Discrete random variables

Definition 1.1 Let X, Y be two discrete RVs on Ω. We call the function $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ **joint probability mass function**, short **joint PMF**, of X, Y, if it holds

$$\begin{aligned} p(x, y) &:= P(X = x, Y = y) \\ &= P(\{\omega \in \Omega | X(\omega) = x\} \cap \{\omega \in \Omega | Y(\omega) = y\}) \end{aligned}$$

Example 1.1 (Three coin tosses) We continue our coin toss examples. This time we have a chance experiment with three independent coin tosses, leading to the sample space $\Omega = \{HHH, HHT, HTT, \ldots\}$. We introduce two random variables

$$X : \Omega \to \mathbb{R}, \quad Y : \Omega \to \mathbb{R},$$

where the first random variable gives the number of "heads" in the first two tosses and the second RV gives the number of "heads" in the last two tosses.[2)]

We would like to compute their joint PMF:

$$\begin{aligned} p(0, 0) &= P(X = 0, Y = 0) = P(\{\omega \in \Omega | X(\omega) = 0\} \cap \{\omega \in \Omega | Y(\omega) = 0\}) \\ &= P(\{TTH, TTT\} \cap \{HTT, TTT\}) = P(\{TTT\}) = \frac{1}{8} \\ &\ldots \end{aligned}$$

Definition 1.3 Let X, Y be two discrete RVs on a common sample space Ω with joint PMF p. The **marginal PMF of** X is given by

$$p_X(x) = \sum_{y \in R_Y} p(x, y),$$

and similarly the **marginal PMF of** Y is given by

$$p_Y(y) = \sum_{x \in R_X} p(x, y).$$

Theorem 1.1 Let (X, Y) be discrete RVs with joint PMF p, and let further $g : R_X \times R_Y \to \mathbb{R}$ be given. We can compute the expectation value / expectation / mean of $g(X, Y)$, if it exists, by

$$\mathrm{E}\left(g(X, Y)\right) = \sum_x \sum_y g(x, y) p(x, y).$$

Existence is guaranteed, if $\sum_x \sum_y |g(x, y)| p(x, y) < \infty$.

Definition 1.4 Let (X, Y) be discrete RVs with joint PMF p. We define the **conditional distribution of** X **given** $Y = y$ by

$$p(x|y) = P(X = x | Y = y) = \frac{p(x, y)}{p_Y(y)}.$$

Furthermore, we define the **conditional expectation of** X **given** $Y = y$ by

$$\mathrm{E}\left(X|y\right) = \mathrm{E}\left(X|Y = y\right) = \sum_x x\, p(x|y) = \frac{\sum_x x\, p(x, y)}{p_Y(y)} = \frac{\sum_x x\, p(x, y)}{\sum_x p(x, y)}.$$

1 Multivariate discrete random variables

Let $X : \Omega \to \mathbb{Z}$ and $Y : \Omega \to \mathbb{Z}$ be discrete random variables with joint PMF

$$p(x, y) = \begin{cases} cxy & \text{if } 1 \leq x \leq y \leq 3 \\ 0 & \text{else} \end{cases}$$

1. Find the normalizing constant c.
2. Are X, Y independent? Prove your claim.
3. Find the expectations of X,Y , XY .

1) $\sum_{x,y} p(x,y) \overset{!}{=} 1 \Rightarrow c = ?$ $\{1,2,3\} \times \{1,2,3\}$

$p(1,1) + p(1,2) + p(1,3) + p(2,2) + p(2,3) + p(3,3)$

$= c + 2c + 3c + 4c + 6c + 9c$

$= 25 c \overset{!}{=} 1 \Rightarrow c = \frac{1}{25}$

2) What to show: X, Y are dependent

independent: $p_X(x) p_Y(y) = p(x,y) \ \forall x,y$

$p_X(x) = \sum_y p(x,y)$

$p_X(1) = \sum_y p(1,y) = p(1,1) + p(1,2) + p(1,3)$

$= \frac{1}{25} + \frac{2}{25} + \frac{3}{25} = \frac{6}{25}$

$p_Y(1) = \sum_x p(x,1) = p(1,1) = \frac{1}{25}$

$p_X(1) \cdot p_Y(1) = \frac{6}{25} \cdot \frac{1}{25} \neq \frac{1}{25} = p(1,1)$ $\Rightarrow$ dependent

3) $E(X) = \sum_{x,y} x \, p(x,y)$

$= 1 \cdot p(1,1) + 1 \cdot p(1,2) + 1 \cdot p(1,3) + 2 \cdot p(2,2) + 2 \cdot p(2,3)$
$+ 3 \cdot p(3,3)$

$= 53/25$

$E(Y) = \sum_{x,y} y \cdot p(x,y) = \underline{\quad\quad} = 70/25$

$1 \cdot p(1,1) + 2 \cdot p(1,2) + 3 \, p(1,3)$
$+ 2 \cdot p(2,2) + 3 \, p(2,3) + 3 \cdot p(3,3)$

$= 1 \cdot \frac{1}{25} + 2 \cdot \frac{2}{25} + 3 \cdot \frac{3}{25} + 2 \cdot \frac{4}{25}$
$+ 3 \cdot \frac{6}{25} + 3 \cdot \frac{9}{25}$

$E(XY) = \sum_{x,y} x \cdot y \, p(x,y) = \underline{\quad\quad} = \frac{\cancel{14}\,7}{25}$

$1 \cdot 1 \cdot p(1,1) + 1 \cdot 2 \cdot p(1,2) + 1 \cdot 3 \cdot p(1,3)$
$+ 2 \cdot 2 \cdot p(2,2) + 2 \cdot 3 \cdot p(2,3) + 3 \cdot 3 \cdot p(3,3)$

$=$

Continuous random variables

Definition 1.11 Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be an n-dimensional random vector of continuous RVs $X_i : \Omega \to \mathbb{R}$. We call the function $\rho : \mathbb{R}^n \to \mathbb{R}$ **joint density**, short **density**, of X, if it holds

$$\begin{aligned} &P(a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n) \\ &= \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \rho(x_1, \dots, x_n) dx_1 \cdots dx_n \end{aligned}$$

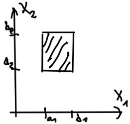
Theorem 1.3 Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be an n-dimensional random vector of continuous RVs with joint density ρ. We are further given a function $g : \mathbb{R}^n \to \mathbb{R}$. We can compute the **expectation value** / **expectation** / **mean** of $g(X_1, \dots, X_n)$, if it exists, by

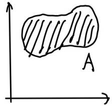
$$\mathrm{E}\left(g(X_1, \dots, X_n)\right) = \int_{\mathbb{R}^n} g(x_1, \dots, x_n) \rho(x_1, \dots, x_n) dx_1 \cdots dx_n \,.$$

Existence is guaranteed, if $\int_{\mathbb{R}^n} |g(x_1, \dots, x_n)| p(x_1, \dots, x_n) dx_1 \cdots dx_n < \infty$.

Theorem 1.4 (Bayes) Let (X, Y) be continuous RVs with joint density ρ. For all $x, y \in \mathbb{R}$, such that $\rho_X(x), \rho_Y(y) > 0$, it holds

$$\rho(y|x) = \frac{\rho(x|y)\rho_Y(y)}{\rho_X(x)} \,.$$





Definition 1.13 Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be an n-dimensional random vector of continuous RVs with joint density ρ. We are further given $1 \leq p < n$. The **marginal joint density** of $(X_1, \dots, X_p)$ is defined by

$$\rho_{1,2,\dots,p}(x_1, \dots, x_p) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho(x_1, \dots, x_n) dx_{p+1} \cdots dx_n \,.$$

Definition 1.14 Let (X, Y) be continuous RVs with joint density ρ. We define the **conditional density of X given $Y = y$** by

$$\rho(x|y) = \rho(x|Y = y) = \frac{\rho(x, y)}{\rho_Y(y)} \,,$$

for all y such that $\rho_Y(y) > 0$. Furthermore, we define the **conditional expectation of X given $Y = y$** by

$$\mathrm{E}\left(X|y\right) = \mathrm{E}\left(X|Y = y\right) = \int_{-\infty}^{\infty} x\, \rho(x|y) dx = \frac{\int_{-\infty}^{\infty} x\, \rho(x, y) dx}{\int_{-\infty}^{\infty} \rho(x, y) dx} \,,$$

for all y such that $\rho_Y(y) > 0$.

2 Multivariate continuous random variable

Let (X, Y) be two continuous RVs with joint density

$$\rho(x,y) = \begin{cases} cxy & \text{if } x, y \in [0,1] \\ 0 & \text{else} \end{cases}.$$

1. Find the normalizing constant c.
2. Find the marginal densities and expectations of X, Y.
3. Find the conditional expectation of X given $Y = y$.

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,y)\,dx\,dy \overset{!}{=} 1$

$\int_0^1\int_0^1 \rho(x,y)\,dx\,dy = \int_0^1\int_0^1 cxy\,dx\,dy = c\int_0^1\left(\int_0^1 xy\,dx\right)dy$

$= c\int_0^1 y\int_0^1 x\,dx\,dy = c\int_0^1 y\left[\frac{1}{2}x^2\right]_0^1 dy$

$= c\int_0^1 \frac{1}{2}y\,dy = c\,\frac{1}{2}\left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{4}c \overset{!}{=} 1$

$\Rightarrow c = 4$

2) $\rho_X(x) = \int_{-\infty}^{\infty} \rho(x,y)\,dy = \int_0^1 4\cdot xy\,dy = \left[4x\frac{1}{2}y^2\right]_0^1 = 2x$

$\rho_Y(y) = \int_{-\infty}^{\infty} \rho(x,y)\,dx = \int_0^1 4xy\,dx = \left[4y\frac{1}{2}x^2\right]_0^1 = 2y$

$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,\rho(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot 4x\,dx\,y\,dy = \int_0^1\int_0^1 4x^2\,dx\,y\,dy$

$\int_0^1 \left[4\frac{1}{3}x^3\right]_0^1 y\,dy = \int_0^1 \frac{4}{3}y\,dy = \left[\frac{4}{3}\cdot\frac{1}{2}y^2\right]_0^1 = \frac{2}{3}$

$E[Y] = \frac{2}{3}$

3) $E[X \mid Y = y] = \int_{-\infty}^{\infty} x\,\rho(x|y)\,dx \int_{-\infty}^{\infty} x\,\frac{\rho(x,y)}{\rho_Y(y)}\,dx = \_\_ = \frac{2}{3}$

$E[Y \mid X = x]$ $\qquad \rho(x,y)$