MSAI Probability

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Home Assignment 1-3

1. (2 points) **Find the probability that in a class of n students, at least two share the same birthday. Use counting rules. Find a numerical answer for n = 50.**

   Solution:

   1. **Probability that no two students share same birthday:**

      $$\text{P(no shared birthday)} = \frac{FavourableOutcome}{TotalOutcome}$$
      $$\text{Total Outcome} = 365!$$

      As each of the 50 people will have birthday on a unique day so
      Favourable Outcome $= 365 \times 364 \times 363 \ldots (365 - n + 1)$
      where n = total number of students = 50

      $$\text{So, P} = \frac{365 \times 364 \ldots 316}{365!} = 0.02$$

   2. **Probability that atleast two students share same birthday:**

      P(alteast two share same birthday) = 1 - P(no shared birthday)
      P(alteast two share same birthday) = 1 - 0.029
      P(alteast two share same birthday) = 0.97

   So , for a class of 50 the probability of atleast two students share the same birthday is 97%.

2. (2 points) **Prove that**

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{m+n}{k}$$

Solution: Using the binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$(1+x)^n (1+x)^m = \sum_{j=0}^{n} \binom{n}{j} x^j \sum_{l=0}^{m} \binom{m}{l} x^l$$

$$= \sum_{j=0}^{n} \sum_{l=0}^{m} \binom{n}{j} \binom{m}{l} x^{j+l}$$

As

$$l = k - j$$

$$\sum_{j=0}^{n} \sum_{k=j}^{j+m} \binom{n}{j} \binom{m}{k-j} x^k$$

$$\sum_{k=0}^{n+m} \sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} x^k \tag{1}$$

Inserting equation as in the question

$$(1+x)^{m+n} = \sum_{k=0}^{n+m} \binom{m+n}{k} x^k \tag{2}$$

Comparing eq(1) and eq(2) gives $\boxed{\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}}$

3. (2 points) **A fair die is rolled n times. What is the probability that at least 1 of the 6 values never appears? Give a formula answer with derivation. Hint: use the inclusion-exclusion formula.**

   Solution:

   Using inclusion-exclusion formula:

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6) = \sum_{i=1}^{6} P(A_i) - \sum_{i=1,j=1}^{6} P(A_i \cap A_j) + \ldots + (-1)^{6+1} P(A_1 \cap A_2 \cap \ldots \cap A_6)$$

   The probability that i doesn't appear

$$P(A_i) = \left(\frac{5}{6}\right)^n$$

   Probability that i and j doesn't appear

$$P(A_i \cap A_j) = \left(\frac{4}{6}\right)^n = \left(\frac{2}{3}\right)^n$$

   Similarly,

$$P(A_i \cap A_j \cap A_k) = \left(\frac{3}{6}\right)^n = \left(\frac{1}{2}\right)^n$$

$$P(A_i \cap A_j \cap A_k \cap A_l) = \left(\frac{2}{6}\right)^n = \left(\frac{1}{3}\right)^n$$

$$P(A_i \cap A_j \cap A_k \cap A_l \cap A_m) = \left(\frac{1}{6}\right)^n$$

$$P(A_1 \cap \ldots \cap A_6) = 0$$

   Then we have,

$$P = \binom{6}{1}\left(\frac{5}{6}\right)^n - \binom{6}{2}\left(\frac{2}{3}\right)^n + \binom{6}{3}\left(\frac{1}{2}\right)^n - \binom{6}{4}\left(\frac{1}{3}\right)^n + \binom{6}{5}\left(\frac{1}{6}\right)^n$$

$$P = 6\left(\frac{5}{6}\right)^n - 15\left(\frac{2}{3}\right)^n + 20\left(\frac{1}{2}\right)^n - 15\left(\frac{1}{3}\right)^n + 6\left(\frac{1}{6}\right)^n$$

4. (3 points) **There are two baskets. The first basket contains one white ball, the second basket contains one black ball. One basket is chosen randomly and a white ball is put into the chosen basket. The balls in this basket are shuffled. Then one ball is extracted from this basket. This ball turns out to be white. What is the posterior probability that the second ball drawn from this basket is also white?**

   Solution:

   Probability of Basket 1 = $\frac{1}{2}$

   Probability of Basket 2 = $\frac{1}{2}$

   Lets see the probability of extraction of white ball from each basket

   P(White | Basket 1) = $\frac{2}{2}$

   P(White | Basket 2) = $\frac{1}{2}$

   According to Law of Total Probability:

   P(White) = P(White | Basket 1) P(Basket 1) + P(White | Basket 2) P(Basket 2)

   P(White) = $\frac{1}{2} \times \frac{2}{2} + \frac{1}{2} \times \frac{1}{2}$

   P(White) = $\frac{3}{4}$

   Applying Bayes Theorem gives:

$$P(Basket1|White) = \frac{P(White|Basket1) \times P(Basket1)}{P(White)}$$

$$P(Basket1|White) = \frac{\frac{2}{2} \times \frac{1}{2}}{\frac{3}{4}}$$

$$P(Basket1|White) = \frac{2}{3}$$

5. (2 points) **If you get a positive result on a COVID test that only gives a false positive with probability 0.001 (true positive with probability 0.999), what's the chance that you've actually got COVID, if**

   1. **The prior probability that a person has COVID is 0.01**
   2. **The prior probability that a person has COVID is 0.0001**

   Solution: We can solve this using Bayes theorem:

$$P(Covid|Positive) = \frac{P(Positive|Covid)P(Covid)}{P(Positive)}$$

   P(Poisitive | Covid) = 0.999

   P(Positive | No Covid) = 0.001

   P(No Covid) = 1 - P(Covid)

   a) **Prior Probability P(Covid) = 0.01**

   P(Positive) = P(Poisitive | Covid) P(Covid) + P(Positive | No Covid) P(No Covid)

   P(Positive) = $(0.999 \cdot 0.01) + (0.001 \cdot (1 - 0.01))$

   P(Positive) = 0.01098

   P(Covid | Positive) = $\frac{P(Positive|Covid)P(Covid)}{P(Positive)}$

   P(Covid | Positive) = $\frac{0.999 \times 0.01}{0.01098}$

P(Covid | Positive) = 0.901 = 90%

b) **Prior Probability P(Covid) = 0.0001**

P(Positive) = P(Poisitive | Covid) P(Covid) + P(Positive | No Covid) P(No Covid)

P(Positive) = $(0.999 \cdot 0.0001) + (0.001 \cdot (1 - 0.0001)$

P(Positive) = 0.0010998

P(Covid | Positive) = $\frac{P(Positive|Covid)P(Covid)}{P(Positive)}$

P(Covid | Positive) = $\frac{0.999 \cdot 0.0001}{0.0010998}$

P(Covid | Positive) = 0.0908 = 9.08%

6. (6 points) Use the result of Problem 1 and Python to create a plot of this probability, and note after which n the probability becomes more than 50%.

Solution:

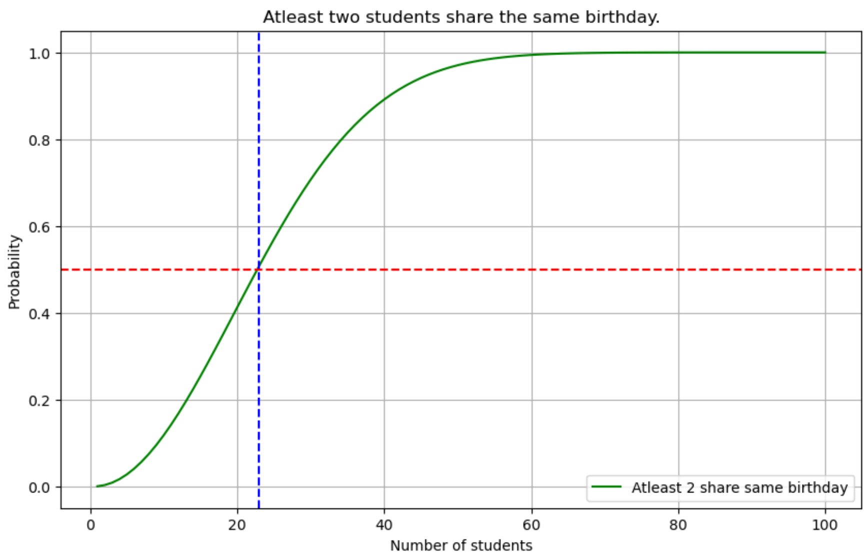

Figure 1: Probability of at least two students share the same birthday

From the graph plotted, the probability that at least two students share the same birthday exceeds 50% when there are **23 students in the class of more.**

Code:

Click here to see code

7. (2 bonus points) Use the result of Problem 3 and Python to create a plot of this probability, and note after which n the probability becomes less than 1%.

   Solution:

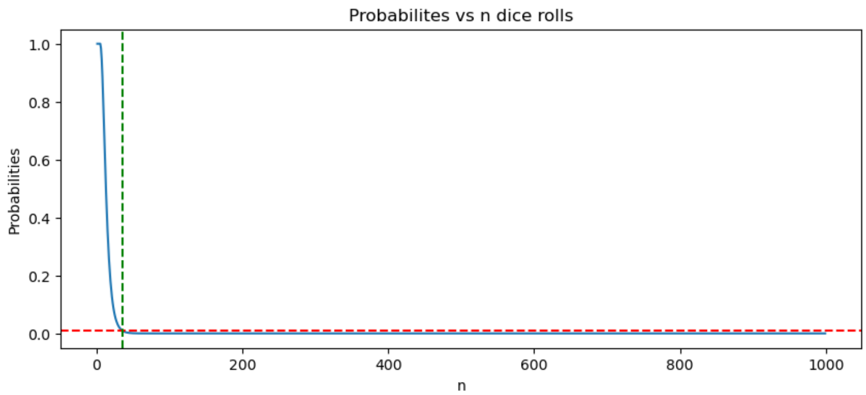

Figure 2: At nth roll probability becomes less than 1

The probability at which n becomes less that 1% **is 36**. I have demonstrated it inside the code.

Code:

Click here to see code.

8. (2 bonus points) **The cloakroom of a theater has randomly permuted all n visitors' hats. Find the probability that at least one visitor gets his hat. Give a formula answer with derivation. Given n = 4, give a number answer. Hint: use inclusion-exclusion formula**

   Solution:

   Using inclusion-exclusion formula:

$$P(A_1 \cup A_2 \ldots_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i \leq j \leq n}^{6} P(A_i \cap A_j) + \ldots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$\mathbf{P}(A_i)$:

A_i gets its own hat $(n-1)!$ ways to arrange (n-1) hats

$$\frac{(n-1)!}{n!} = \frac{1}{n}$$

$\mathbf{P}(A_i \cap A_j)$: The probability that both i and j visitors get there own hat is n for i_{th} visitor and $(n-1)$ for j_{th} visitor is

$$\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$\mathbf{P}(A_i \cap A_j \cap A_k)$:

The probability that both i,j and k visitors get there own hat is n for i_{th} visitor, $(n-1)$ for j_{th} visitor and $(n-2)$ for k_{th} visitor is

$$\frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

Similarly,

Probability that at least j visitors get there hats are

$$\frac{(n-j)!}{n!} = \frac{1}{n(n-1).....(n-j+1)}$$

As n = 4:

P(at least one gets his hat) = $\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = 0.625$

9. (2 bonus points) **You are the contestant on the Monty Hall show. Monty is trying out a new version of his game, with rules as follows. You get to choose one of three doors. One door has a car behind it, another has a computer, and the other door has a goat (with all permutations equally likely). Monty, who knows which prize is behind each door, will open a door (but not the one you chose) and then let you choose whether to switch from your current choice to the other unopened door. Suppose that Monty always opens the door that reveals your less preferred prize out of the two alternatives, e.g., if he is faced with the choice between revealing the goat or the computer, he will reveal the goat. Monty opens a door, revealing a goat. Given this information, should you switch? If you do switch, what is your probability of success in getting the car?**

   Solution: Given the conditions:

   1. If the player chooses car Monty revals goat
   2. If the player chooses goat Monty reveals computer
   3. If the player choose computer Monty reveals

   In beginning

   1. P(Car) = 1/3
   2. P(Computer) = 1/3
   3. P(Goat) = 1/3

   As Monty has already reveled the goat

   1. Player can choose a car
   2. Player can choose a computer

   Now in beginning

   1. Player has chooses car switching leads to a computer So
      P = 1/3
   2. Player has chosen computer and switching leads to a car
      P = 1/3

   Now as the goat has been revealed

   Probability of getting the car now is between **car** and **computer**

   P = 1/2

   So now the probability of success in getting the car is **P = 1/2 after switching**.