University of California, Santa Barbara

Department of Mechanical Engineering

Level set methods and their applications

HW#2

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1. Problem Description

In this homework we aim to implement the **Hamilton-Jacobi ENO** scheme to solve the famous level set equation, which is in nature the advection equation as the mass or concentration or density conservation laws. First, we are going to solve it in one dimension, then we do the dimension by dimension simulation in 2D. Below is the equation we want to solve:

$$\phi_t + \mathbf{u}_{ext} \cdot \nabla\phi = 0$$

In level set methods manner, the advection equation will be a function which evolves in spatial domain with the external velocity. We will **capture** the interface (the zero-level set of the function), instead of **tracking** it in time.

2. Hamilton-Jacobi ENO scheme

The idea is that since the **characteristic lines** of a PDE define the direction that the information carries on, the velocity will decide which of the ${\varphi_x}^-$ or ${\varphi_x}^+$ we need to calculate for approximating the derivative of φ with respect to x. This is basically the idea of **upwind differencing**.

We will extrapolate an interpolant through the nodes on the spatial domain and find the derivative of that interpolant. This is going to be done as bellow:

$$\phi(x) = Q_0(x) + Q_1(x) + Q_2(x) + Q_3(x)$$

that can be differentiated and evaluated at x_i to find $\varphi(x_i)$ and $\varphi(x_i)'$ That is, we use:

$$\phi_x(x_i) = Q_1'(x_i) + Q_2'(x_i) + Q_3'(x_i)$$

we use the smoothest possible polynomial interpolation to find ¢ and then differentiate to get ¢x. Because, between all the possible interpolants the one which less smooth, may have singularities, steepness, discontinuities. So, when we face two choice for calculating the derivatives of φ, we go with the smaller one.

In our case, since the velocity in x and y direction is positive, the zeroth, first, second, and third divided differences will be as follows:

$$D_i^0\phi = \phi_i$$

$$D_{i+1/2}^1\phi = \frac{D_{i+1}^0\phi - D_i^0\phi}{\triangle x}$$

$$D_i^2\phi = \frac{D_{i+1/2}^1\phi - D_{i-1/2}^1\phi}{2\triangle x}$$

$$D_{i+1/2}^3\phi = \frac{D_{i+1}^2\phi - D_i^2\phi}{3\triangle x}$$

These values will be used in linear extrapolation as the coefficients of the polynomial. We will go with the zeroth and first divided differences, which make the method to be one order accurate.

3. Results for the first part

The plots for the first part for three different times for a wave going to the left and the wave going to the right are as bellow.

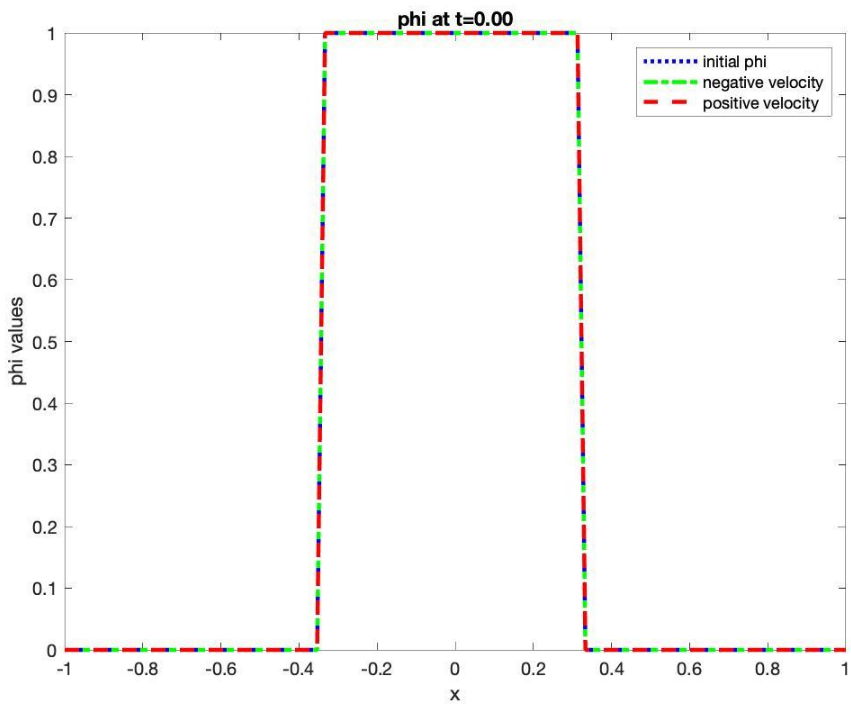

Figure 1: solution of advection equation at time = 0. The three plots are the same.

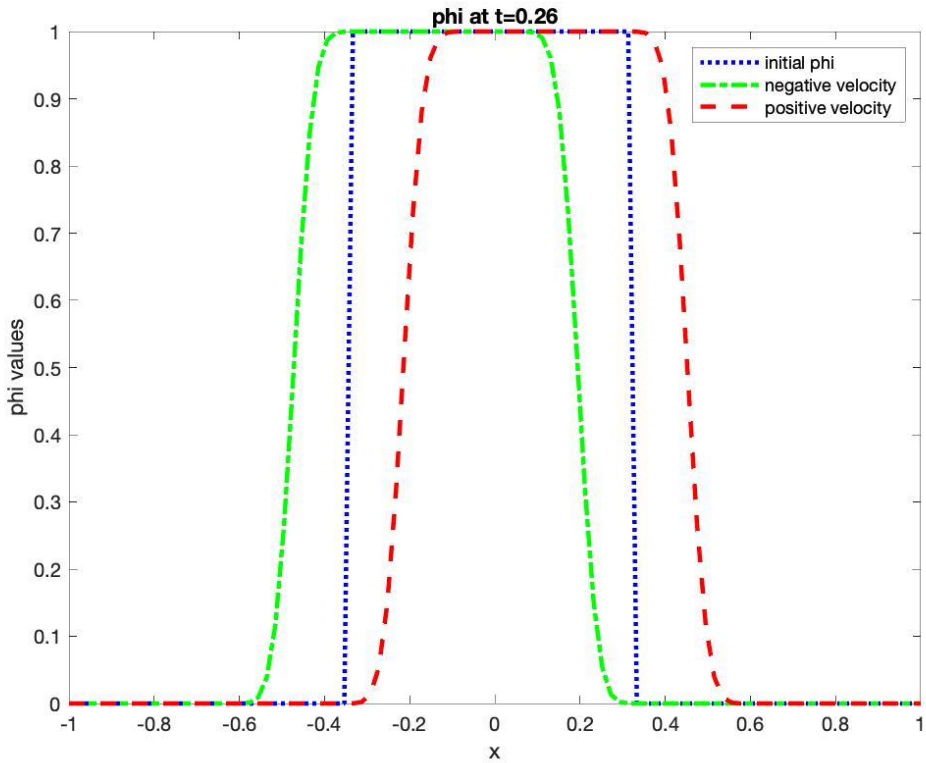

Figure 2: solution of advection equation at time = 0.26. The positive velocity makes the level set function evolve from left to right, which means slope of characteristic lines is positive and the information goes from left to right. In this case, we have a 1D (in space) problem but the level set function is a 2D function, which shows the wave going to the right. The opposite is true for the negative velocity.

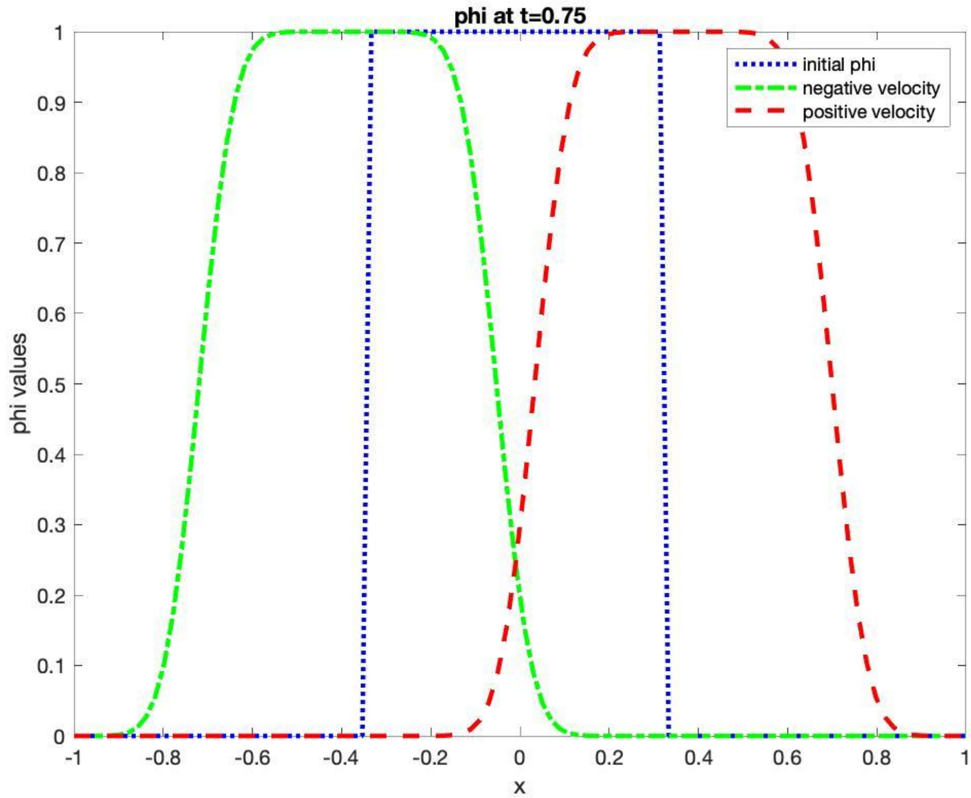

Figure 3: solution of advection equation at time = 0.75. As it can be seen, the sharpness of the initial function has been smoothed out using the first order HJ-ENO scheme or upwind differencing.

4. Results for the second part

We will expect to see a 2D wave moving in the velocity direction. As we saw in the previous part, the shape of the wave was maintained but it was moving in the x-direction as time evolve.

The same thing is true in 2D, we will have a 3D level set function. If we look at a contour of this function meaning a constant value of the function (for the level set method, it's always the zero-level set contour), the **shape is going to stay the same**, but it's **moving** in x and y direction.

The **linear extrapolation** idea, will help us determining the boundary condition. Since the velocity in this problem is positive in both directions, we will implement the code for the second to the end point of the spatial grid. The first node has to be calculated using an interpolant going through the second and third nodes. The **boundary condition** will be:

$$\varphi(1, j) = \varphi(2, j) + \frac{\varphi(3, j) - \varphi(2, j)}{\Delta x}(x(1) - x(2))$$

$$\varphi(i, 1) = \varphi(i, 1) + \frac{\varphi(i, 3) - \varphi(i, 2)}{\Delta y}(y(1) - y(2))$$

It is necessary to mention that the figures bellow is for **the $-\varphi$ function** for better visualization. The results are matched with what we have expected:

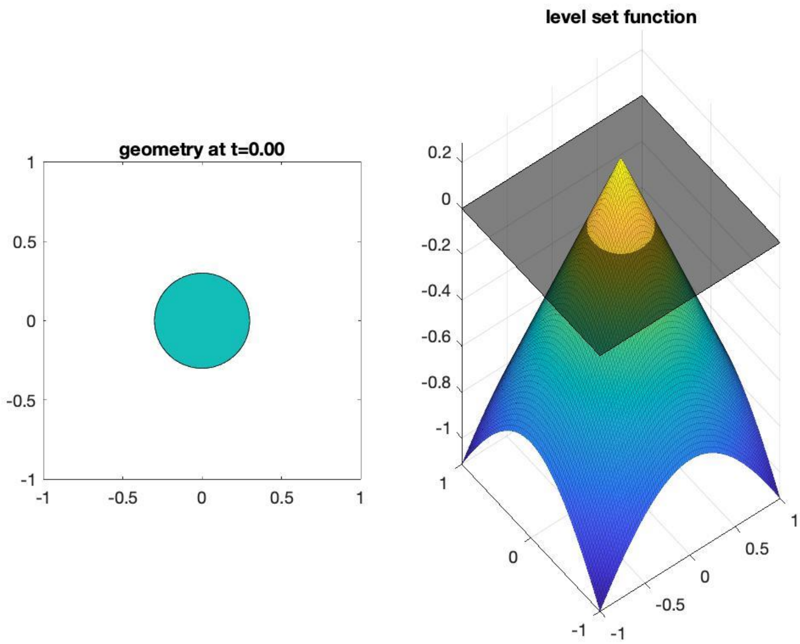

Figure 4: Right plot shows the level set function and a z=0 surface. The left plot shows the intersection of these two surfaces (zero level set) initially.

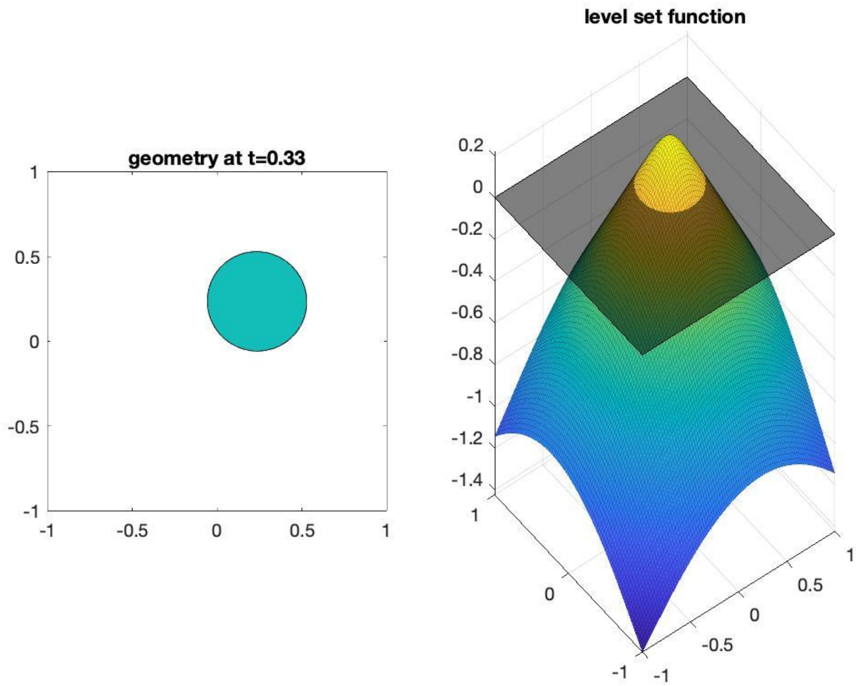

Figure 5: zero-level-set contour and the level set function at t = 0.33.

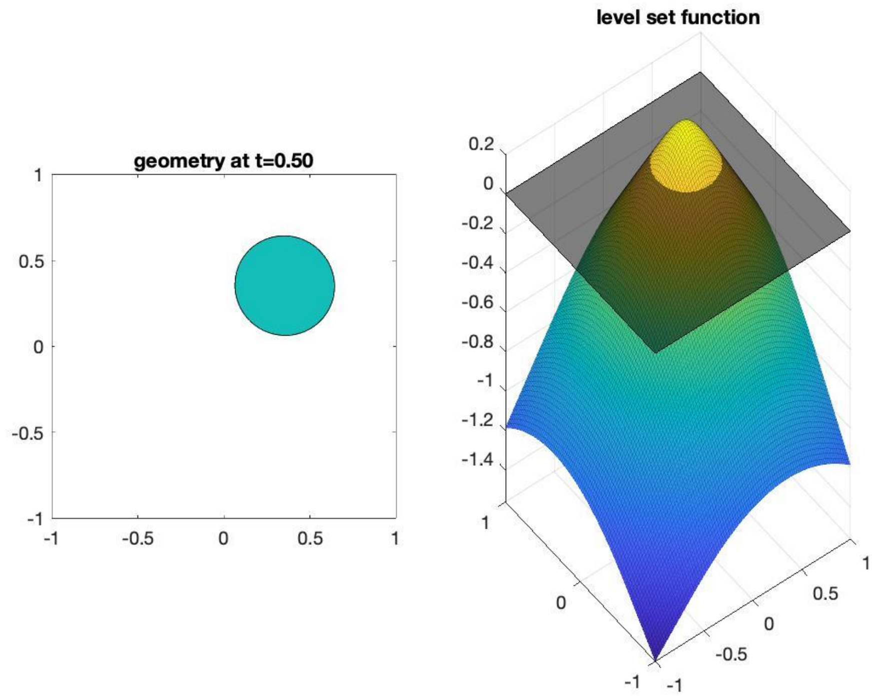

Figure 6: zero-level-set contour and the level set function at t = 0.5, the change of the location of the 2D wave is represented as in the left plot, conceptually, it is equal to capture the zero-level-set of a 3D function which evolves with an external velocity.

5. Conclusion

In this homework, we implemented the first-order Hamilton-Jacobi ENO scheme for solving the advection equation. Since the velocities in x and y direction were positive, we calculated the ${\varphi_x}^-$, so starting by the $i-1$ node on the grid, won't let us do the computation for the $i=1$ node. That's where we need to impose the boundary condition, which was done by linear extrapolation.

The results of both parts were consistent with what we expect from theory and intuition. We don't see any change in shape of the contour but it changes its location.

There is another point to make. Since this function has an absolute value of 1 in its gradient, and the velocity is 1 in magnitude, we don't need to be worry for the appearance of the level set function (it's not that messed up!).