

CH-231-A

Algorithms and Data Structures

ADS

Lecture 11

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Fibonacci Numbers (1)

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

n 0 1 2 3 4 5 ...

Output: return the n -th Fibonacci number

Fibonacci Numbers (2)

Naive algorithm:

Implement the recursion as in the definition.

Fibonacci(n)

1 **if** (n < 2)

2 **return** n

3 **else**

4 **return** Fibonacci (n - 1) + Fibonacci (n - 2)

$$T(n) = T(n - 1) + T(n - 2) + \Theta(1)$$

Fibonacci Numbers (3)

$$T(n) = T(n-1) + T(n-2) + \Theta(1)$$

Lower bound:

$$T(n-1) \approx T(n-2)$$

$$T(n) = 2T(n-2) + \Theta(1)$$

$$T(n) = 4T(n-4) + \Theta(1)$$

$$T(n) = 8T(n-6) + \Theta(1)$$

...

$$T(n) = 2^k T(n-2k) + \Theta(1) \implies n-2k = 0 \implies k = n/2$$

$$T(n) = 2^{n/2} T(0) + \Theta(1)$$

$$\implies T(n) = \Omega(2^{n/2}), \text{ i.e., exponential time}$$

Fibonacci Numbers (4)

Side note: A tighter lower bound is the following

$$T(n) = \Omega(\Phi^n),$$

because #leaves in rec. tree = $\text{Fibonacci}(n) \times \Theta(1)$
where Φ is the golden ratio

$$\phi = (1 + \sqrt{5})/2$$

The closed form for $\text{Fibonacci}(n)$ explains the time complexity from above.

Fibonacci Numbers (5)

Bottom up approach:

Avoid recursion, i.e.,
compute $F_0, F_1, F_2, \dots, F_n$ in the given order instead, forming each number by summing the two previous.

$$T(n) = \Theta(n).$$

Fibonacci Numbers (6)

Closed form (rounded to next integer):

$$F_n = \Phi^n / \sqrt{5} \text{ where } \Phi = (1 + \sqrt{5})/2$$

(proof by induction).

Compute by "Power of a number" recursion.

$$T(n) = \Theta(\lg n)$$

But: numerical problems may occur (floating-point arithmetic).

Fibonacci Numbers (7)

Matrix representation:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

(proof by induction).

Compute by "Power of a number" recursion
(using a generalization to 2x2 matrices).

$$T(n) = \Theta(\lg n)$$

And: uses integers only (no floating-point errors).

Maximum-Subarray Problem (1)

- ▶ Motivation scenario: buy & sell stock
- ▶ **Input:** a sequence of numbers
- ▶ **Output:** subsequence that results in the highest profit



Maximum-Subarray Problem (2)

Brute-Force algorithm:

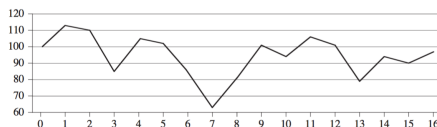
- ▶ Try every pair of days.

Number of pairs:

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Maximum-Subarray Problem (3)

- ▶ Transformation:
 - ▶ Consider daily change in price instead



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	-3	-25	20	-3	-16	-23	18	20	-7	12	-5	-22	15	-4	7

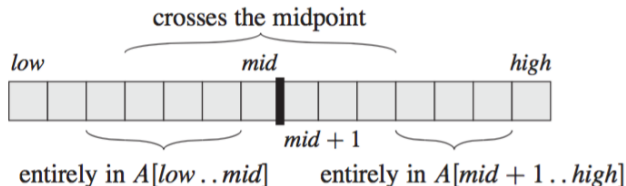
- ▶ Maximum-subarray problem:
 - ▶ **Input:** a sequence of numbers $\langle a_1, a_2, \dots, a_n \rangle$ of positive and negative numbers (otherwise it does not make sense)
 - ▶ **Output:** contiguous, non-empty subsequence $\langle a_i, a_{i+1}, \dots, a_j \rangle$ with $i \geq 1, j \leq n$,

such that $\sum_{k=i}^j a_k$ is maximized

Maximum-Subarray Problem (4)

Divide & Conquer:

- ▶ Idea: Divide A into 2 pieces \implies maximum subarray is either
 - ▶ entirely in the subarray $A[\text{low} \dots \text{mid}]$, or
 - ▶ entirely in the subarray $A[\text{mid} + 1 \dots \text{high}]$, or
 - ▶ crossing the midpoint



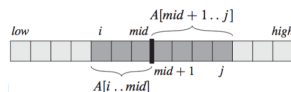
Maximum-Subarray Problem (5)

FIND-MAX-CROSSING-SUBARRAY($A, low, mid, high$)

```

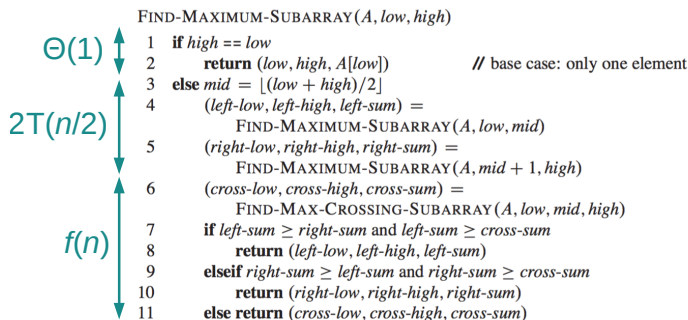
1   $left-sum = -\infty$ 
2   $sum = 0$ 
3  for  $i = mid$  downto  $low$ 
4       $sum = sum + A[i]$ 
5      if  $sum > left-sum$ 
6           $left-sum = sum$ 
7           $max-left = i$ 
8   $right-sum = -\infty$ 
9   $sum = 0$ 
10 for  $j = mid + 1$  to  $high$ 
11      $sum = sum + A[j]$ 
12     if  $sum > right-sum$ 
13          $right-sum = sum$ 
14          $max-right = j$ 
15 return ( $max-left, max-right, left-sum + right-sum$ )

```



Time complexity: $\Theta(n)$

Maximum-Subarray Problem (6)



► Recurrence:

$$T(n) = 2T(n/2) + \Theta(n)$$

► Solution:

$$a = 2, b = 2, n^{\log_b a} = n, f(n) = \Theta(n) \implies \text{Case 2}$$

$$\text{Thus, } T(n) = \Theta(n \lg n)$$

Matrix Multiplication

Problem: Input A, B , Output C

$$\left. \begin{array}{l} A = [a_{ij}], B = [b_{ij}]. \\ C = [c_{ij}] = A \cdot B. \end{array} \right\} i, j = 1, 2, \dots, n.$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$

Matrix Multiplication: Standard Algorithm

Standard algorithm:

```
1  for i = 1 to n do
2      for j = 1 to n do
3          c[i][j] = 0
4          for k = 1 to n do
5              c[i][j] = c[i][j] + a[i][k] * b[k][j]
```

Time complexity: $T(n) = \Theta(n^3)$

Matrix Multiplication: Divide & Conquer (1)

Divide & Conquer:

Idea: $n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

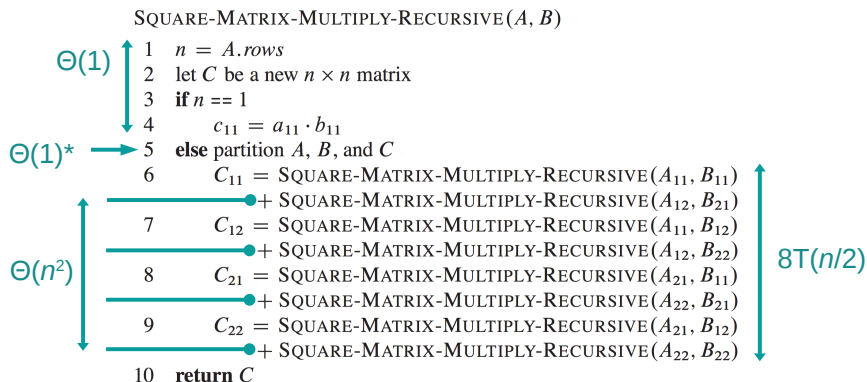
$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

Combining subproblem solutions:

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array} \right\} \begin{array}{l} 8 \text{ mults of } (n/2) \times (n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} \end{array}$$

Matrix Multiplication: Divide & Conquer (2)



* with index calculations,
otherwise $\Theta(n^2)$ for copying entries

Matrix Multiplication: Divide & Conquer (3)

Recurrence:

$$T(n) = 8T(n/2) + \Theta(n^2)$$

#subproblems subproblem size work dividing and combining

$$n^{\log_b a} = n^{\log_2 8} = n^3, f(n) = \Theta(n^2)$$

meaning that $f(n) = \Theta(n^{3-\epsilon})$, where $\epsilon = 1$

Case 1: $T(n) = \Theta(n^3)$.

1

Not better than the standard algorithm.

Matrix Multiplication: Conclusions

- ▶ Lessons learned:
 - ▶ #additions goes away
(constant factor)
 - ▶ #multiplications not \Rightarrow recursive case
(they make the tree "bushy")
- ▶ What to do?
 - ▶ Try to reduce #multiplications
 - ▶ It is ok to have more additions