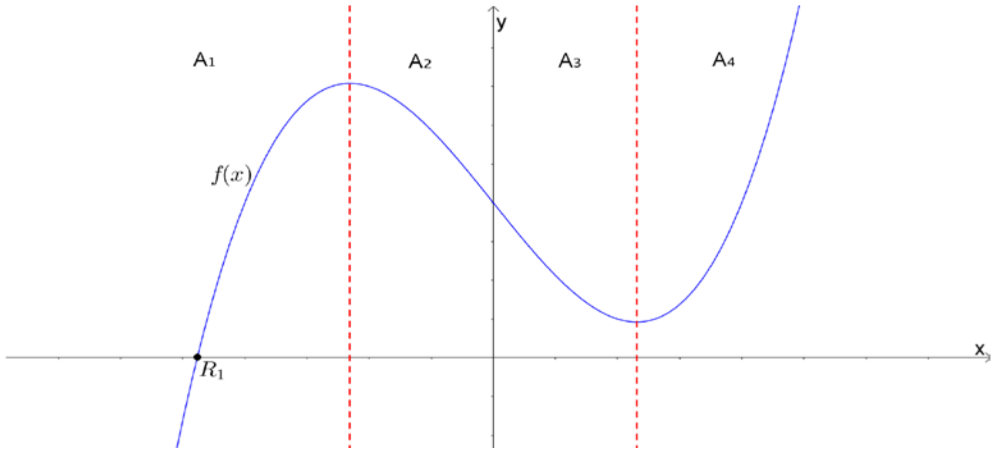


Figure 7

Possible tangents for each part with increment $x = 0.1$ will be illustrated in figures 7.1, 7.2, 7.3 and 7.4

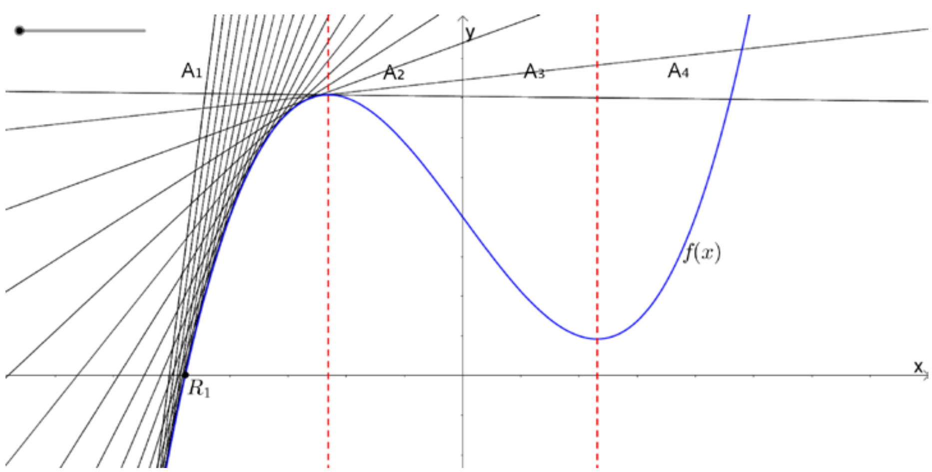


Figure 7.1

From observation of figure 7.1, first estimate in A_1 would give second estimate only in A_1 again.

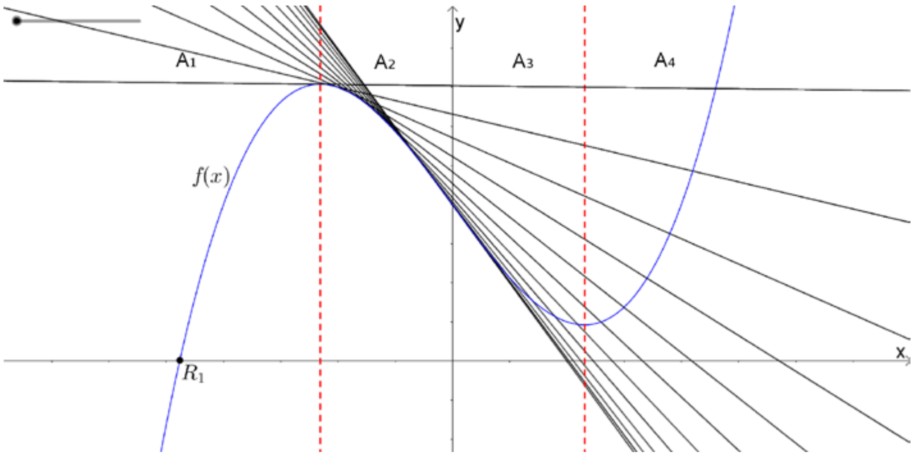


Figure 7.2

From observation of figure 7.2, first estimate in A_2 would give second estimate in any of A_3 or A_4

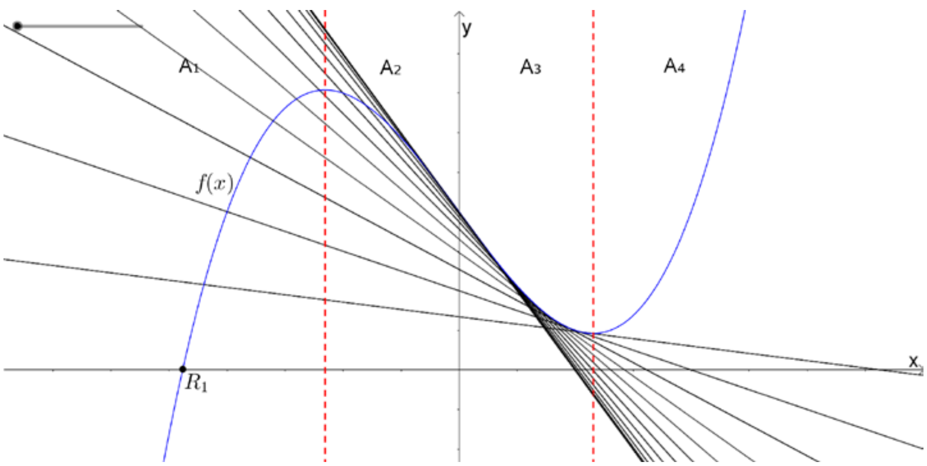


Figure 7.3

From observation of figure 7.3, first estimate in A_3 would give second estimate in any of A_3 or A_4

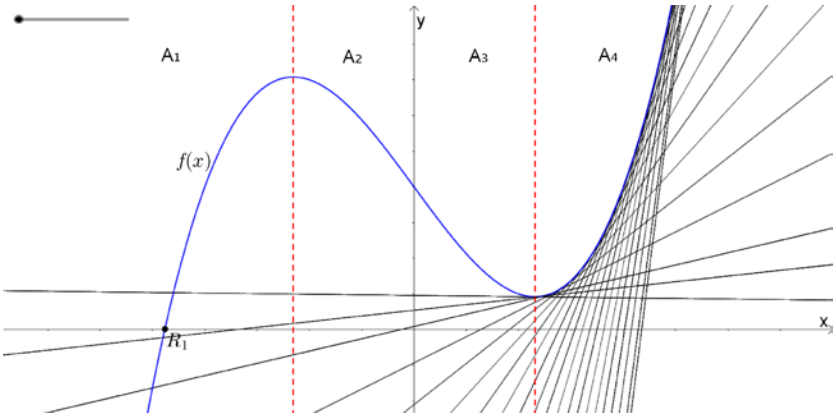


Figure 7.4

From observation of figure 7.4, first estimate in A_4 would give second estimate in any of the four parts.

Graphing tangents in different parts showed that estimates in A_1 will approach the real root R_1.

The observations could be summarized in the following table.

First estimate	Second estimate
In A_1	In A_1
In A_2	In A_3
	In A_4
In A_3	In A_3
	In A_4
In A_4	In A_1
	In A_2
	In A_3
	In A_4

Observing the table above, two possible ways are found that could lead to an infinite loop:

1) Estimate in A_2 leading to A_4, and then from A_4 back to A_2.

2) Estimate in A_3 leading to A_4, and then from A_4 back to A_3.

Since there are two possible ways to form similar triangles; this would be a reason why two infinite loops exist for this case. To conclude, there are two and only two possible infinite loops that exist for third degree polynomials in the form: $f(x) = a(x-k)(x^2 + px + q)$.

For functions in the form: $f(x) = a(x-k)(x-m)(x-n)$. Third degree polynomials having three distinct real roots at: $x = k$, $x = m$ and $x = n$. Applying equation 1 on two examples: $f_7(x) = x^3 - 4x^2 + 2x + 3$ and $f_8(x) = x^3 - 3x + 1$, and using technology to solve for *a*, the following table is obtained.

$f_7(x) = x^3 - 4x^2 + 2x + 3$
$\left\{ a = \dfrac{-\sqrt{5}+1}{2}, a = 0.6204009602042, a = \dfrac{\sqrt{5}+1}{2}, a = 2.246045304457, a = 3 \right\}$
$f_8(x) = x^3 - 3x + 1$
$\{a = -1.879385241572, a = -0.6485874050814, a = 0.3472963553339, a = 0.8893406566662, a = 1.532088886238\}$

It can be seen that there are five real solutions for *a,* which are: the three real roots of the function and two estimates leading to an infinite loop. Hence, it suggests that for third degree polynomials in this case, there exists only one possible infinite loop. Geometrically, in order for an infinite loop to exist, one pair of similar triangles must exist. Using technology to draw the triangles of the following function $f_7(x) = x^3 - 4x^2 + 2x + 3$, figure 8 below is obtained.

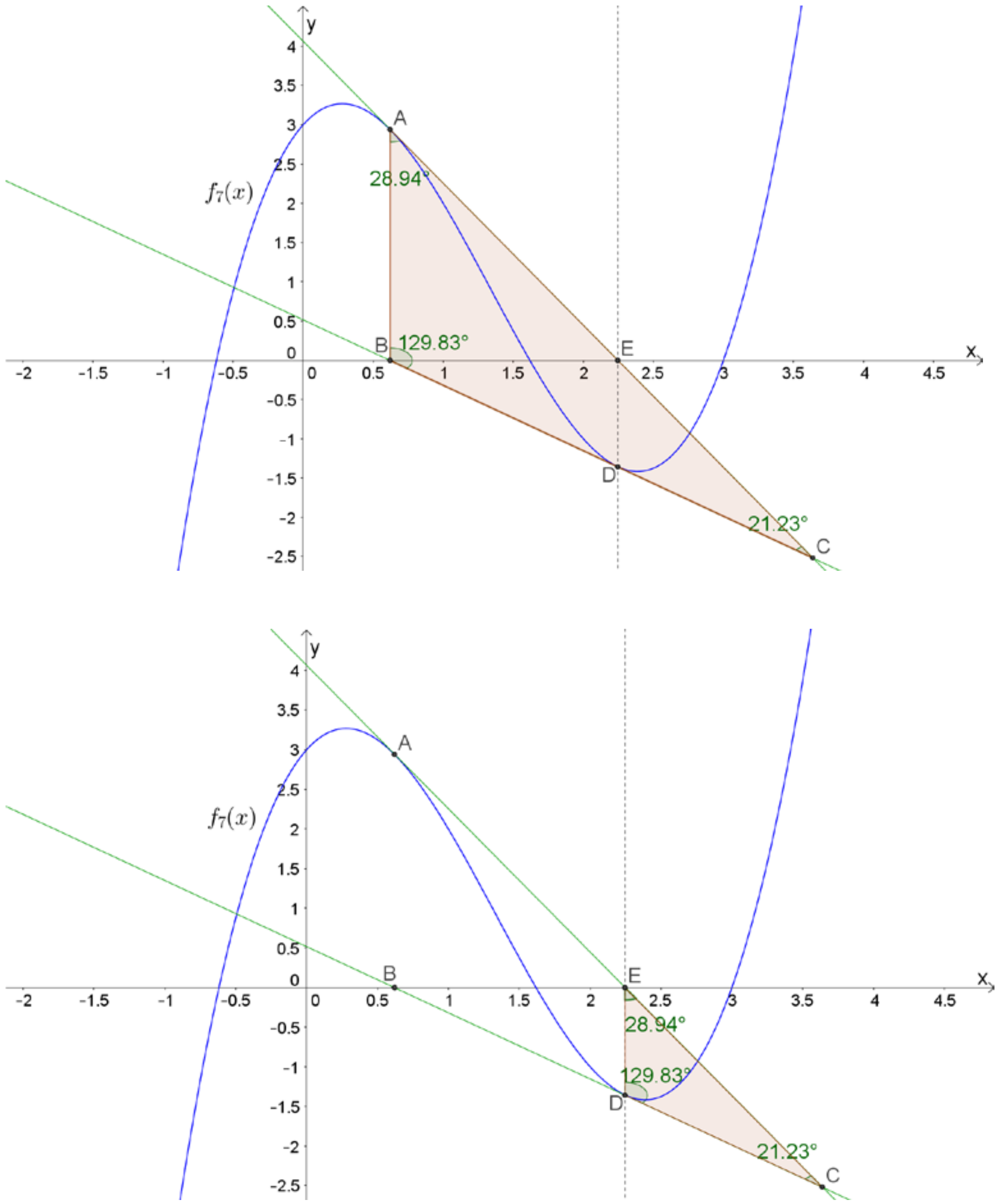


Figure 8

Since AB and ED are vertical, $\angle ABC$ and $\angle EDC$ are corresponding angles, hence $\angle ABC = \angle EDC$. On the other hand, $\angle ACB$ and $\angle ECD$ are the same angle, and therefore equal as well. Consequently, as shown in figure 8, the triangles are indeed similar.

Studying this case in more depth, consider the following graph of *f(x)* with R_1, R_2 and R_3 representing the three real roots, and A_1, A_2, A_3 and A_4 the four different parts of *f(x)* below.

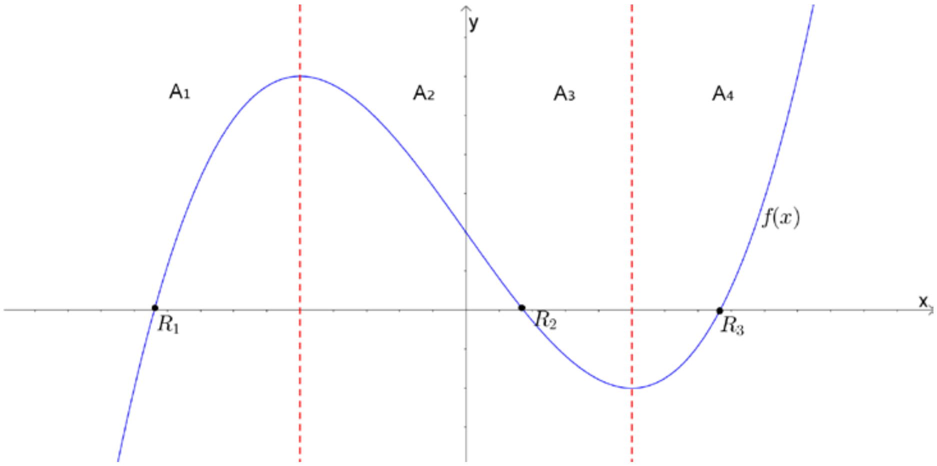


Figure 9

Possible tangents for each part with increment $x = 0.1$ will be illustrated in figures 9.1, 9.2, 9.3 and 9.4

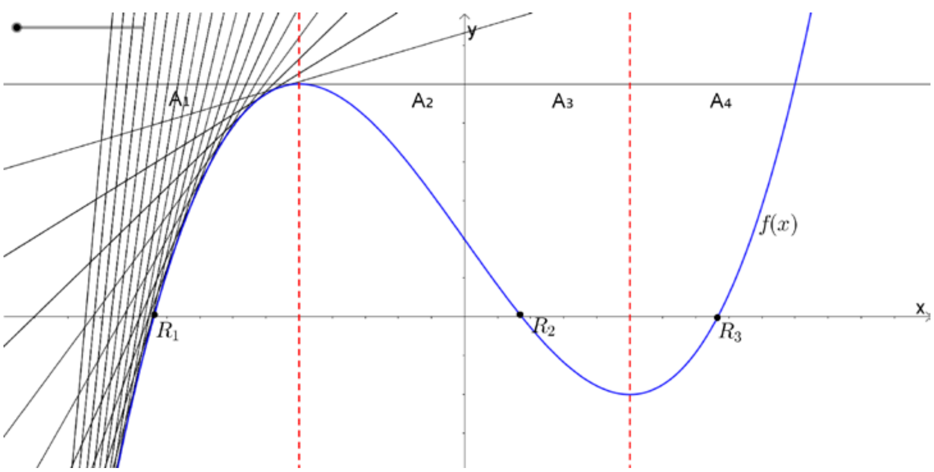


Figure 9.1

From observation of figure 9.1, first estimate in A_1 would give second estimate only in A_1 again.

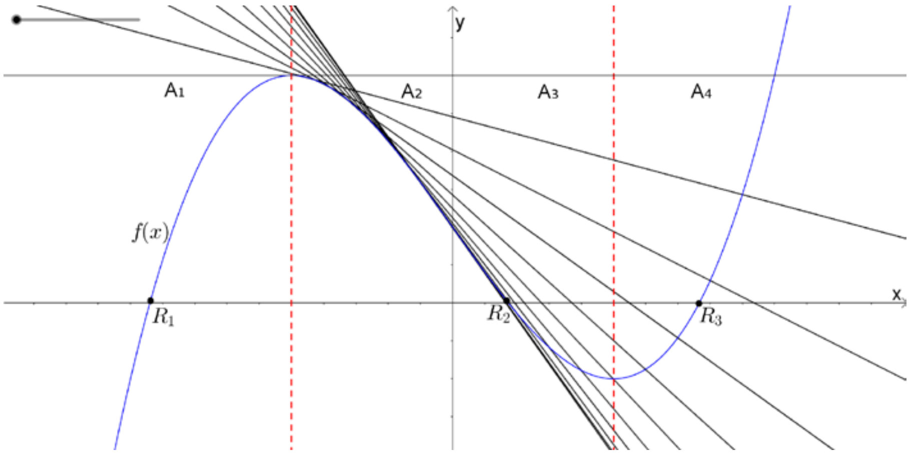


Figure 9.2

From observation of figure 9.2, first estimate in A_2 would give second estimate in any of A_3 or A_4

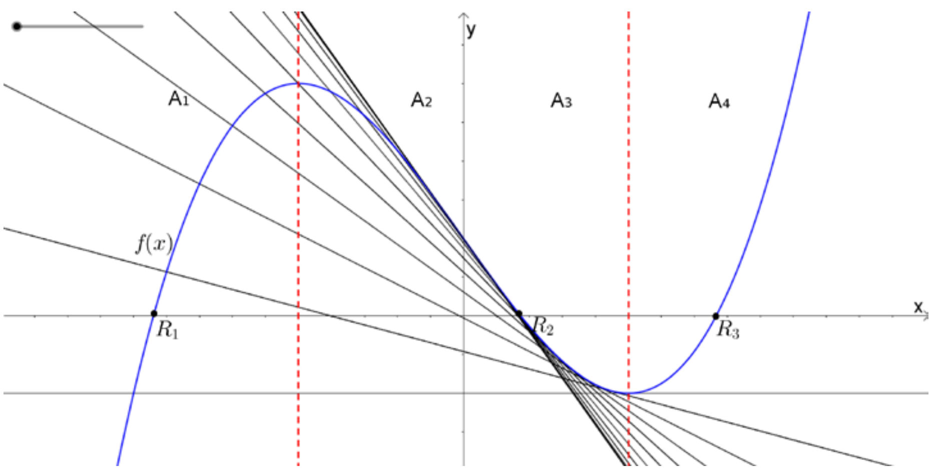


Figure 9.3

From observation of figure 9.3, first estimate in A_3 would give second estimate in any of A_1, A_2 or A_3.

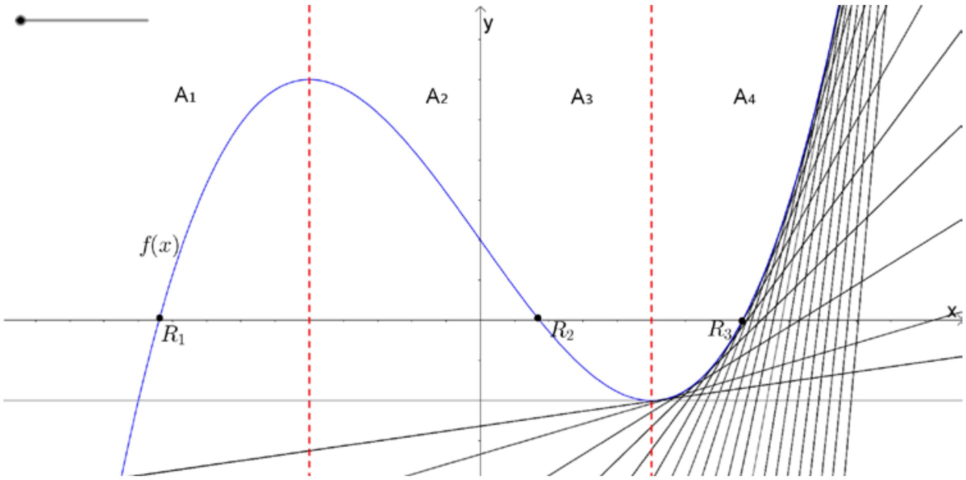


Figure 9.4

From observation of figure 9.4, first estimate in A_4 would give second estimate only in A_4 again.

Graphing tangents in different parts showed that: estimates in A_1 will approach the real root R_1, estimates in A_2 and A_3 will approach the real root R_2 and estimates in A_4 will approach the real root R_3.

The observations could be summarized in the following table.

First estimate	**Second estimate**
In A_1	In A_1
In A_2	In A_3
	In A_4
In A_3	In A_1
	In A_2
In A_4	In A_4

Observing the table above, the only possible way that is found that could lead to an infinite loop is: Estimate in A_2 leading to A_3, and then from A_3 back to A_2. Which means, only one pair of similar triangles that exist. It should be noted that the point of inflexion of the function being above or below the *x*-axis will not affect the results in any way. Also reflection of the function along either the *x*-axis or the *y*-axis would not affect the similar triangles that may have been formulated if infinite loops were to exist. Thus, it can be concluded that, there is one and only one possible infinite loop that exist for third degree polynomials in the form: $f(x)=a(x-k)(x-m)(x-n)$.

Conclusion

Solving equation 1: $\frac{f(a)}{f'(a)}+\frac{f(a-\frac{f(a)}{f'(a)})}{f'(a-\frac{f(a)}{f'(a)})}=0$ gives the real roots and the estimates that form infinite loop(s). In fact, it gives all values of *x* for which where $x_1=x_3$; either when $x_1=x_2=x_3$ (which is the real root) or $x_3=x_2-\frac{f(x_2)}{f'(x_2)}=x_1$ (which is an infinite loop).

Solving equation 1 results in solving a ninth degree polynomial, since the function itself was cubic, and substituted in a cubic. When higher polynomials of *n*th degree are studied, it should be noted that equation 1 would be of (n^2)th degree. However, the number of real solutions for this equation will need to be investigated.

After solving equation 1 for different cases of third degree functions, it is found that the maximum number of possible real solutions of this equation for any third degree polynomial is five. The number of infinite loops, which is justified by the findings, ranges from zero to two. The following table summarizes the number of infinite loop(s) for each third degree polynomial case.

Third degree case	**Roots**	**Number of infinite loops**
$f(x) = a(x-k)^3 + d$	Three identical	0
$f(x) = a(x-k)^2(x-m)$	One real and two identical	0
$f(x) = a(x-k)(x-m)(x-n)$	Three distinct	1
$f(x) = a(x-k)(x^2 + px + q)$	One real and an irreducible real quadratic	2

In all cases, for one infinite loop to exist, there has to be a possibility of forming one pair of similar triangles, the first formed between: the point of intersection of the first tangent and parabola, *x*-intercept of the second tangent and point of intersection of the two tangents, while the second triangle formed between: the point of intersection of the second tangent and parabola, *x*-intercept of the first tangent and point of intersection of the two tangents. Similarly, further investigation can be made for higher polynomials or other functions, where each must be divided into different parts; parts in which the boundaries are either: the real roots, points of inflexion or stationary points. Then a table of the position of possible *x*-intercepts of tangents must be created to generalize the possible similar triangles that can exist. The number of infinite loops that exist for any function depends on the number of similar pairs of triangles that can exist.

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