Homework 2

Rotation Matrices

1) 2D rotation: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Commutative: $R_1 R_2 = R_2 R_1 \Rightarrow$

$$\Rightarrow R_1 R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$R_2 R_1 = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$\Rightarrow R_1 R_2 = R_2 R_1$

3D rotation: $R_z = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$R_{z1} R_{z2} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta & 0 & 0 \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{z2} R_{z1} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\beta\sin\alpha - \sin\beta\cos\alpha & 0 & 0 \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\alpha\sin\beta + \cos\alpha\cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow R_{z1} R_{z2} \neq R_{z2} R_{z1}$$

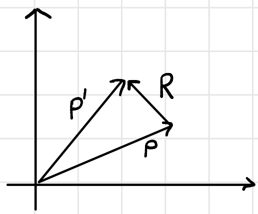
2) $P = \begin{bmatrix} \rho \\ \theta \end{bmatrix}$ $\quad P_o = \begin{bmatrix} \rho \\ \theta \\ 1 \end{bmatrix}$ $\quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{bmatrix}$

$$M P_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ \theta \\ 1 \end{bmatrix} = \begin{bmatrix} \rho \\ \theta + \phi \\ 1 \end{bmatrix}$$

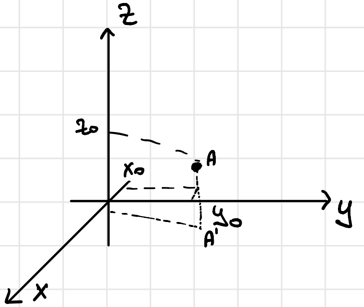
in polar coordinates: $\quad x = \rho\cos(\theta + \phi)$

$$y = \rho\sin(\theta + \phi)$$

3)



4)



$A = (x_0, y_0, z_0) \quad A' = (x_0, y_0, -z_0)$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ -z_0 \end{bmatrix}$$

5)
$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(-30^\circ) = \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6) $A = \begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}$

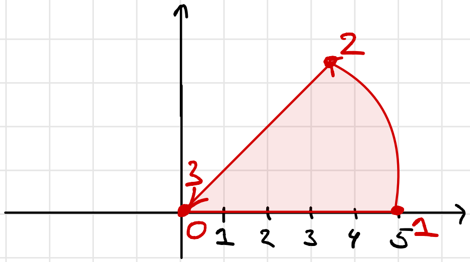
$\det(A) = 3\cdot(-3) - 2(-1) = -9+2 = -7$

$tr(A) = 3 + (-3) = 0$

A square matrix is a rotation matrix if and only if $R^T = R^{-1}$ and $\det(R) = 1$.

However, $\det(A) \neq 1$

Robot Motion



$$P_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & -2.5\sqrt{2} \\ 0 & 1 & -2.5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \qquad P_f = T_2(R(T_1 P_0))$$

$$T_1 P_0 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$R(T_1 P_0) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 1 \end{bmatrix}$$

$$T_2(R(T_1 P_0)) = \begin{bmatrix} 1 & 0 & -2.5\sqrt{2} \\ 0 & 1 & -2.5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$