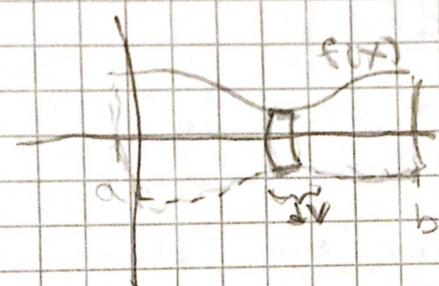
Solutions HW 8

1) $\int_{-1}^{0} \ln(x+1)\,dx = \int_{0}^{1} 1 \cdot \ln(u)\,du = u\ln(u)\Big|_{0}^{1} - \int_{0}^{1} \frac{u}{u}\,du$

$$= u(\ln(u)-1)\Big|_{0}^{1} = (x+1)(\ln(x+1)-1)\Big|_{-1}^{0} = \underbrace{(x+1)\ln(x+1)}_{\substack{\ln(1)=0 \text{ at } x=0 \\ x+1=0 \text{ at } x=-1}} - x - 1\Big|_{-1}^{0}$$

$$= 0 - 1 - (1-1) = -1$$

2)



Divide into small disks

Volume of disks: $\pi f(x)^2 \, dx$
(f(x) is the radius)

Total Volume $V = \int dV = \int_a^b \pi f(x)^2 \, dx$

b) $y = \frac{1+x^2}{2} \Rightarrow V = \int_0^1 \pi \left(\frac{1+x^2}{2}\right)^2 dx = \frac{\pi}{4}\int_0^1 1+2x^2+x^4 \, dx$

$$= \frac{\pi}{4}\left(x+\frac{2x^3}{3}+\frac{x^5}{5}\right)\Big|_0^1 = \frac{\pi}{4}\left(1+\frac{2}{3}+\frac{1}{5}\right) = \frac{7\pi}{15}$$

3. A force equal and opposite to the force exerted by the spring needs to be applied to expand it from its equilibrium:

$$\int_0^\ell \tilde{F}(x)\,dx = \int_0^\ell -F(x)\,dx = \int_0^\ell kx\,dx = \frac{kx^2}{2}\Big|_0^\ell = \frac{k\ell^2}{2}$$

(Note: $-\frac{k\ell^2}{2}$ would be the work done by the spring when expanded from its equilibrium)

b) $\frac{1}{\sqrt{1}} \geq \frac{1}{\sqrt{1+x^4}}$ for $0 \leq x \leq 1$ and $\frac{1}{\sqrt{x^4}} \geq \frac{1}{\sqrt{1+x^4}}$ for $x > 1$

Then we can bound: $\int_0^\infty \frac{1}{\sqrt{1+x^4}} \leq \int_0^1 \frac{1}{\sqrt{1}}\,dx + \int_1^\infty \frac{1}{\underbrace{\sqrt{x^4}}_{= x^2}}$

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{1+x^4}} \leq \int_0^1 1\,dx + \int_1^\infty \frac{1}{x^2}\,dx = x\Big|_0^1 - \frac{1}{x}\Big|_1^\infty = 1 - (0-1) = 2$$

$\Rightarrow \int_0^\infty \frac{1}{\sqrt{1+x^4}} \leq 2$, so it converges since $\frac{1}{\sqrt{1+x^4}}$ is a bounded and increasing function