$\sqrt{2}\times\sqrt{2}\times\sqrt{2}$

$2\sqrt{2}$

Homework 5 Solutions

Problem 1

(a) $xy^2 = 3x + y \Rightarrow \dfrac{d}{dx}(xy^2) = \dfrac{d}{dx}(3x) + \dfrac{d}{dx}(y)$

$\Rightarrow y^2 + x \cdot \dfrac{d}{dx}(y^2) = 3 + \dfrac{dy}{dx} \Rightarrow y^2 + 2y \cdot \dfrac{dy}{dx} \cdot x = 3 + \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}(2xy - 1) = 3 - y^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{3 - y^2}{2xy - 1}$

at $(2,2)$: $\dfrac{dy}{dx} = -\dfrac{1}{7}$ and using $y - y_1 = m(x - x_1)$:

$y - 2 = -\dfrac{1}{7}(x - 2) \Rightarrow y - 2 = -\dfrac{1}{7}x + \dfrac{2}{7}$

$\therefore y = -\dfrac{1}{7}x + \dfrac{16}{7}$ or ~~$7y + x = 16$~~ //

(b) $y^{1/2} \cdot x^{3/2} + x \cdot y^{1/3} = 12$

$\therefore \dfrac{d}{dx}(y^{1/2}) \cdot x^{3/2} + \dfrac{d}{dx}(x^{3/2}) \cdot y^{1/2} + \dfrac{d}{dx}(x) \cdot y^{1/3} + \dfrac{d}{dx}(y^{1/3}) \cdot x = 0$

$\therefore \left(\dfrac{1}{2} y^{-1/2} \cdot x^{3/2} \cdot \dfrac{dy}{dx}\right) + \left(y^{1/2} \cdot \dfrac{3}{2} \cdot x^{1/2}\right) + \left(y^{1/3}\right) + \left(\dfrac{1}{3} \cdot y^{-2/3} \cdot x \cdot \dfrac{dy}{dx}\right) = 0$

$\therefore \dfrac{dy}{dx}\left(\dfrac{y^{-1/2} \cdot x^{3/2}}{2} + \dfrac{x \cdot y^{-2/3}}{3}\right) = -y^{1/3} - \left(\dfrac{3 \cdot x^{1/2} \cdot y^{1/2}}{2}\right)$

$\therefore \dfrac{dy}{dx} = \dfrac{-3\left(2y^{1/3} + 3y^{1/2} \cdot x^{1/2}\right)}{3y^{-1/2} \cdot x^{3/2} + 2xy^{-2/3}}$

at $(2,8)$: $\dfrac{dy}{dx} = -12$ | $\therefore y - 8 = -12(x - 2)$

$\therefore \quad y - 8 = -12x + 24 \quad \Rightarrow \quad y = -12x + 32$ //

or $\cancel{y + 12x = 32}$

(c) $\frac{d}{dy}(xy^2) = \frac{d}{dy}(3x + y) \Rightarrow 2xy + y^2 \cdot \frac{dx}{dy} = 3 \cdot \frac{dx}{dy} + 1$

$\Rightarrow \frac{dx}{dy}(y^2 - 3) = 1 - 2xy \Rightarrow \frac{dx}{dy} = \frac{1 - 2xy}{y^2 - 3}$

at $(2, 2)$; $\frac{dx}{dy} = \frac{1-8}{1} = -7$

but $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{-7} = -\frac{1}{7}$ $\therefore$, same tangent!

Here, using $x - x_1 = m_2(y - y_1) \Rightarrow x - 2 = -7(y - 2)$

$\therefore \quad x - 2 = -7y + 14 \Rightarrow x = -7y + 16$ //

Problem 2

(a) ~~Implementing~~ ~~derivatives~~ $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ where $\frac{dV}{dt} = 0.001\pi$

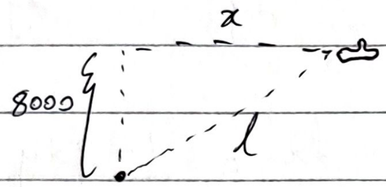
$V(r) = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV(r)}{dr} = 4\pi r^2 \Rightarrow \frac{dr}{dV} = \frac{1}{4\pi r^2}$

$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot 0.001\pi$

at $r = 0.2$ m,

$$\frac{dr}{dt} = \frac{1}{4\cancel{\pi}(0.2)^2} \cdot 0.001\cancel{\pi} = 6.25 \times 10^{-3} \text{ m per second}$$

(b)



since this value does NOT change as time passes

By pythagoras theorem, $l^2 = 8000^2 + x^2$

Taking derivatives w.r.t. time, $2l \cdot \frac{dl}{dt} = 0 + 2x \cdot \frac{dx}{dt}$

$$\therefore \frac{dl}{dt} = \frac{x}{l} \cdot \frac{dx}{dt}$$

$x = 500 \times 60 = 30\,000$ m, $l = \sqrt{8000^2 + 30000^2}$
$= 31048.35$ m

and $\frac{dx}{dt} = 500 \text{ ms}^{-1}$

$$\therefore \frac{dl}{dt} = \frac{30\,000}{31048.35} \cdot 500$$

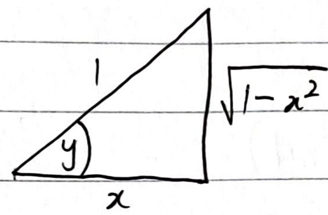
$$= 483 \text{ ms}^{-1}$$

Problem 3

(a) $\arccos(x) = y$ and $x = \cos y$

$\frac{dx}{dy} = -\sin y$ and $\therefore,\ \frac{dy}{dx} = -\frac{1}{\sin y}$

Since $\cos y = x$:



$\therefore \sin y = \sqrt{1-x^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ //

(b) $f(x) = 2x^3 - 6x + 9$

$f'(x) = 6x^2 - 6$

At critical points, $f'(x) = 0$ and $\therefore$, $6x^2 - 6 = 0$

$\therefore x^2 = 1$ and $x = \pm 1$.

Now, checking for maxima and minima, $f''(x) = 12x$

at $x = 1$, $f''(+1) = 12$. Since $f''(1)$ is (+)ve, this would yield a <u>minimum</u>.

at $x = -1$, $f''(-1) = -12$. Since $f''(-1)$ is (−)ve, this would yield a <u>maximum</u>.

(c) $g(x) = 2x^3 + 6x + 9$
$g'(x) = 6x^2 + 6$

at $g'(x) = 0$, $6x^2 + 6 = 0$ and $x^2 = -1$
$x = \pm j$

Since the roots are complex, we can establish here that there are <u>no</u> critical points for this function.

(d) $h(t) = \sin(\omega t)$
$h'(t) = \omega . \cos(\omega t)$

At $h'(t) = 0$, $\omega . \cos(\omega t) = 0$; $\cos(\omega t) = 0$

$\therefore \omega t = \frac{\pi}{2}(2n+1)$ and $t = \frac{\pi}{2\omega} . (2n+1)$ where $n = 0, \pm 1, \pm 2, \ldots$

$h''(t) = -\omega^2 . \sin(\omega t)$

$\therefore h''(t) = -\omega^2 . \sin\left(\omega . \frac{\pi}{2\omega} . (2n+1)\right) = -\omega^2 . \sin\left(\frac{\pi}{2}(2n+1)\right)$

Hence, for $n = 0$, we have a maximum point.

All odd n's yield minimas.
All even n's yield maximas

Hence, this function alternates between maximas and minimas.