Final 2020

(4) $$\underbrace{\left(1+\frac{dy}{dx}\right)}_{\text{inner derivative}} \cos(x+y) = 2y\frac{dy}{dx}\cos x + y^2(-\sin x)$$

$x=0,\ y=0$

$$\left(1+\frac{dy}{dx}\right)\underbrace{\cos 0}_{=1} = 0+0 \quad\Rightarrow\quad \frac{dy}{dx} = -1$$

$\Rightarrow$ The tangent line equation is $y-0 = \frac{dy}{dx}(x-0)$

for point $(0,0)$ $\Rightarrow$ $y = -1\cdot x = -x$

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(5)

a) $$\int_0^1 \ln x \, dx = \underbrace{x \ln x \Big|_0^1}_{=0-0=0} - \underbrace{\int_0^1 x \frac{1}{x} dx}_{=x|_0^1 = 1} = -1$$

use integration by parts with

$u' = 1 \Rightarrow u = x$
$v = \ln x \Rightarrow v' = \frac{1}{x}$

b) $$\int \frac{x^2+1}{x^2-1} dx$$

$$(x^2+1):(x^2-1) = 1 + \frac{2}{x^2+1}$$
$$\frac{-(x^2-1)}{2}$$

Partial fractions: $$\frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{x^2-1}$$

$0 \cdot x = Ax + Bx \quad \Rightarrow \quad A + B = 0$

$2 = -A + B \quad \Rightarrow \quad A = -1 \,,\, B = 1$

$$\Rightarrow \int \frac{x^2+1}{x^2-1}\,dx = \int \left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right) dx$$

$$= x - \ln|x+1| + \ln|x-1| + C$$

$$= x + \ln\left|\frac{x-1}{x+1}\right| + C$$

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⑥ $$\int_0^\infty \frac{\ln x + e^x}{1+x^2}\,dx$$

The integrand has a vertical asymptote at $x=0$. (because of $\ln x$)

So let's check $x \searrow$ and $x \to \infty$ independently:

$$I_1 = \int_0^1 \frac{\ln x + e^{-x}}{1+x^2}\,dx = \int_0^1 \frac{\ln x}{1+x^2}\,dx + \int_0^1 \frac{e^{-x}}{1+x^2}\,dx$$

cont. on $[0,1]$, so no issue here!

Now $$\int_0^1 \frac{\ln x}{1+x^2}\,dx > \int_0^1 \ln x\,dx = -1 \quad \text{by ⑤ a)}$$

As $\frac{\ln x}{1+x^2}$ does not change sign on $[0,1]$,

I_1 converges. (It's bound by 0 and -1)

$$I_2 = \int_e^\infty \frac{\ln x + e^{-x}}{1+x^2}\,dx \le \int_e^\infty \frac{\ln x + e^{-x}}{x^2}\,dx$$

$$\le \int_e^\infty \frac{\ln x + \ln x}{x^2}\,dx$$

since $\ln x \ge e^{-x}$ for $x \ge e$ (actually $\ln x > e^{-x}$ for $x \ge e$)

$$= 2\int_e^\infty \frac{\ln x}{x^2}\,dx \quad \left[\begin{array}{l} u = \ln x \\ u' = \frac{1}{x} \\ v' = \frac{1}{x^2} \\ v = -\frac{1}{x} \end{array}\right.$$

$$= 2\int_e^\infty \ln x \cdot \frac{1}{x^2}\,dx = 2\cdot\underbrace{\left(\underbrace{\left[-\frac{\ln x}{x}\right]_e^\infty}_{\substack{\text{converges}\\ \text{for } x\to\infty}} - \underbrace{\int_e^\infty -\frac{1}{x}\cdot\frac{1}{x}\,dx}_{\substack{\text{converges since}\\ \int_e^\infty \frac{1}{x^\alpha}dx \text{ converges}\\ \text{for } \alpha > 1}}\right)}_{\text{converges}}$$

As the integrand does not switch sign on $[e, \infty)$, I_2 also converges.

$$I_3 = \underbrace{\int_1^e \frac{\ln x + e^{-x}}{1+x^2}\,dx}_{\substack{\text{converges, as no discontinuities}\\ \text{and fixed bounds that are not discontinuities.}}}$$

$\Rightarrow \quad I_1 + I_2 + I_3$ converges

⑦

a) $$\int_{y_0=2}^{y(t)} \frac{dy}{y^3} = \int_{t_0=0}^{t} t^3\,dt \qquad \text{as} \quad y_0 = y(t_0) = y(0) = 2$$

$$\Rightarrow -\frac{1}{2} y^{-2} \Big|_2^{y(t)} = \frac{1}{4} t^4 \Big|_0^t$$

$$\Rightarrow \frac{1}{8} - \frac{1}{2}\frac{1}{y^2(t)} = \frac{1}{4} t^4 \quad \Rightarrow \quad \frac{1}{4} - \frac{1}{2} t^4 = \frac{1}{y^2(t)}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{\frac{1}{4} - \frac{1}{2} t^4}}$$

(choose pos. root to match initial condition

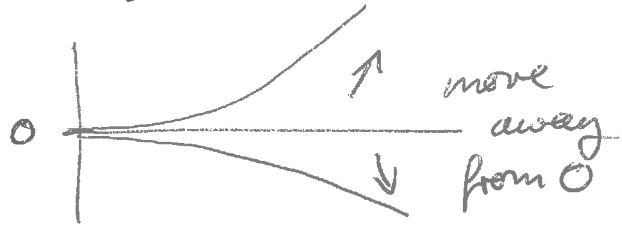
$$y(0) = \frac{1}{\sqrt{\frac{1}{4}}} = 2$$
)

b) For $t > 0$, $\frac{dy}{dt} = 0$ iff $y = 0$

And $\frac{dy}{dt} > 0$ for $y > 0$

$\frac{dy}{dt} < 0$ for $y < 0$

$\Bigg\}$ $y = 0$ equilibrium point and unstable



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⑧

$$\|u\|^2 \cdot \|v\|^2 = (u \circ v)^2 + \|u \times v\|^2$$

$u \circ v = \|u\|\,\|v\| \cos\Theta$

Θ angle between u and v

$\|u \times v\| = \|u\|\|v\| \sin\Theta$

$$\Rightarrow (u \circ v)^2 + \|u \times v\|^2 = \|u\|^2\|v\|^2 \overbrace{(\cos^2\Theta + \sin^2\Theta)}^{=1} = \|u\|^2\,\|v\|^2 \checkmark$$

(9)

$$\left(\begin{array}{cccc|c} 2 & 0 & 2 & 4 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 2 & -1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 3 & -3 \end{array}\right) \xrightarrow[\substack{R3 - R1 \to R3 \\ R4 - \frac{R1}{2} \to R4}]{\frac{R1}{2} \to R1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow[-R4 + R2 \to R4]{R3 + R2 \to R3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Check (not required!)

$$A \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 + 0 \\ 0 + -2 \\ -2 + 2 \\ -1 + 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ -3 \end{pmatrix} = b \quad ✓$$

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 - 2 \\ 0 - 0 \\ 2 - 2 \\ 1 - 1 \end{pmatrix} = 0 \quad ✓$$

$$A \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 + 0 - 4 \\ 0 + 1 - 1 \\ 4 - 1 - 3 \\ 2 + 1 - 3 \end{pmatrix} = 0 \quad ✓$$

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a) $$L(\lambda y+\mu x)=L\begin{pmatrix}\lambda y_1+\mu x_1\\ \vert\\ \lambda y_5+\mu x_5\end{pmatrix}=\begin{pmatrix}0\\ \lambda y_1+\mu x_1\\ \vert\\ \lambda y_4+\mu x_4\end{pmatrix}$$

$$=\lambda\begin{pmatrix}0\\ y_1\\ \vert\\ y_4\end{pmatrix}+\mu\begin{pmatrix}0\\ x_1\\ \vert\\ x_4\end{pmatrix}=\lambda Ly+\mu Lx \quad \checkmark$$

b) $$S=\begin{array}{c}\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}\\ \begin{array}{ccccc}\uparrow&\uparrow&\uparrow&\uparrow&\\ e_2&e_3&e_4&e_5&\end{array}\end{array}$$

$$e_2=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\leftarrow 1 \text{ at second position, } e_{\underline{\underline{2}}}$$

c) We immediately see that columns 1–4 have pivots,

so

$$\text{Range } S=\text{span}\{e_2,e_3,e_4,e_5\}$$

$$\text{Ker } S=\text{span}\{e_5\} \leftarrow$$

since pivot is missing in fifth column! where all entries are zero, and we have to replace the missing pivot with −1

constant included in <u>span</u>

$$\Rightarrow \begin{pmatrix}0\\0\\0\\0\\-1\end{pmatrix}=\underset{\downarrow}{-1}\underbrace{\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}}_{e_5}$$

d) Rank-nullity theorem for an $n \times m$ matrix S says that:

$$\text{rank } S + \text{nullity } S = m$$

or $$\dim \text{Range } S + \dim \text{Ker } S = m$$

Here: $$4 + 1 = 5$$ ✓