Solutions - Homework 3

①

(a) $\lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta^2} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} = \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)^2 = 1^2 = 1$

(b) $\frac{3x^2 - 12x + 9}{x^2 - 5}$

To compute horizontal asymptotes, divide the whole equation by the greatest power of x in the denominator and take limits of x to ∞

$$\lim_{x \to \infty} \frac{\cancel{x^2}3 - \frac{12}{x} + \frac{9}{x^2}}{1\cancel{x^2} - \frac{5}{x^2}} = \frac{3}{1} = 3$$

$\therefore$ $y = 3$ is a horizontal asymptote

To compute vertical asymptotes, check for which values of x the dominator is zero.

$x^2 - 5 = 0 \quad \therefore \quad x = \pm\sqrt{5}$

$\therefore$ $x = \sqrt{5}$ and $x = -\sqrt{5}$ are vertical asymptotes.

② $\lim_{x \to 0} x^n . \cos\left(\frac{1}{x^n}\right)$: since $|\cos(\phi)| \leq 1$, if we let $\phi = \frac{1}{x^n}$

$$\left|x^n . \cos\left(\frac{1}{x^n}\right)\right| \leq |x^n|$$

By squeeze law $\lim_{x \to 0} -|x^n| \leq \lim_{x \to 0} x^n . \cos\left(\frac{1}{x^n}\right) \leq \lim_{x \to 0} |x^n|$

$$\Rightarrow 0 \leq \lim_{x \to 0} x^n . \cos\left(\frac{1}{x^n}\right) \leq 0$$

$$\Rightarrow \lim_{x \to 0} x^n . \cos\left(\frac{1}{x^n}\right) = 0 \quad //$$

(b) $\lim_{x \to 0} x^2 . e^{\sin(\frac{1}{x})}$

$$\Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad \Rightarrow e^{-1} \leq e^{\sin(\frac{1}{x})} \leq e^{1}$$

$$\Rightarrow x^2 . e^{-1} \leq x^2 . e^{\sin(\frac{1}{x})} \leq x^2 . e$$

$$\Rightarrow \lim_{x \to 0} . x^2 . e^{-1} \leq \lim_{x \to 0} x^2 . e^{\sin(\frac{1}{x})} \leq \lim_{x \to 0} x^2 . e$$

$$\Rightarrow 0 \leq \lim_{x \to 0} x^2 . e^{\sin(\frac{1}{x})} \leq 0$$

$$\Rightarrow \lim_{x \to 0} x^2 . e^{\sin(\frac{1}{x})} = 0 \quad //$$

3 The solutions to the equation are equal to the roots of $f(x) = e^x + x + 2 - \cos(x)$

By inspection : $f(-10) = e^{-10} - 10 + 2 - \cos(-10)$

$$< 1 - 10 + 2 + 1 = -6 < 0$$

$\therefore f(-10) < 0$

and $f(10) = e^{10} + 10 + 2 - \cos(10) > 0$

$\therefore f(10) > 0$

Therefore, since the function is continuous, by the Intermediate Value Theorem, we know that there exists an x_0 in $(-10, 10)$ such that $f(x_0) = 0$

$\Rightarrow$ which is a solution to the equation

Bonus:

Can be shown by analysing the functions $f_1(x) = e^x$ and $f_2(x) = -x - 2$ and their intersection.

For a more rigorous way, derivatives can be considered. The derivative is strictly positive, so the function is monotonically increasing (strictly). e^x + 1 is strictly positive everywhere.