

KHOVANOV HOMOLOGY AND THE SLICE GENUS

JACOB RASMUSSEN

ABSTRACT. We use Lee’s work on the Khovanov homology to define a knot invariant s . We show that $s(K)$ is a concordance invariant and that it provides a lower bound for the slice genus of K . As a corollary, we give a purely combinatorial proof of the Milnor conjecture.

1. INTRODUCTION

In [6], Khovanov introduced an invariant of knots and links, now widely known as the Khovanov homology. This invariant takes the form of a graded homology theory $\text{Kh}(L)$, whose graded Euler characteristic is the unnormalized Jones polynomial of L . In [9], Lee showed that $\text{Kh}(L)$ is naturally viewed as the E_2 term of a spectral sequence which converges to $\mathbf{Q} \oplus \mathbf{Q}$. In this paper, we use this spectral sequence to define a knot invariant $s(K)$. The definition of $s(K)$ was motivated by a similar invariant $\tau(K)$ which is defined using knot Floer homology [15], [17]. In fact, the similarities between the two invariants extend far beyond their manner of definition.

Our main result is that the invariant s gives a lower bound for the slice genus:

Theorem 1.

$$|s(K)| \leq 2g_*(K)$$

where $g_*(K)$ denotes the slice genus.

In fact,

Theorem 2. *The map s induces a homomorphism from $\text{Conc}(S^3)$ to \mathbf{Z} , where $\text{Conc}(S^3)$ denotes the concordance group of knots in S^3 .*

For alternating knots, $s(K)$ does not provide any new information about $g_*(K)$:

Theorem 3. *If K is an alternating knot, then $s(K)$ is equal to the classical knot signature $\sigma(K)$.*

There is, however, a class of knots for which $s(K)$ gives much better — indeed, sharp — information. We say that a knot is *positive* if it admits a planar diagram with all positive crossings.

Theorem 4. *If K is a positive knot,*

$$s(K) = 2g_*(K) = 2g(K)$$

where $g(K)$ is the ordinary genus of K .

As a corollary, we get a Khovanov homology proof of following result, which was first proved by Kronheimer and Mrowka using gauge theory [8]:

The author is supported by an NSF Postdoctoral fellowship.

Corollary 1. (*The Milnor Conjecture*) *The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.*

As the reader familiar with knot Floer homology will have already noted, the theorems above all hold with $2\tau(K)$ in place of $s(K)$. (See [15] for the first three, and [11] and [18] for the final one.) Indeed, the equality $s(K) = 2\tau(K)$ holds in all cases for which the author knows the value of $\tau(K)$. Based on these observations, we make the following (perhaps optimistic)

Conjecture. *For any knot $K \subset S^3$, $s(K) = 2\tau(K)$.*

Readers familiar with the Khovanov homology may also have observed that the notation $s(K)$ has already been used by Bar-Natan [1] to describe an apparent knot invariant which appears in one of his “phenomenological conjectures.” This is no coincidence. Indeed, the author’s interest in the subject was first aroused by the observation that Bar-Natan’s s appeared to give a lower bound for the slice genus. Although we are unable to prove that the $s(K)$ defined here is the same as that determined by Bar-Natan’s conjecture, we do give a fairly general condition (at least for small knots) under which the two agree.

The remainder of the paper is organized as follows. In section 2, we review the Khovanov complex and Lee’s construction of a spectral sequence from it. In section 3, we define s and show that it behaves nicely with respect to the structure of the concordance group. Section 4 is devoted to the proof of Theorem 1. In section 5, we prove Theorems 3 and 4, and discuss the relationship between $s(K)$ and $\tau(K)$ in more detail. Finally, section 6 contains proofs of some technical results establishing the invariance of Lee’s spectral sequence, which are needed in section 2.

Finally, we take this opportunity to fix two conventions which we will use throughout. First, we will always work with \mathbf{Q} coefficients. Although Khovanov’s complex can be defined with coefficients in \mathbf{Z} , Lee’s theorem (Theorem 2.2) does not hold in this context. Second, we will often abuse our notation, letting L refer both to a planar diagram of a link and to the underlying link itself. The reader should have little trouble determining from context which meaning is intended.

Acknowledgements: The author would like to thank Peter Kronheimer, Peter Ozsváth, and Zoltan Szabó for many helpful conversations and for encouraging him to pursue this problem.

2. REVIEW OF KHOVANOV HOMOLOGY

In this section, we briefly recall the construction of the Khovanov complex [6] and Lee’s extension of it [9].

2.1. The cube of resolutions. Given a link diagram L with crossings labeled 1 through k , we can form the cube of all possible resolutions of L . This is a k -dimensional cube with its vertices and edges decorated by 1-manifolds and cobordisms between them. More specifically, each crossing of L can be resolved in two different ways, as illustrated in Figure 1. To each vertex v of the cube $[0, 1]^k$, we associate the planar diagram D_v obtained by resolving the i -th crossing of L according to the i -th coordinate of v . Then D_v is a planar diagram without crossings, so it is a disjoint union of circles.

Let e be an edge of the cube. The coordinates of its two ends differ in one component — say the l -th. We call the end which has a 0 in this component the *initial end*, and denote it by $v_e(0)$. The other end is called the *terminal end*, written $v_e(1)$. We assign to e the cobordism $S_e : D_{v_e(0)} \rightarrow D_{v_e(1)}$, which is a product cobordism except in a neighborhood of the l -th crossing, where it is the obvious saddle cobordism between the 0 and 1-resolutions.

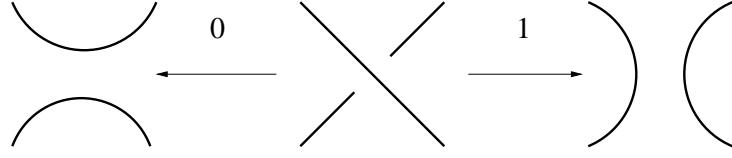


FIGURE 1. 0– and 1–resolutions of a crossing.

The Khovanov complex is constructed by applying a $1 + 1$ dimensional TQFT \mathcal{A} to the cube of resolutions. In other words, one replaces each vertex v by the group $\mathcal{A}(D_v)$, and each edge e by the map $\mathcal{A}(S_e)$. The underlying group of $CKh(L)$ is the direct sum of the groups $\mathcal{A}(D_v)$ for all vertices v , and the differential on the summand $\mathcal{A}(D_v)$ is a sum of the maps $\mathcal{A}(S_e)$ for all edges e which have v as their initial end. More precisely, for $x \in \mathcal{A}(D_v)$

$$(1) \quad d(x) = \sum_{i=1}^{c_0(v)} (-1)^{s(e_i)} \mathcal{A}(S_{e_i}).$$

Here $c_0(v)$ is the number of crossings in v which have a 0-resolution, and e_i is the edge which corresponds to changing the i -th such crossing to a 1-resolution. The signs $(-1)^{s(e_i)}$ are chosen in such a way that $d^2 = 0$. (There are many different ways to do this, but it is easy to see that they all give rise to isomorphic chain complexes.) The *homological grading* of an element $x \in \mathcal{A}(D_v)$ is defined to be $\text{gr}(v) = |v| - n_-$, where $|v|$ is the number of 1's in the coordinates of v and n_- is the number of negative crossings in the diagram for L . Note that d increases the homological grading by 1 — strictly speaking, the Khovanov homology is a cohomology theory!

2.2. Khovanov’s TQFT. We now give a more explicit description of the TQFT \mathcal{A} . Let V be a vector space spanned by two elements, \mathbf{v}_+ and \mathbf{v}_- . The vector space associated by \mathcal{A} to a single circle is defined to be V , so that if D is a diagram composed of n disjoint circles, $\mathcal{A}(D) = V^{\otimes n}$. Thus we can think of $CKh(L)$ as being the vector space spanned by the space of “states” for L , where a state consists of a complete resolution of L , together with a labeling of each component of the resolution by either \mathbf{v}_+ or \mathbf{v}_- .

The cobordisms S_e come in two forms: either two circles can merge into one, or one can split into two. In the first case, $\mathcal{A}(S_e)$ is given by a map $m: V^{\otimes 2} \rightarrow V$, where the two factors in the tensor product correspond to the labels on the two circles that merge, and the copy of V in the image corresponds to the label on the single resulting circle. Likewise, in the second case, $\mathcal{A}(S_e)$ is given by a map $\Delta: V \rightarrow V^{\otimes 2}$. The formulas for these maps are

$$(2) \quad \begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- \\ m(\mathbf{v}_- \otimes \mathbf{v}_-) &= 0. \end{aligned}$$

For reference, we also record two other maps ι and ϵ used to define \mathcal{A} . These maps are not needed at the moment, but they make an appearance in section 4 when we study cobordisms. Corresponding to the addition of a 0-handle (the birth of a circle in a diagram), there is a map $\iota: \mathbf{Q} \rightarrow V$, and corresponding to the addition of a two handle (the death of a circle) there is a map $\epsilon: V \rightarrow \mathbf{Q}$. These maps are given by

$$\begin{aligned} \epsilon(\mathbf{v}_-) &= 1 & \iota(1) &= \mathbf{v}_+ \\ \epsilon(\mathbf{v}_+) &= 0. \end{aligned}$$

\mathcal{A} is especially nice because it is a graded TQFT. We define a grading p on V by setting $p(\mathbf{v}_\pm) = \pm 1$ and extend it to $V^{\otimes n}$ by $p(v_1 \otimes v_2 \otimes \dots \otimes v_n) = p(v_1) + p(v_2) + \dots + p(v_n)$. Then it is easy to see that if \mathbf{v} is a homogenous element of $V^{\otimes n}$, $p(S_e(\mathbf{v})) = p(\mathbf{v}) - 1$. Next, we define a grading q on $CKh(L)$ by $q(\mathbf{v}) = p(\mathbf{v}) + \text{gr}(\mathbf{v}) + n_+ - n_-$, where n_\pm are the number of positive and negative crossings in the diagram L . (The term $n_+ - n_-$ is included so that the q -grading remains invariant for different diagrams of the same knot.) Then $q(d(\mathbf{v})) = q(\mathbf{v})$, so $CKh(L)$ splits into a direct sum of complexes, one for each q grading. In fact, its graded Euler characteristic is the unnormalized Jones polynomial of L , but we will not make use of this here.

In [6], Khovanov proves that the homology of $CKh(L)$ (thought of as a bigraded group) is an invariant of the underlying link L . We denote this homology group by $Kh(L)$.

2.3. Lee's TQFT. In [9], Lee considers a similar construction, but with another TQFT \mathcal{A}' in place of \mathcal{A} . The underlying vector spaces for these two TQFT's are the same, but the maps $m': V \otimes V \rightarrow V$ and $\Delta': V \rightarrow V \otimes V$ induced by cobordisms are slightly different. They are given by

$$(3) \quad \begin{aligned} m'(\mathbf{v}_+ \otimes \mathbf{v}_+) &= m'(\mathbf{v}_- \otimes \mathbf{v}_-) = \mathbf{v}_+ & \Delta'(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ m'(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m'(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_- & \Delta'(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- + \mathbf{v}_+ \otimes \mathbf{v}_+. \end{aligned}$$

(The maps ι and ϵ corresponding to the addition of 0 and 2-handles are the same as for \mathcal{A} .) We denote the resulting complex by $CKh'(L)$ and its homology by $Kh'(L)$.

Using the obvious identification between the underlying groups of $CKh(L)$ and $CKh'(L)$, we can define a q -grading on the latter group as well. It is clear from equation 3 that this grading does not behave quite so well with respect to the differential d' . Indeed, $\Delta'(\mathbf{v}_-)$ is not even homogenous. It is easy to see, however, that if $\mathbf{v} \in CKh'(L)$ is a homogenous element, then the q -grading of every monomial in $d'(\mathbf{v})$ is greater than or equal to the q -grading of \mathbf{v} . In other words, the q -grading defines a filtration on the complex $CKh'(L)$. This fact leads to the following theorem, which is implicit in [9]:

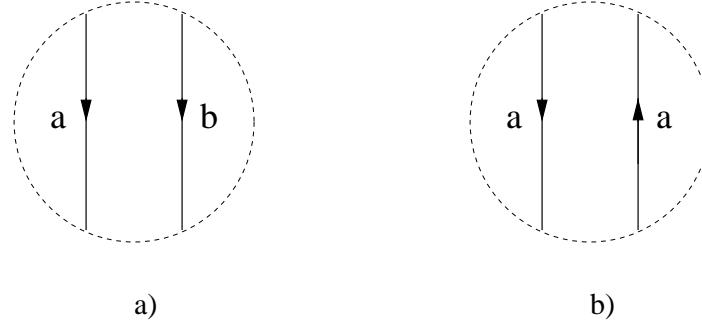
Theorem 2.1. *There is a spectral sequence with E_2 term $Kh(L)$ which converges to $Kh'(L)$. The E_2 and higher terms of this spectral sequence are invariants of the link L .*

The first part of the theorem is more or less immediate from the observations above. The filtration on CKh' gives rise to a spectral sequence converging to Kh' . The differential on its E_1 term is the part of d' which preserves (rather than raises) the q -grading. Comparing equations 2 and 3, we see that the E_1 term is the complex CKh .

To prove the second statement, we check that the spectral sequence is invariant under the Reidemeister moves. Suppose L and \tilde{L} are two diagrams related by the i -th Reidemeister move. In [9], Lee defines maps $\rho'_i: CKh'(L) \rightarrow CKh'(\tilde{L})$ which induce isomorphisms on homology. In section 6, we show that these maps induce isomorphisms on E_2 terms of spectral sequences, thus completing the proof of the theorem.

2.4. Calculation of Kh' . Lee's second major result is that the homology group $Kh'(L)$ is surprisingly simple. To show this, she introduces a new basis $\{\mathbf{a}, \mathbf{b}\}$ for V , where $\mathbf{a} = \mathbf{v}_- + \mathbf{v}_+$ and $\mathbf{b} = \mathbf{v}_- - \mathbf{v}_+$. With respect to this new basis, the maps m' and Δ' are given by

$$\begin{aligned} m'(\mathbf{a} \otimes \mathbf{a}) &= 2\mathbf{a} & \Delta'(\mathbf{a}) &= \mathbf{a} \otimes \mathbf{a} \\ m'(\mathbf{a} \otimes \mathbf{b}) &= m'(\mathbf{b} \otimes \mathbf{a}) = 0 & \Delta'(\mathbf{b}) &= \mathbf{b} \otimes \mathbf{b} \\ m'(\mathbf{b} \otimes \mathbf{b}) &= -2\mathbf{b} \end{aligned}$$

FIGURE 2. Local behavior of the state \mathfrak{s}_o .

and the maps ϵ' and ι' are given by

$$\epsilon'(\mathbf{a}) = \epsilon'(\mathbf{b}) = 1 \quad \iota'(1) = (\mathbf{a} - \mathbf{b})/2$$

Using this basis, she proves

Theorem 2.2. (*Theorem 5.1 of [9]*) $\text{Kh}'(L)$ has rank 2^n , where n is the number of components of L .

Indeed, Lee exhibits an explicit bijection between the set of possible orientations for L and a set of generators of $\text{Kh}'(L)$, which we refer to as *canonical generators*. This bijection may be described as follows. Given an orientation o of L , let D_o be the corresponding oriented resolution. We label the circles in D_o with \mathbf{a} 's and \mathbf{b} 's according to the following rule. To each circle C we assign a mod 2 invariant, which is the mod 2 number of circles in D_o which separate it from infinity. (In other words, draw a ray in the plane from C to infinity, and take the number of other times it intersects the other circles, mod 2.) To this number, we add 1 if C has the standard (counterclockwise) orientation, and 0 if it does not. Label C by \mathbf{a} if the resulting invariant is 0, and by \mathbf{b} if it is 1. We denote the resulting state by \mathfrak{s}_o .

The name “canonical generator” is justified by the following result, whose proof is given in section 6.

Proposition 2.3. Suppose L and \tilde{L} are related by the i -th Reidemeister move. Then an orientation o on L induces an orientation \tilde{o} on \tilde{L} , and $\rho'_{i*}([\mathfrak{s}_o])$ is a nonzero multiple of $[\mathfrak{s}_{\tilde{o}}]$.

We end this section with an elementary but important observation.

Lemma 2.4. (*Coherent orientations*) Suppose there is a region in the state diagram for \mathfrak{s}_o containing exactly two segments, as shown in Figure 2. Then either the orientations of the two are the same and the labels are different (like part a of the figure) or the orientations are different and the labels are the same (like part b).

Proof. We consider three possible cases: either the two segments belong to the same circle in D_o , or they belong to two circles, one of which is contained inside the other, or they belong to two circles, neither of which is contained inside the other. In each case, it is easy to verify that the claim holds. \square

Corollary 2.5. If two circles in the state diagram for \mathfrak{s}_o share a crossing, they have different labels.

3. DEFINITION AND BASIC PROPERTIES OF THE INVARIANT

Let K be a knot in S^3 . By Theorems 2.1 and 2.2, we know that there is a spectral sequence associated to K which converges to $\mathbf{Q} \oplus \mathbf{Q}$. This spectral sequence is a relatively complicated object, but we can extract some simpler invariants of K from it. Let s_{\max} and s_{\min} (with $s_{\max} \geq s_{\min}$) be the q -gradings of the two surviving copies of \mathbf{Q} which remain in the E_∞ term of the spectral sequence. Like all q -gradings for a knot, s_{\max} and s_{\min} are odd integers. Since the isomorphism type of the spectral sequence is an invariant of K , s_{\max} and s_{\min} are invariants as well.

Before making this definition formal, we digress to establish some terminology related to filtrations. Suppose C is a chain complex. A *finite length filtration* of C is a sequence of subcomplexes

$$0 = C_n \subset C_{n-1} \subset C_{n-2} \subset \cdots \subset C_m = C.$$

To such a filtration, we associate a *grading* defined as follows: $x \in C$ has grading i if and only if $x \in C_i$ but $x \notin C_{i-1}$. If $f : C \rightarrow C'$ is a map between two filtered chain complexes, we say that f respects the filtration if $f(C_i) \subset C'_i$. More generally, we say that f is a *filtered map of degree k* if $f(C_i) \subset C'_{i+k}$.

A filtration $\{C_i\}$ on C induces a filtration $\{S_i\}$ on $H_*(C)$ defined as follows: a class $[x] \in H_*(C)$ is in S_i if and only if has a representative which is an element of C_i . If $f : C \rightarrow C'$ is a filtered chain map of degree k , then it is easy to see that the induced map $f_* : H_*(C) \rightarrow H_*(C')$ is also filtered of degree k .

A finite length filtration $\{C_i\}$ on C induces a spectral sequence, which converges to the associated graded group of the induced filtration $\{S_i\}$. In other words, the group which survives at grading i in the spectral sequence is naturally identified with the group S_i/S_{i+1} .

Let us denote by s the grading on $Kh'(K)$ induced by the q -grading on $CKh'(K)$. Then the informal definition above is equivalent to

Definition 3.1.

$$\begin{aligned} s_{\min}(K) &= \min\{s(x) \mid x \in Kh'(K), x \neq 0\} \\ s_{\max}(K) &= \max\{s(x) \mid x \in Kh'(K), x \neq 0\} \end{aligned}$$

Since Kh of the unknot U has rank two and is supported in q -gradings ± 1 , we have $s_{\max}(U) = 1$, $s_{\min}(U) = -1$.

Another proof that s_{\max} and s_{\min} are knot invariants could be given using

Proposition 3.2. *The maps ρ'_{i*} and $(\rho'_{i*})^{-1}$ both respect the induced filtration s on Kh' .*

The proof may be found in section 6.

3.1. The invariant s . Our first task in this section is to prove

Proposition 3.3.

$$s_{\max}(K) = s_{\min}(K) + 2$$

which justifies

Definition 3.4.

$$s(K) = s_{\max}(K) - 1 = s_{\min}(K) + 1$$

Since s_{\max} and s_{\min} are odd, $s(K)$ is always an even integer.

Before proving the proposition, we need some preliminary results.

$$\begin{array}{c}
CKh' (\boxed{K_1} \sqcup \boxed{K_2}) \\
\downarrow \\
CKh' (\boxed{K_1} \times \boxed{K_2}) = CKh' (\boxed{K_1} \# \boxed{K_2})
\end{array}$$

FIGURE 3. A short exact sequence for $CKh'(K_1 \# K_2)$.

Lemma 3.5. *Let n be the number of components of L . There is a direct sum decomposition $Kh'(L) \cong Kh'_o(L) \oplus Kh'_e(L)$, where $Kh'_o(L)$ is generated by all states with q -grading congruent to $2+n \pmod{4}$, and $Kh'_e(L)$ is generated by all states with q -grading congruent to $n \pmod{4}$. If o is an orientation on L , then $\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$ is contained in one of the two summands, and $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$ is contained in the other.*

Proof. Following Lee [9], we write

$$\begin{aligned}
m' &= m + \Phi_m \\
\Delta' &= \Delta + \Phi_\Delta
\end{aligned}$$

where m and Δ preserve the q -grading and Φ_m and Φ_Δ raise it by 4. This proves the first statement.

For the second, let $\iota: CKh'(L) \rightarrow CKh'(L)$ be the map which acts by the identity on CKh'_e and by multiplication by -1 on CKh'_o . We claim that $\iota(\mathfrak{s}_o) = \pm \mathfrak{s}_{\bar{o}}$. To see this, we define a new grading on V with respect to which \mathbf{v}_- has grading 0 and \mathbf{v}_+ has grading 2. Let $i: V \rightarrow V$ be given by $i(\mathbf{v}_-) = \mathbf{v}_-$, $i(\mathbf{v}_+) = -\mathbf{v}_+$, so that $i(\mathbf{a}) = \mathbf{b}$ and $i(\mathbf{b}) = \mathbf{a}$. Then the induced map $i^{\otimes n}: V^{\otimes n} \rightarrow V^{\otimes n}$ acts as the identity on elements whose new grading is congruent to 0 mod 4 and as multiplication by -1 on elements whose new grading is congruent to 2 mod 4. The new grading differs from the q -grading on D_o by an overall shift, so

$$\iota(\mathfrak{s}_o) = \pm i^{\otimes n}(\mathfrak{s}_o) = \pm \mathfrak{s}_{\bar{o}}$$

It follows that $\mathfrak{s}_o + \iota(\mathfrak{s}_o) = \mathfrak{s}_o \pm \mathfrak{s}_{\bar{o}}$ is contained in one summand, while $\mathfrak{s}_o - \iota(\mathfrak{s}_o) = \mathfrak{s}_o \mp \mathfrak{s}_{\bar{o}}$ is contained in the other. \square

Corollary 3.6.

$$s(\mathfrak{s}_o) = s(\mathfrak{s}_{\bar{o}}) = s_{\min}(K)$$

Corollary 3.7. $s_{\max}(K) > s_{\min}(K)$.

Proof. Since $CKh'(K)$ decomposes as a direct sum, its affiliated spectral sequence decomposes too. The homology of each summand is \mathbf{Q} , so each must account for one of the surviving terms in the spectral sequence. The two summands are supported in different q -gradings, so the surviving terms must have different q -gradings as well. \square

Lemma 3.8. *For knots K_1, K_2 , there is a short exact sequence*

$$0 \longrightarrow Kh'(K_1 \# K_2) \xrightarrow{p_*} Kh'(K_1) \otimes Kh'(K_2) \xrightarrow{\partial} Kh'(K_1 \# K_2) \longrightarrow 0$$

The maps p_* and ∂ are filtered of degree -1 .

Proof. Consider the diagram for $K_1 \# K_2$ shown in Figure 3. From it, we get a short exact sequence

$$0 \longrightarrow CKh'(D_1)\{1\} \longrightarrow CKh'(D_2) \xrightarrow{p} CKh'(D_3) \longrightarrow 0$$

where D_1 and D_2 are both diagrams for $K_1 \# K_2$, and D_3 is a diagram for the disjoint union $K_1 \coprod K_2$. Since $Kh'(K_1 \# K_2)$ has rank two and $Kh'(K_1 \coprod K_2) \cong Kh'(K_1) \otimes Kh'(K_2)$ has rank four, the resulting long exact sequence must split, giving the short exact sequence of the lemma. It is clear that the maps p_* and ∂ are filtered of some degree, which can be worked out by considering (for example) the case $K_1 = K_2 = U$. \square

Proof. (of Proposition 3.3.) Consider the exact sequence of the previous lemma with $K_1 = K$ and K_2 the unknot. Denote the canonical generators of K by \mathfrak{s}_a and \mathfrak{s}_b , according to their label near the connected sum point, and the canonical generators of U by \mathbf{a} and \mathbf{b} . Without loss of generality, we may assume that $s(\mathfrak{s}_a - \mathfrak{s}_b) = s_{\max}(K)$. From Figure 3, we see that $\partial((\mathfrak{s}_a - \mathfrak{s}_b) \otimes \mathbf{a}) = \mathfrak{s}_a$. Since ∂ is a filtered map of degree -1 , we conclude that

$$\begin{aligned} s((\mathfrak{s}_a - \mathfrak{s}_b) \otimes \mathbf{a}) &\leq s(\mathfrak{s}_a) + 1 \\ s_{\max}(K) - 1 &\leq s_{\min}(K) + 1 \end{aligned}$$

Since we already know that $s_{\max}(K) \neq s_{\min}(K)$, this gives the desired result. \square

3.2. Properties of s . We check that s behaves nicely with respect to mirror image and connected sum.

Proposition 3.9. *Let \overline{K} be the mirror image of K . Then we have*

$$\begin{aligned} s_{\max}(\overline{K}) &= -s_{\min}(K) \\ s_{\min}(\overline{K}) &= -s_{\max}(K) \\ s(\overline{K}) &= -s(K) \end{aligned}$$

Proof. Suppose that C is a filtered complex with filtration $C = C_0 \supset C_1 \supset \dots \supset C_n = \{0\}$. Then the dual complex C^* has a filtration $\{0\} = C_0^* \subset C_{-1}^* \subset \dots \subset C_{-n}^* = C^*$, where $C_{-i}^* = \{x \in C^* \mid \langle x, y \rangle = 0, \forall y \in C_i\}$.

To prove the proposition, we observe that the filtered complex $CKh'(\overline{K})$ is isomorphic to $(CKh'(K))^*$. Indeed, it is easy to see from equation 3 that there is an isomorphism

$$r: (V, m', \Delta') \rightarrow (V^*, \Delta'^*, m'^*)$$

which sends \mathbf{v}_{\pm} to \mathbf{v}_{\mp}^* . Then if \mathbf{v} is a state of the diagram \overline{K} , we define $R(\mathbf{v})$ to be state of K obtained by applying r all the labels of \mathbf{v} . It is straightforward to check that the map $R: CKh'(\overline{K}) \rightarrow (CKh'(K))^*$ is the desired isomorphism. (Compare with section 7.3 of [6], where it is shown that $CKh(\overline{K}) \cong (CKh(K))^*$.)

We now appeal to the following general result, whose proof is left to the reader:

Lemma 3.10. *If C_1 and C_2 are dual filtered complexes over a field, then their associated spectral sequences E_n^1 and E_n^2 are dual, in the sense that $E_n^1 \cong (E_n^2)^*$.*

Thus if the two surviving generators in E_{∞}^1 have filtration gradings s_{\min} and s_{\max} , the surviving generators in E_{∞}^2 will have gradings $-s_{\max}$ and $-s_{\min}$. \square

Proposition 3.11.

$$s(K_1 \# K_2) = s(K_1) + s(K_2)$$

Proof. We use the short exact sequence of Lemma 3.8. Denote the canonical generators of K_i by \mathfrak{s}_a^i and \mathfrak{s}_b^i , according to their label near the connected sum point. It is not difficult to see that $Kh'(K_1 \# K_2)$ has a canonical generator \mathfrak{s}_o which maps to $\mathfrak{s}_a \otimes \mathfrak{s}_b$ under p_* . Thus

$$\begin{aligned} s(\mathfrak{s}_o) - 1 &\leq s(\mathfrak{s}_a^1 \otimes \mathfrak{s}_b^2) \\ s_{\min}(K_1 \# K_2) - 1 &\leq s_{\min}(K_1) + s_{\min}(K_2) \end{aligned}$$

Applying the same argument to \overline{K}_1 and \overline{K}_2 , and using the fact that $s_{\min}(K) = -s_{\max}(K)$, we see that

$$\begin{aligned} s_{\max}(K_1 \# K_2) + 1 &\geq s_{\max}(K_1) + s_{\max}(K_2) \\ s_{\min}(K_1 \# K_2) + 3 &\geq s_{\min}(K_1) + s_{\min}(K_2) + 4 \end{aligned}$$

Thus

$$\begin{aligned} s_{\min}(K_1 \# K_2) &= s_{\min}(K_1) + s_{\min}(K_2) + 1 \\ s_{\max}(K_1 \# K_2) &= s_{\max}(K_1) + s_{\max}(K_2) - 1. \end{aligned}$$

This proves the claim. \square

4. BEHAVIOR UNDER COBORDISMS

Let L_0 and L_1 be two links in \mathbf{R}^3 . An oriented cobordism from L_0 to L_1 is a smooth, oriented, compact, properly embedded surface $S \subset \mathbf{R}^3 \times [0, 1]$ with $S \cap (\mathbf{R}^3 \times \{i\}) = L_i$. In this section, we define and study a map $\phi_S: Kh'(L_0) \rightarrow Kh'(L_1)$ induced by such a cobordism. Our construction follows section 6.3 of [6], where Khovanov describes a similar map for the homology theory Kh .

4.1. Elementary cobordisms. Following Khovanov, we decompose the cobordism S into a series of elementary cobordisms, each represented by a single move from one planar diagram to another. (See [3] for a more detailed treatment of this material.) For $i \in [0, 1]$, let

$$\begin{aligned} L_i &= S \cap (\mathbf{R}^3 \times \{i\}) \\ S_i &= S \cap (\mathbf{R}^3 \times [0, i]). \end{aligned}$$

After a small isotopy of S , we can assume that L_i is a link in \mathbf{R}^3 for all but finitely many values of i . The orientation on S restricts to an orientation on S_i , which in turn determines an orientation on L_i . We denote this orientation by o_i . (Note that with this convention, o_0 is the reverse of the orientation induced on L_0 by S .)

Next, we fix a projection $p: \mathbf{R}^3 \rightarrow \mathbf{R}^2$. After a further small isotopy of S , we can assume that p defines a regular projection of L_i for all but finitely many values of i , and that this set of special values is disjoint from the first set where L failed to be a link. The isotopy type of the oriented planar diagram L_i remains constant except when L passes through one of the two types of special values, where it changes by some well-defined local move. Each of these moves corresponds to an elementary cobordism, so we can write the whole cobordism S as a composition of elementary cobordisms.

The necessary moves may be subdivided into two types: Reidemeister moves and Morse moves. There is one Reidemeister-type move for each of the ordinary Reidemeister moves, as well as one for each of their inverses. These moves do not change the topology of the surface S_i . The Morse moves correspond to the addition of a 0, 1 or 2-handle to S_i . They are illustrated in Figure 4.

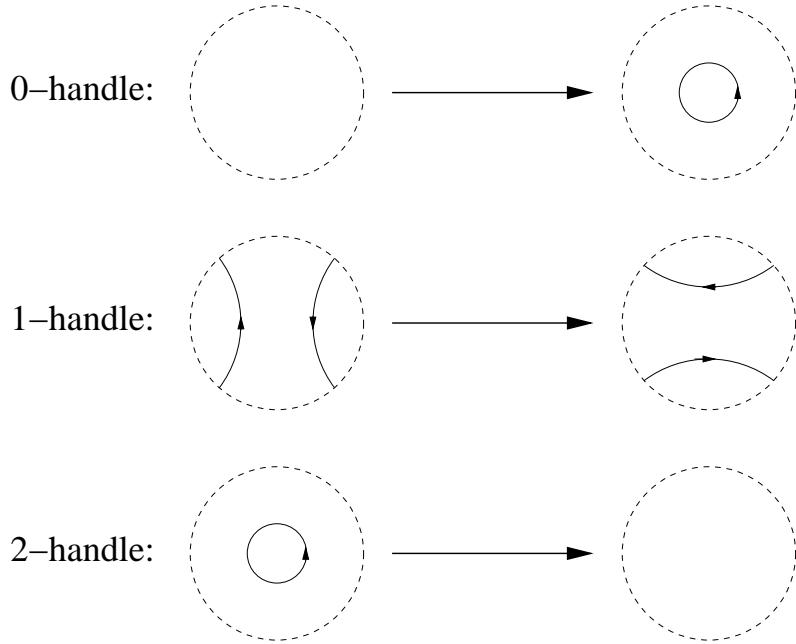


FIGURE 4. Local pictures for Morse moves.

4.2. Induced Maps. Given a cobordism S from L_0 to L_1 , we want to assign to it an induced map $\phi_S : Kh'(L_0) \rightarrow Kh'(L_1)$ which respects the filtration on Kh' . In addition, we would like this assignment to be functorial, in the sense that if S is the composition of two cobordisms S_1 and S_2 , ϕ_S is the composition of ϕ_{S_1} and ϕ_{S_2} . Thus it suffices to consider the case when S is an elementary cobordism.

Suppose that S is an elementary cobordism corresponding to the i -th Reidemeister move or its inverse. Then we define ϕ_S to be ρ'_{i*} or its inverse. By Proposition 3.2, this is a filtered map of degree 0. If S is an elementary cobordism corresponding to a Morse move, then we take ϕ_S to be the map induced by $\psi : CKh'(L_0) \rightarrow CKh'(L_1)$, where ψ is the result of applying the TQFT \mathcal{A}' to the corresponding map of cubes. In other words, if the move corresponds to the addition of a 0-handle or a 2-handle, we apply ι' or ϵ' , respectively, to the summand at each vertex of the cube. If it corresponds to the addition of a 1-handle, we apply either m' or Δ' , depending on whether the move results in a merge or a split at the vertex in question. It is easy to see that ϕ_S is a filtered map of degree 1 for a 0- or 2-handle addition and degree -1 for a 1-handle.

In general, given a cobordism S , we decompose it as a union of elementary cobordisms: $S = S_1 \cup S_2 \dots \cup S_k$ and define the induced morphism $\phi_S : Kh'(L_0) \rightarrow Kh'(L_1)$ to be the composition $\phi_{S_k} \circ \dots \circ \phi_{S_1}$, which is a filtered map of degree $\chi(S)$. We expect that the map ϕ_S will depend only on the isotopy class of S rel ∂S (c.f [5], where an analogous result is proved for the Khovanov homology), but since we do not need this fact, we will not pursue it here.

4.3. Canonical generators. The maps ϕ_S behave nicely with respect to canonical generators.

Proposition 4.1. *Suppose S is an oriented cobordism from L_0 to L_1 which is weakly connected, in the sense that every component of S has a boundary component in L_0 . Then $\phi_S([\mathfrak{s}_{o_0}])$ is a nonzero multiple of $[\mathfrak{s}_{o_1}]$.*

Remark: Some sort of connectedness hypothesis is clearly necessary for the proposition to hold. For example, if we take S to be the union of a product cobordism and a trivially embedded sphere, the induced map on Kh' is the zero map.

Proof. In fact, we will prove a slightly stronger statement. Suppose i is a regular value for the cobordism S , so that L_i is a link. We divide the components of S_i into two sorts: those of the *first type*, which have a boundary component in L_0 , and those of the *second type*, which do not. We say that an orientation o on S_i is *permissible* if it agrees with the orientation of S on components of the first type. (Here and in what follows, we use o_I to denote both a permissible orientation on S_i and the orientation it induces on L_i .) We claim that

$$\phi_{S_i}([\mathfrak{s}_{o_0}]) = \sum_I a_I [\mathfrak{s}_{o_I}]$$

where $\{o_I\}$ runs over the set of permissible orientations on S_i and each coefficient a_I is nonzero. Note that the weak connectivity hypothesis implies that there is only one permissible orientation on S_1 , so the proposition is implied by the claim.

To prove the claim, it suffices to check that if it holds for S_i , then it holds for $S_{i'}$ as well, where $S_{i'}$ is the composition of S_i with a single elementary cobordism S_e . If this cobordism corresponds to a Reidemeister type move, this is a straightforward consequence of Proposition 2.3. Below, we check that it holds for each of the Morse-type moves as well.

0-Handle Move: In this case, $\phi_{S_e}(\mathfrak{s}_{o_I}) = \mathfrak{s}_{o_I} \otimes \frac{1}{2}(\mathbf{a} - \mathbf{b})$, where the second factor in the tensor product refers to the labels on the newly created circle. $S_{i'}$ has a new component of the second type — namely, the disk bounded by the new circle — and $\mathfrak{s}_{o_I} \otimes \mathbf{a}$ and $\mathfrak{s}_{o_I} \otimes \mathbf{b}$ are the canonical generators corresponding to the two possible orientations on $S_{i'}$ which agree with o_I on all components other than the new one.

1-Handle Move: Suppose that the orientation o_I is actually the orientation o_i induced by S_i . Then the two strands involved in the move have opposite orientations, so by Lemma 2.4, they must have the same label. Since

$$\begin{array}{ll} m'(\mathbf{a} \otimes \mathbf{a}) = 2\mathbf{a} & \Delta'(\mathbf{a}) = \mathbf{a} \otimes \mathbf{a} \\ m'(\mathbf{b} \otimes \mathbf{b}) = -2\mathbf{b} & \Delta'(\mathbf{b}) = \mathbf{b} \otimes \mathbf{b} \end{array}$$

we see that $\phi_{S_e}(\mathfrak{s}_{o_i})$ is a nonzero multiple of $\mathfrak{s}_{o_{i'}}$,

More generally, the orientation o_I is either compatible with some orientation o_e on S_e , or it is not. In the former case, the two strands involved in the move point in opposite directions and have the same label, and $\phi_{S_e}(\mathfrak{s}_{o_I})$ is a nonzero multiple of $\mathfrak{s}_{o'_I}$ where o'_I is the orientation induced on $L_{i'}$ by o_e . In the latter case, the two strands point in the same direction and have different labels, so $\phi_{S_e}(\mathfrak{s}_{o_I}) = 0$.

Now we consider what happens to the components of S_i during the move. If the move splits one component of L_i into two components of $L_{i'}$, then the number and type of components of S_i remains constant. In this case, the set of permissible orientations on S_i is naturally identified with the set of permissible orientations on $S_{i'}$. There is always an orientation on S_e compatible with o_I , and $\phi_{S_e}(\mathfrak{s}_{o_I})$ is a nonzero multiple of $\mathfrak{s}_{o'_I}$.

On the other hand, if the move merges two components of L_i into one component of $L_{i'}$, there are several possibilities to consider. If the merge involves only a single component of S_i , the situation is like the one above: there is always an orientation on S_e compatible with

o_I , and $\phi_{S_e}(\mathfrak{s}_{o_I})$ is a nonzero multiple of $\mathfrak{s}_{o'_I}$. The same argument applies when S_e merges two components of S_i , both of which are of the first type.

Finally, suppose the merge joins two components of S_i , at least one of which is of the second type. Then the set of permissible orientations on $S_{i'}$ is only half as large as the set of permissible orientations on S_i . If o_I extends to a permissible orientation o'_I on $S_{i'}$, $\phi_{S_e}(\mathfrak{s}_{o_I}) = \mathfrak{s}_{o'_I}$, while if it does not, $\phi_{S_e}(\mathfrak{s}_{o_I}) = 0$.

2-Handle Move: In this case, a permissible orientation o_I on S_i extends to a unique permissible orientation o'_I on $S_{i'}$. Since $\epsilon'(\mathbf{a}) = \epsilon'(\mathbf{b}) = 1$, $\phi_{S_e}(\mathfrak{s}_{o_I}) = \mathfrak{s}_{o'_I}$. To prove the claim, it suffices to show that two permissible orientations on S'_i cannot induce the same orientation on $L_{i'}$. But if this were the case, S_i would have a closed component, contradicting the hypothesis that S is weakly connected. \square

Corollary 4.2. *If S is a connected cobordism between knots K_0 and K_1 , then ϕ_S is an isomorphism.*

Proof. Fix an orientation o on S . Then $\{\mathfrak{s}_{o_0}, \mathfrak{s}_{\bar{o}_0}\}$ is a basis for $Kh'(K_1)$. Its image under ϕ_S is $\{k_1 \mathfrak{s}_{o_1}, k_2 \mathfrak{s}_{\bar{o}_1}\}$ ($k_1, k_2 \neq 0$), which is a basis for $Kh'(K_2)$. \square

4.4. The slice genus. We can now prove the first two theorems from the introduction.

Proof. (of Theorem 1.) Suppose $K \subset S^3$ bounds an oriented surface of genus g in B^4 . Then there is an orientable connected cobordism of Euler characteristic $-2g$ between K and the unknot U in $\mathbf{R}^3 \times [0, 1]$. Let $x \in Kh'(K) - \{0\}$ be a class for which $s(x)$ is maximal. Then $\phi_S(x)$ is a nonzero element of $Kh'(U)$. Now ϕ_S is a filtered map with filtered degree $-2g$, so

$$s(\phi_S(x)) \geq s(x) - 2g.$$

On the other hand, $s_{\max}(U) = 1$, so

$$s(\phi_S(x)) \leq 1.$$

It follows that $s(x) \leq 2g + 1$, so $s_{\max}(K) \leq 2g + 1$ and $s(K) \leq 2g$. To show that $s(K) \geq -2g$, we apply the same argument to \overline{K} (which bounds a surface \overline{S} of genus g) and use the fact that $s(\overline{K}) = -s(K)$. \square

Proof. (of Theorem 2.) If K_1 and K_2 are concordant, then $K_1 \# \overline{K_2}$ is slice, so

$$0 = s(K_1 \# \overline{K_2}) = s(K_1) - s(K_2).$$

Thus s gives a well-defined map from $\text{Conc}(S^3)$ to \mathbf{Z} . That this map is a homomorphism is immediate from Propositions 3.9 and 3.11. \square

Corollary 4.3. *Suppose K_+ and K_- are knots that differ by a single crossing change — from a positive crossing in K_+ to a negative one in K_- . Then*

$$s(K_-) \leq s(K_+) \leq s(K_-) + 1$$

Proof. In [11], Livingston shows that this skein inequality holds for any knot invariant satisfying the properties of Theorems 1 and 2. \square

5. COMPUTATIONS AND RELATIONS WITH OTHER INVARIANTS

Although the invariant $s(K)$ is algorithmically computable from a diagram of K , it is impossible to compute by hand for all but the smallest knots. In this section, we describe some techniques which enable us to efficiently compute s .

5.1. Using Kh . For many knots, it is a simple matter to compute $s(K)$ from the ordinary Khovanov homology $\text{Kh}(K)$. Although $\text{Kh}(K)$ is also hard to compute by hand, there are already a number of computer programs available for this purpose, including Bar-Natan's pioneering program [1] and a more recent, faster program written by Shumakovitch [19].

In [1], Bar-Natan made the following observation, based on his computations of Kh for knots with 10 and fewer crossings.

Conjecture. (Bar-Natan) *The graded Poincaré polynomial $P_{\text{Kh}}(K)$ of $\text{Kh}(K)$ has the form*

$$P_{\text{Kh}}(K) = q^{s(K)}(q + q^{-1}) + (1 + tq^4)Q_{\text{Kh}}(K)$$

where $Q_{\text{Kh}}(K)$ is a polynomial with all positive coefficients.

In [9], Lee showed that this conjecture holds whenever her spectral sequence for Kh' converges after the E_2 term. In this case, it is easy to see that the invariant $s(K)$ is equal to the exponent $s(K)$ which appears in Bar-Natan's conjecture.

To see how widely applicable this condition is, we introduce the notion of the homological width of a knot.

Definition 5.1. *If K is a knot, let $\mu(K) = \{a - 2b \mid q^a t^b \text{ is a monomial in } P_{\text{Kh}}(K)\}$. The width $W(K)$ is one more than the difference between the maximum and minimum elements of $\mu(K)$.*

In other words, $W(K)$ is the number of diagonals in the convex hull of the support of $\text{Kh}(K)$.

Proposition 5.2. *If $W(K) \leq 3$, then the spectral sequence for $\text{Kh}'(K)$ converges after the E_2 term, and our $s(K)$ is the same as Bar-Natan's.*

Proof. Suppose $W(K)$ has width ≤ 3 . Then if x is an element of $\text{Kh}'(K)$ with q -grading a and homological grading b , the minimum possible q -grading of an element with homological grading $b - 1$ is $a - 6$. Since the differential d_n on the E_n term of the spectral sequence lowers the q -grading by $4(n - 1)$, d_n must be trivial for all $n \geq 3$. \square

Theorem 3 follows from this fact, since Lee has shown [10] that if K is an alternating knot, then it has width two and Bar-Natan's s is equal to $\sigma(K)$.

The proposition also applies to many non-alternating knots. Indeed, using Shumakovitch's tables and a computer, it is straightforward to check that there are only four knots with 13 or fewer crossings whose width is greater than three. Inspecting Kh of these four exceptions, one sees that in each case, the spectral sequence must converge after the E_2 term. Thus for all knots with 13 or fewer crossings, the value of $s(K)$ agrees with the value of Bar-Natan's s tabulated in [1] and [19]. Below, we list those knots of 11 crossings or fewer for which $s(K) \neq \sigma(K)$. There are 22 such knots, and $|s(K)| > |\sigma(K)|$ (and thus provides a better bound on the slice genus) for precisely half of them.

K	$s(K)$	$\sigma(K)$	K	$s(K)$	$\sigma(K)$	K	$s(K)$	$\sigma(K)$
9_{42}	0	2	11_{n9}	6	4	11_{n70}	2	4
10_{132}	-2	0	11_{n12}	2	0	11_{n77}	8	6
10_{136}	0	2	11_{n19}	-2	-4	11_{n79}	0	2
10_{139}	8	6	11_{n20}	0	-2	11_{n92}	0	-2
10_{145}	-4	-2	11_{n24}	0	2	11_{n96}	0	2
10_{152}	-8	-6	11_{n31}	4	2	11_{n138}	0	2
10_{154}	6	4	11_{n38}	0	2	11_{n183}	6	4
10_{161}	-6	-4						

Knots with 10 or fewer crossings are labeled according to their numbering in Rolfsen, while those with 11 crossings use the *Knotscape* numbering. The values of the signature are taken from [2]. All of the knots in the table have a homological width of 3, which raises the following question: if K has homological width 2 (*i.e.* is H-thin in the terminology of [7]), must $s(K) = \sigma(K)$?

5.2. Positive knots. If K is a positive knot, $s(K)$ can be computed directly from the definition. To see this, consider a canonical generator \mathfrak{s}_o for a positive diagram of K . Since each crossing of K is positive, its oriented resolution is the 0-resolution. Thus the state \mathfrak{s}_o lives in the extreme corner of the cube of resolutions: it has homological grading 0, and there are no generators in $CKh'(K)$ with homological grading -1 . It follows that the only class homologous to \mathfrak{s}_o is \mathfrak{s}_o itself, so

$$s_{\min}(K) = s([\mathfrak{s}_o]) = q(\mathfrak{s}_o)$$

To compute $q(\mathfrak{s}_o)$, we change back to the basis $\{\mathbf{v}_-, \mathbf{v}_+\}$. In the expansion of \mathfrak{s}_o with respect to this basis, there is a unique state with minimal q -grading, namely, the state in which every circle of the oriented resolution is labeled with a \mathbf{v}_- . If the positive diagram of K has n crossings, and its oriented resolution has k circles, then

$$\begin{aligned} q(\mathfrak{s}_o) &= p(\mathfrak{s}_o) + \text{gr}(\mathfrak{s}_o) + n_+ - n_- \\ &= -k + 0 + n - 0 \end{aligned}$$

so

$$s(K) = -k + n + 1$$

On the other hand, Seifert's algorithm gives a Seifert surface S for K with euler characteristic $k - n$, so

$$2g(K) \leq 2g(S) = n - k + 1 = s(K) \leq 2g_*(K)$$

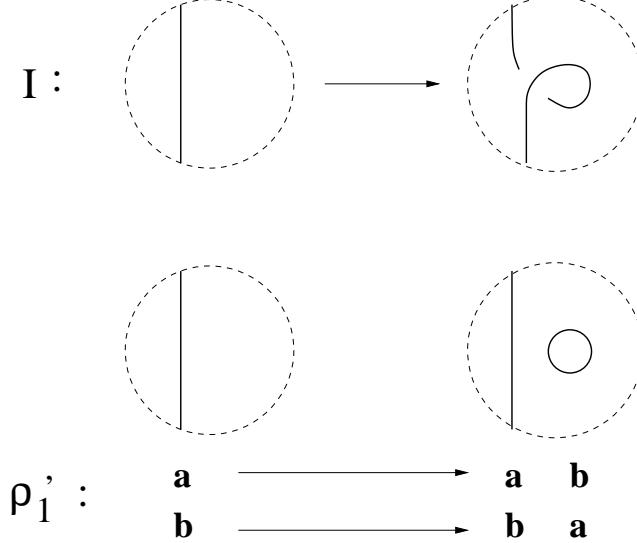
Since $g_*(K) \leq g(K)$, the inequalities above must all be equalities. This completes the proof of Theorem 4.

5.3. Comparison with τ . We end this section by commenting on the conjecture relating s and τ which was stated in the introduction. In addition to the fact that the two invariants share the properties of Theorems 1 through 4, there is a good deal of numerical evidence supporting the conjecture. Recently, a fair amount of work has been done on the problem of computing τ for knots with 10 and fewer crossings. Combining the results of [4], [11], [13], [14], and [15] with some unpublished computations of the author, it appears that the value of τ has been determined for all but two knots of 10 crossings and fewer. (The exceptions are 10_{141} and 10_{150} .) For all of these knots, $s = 2\tau$. The equality can also be checked on certain special classes of knots, such as the pretzel knots of [16]. If the conjecture were true, it would make many computations in knot Floer homology easier. (For example, with our current technology, it seems like quite a laborious project to compute τ for all 11-crossing non-alternating knots.) Even if it is not true, we hope that the remarkable similarity between the two theories will have an enlightening explanation.

6. REIDEMEISTER MOVES

In this section, we prove the results involving Reidemeister moves which were stated in section 2 and 3.

Proof. (of Theorem 2.1.) The proof that the desired spectral sequence exists was sketched in section 2. To prove its invariance, we use the following basic lemma, whose proof may be found in [12], Proposition 3.2.

FIGURE 5. The Reidemeister I move and the map ρ'_1 .

Lemma 6.1. Suppose $F: C_1 \rightarrow C_2$ is a map of filtered complexes which respects the filtrations. Then F induces maps of spectral sequences $F_n: E_n^1 \rightarrow E_n^2$, and if F_n is an isomorphism, F_m is an isomorphism for all $m \geq n$.

In section 4 of [9], Lee proves the invariance of Kh' by checking its invariance under the three Reidemeister moves. For each move, she exhibits a chain map between the complexes associated to the link diagram before and after the move. To prove the theorem, it suffices to check that these maps respect the q -filtration, and that they induce isomorphisms on the E_2 terms. The latter claim is straightforward, since in each case the induced maps on the E_1 terms are identical to the maps used in section 5 of [6] to prove invariance of Kh . Below, we sketch the proof of invariance for each move and explain why the maps in question respect the filtrations. For full details, we refer the reader to [6] and [9].

Reidemeister I Move: Let \tilde{L} be the diagram L with an additional left-hand curl added in. Then $CKh'(\tilde{L})$ can be decomposed as a direct sum $X_1 \oplus X_2$, where X_2 is acyclic and X_1 is isomorphic to $CKh'(L)$ via the map $\rho'_1: CKh'(L) \rightarrow X_1$ illustrated in Figure 5. In terms of the basis $\{\mathbf{v}_\pm\}$, we have

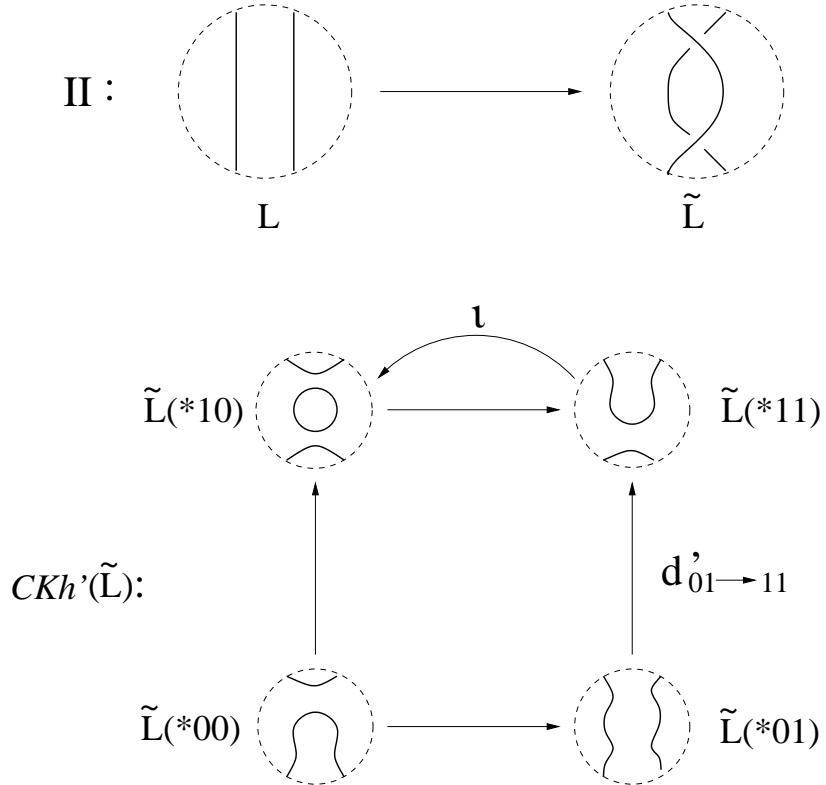
$$\begin{aligned}\rho'_1(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- - \mathbf{v}_+ \otimes \mathbf{v}_+ \\ \rho'_1(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- - \mathbf{v}_- \otimes \mathbf{v}_+\end{aligned}$$

The corresponding map ρ_1 in [6] is given by

$$\begin{aligned}\rho_1(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- \\ \rho_1(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- - \mathbf{v}_- \otimes \mathbf{v}_+\end{aligned}$$

so ρ'_1 is filtration non-decreasing, and its induced map on E_1 terms is ρ_1 .

Remark: There is another version of the first Reidemeister move, corresponding to the addition of a right-hand curl. Although it is not difficult to define an appropriate map ρ'_1 for this move directly, for the sake of brevity we adopt the solution of [1] and [9] and define

FIGURE 6. The Reidemeister II move and the maps ι and $d'_{01 \rightarrow 11}$.

it to be the composition of maps induced by an appropriate Reidemeister II move followed by a Reidemeister I move.

Reidemeister II Move: Let L and \tilde{L} be as shown in figure 6. In this case, $CKh'(\tilde{L})$ can be decomposed as a direct sum $X_1 \oplus X_2 \oplus X_3$, where X_2 and X_3 are acyclic and there is an isomorphism $\rho'_2 : CKh'(L) \rightarrow X_1$, which is given by

$$\rho'_2(z) = (-1)^{\text{gr}(z)}(z + \iota(d'_{01 \rightarrow 11}(z)))$$

The maps ι and $d'_{01 \rightarrow 11}$ are shown in the figure. The isomorphism ρ_2 in [6] has the same form, but with $d_{01 \rightarrow 11}$ in place of $d'_{01 \rightarrow 11}$. Since $d - d'$ is strictly filtration increasing, it follows that ρ'_2 is filtration non-decreasing, and its induced map on E_1 terms is ρ_2 .

Reidemeister III Move: Let L and \tilde{L} be as shown in Figure 7. Then there are direct sum decompositions

$$CKh'(L) \cong X_1 \oplus X_2 \oplus X_3$$

$$CKh'(\tilde{L}) \cong \tilde{X}_1 \oplus \tilde{X}_2 \oplus \tilde{X}_3$$

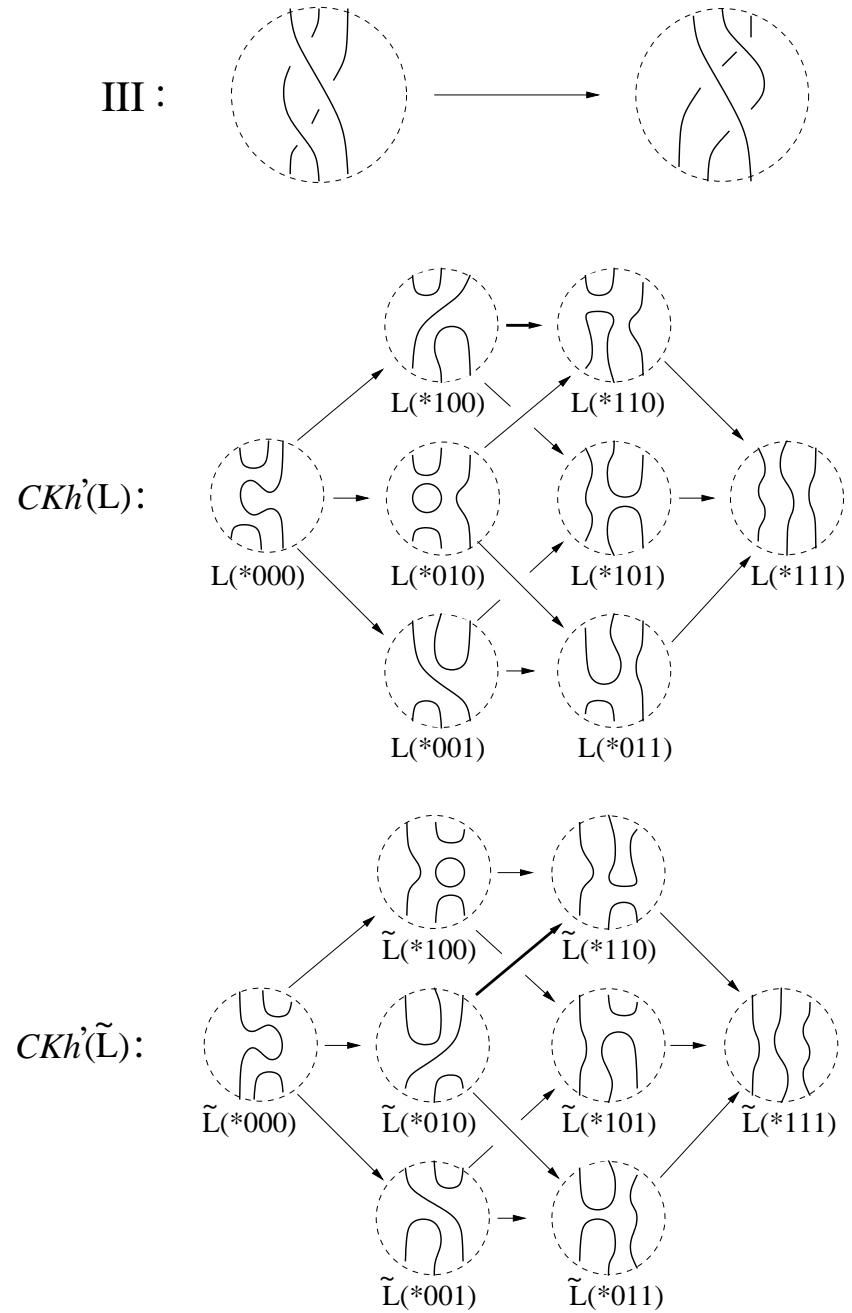


FIGURE 7. The Reidemeister III move. The relevant components of the differentials ($d'_{100 \rightarrow 110}$ and $d'_{010 \rightarrow 110}$) are marked in bold.

where X_2, X_3, \tilde{X}_2 , and \tilde{X}_3 are acyclic and there is an isomorphism $\rho'_3: X_1 \rightarrow \tilde{X}_1$. To describe X_1 and \tilde{X}_1 , we first define maps

$$\begin{aligned}\beta' &: CKh'(L(*100)) \rightarrow CKh'(L(*010)) \\ \tilde{\beta}' &: CKh'(\tilde{L}(*010)) \rightarrow CKh'(\tilde{L}(*100))\end{aligned}$$

by

$$\begin{aligned}\beta' &= \iota \circ d'_{100 \rightarrow 110} \\ \tilde{\beta}' &= \iota \circ d'_{010 \rightarrow 110}\end{aligned}$$

Then

$$\begin{aligned}X_1 &= \{x + \beta'(x) + y \mid x \in CKh'(L(*100)), y \in CKh'(L(*1))\} \\ \tilde{X}_1 &= \{x + \tilde{\beta}'(x) + y \mid x \in CKh'(\tilde{L}(*010)), y \in CKh'(\tilde{L}(*1))\}\end{aligned}$$

and

$$\rho'_3(x + \beta'(x) + y) = x + \tilde{\beta}'(x) + y.$$

The isomorphism ρ_3 in [6] is defined similarly, except that it uses d instead of d' to define maps β and β' . Since d' does not increase the q -grading, we clearly have $q(\beta'(x)) \geq q(x)$. From this, it follows that ρ'_3 does not decrease the q -grading. Since $d - d'$ strictly increases the q -grading, the map induced on E_1 terms by ρ'_3 is equal to ρ_3 . To finish the proof, we apply Lemma 6.1 three times: first to the inclusions $X_1 \hookrightarrow CKh'(L)$ and $\tilde{X}_1 \hookrightarrow CKh'(\tilde{L})$, and then to the map ρ'_3 .

□

Proof. (of Proposition 2.3.) We check the claim directly for each Reidemeister move:

Reidemeister I Move: In this case, it is easy to see that $\rho'_1(\mathfrak{s}_o) = \mathfrak{s}_{\tilde{o}}$.

Reidemeister II Move: Suppose that the two strands in L point in the same direction. Then by Lemma 2.4, they have different labels, so $d'_{01 \rightarrow 11}(\mathfrak{s}_o) = 0$. The oriented resolution of \tilde{L} is contained in $CKh'(\tilde{L}(*01)) \cong CKh'(L)$, so $\rho'_2(\mathfrak{s}_o) = (-1)^0(\mathfrak{s}_{\tilde{o}}) = \mathfrak{s}_{\tilde{o}}$.

Now suppose the two strands point in different directions, so that they have the same label. Let us assume for the moment that this label is **a**. Then we define $\mathfrak{s}_{ij} \in Kh'(\tilde{L}(*ij))$ be the state which is identical to \mathfrak{s}_o outside the area where the move takes place and has all components inside the area of the move labeled with an **a**. Then a direct computation shows that either

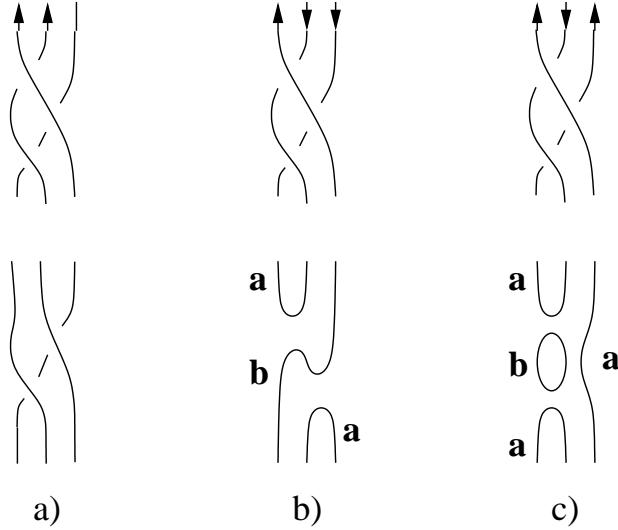
$$\begin{aligned}\rho'_2(\mathfrak{s}_o) &= \mathfrak{s}_{\tilde{01}} + \frac{1}{2}(\mathfrak{s}_{\tilde{10}} - \mathfrak{s}_{\tilde{o}}) \\ &= -\frac{1}{2}(\mathfrak{s}_{\tilde{o}} + d'(\mathfrak{s}_{\tilde{00}}))\end{aligned}$$

if the two strands belong to the same component, or

$$\begin{aligned}\rho'_2(\mathfrak{s}_o) &= \mathfrak{s}_{\tilde{01}} + (\mathfrak{s}_{\tilde{10}} - \mathfrak{s}_{\tilde{o}}) \\ &= -(\mathfrak{s}_{\tilde{o}} + d'(\mathfrak{s}_{\tilde{00}}))\end{aligned}$$

if they belong to different components. This proves the claim in the case where both strands are labeled with an **a**. We leave it to the reader to check that a similar argument applies when they are both labeled with a **b**.

Reidemeister III Move: Here there are three cases to consider. First, suppose that the two overlying strands in L are oriented as shown in Figure 8a. Then $\mathfrak{s}_o \in CKh'(L(*1))$, and it is easy to see that $\rho'_3(\mathfrak{s}_o) = \mathfrak{s}_{\tilde{o}}$.

FIGURE 8. Possible orientations for L and their respective canonical generators.

Next, suppose that the three strands are oriented as shown in Figure 8b. Then $\mathfrak{s}_o \in CKh'(L(*100))$ and $\mathfrak{s}_{\bar{o}} \in CKh'(\tilde{L}(*010))$. Clearly $\beta'(\mathfrak{s}_o) = \tilde{\beta}'(\mathfrak{s}_{\bar{o}}) = 0$, so $\mathfrak{s}_o \in X_1$ and $\mathfrak{s}_{\bar{o}} \in \tilde{X}_1$. Again, it follows that $\rho'_3(\mathfrak{s}_o) = \mathfrak{s}_{\bar{o}}$.

Finally, suppose the strands are oriented as shown in Figure 8c. In this case, the oriented resolution of L is in $L(*010)$, and the oriented resolution of \tilde{L} is in $\tilde{L}(*100)$. Inside the region under consideration, \mathfrak{s}_o looks like the state of Figure 8c (perhaps with **a**'s and **b**'s reversed.) Our first step is to exhibit some $\mathfrak{t} \in X_1$ which is homologous to \mathfrak{s}_o . As before, we denote by \mathfrak{s}_{ijk} the unique state of $L(*ijk)$ which is the same as \mathfrak{s}_o outside the area of the Reidemeister move and has all its components inside this area labeled by **a**'s.

Assume for the moment that all three strands shown in $L(*000)$ belong to different components. In this case, we can take

$$\mathfrak{t} = \mathfrak{s}_o - 2\mathfrak{s}_{100} - \mathfrak{s}_{010} - 2\mathfrak{s}_{001} = \mathfrak{s}_o - d'(\mathfrak{s}_{000}).$$

Indeed, $\beta'(-2\mathfrak{s}_{100}) = \mathfrak{s}_o - \mathfrak{s}_{010}$ and $\mathfrak{s}_{001} \in CKh'(L(*1))$, so $\mathfrak{t} \in X_1$. Then

$$\begin{aligned} \rho'_3(\mathfrak{t}) &= -2\mathfrak{s}_{\bar{0}\bar{1}\bar{0}} - 2\tilde{\beta}'(\mathfrak{s}_{\bar{0}\bar{1}\bar{0}}) - 2\mathfrak{s}_{\bar{0}\bar{0}\bar{1}} \\ &= -2\mathfrak{s}_{\bar{0}\bar{1}\bar{0}} - 2\mathfrak{s}_{100} + 2\mathfrak{s}_{\bar{o}} - 2\mathfrak{s}_{001} \\ &= 2\mathfrak{s}_{\bar{o}} - d'(\mathfrak{s}_{000}) \end{aligned}$$

which proves the claim.

We leave it to the reader to check that a similar argument applies to each of the four other ways in which the segments outside the area of the move can be connected, as well as when the roles of **a** and **b** are reversed. In each case, it is not difficult to verify that $\rho'_{3*}([\mathfrak{s}_o])$ is one of $\pm[\mathfrak{s}_{\bar{o}}]$, $\pm 2[\mathfrak{s}_{\bar{o}}]$, or $\pm \frac{1}{2}[\mathfrak{s}_{\bar{o}}]$. \square

Proof. (of Proposition 3.2.) In the case of ρ'_{1*} and ρ'_{2*} , the claim is immediate, since these maps are induced by filtered chain maps. For the others, we use the following

Lemma 6.2. Suppose $f: C_1 \rightarrow C_2$ is a map of filtered chain complexes with the property that the induced map of spectral sequences $f_2: E_1^2 \rightarrow E_2^2$ is an isomorphism. Then f_*^{-1} is a filtered map with respect to the induced filtrations on $H_*(C_1)$ and $H_*(C_2)$.

Proof. Since f_2 is an isomorphism, f_∞ (the induced map on filtered graded) is as well. It follows that f_* is an isomorphism. Suppose f_*^{-1} does not respect the filtration. Then there must be some $\mathbf{v} \in H_*(C_1)$ whose filtration is strictly increased by f_* . But this contradicts the fact that f_∞ is an isomorphism. \square

The remaining cases now follow easily from the results used in the proof of Theorem 2.1. Indeed, ρ'_1 and ρ'_2 both induce isomorphisms of E_2 terms, and $\rho'_{3*} = \iota_{1*} \circ \psi_* \circ \iota_{2*}^{-1}$, where ι_1 , ι_2 , and ψ all induce isomorphisms of E_2 terms. \square

REFERENCES

- [1] D. Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Alg. Geom. Top.*, 2:337–370, 2002.
- [2] D. Bar-Natan. The Knot Atlas. www.math.toronto.edu/~drorbn/KAtlas/index.html, 2003.
- [3] J. S. Carter and M. Saito. Reidemeister moves for surface isotopies and their interpretation as moves to movies. *J. Knot Theory Ramifications*, 2:251–284, 1993.
- [4] H. Goda, H. Matsuda, and T. Morifuji. Knot Floer homology of (1,1)-knots. math.GT/0311084, 2003.
- [5] M. Jacobsson. An invariant of link cobordisms from Khovanov's homology theory. math.GT/0206303, 2002.
- [6] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101:359–426, 2000.
- [7] M. Khovanov. Patterns in knot cohomology I. math.QA/0201306, 2002.
- [8] P. B. Kronheimer and T. S. Mrowka. Gauge theory for embedded surfaces. I. *Topology*, 32:773–826, 1993.
- [9] E. S. Lee. Khovanov's invariants for alternating links. math.GT/0210213, 2002.
- [10] E. S. Lee. The support of Khovanov's invariants for alternating knots. math.GT/0201105, 2002.
- [11] C. Livingston. Computations of the Ozsvath-Szabo knot concordance invariant. math.GT/0311036, 2003.
- [12] J. McCleary. *User's Guide to Spectral Sequences*. Mathematics Lecture Series, 12. Publish or Perish Inc, 1985.
- [13] P. Ozsváth and Z. Szabó. Heegaard Floer homology and alternating knots. *Geom. Topol.*, 7:225–254, 2002. math.GT/0209149.
- [14] P. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. math.GT/0209056, 2002.
- [15] P. Ozsváth and Z. Szabó. Knot Floer homology and the four-ball genus. *Geom. Topol.*, 7:615–639, 2003. math.GT/0301026.
- [16] P. Ozsváth and Z. Szabó. Knot Floer homology, genus bounds, and mutation. math.GT/0303225, 2003.
- [17] J. Rasmussen. Floer homology and knot complements. math.GT/0306378, 2003.
- [18] L. Rudolph. Positive links are strongly quasipositive. *Geom. Topol. Monogr.*, 2:555–562, 1999.
- [19] A. Shumakovitch. KhoHo pari package. www.geometrie.ch/KhoHo/, 2003.

PRINCETON UNIVERSITY DEPT. OF MATHEMATICS, PRINCETON, NJ 08544
E-mail address: jrasmus@math.princeton.edu