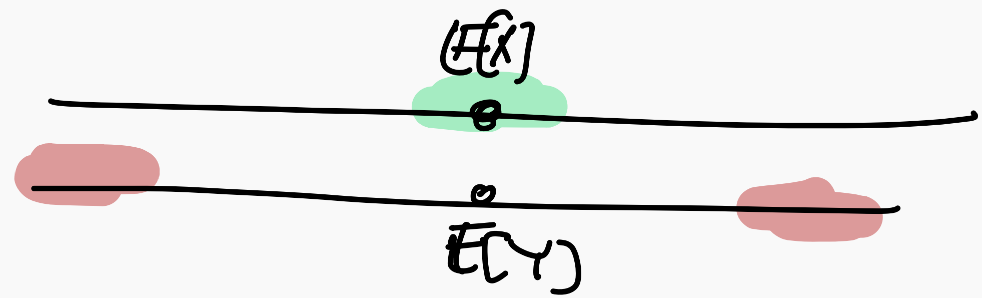
Definition
The variance of a random variable X is defined by

$$\mathrm{Var}\,[X] = \mathbb{E}\left[(X - \mathbb{E}\,[X])^2\right].$$

$$\mathbb{E}[X] = \sum p_i x_i$$

$$\mathbb{E}\left[X - \mathbb{E}[X]\right] = 0.$$



Alternative formula for the variance

Theorem

The variance of a random variable X is given by

$$\mathrm{Var}\left[X\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 .$$

Set $\mu = \mathbb{E}[X]$

definition $\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$

$\mathbb{E}[\mu^2]$

$= \mathbb{E}[X^2] - 2\mu\,\mathbb{E}[X] + \mu^2\,\mathbb{E}[1]$ $\quad \mathbb{E}[1] = 1 . 1 = 1$

c

$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 . 1$

$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 = \mathbb{E}[X^2] - \mu^2$

C constant $\mathbb{E}[C] = C$

$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 .$

Properties of the variance

Theorem
Let X, Y be random variables and c a constant. We have

1. $\mathrm{Var}[cX] = c^2\mathrm{Var}[X]$,
2. $\mathrm{Var}[X] \geq 0$, *with equality when $X = c$ for a constant c.*
3. $\mathrm{Var}[X+a] = \mathrm{Var}[X]$, translation invariance.

$$\mathrm{Var}[cX] = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2$$
$$= \mathbb{E}[c^2X^2] - c^2\mathbb{E}[X]^2$$
$$= c^2\mathbb{E}[X^2] - c^2\mathbb{E}[X]^2 = c^2(\mathbb{E}[X^2] - \mathbb{E}(X)^2)$$
$$= c^2\,\mathrm{Var}[X].$$

$\mu = \mathbb{E}[X]$

$$\mathrm{Var}[X] = \mathbb{E}[\underbrace{(X-\mu)^2}_{\geq 0}] \geq 0.$$

$$\boxed{\mathbb{E}[X^2] \geq \mathbb{E}[X]^2}$$

$$\mathbb{E}[X+a] = \mathbb{E}[X] + \mathbb{E}[a] = a + \mathbb{E}[X]$$

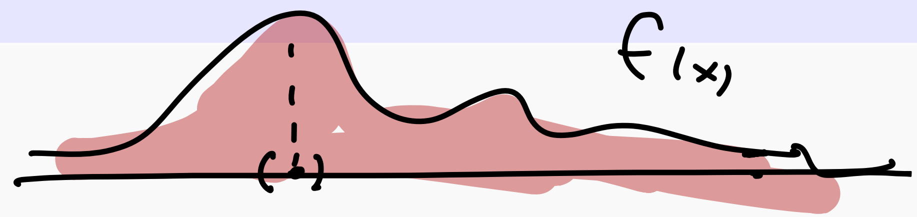
$$\mathrm{Var}[X+a] = \mathbb{E}\Big((X+\cancel{a}) - \underbrace{\mathbb{E}(X+a)}_{\mathbb{E}[X]+\cancel{a}}\Big)^2 = \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathrm{Var}[X].$$

Expected value of continuous random variable

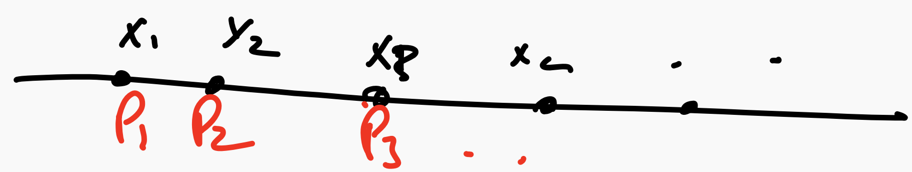
Definition
Let X be a continuous random variable with the probability density function $f(x)$. The expected value of X is defined by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

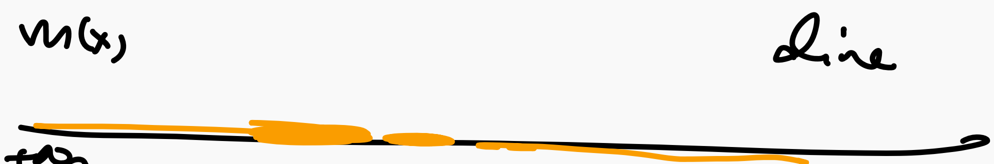
why is this a good definition?



$$\mathbb{E}[X] = \sum x_i \cdot p_i$$



<u>physics</u>



$$\int_{-\infty}^{+\infty} x \cdot m(x) dx = \underline{\text{Center of mass}}$$

mass density

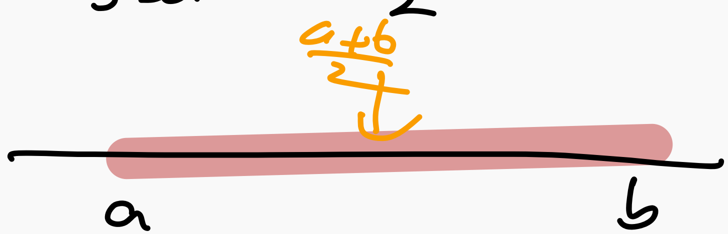
Example

Let X have uniform distribution over the interval $[a, b]$. Compute $\mathbb{E}[X]$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{b+a}{2}$$

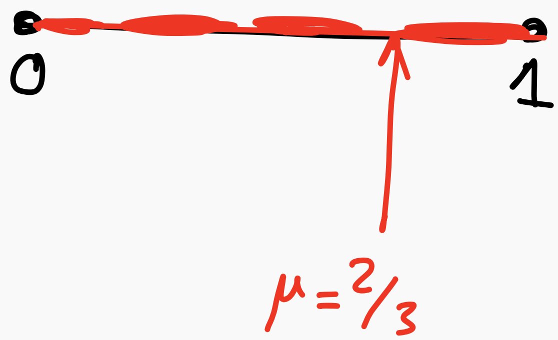


Example

Suppose X is a continuous with the density function

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } \textit{otherwise} \end{cases}$$

Find $\mathbb{E}[X]$.



$$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_0^1 2x^2\, dx$$

$$= \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

Example
Let X have the Cauchy distribution given by

$$f_X(t) = \frac{1}{\pi(1+x^2)}.$$

Find $\mathbb{E}[X]$.

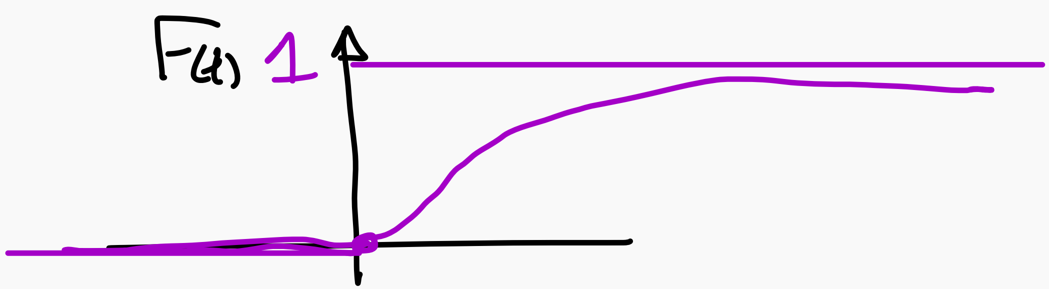
Theorem

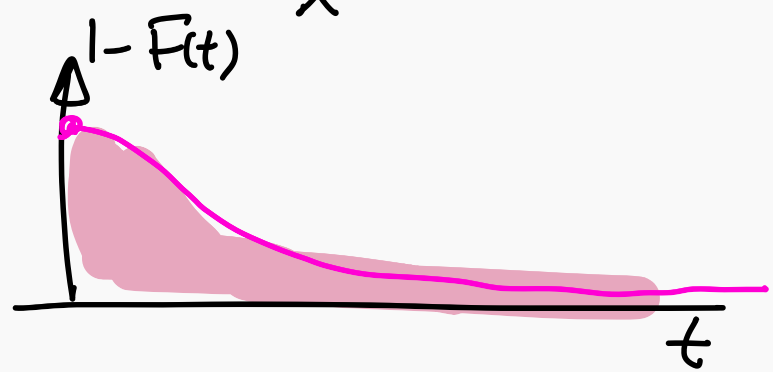
Let X be a non-negative continuous random variable with the distribution function $F_X(t)$. Then

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X \geq t] \; dt = \int_0^{\infty} (1 - F_X(t)) \; dt.$$

X continuous

$F_X(t) = \mathbb{P}[X \leq t]$ $\quad$ $\mathbb{P}[X \geq t] = 1 - F_X(t)$





Example

Find the expected value of an exponential random variable.

$$X \qquad f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases} \qquad F(t) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$\mathbb{E}[X] = \int_0^\infty \lambda \underbrace{x}_{u} \, \underbrace{e^{-\lambda x}}_{dv} \, dx$$

integration by parts.

$$\mathbb{E}[X] = \int_0^\infty (1 - F(t))\, dt = \int_0^\infty \left(1 - (1 - e^{-\lambda t})\right) dt$$

$$= \int_0^\infty e^{-\lambda t}\, dt = \left.\frac{e^{-\lambda t}}{-\lambda}\right|_0^\infty = \frac{1}{\lambda}.$$

Theorem

Let X be a continuous random variable with the density function $f(x)$. For any continuous function $h : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}\left[h(X)\right] = \int_{-\infty}^{\infty} h(x)f(x)\ dx.$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

X random variable with density $f(x)$

$$X^2 = h(X) \quad h(x) = x^2$$

$$\sin X = g(X) \quad g(x) = \sin x$$

Example

Let X have the uniform distribution over the interval $[0, \pi]$. Compute $\mathbb{E}[\sin X]$.

$$\mathbb{E}[\sin X] = \int_0^{\pi} \sin x \cdot \frac{1}{\pi}\, dx = \frac{1}{\pi}\left(-\cos x \Big|_0^{\pi}\right.$$

$$f(x) = \begin{cases} \frac{1}{\pi} & 0 \leq x \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

$$= \frac{1}{\pi}(1+1) = \frac{2}{\pi}.$$

The expected value of log-normal distributions

Let $Y = e^X$, where X has the $N(0,1)$ distribution. Find $\mathbb{E}[Y]$.

Example

Let X be a continuous random variable with the uniform distribution on $[0, 1]$. Find $\mathrm{Var}[X]$.

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathbb{E}[X] = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Expected value as the value of a game

Expected value as the value of a game

Consider a lottery with the possible outcomes

$$x_1, \ldots, x_n$$

attained with probabilities $p_1, \ldots, p_n$.

Ex.

	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
X	1	2	3	4	5	6
Payoff	1	4	9	16	25	-15

$$E[X] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{6}\cdot 16 + \frac{1}{6}\, 25 - \frac{1}{6}\cdot 15 > 0$$

$$= \frac{1}{6}(40) = \frac{40}{6} = \frac{20}{3} = 6.66.$$

Expected value as the value of a game

Consider a lottery with the possible outcomes

$$x_1, \ldots, x_n$$

attained with probabilities $p_1, \ldots, p_n$.

The expected value of the lottery is given by

$$E := p_1 x_1 + \cdots + p_n x_n.$$

Decision making using expected payoff

The idea of utility and its history

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The idea of utility goes back to Daniell Bernoulli (1738).

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Suppose a coin is thrown until a head occurs. If this happens at round n for the first time then the player gets 2^n dollars.

	1/2	1/4	1/8	1/16	1/32	...
Round	1	2	3	4	5	...
Payoff	2	4	8	16	32	...

$$\mathbb{E}[\text{Payoff}] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 16 \cdot \frac{1}{16} + \dots$$

$$= 1 + 1 + 1 + 1 + \dots = \infty.$$

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If head occurs in the first round, the player get 2 dollars.

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Suppose a coin is thrown until a head occurs. If this happens at round n for the first time then the player gets 2^n dollars.

If head occurs in the first round, the player get 2 dollars.

If tails occurs in the first round, and heads in the second round then the player get 4 dollars.

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How much are you willing to pay to play this game?

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The value of the game is

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots = \infty$$

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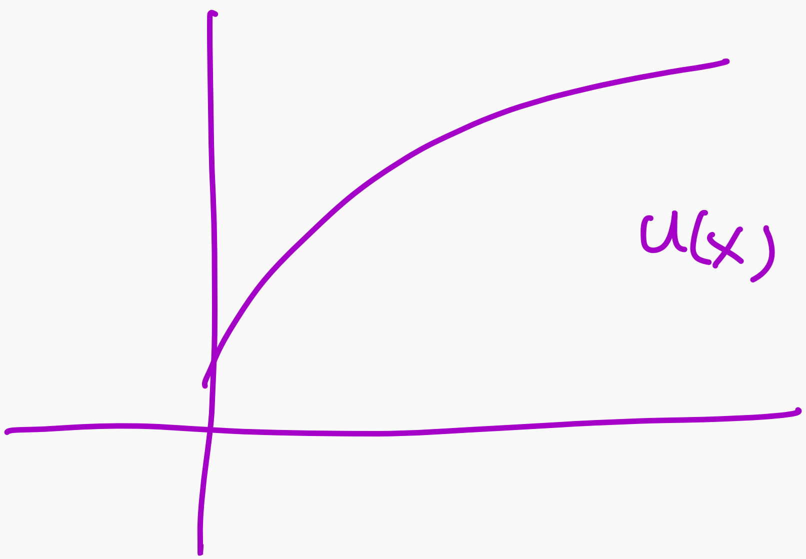
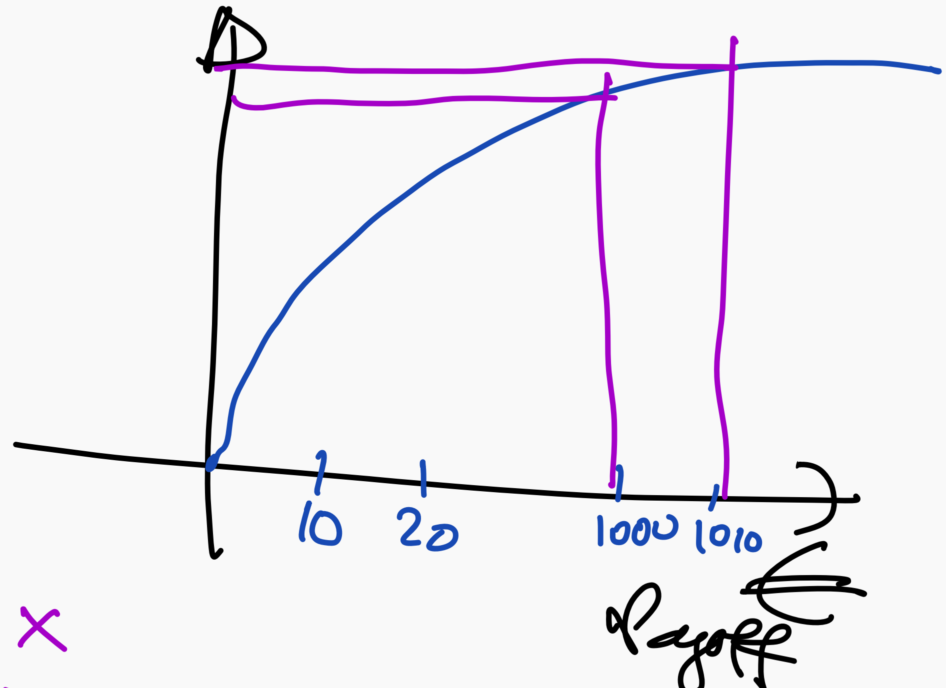
In reality people only pay around 3-4 dollars to play this game.

Bernoulli: it is not the absolute value of money but its utility

$\mathbb{E}[\text{Payoff}] \rightsquigarrow$

$\mathbb{E}[\underline{\underline{\text{utility}}}(\text{Payoff})]$





$u(x) = \log_{10} x$

$+1$

$10 \rightarrow 100$

$100 \rightarrow 1000$

Solution to the paradox

If the utility is

$$u(x) = 2\log_2 x$$

then:

1. With probability $1/2$, one gets $u(2) = \log_2 2 = 2$
2. With probability $1/4$, one gets $u(4) = \log_2 4 = 4$
3. With probability $1/8$, one gets $u(8) = \log_2 8 = 6$.

So the expected utility is

$$\frac{2}{2} + \frac{4}{4} + \frac{6}{8} + \frac{8}{16} + \cdots = 4.$$

	1/2	1/4	1/8	1/16	1/32	. .
Round	1	2	3	4	5	. .
Payoff	2	4	8	16	32	. .
	2	3	4	5	6	. .

$$\frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$$

$=$ finite sum

Utility function and attitude towards risk

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Suppose you are offered the choice between the following alternatives:

1. 1 dollars.
2. first throwing a fair coin. If the outcomes is heads 2 dollars and if the outcome is tails zero.

Utility function and attitude towards risk

Suppose you are offered the choice between the following alternatives:

1. 1 dollars.
2. first throwing a fair coin. If the outcomes is heads 2 dollars and if the outcome is tails zero.

1. 100000 dollars.
2. first throwing a fair coin. If the outcomes is heads 200000 dollars and if the outcome is tails zero.

Suppose $A_1, A_2, \dots, A_n$ denotes possible options in a choice decision making. A lottery is a a lottery is a scenario where each outcome will happen with a given probability, We write $L = (A_1, p_1; \dots, A_n; p_n)$.

For two lotteries L and M we write $M > L$ if M is preferable to L and $M \sim L$ if the decision maker is indifferent between M and L.

Combination of lotteries:

$$A_1 > A_2 > A_3 > \dots A_{n-1} > A_n$$

$$\begin{array}{ccccc} A_1, & A_5 & \text{vs.} & A_2 & A_3 \\ \underbrace{1/2} & 1/2 & > & 1/3 & 2/3 \\ 0.9 & 0.10 & < & & \end{array}$$

Axioms

1. For any two lotteries L and M we have $L < M$, $M < L$ or $M \sim L$.
2. If $L < M$ and $M < N$ then $L < N$.
3. If $L < M < N$ then there exists $p \in [0, 1]$ such that $pL + (1-p)N \sim M$.
4. For any N and $0 < p \leq 1$ we have $L < M$ iff $pL + (1-p)N \leq pM + (1-p)N$.

Von Neumann–Morgenstern utility theorem

For any preference satisfying axioms 1-4 above, there exists a function $u : \{A_1, \ldots, A_n\} \to \mathbb{R}$ such that $L < M$ iff $\mathbb{E}[u(L)] \leq \cancel{eu(M)}$.

$\mathbb{E}[u(M)]$