Robotics

PS01 Solution

Problem 1

Given the homogeneous matrix A with

$$A = \begin{pmatrix} 0.866 & -0.433 & -0.250 & 2 \\ 0 & -0.5 & 0.866 & -4 \\ -0.5 & -0.75 & -0.433 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What is the rotation matrix part R_A of A? Is it a right- or a left-handed rotation?

What is the inverse A^{-1} of A (use an as simple as possible computation)?

Problem 1

rotation part of A

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{matrix}} & \begin{matrix} 2 \\ -4 \\ 1 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{pmatrix} \qquad R_A = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}$$

Problem 1

handedness: $\det(R_A)$

$$R_A = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}$$

$$\left(\begin{array}{ccc|cc} 0.866 & -0.433 & -0.250 & 0.866 & -0.433 \\ 0 & -0.5 & 0.866 & 0 & -0.5 \\ -0.5 & -0.75 & -0.433 & -0.5 & -0.75 \end{array}\right)$$

rule of Sarrus

(0.866 * -0.5 * -0.433)+(-0.433 * 0.866 * -0.5)+(-0.25 * 0 * -0.75)
– (-0.25*-0.5*-0.5) – (0.866*0.866*-0.75) – (-0.433*0*-0.433)
= ~1

$\det(R_A) = 1$ => right handed

Problem 1

note

common abbreviation: sin() = s, cos() = c

$$R_Y = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_P = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}, R_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

Problem 1

note: right-handed rotation matrices

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_y = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}, R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

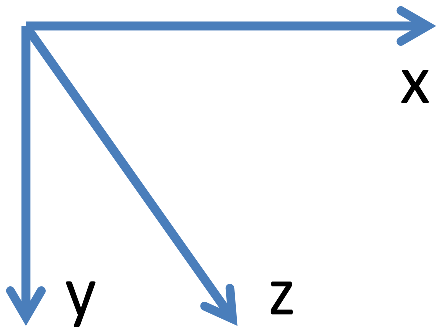
$$R_y = \left(\begin{array}{ccc|cc} c\beta & 0 & s\beta & c\beta & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -s\beta & 0 & c\beta & -s\beta & 0 \end{array}\right)$$

same for x & z

$$\det(R_y) = c^2\beta + s^2\beta = 1$$

Problem 1

note: Computer graphics and simulation often left-handed



- z points out of monitor
- "typically" reflection on y-axis

$$p^{LH} = A^{reflect} p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$

Problem 1

note: "left-handed" rotation matrices

$$R_y^{LH} = A_y^{reflect} R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & -1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}$$

$$R_y^{LH} : \left(\begin{array}{ccc|cc} c\beta & 0 & s\beta & c\beta & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -s\beta & 0 & c\beta & -s\beta & 0 \end{array}\right)$$

same for x & z

$$\det(R_y^{LH}) = -c^2\beta - s^2\beta = -(c^2\beta + s^2\beta) = -1$$

Problem 1

note: “left-handed” rotation matrices

$$R_z^{LH} = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ -s\lambda & -c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_z^{LH} = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & -1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}, R_z^{LH} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c\alpha & s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

note: typically always standard rotation matrices (det=1)

- in both right/left handed coordinate systems
- RH frame: positive angle = counter-clockwise
- LH frame: positive angle = clockwise

Problem 1

rotation part of A
$$R_A = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}$$

homogeneous matrix, respectively rotation matrix:
inverse = transpose

$$R_A^{-1} = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}^{-1} = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}^{T}$$
$$= \begin{pmatrix} 0.866 & 0 & -0.5 \\ -0.433 & -0.5 & -0.75 \\ -0.250 & 0.866 & -0.433 \end{pmatrix}$$

Problem 1

note: inverse rotation = minus angle

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

$$R_x^{-1}(\alpha) = R_x^{T}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & s\alpha \\ 0 & -s\alpha & c\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(-\alpha) & -s(-\alpha) \\ 0 & s(-\alpha) & c(-\alpha) \end{pmatrix} = R_x(-\alpha)$$

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$

same for y & z

Problem 2

Given a Yaw-Pitch-Roll system, i.e., first a rotation by γ around the z-axis, followed by a rotation β around the y-axis, and finally by α around the x-axis.

Now show that this system has a Gimbal lock, i.e., there is a case where one degree of freedom is lost. To ease things, you get the hint that this case happens when $\beta = \pi/2$.

Problem 2

common abbreviation: sin() = s, cos() = c

$$R_Y = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_P = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}, R_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

Yaw, Pitch, Roll system

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2

$$\beta = \pi / 2: \quad s\beta = 1\ , \ \ c\beta = 0$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ s\lambda & c\lambda & 0 \\ -c\lambda & s\lambda & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ c\alpha s\lambda + s\alpha c\lambda & c\alpha c\lambda - s\alpha s\lambda & 0 \\ s\alpha s\lambda - c\alpha c\lambda & s\alpha c\lambda + c\alpha s\lambda & 0 \end{pmatrix}$$

$$s(\alpha \pm \beta) = s\alpha c\beta \pm c\alpha s\beta$$
$$c(\alpha \pm \beta) = c\alpha c\beta \mp s\alpha s\beta$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ s(\alpha + \lambda) & c(\alpha + \lambda) & 0 \\ -c(\alpha + \lambda) & s(\alpha + \lambda) & 0 \end{pmatrix}$$

2 DoF α, λ
reduced to
1 DoF $\omega = \alpha + \lambda$

Problem 3

Given the quaternions $q_1 = (1, (2, 3, 4))$ and $q_2 = (0.4811480, (0.1984591, 0.7246066, 0.4517253))$. Which of the two represents an orientation? And why?

Problem 3

$$q_1 = (1, (2, 3, 4))$$

$$q_2 = (0.4811480, (0.1984591, 0.7246066, 0.4517253))$$

rotation q : norm q = 1

$$\text{norm}\,|q| = \sqrt{q\,\overline{q}} = \sqrt{\overline{q}\,q} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Problem 3

$$
\begin{aligned}
|q_1| &= \sqrt{1^2 + 2^2 + 3^2 + 4^2} \\
&= \sqrt{1+4+9+16} \\
&= \sqrt{30} \\
&\approx 5.477
\end{aligned}
$$

$$
\begin{aligned}
|q_2| &= \sqrt{0.4811480^2 + 0.1984591^2 + 0.7246066^2 + 0.4517253^2} \\
&= \sqrt{0.231503 + 0.039386 + 0.525055 + 0.204056} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

Problem 4

Given point $p = (2, 3, 4)^T$. Use quaternions to rotate it

- by 30^o around the y-axis
- by 30^o around the axis $(1, -1, 3)^T$
- first by 30^o the y-axis, then by 90^o around the axis $(1, -1, 3)^T$

Problem 4

quaternion rotation: $p' = q\ p\ \overline{q}$

$q = (a, (b, c, d)^T)$

conjugate $\overline{q}$ of q: $\overline{q} = (a, -(b, c, d)^T) = (a, (-b, -c, -d)^T)$

Problem 4

quaternion rotation: $p' = q\ p\ \bar{q}$

rotate by angle θ around unit axis $\mathbf{v}$:

use $q = (\cos(\theta / 2),\ \mathbf{v}\sin(\theta / 2))$

note 1: 3D vector often implicitly in quaternion

note 2: always proper rotation as $s^2\alpha + c^2\alpha = 1$

Problem 4

$$q_1 q_2 = (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum v1_i v2_i \qquad \mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} v1_2 v2_3 - v1_3 v2_2 \\ v1_3 v2_1 - v1_1 v2_3 \\ v1_1 v2_2 - v1_2 v2_1 \end{pmatrix}$$

note: option to remember cross product via “dirty” use of rule of Sarrus

$$a \times b : \begin{pmatrix} \text{entry 1-3} \\ \text{a (as row)} \\ \text{b (as row)} \end{pmatrix} = \left(\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}\right) \Rightarrow \begin{array}{l} \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow e_1 \\ \leftarrow e_2 \\ \leftarrow e_3 \end{array}$$

Problem 4: simple example

rotate x=2 by 180 deg around z-axis

$$p = (0, (2,0,0)^T)$$

$$\theta = 180^o, \ \mathbf{v} = (0,0,1)^T \ :$$

$$q = (\cos(90^o), \ (0,0,1)^T \sin(90^o))$$

$$= (0, \ (0,0,1)^T)$$

$$p' = q \ p \ \overline{q}$$

$$= (0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) \cdot (0, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}) \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$$

Problem 4: simple example

$$p' = \boxed{(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) \cdot (0, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix})} \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$$

$$q_1 q_2 = (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum v1_i v2_i$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 2-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} v1_2 v2_3 - v1_3 v2_2 \\ v1_3 v2_1 - v1_1 v2_3 \\ v1_1 v2_2 - v1_2 v2_1 \end{pmatrix}$$

Problem 4: simple example

$$p' = \boxed{(0,\begin{pmatrix}0\\0\\1\end{pmatrix})\cdot(0,\begin{pmatrix}2\\0\\0\end{pmatrix})}\cdot(0,\begin{pmatrix}0\\0\\-1\end{pmatrix})$$

$$(0,\begin{pmatrix}0\\0\\1\end{pmatrix})\cdot(0,\begin{pmatrix}2\\0\\0\end{pmatrix}) = (0\cdot 0 - 0, \left(0\begin{pmatrix}2\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}0\\2\\0\end{pmatrix}\right)) = (0,\begin{pmatrix}0\\2\\0\end{pmatrix})$$

$$(s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\0\\0\end{pmatrix} = 0 \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

Problem 4: simple example

$$p' = (0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) \cdot (0, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}) \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$$

$$= (0, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}) \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$$

$$= (0 \cdot 0 - 0, \left(0 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \right)) = (0, \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix})$$

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \right)$$

Problem 4: rotate by 30 deg around y

$$p = (2,3,4)^T \Leftrightarrow p = (0, (2,3,4)^T)$$

$$\theta_1 = 30^o, \ \mathbf{v}_1 = (0,1,0)^T \ :$$

$$q_1 = (\cos(15^o), \ (0,1,0)^T \sin(15^o))$$

$$= (0.9659, \ (0, 0.2588, 0)^T)$$

Problem 4 : rotate by 30 deg around y

$$p = (0, (2,3,4)^T)$$

$$q_1 = (0.9659,\ (0, 0.2588, 0)^T)$$

$$p_1 = q_1\ p\ \overline{q}_1$$
$$= (0.9659,\ (0,0.2588,0)^T) \cdot (0,(2,3,4)^T) \cdot (0.9659,\ (-0,-0.2588,-0)^T)$$

$$q_1 q_2 = (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum v1_i v2_i$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} v1_2 v2_3 - v1_3 v2_2 \\ v1_3 v2_1 - v1_1 v2_3 \\ v1_1 v2_2 - v1_2 v2_1 \end{pmatrix}$$

Problem 4 : rotate by 30 deg around y

quat.mul. 1: $(0.9659,\ (0,0.2588,0)^T)\cdot(0,(2,3,4)^T)$

intermediate dot & cross product of vector parts:

$(0,0.2588,0)^T \cdot (2,3,4)^T = 0.77645714$

$(0,0.2588,0)^T \times (2,3,4)^T = (1.0352762,0,-0.5176381)^T$

$$
\begin{aligned}
q_1 p &= (s_1, \mathbf{v}_1)(s_2, \mathbf{v}_2) \\
&= (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \\
&= (0 - 0.77645714, (0.9659 \cdot (2,3,4)^T + \mathbf{0} + (1.0352762, 0, -0.5176381)^T)) \\
&= (-0.77645714, (2.9671278, 2.89777748, 3.34606521)^T)
\end{aligned}
$$

Problem 4 : rotate by 30 deg around y

quat.mul. 2: $(-0.7765, (2.9671, 2.8978, 3.3460) \cdot (0.9659,\ (0, -0.2588, 0))$

intermediate dot & cross product of vector parts:

$(2.9671, 2.8978, 3.3460) \cdot (0, -0.2588, 0) = -0.75$

$(2.9671, 2.8978, 3.3460)^T \times (0, -0.2588, 0)^T = (0.8660254, 0, -0.7679492)^T$

$$
\begin{aligned}
(q_1 p)\overline{q}_1 &= (s_1, \mathbf{v}_1)(s_2, \mathbf{v}_2) \\
&= \left(s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2\right) \\
&= (0, (3.7320508, 3, 2.4641016)^T) \\
&\approx (0, (3.73, 3, 2.46)^T)
\end{aligned}
$$

Problem 4 : rotate by 30 deg around (1,-1,3)

$$\theta_2 = 30^o,\ \mathbf{v}_2 = (1,-1,3)^T$$

normalize $\mathbf{v}_2$!!!

$$\left|\mathbf{v}_2\right| = 3.31662479$$

$$\hat{\mathbf{v}}_2 = (0.30151134, -0.301511, 0.90453403)^T$$

$$q_2 \approx (\cos(15^o)),\ \ (0.3015, -0.3015, 0.9045)^T \sin(15^o))$$

$$\approx (0.9659, (0.0780369, -0.0780369, 0.23411063)^T)$$

Problem 4 : rotate by 30 deg around (1,-1,3)

$$p = (0, (2, 3, 4)^T)$$

$$q_2 = (0.9659, (0.0780369, -0.0780369, 0.23411063)^T)$$

$$p_2 = q_2\ p\ \overline{q}_2$$

$$= (0.9659, (0.0780, -0.0780, 0.2341)^T) \cdot (0, (2, 3, 4)^T) \cdot (0.9659, (-0.0780, 0.0780, -0.2341)^T))$$

$$= (-0.8584, (0.9173, 3.0538, 4.2539)^T) \cdot (0.9659, (-0.0780, 0.0780, -0.2341)^T))$$

$$= (0, (-0.093798, 2.76561296, 4.6198038)^T)$$

Problem 4

rotate
- first by 30 deg around y
- then by 30 deg around (1,-1,3)

$$p = (0, (2,3,4)^T)$$

$$q_1 = (0.9659,\ \ (0, 0.2588, 0)^T)$$

$$q_2 = (0.9659, (0.0780, -0.0780, 0.2341)^T)$$

option 1: $p_3 = q_2 \cdot (q_1 \cdot p \cdot \bar{q}_1) \cdot \bar{q}_2$

4 quaternion multiplications

Problem 4

rotate
- first by 30 deg around y
- then by 30 deg around (1,-1,3)

better option 2: chaining

$$q_3 = q_2 \cdot q_1$$

$$p_3 = q_3 \cdot p \cdot \bar{q}_3$$

3 quaternion multiplications

Problem 4

why does chaining work?

$$\overline{q_1 \cdot q_2} = \overline{q}_2 \cdot \overline{q}_1$$

- i.e., order of multiplications is swapped
- proof by using def's of conjugate and quat. mul.

and note that in general: $q_1 \cdot q_2 \neq q_2 \cdot q_1$

(quat.mult. is not commutative)

Problem 4

chaining two quaternion rotations

$$
\begin{aligned}
q_2 \cdot (q_1 \cdot p \cdot \overline{q}_1) \cdot \overline{q}_2 &= q_2 \cdot q_1 \cdot p \cdot \overline{q}_1 \cdot \overline{q}_2 \\
&= q_2 \cdot q_1 \cdot p \cdot \overline{q_2 \cdot q_1} \\
&= q_3 \cdot p \cdot \overline{q}_3 \text{ (with } q_3 = q_2 \cdot q_1 \text{)}
\end{aligned}
$$

Problem 4

rotate

- first by 30 deg around y
- then by 30 deg around (1,-1,3)

option 2: chaining

$$
\begin{aligned}
q_3 &= q_2 \cdot q_1 \\
&= (0.9659, (0.0780, -0.0780, 0.2341)^T) \cdot (0.9659, (0, 0.2588, 0)^T \\
&= (0.9532, (0.0148, 0.1746, 0.2463)^T)
\end{aligned}
$$

$$
\begin{aligned}
p_3 &= (0.9532, (0.0148, 0.1746, 0.2463)^T) \cdot (0, (2,3,4)^T \\
&\quad \cdot (0.9532, (-0.0148, -0.1746, -0.2463)^T \\
&= (0, (1.6027, 3.8155, 3.4457)^T)
\end{aligned}
$$

Problem 5

Use the Rodrigues formula to rotate $p = (2, 3, 4)^T$ by 30^o around the axis $(1, -1, 3)^T$.

Problem 5

- rotate **v** = (2,3,4)
- by angle theta = 30 deg
- around a normalized axis **k**, **k’**=(1,-1,3)

$$\mathbf{v}' = \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)$$

normalize $\mathbf{k}' = (1,-1,3)^T$!!!

$$|\mathbf{k}'| = 3.31662479$$

$$\mathbf{k} = (0.30151134, -0.301511, 0.90453403)^T$$

Problem 5

$$\mathbf{v'} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cos 30^0 + \left(\begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) \sin 30^0 + \begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \left(\begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) (1 - \cos 30^0)$$

$$= \begin{pmatrix} 1.73205081 \\ 2.5980762 \\ 3.46410162 \end{pmatrix} + \begin{pmatrix} -1.9598237 \\ 0.3015113 \\ 0.75377836 \end{pmatrix} + \begin{pmatrix} 0.1339746 \\ -0.133975 \\ 0.40192379 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0937983 \\ 2.765613 \\ 4.61980377 \end{pmatrix}$$