Exercise 2 - Kernel-based model

$T = \{(-2, 3), (0, -1), (3, 4)\}$

a) Gaussian Kernel with $\sigma = 1$

Define:

$$A = \begin{pmatrix} K(x_1, x_1) & K(x_1, x_2) & K(x_1, x_3) \\ K(x_2, x_1) & K(x_2, x_2) & K(x_2, x_3) \\ K(x_3, x_1) & K(x_3, x_2) & K(x_3, x_3) \end{pmatrix}, \quad y = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

where $K(x_i, x_j) = e^{-\frac{\|x_i - x_j\|_2^2}{\sigma^2}}$ with $\sigma = 1$

$$A = \begin{pmatrix} e^{-(-2+2)^2} & e^{-(-2-0)^2} & e^{-(-2-3)^2} \\ e^{-(0+2)^2} & e^{-(0-0)^2} & e^{-(0-3)^2} \\ e^{-(3+2)^2} & e^{-(3-0)^2} & e^{-(3-3)^2} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & e^{-4} & e^{-25} \\ e^{-4} & 1 & e^{-9} \\ e^{-25} & e^{-9} & 1 \end{pmatrix}$$

$A\hat{\alpha} = y \rightarrow$ Solve for $\hat{\alpha} = \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & e^{-4} & e^{-25} & 3 \\ e^{-4} & 1 & e^{-9} & -1 \\ e^{-25} & e^{-9} & 1 & 4 \end{array}\right) \Rightarrow \hat{\alpha} \approx \begin{pmatrix} 3.02 \\ -1.06 \\ 4 \end{pmatrix}$$

Predictor function:

$$f_{\hat{\alpha}}(x) = \hat{\alpha}_1 K(x, x_1) + \hat{\alpha}_2 K(x, x_2) + \hat{\alpha}_3 K(x, x_3)$$

$$f_{\hat{\alpha}}(x) = 3.02\, e^{-(x+2)^2} - 1.06\, e^{-x^2} + 4\, e^{-(x-3)^2}$$

b) Gaussian kernel with ridge regression $\lambda = 1$

$$A = \begin{pmatrix} 1 & e^{-4} & e^{-25} \\ e^{-4} & 1 & e^{-9} \\ e^{-25} & e^{-9} & 1 \end{pmatrix}$$ from the previous task

Solve:

$$(A + \lambda I)\hat{\beta} = y \quad \text{where} \quad \hat{\beta} = \begin{pmatrix} \hat{\beta_1} \\ \hat{\beta_2} \\ \hat{\beta_3} \end{pmatrix} \quad \text{and} \quad \lambda = 1$$

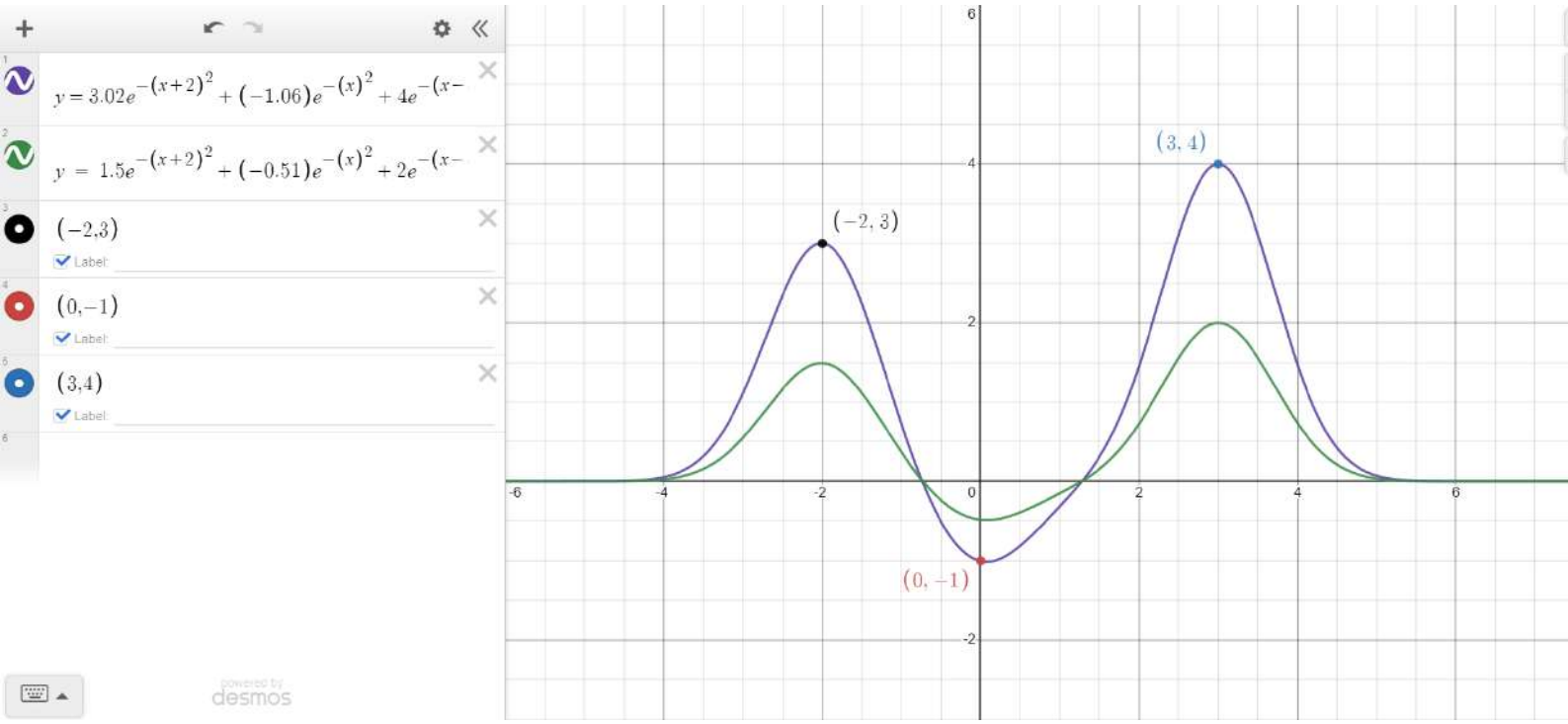
$$A + \lambda I = A + I = \begin{pmatrix} 2 & e^{-4} & e^{-25} \\ e^{-4} & 2 & e^{-9} \\ e^{-25} & e^{-9} & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & e^{-4} & e^{-25} & 3 \\ e^{-4} & 2 & e^{-9} & -1 \\ e^{-25} & e^{-9} & 2 & 4 \end{array}\right) \Rightarrow \hat{\beta} \approx \begin{pmatrix} 1.50 \\ -0.51 \\ 2 \end{pmatrix}$$

Predictor function:

$$f_{\hat{\beta}}(x) = \hat{\beta_1} k(x, x_1) + \hat{\beta_2} k(x, x_2) + \hat{\beta_3} k(x, x_3)$$

$$f_{\hat{\beta}}(x) = 1.5\, e^{-(x+2)^2} - 0.51\, e^{-x^2} + 2\, e^{-(x-3)^2}$$


$y = 3.02e^{-(x+2)^2} + (-1.06)e^{-(x)^2} + 4e^{-(x-}$
$y = 1.5e^{-(x+2)^2} + (-0.51)e^{-(x)^2} + 2e^{-(x-}$
$(-2,3)$
Label:
$(0,-1)$
Label:
$(3,4)$
Label:
powered by desmos
$(-2, 3)$
$(3, 4)$
$(0, -1)$
6
4
2
-2
-6
-4
-2
0
2
4
6