CH-231-A

Algorithms and Data Structures

ADS

Lecture 34

Dr. Kinga Lipskoch

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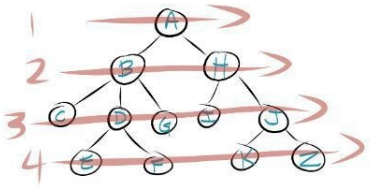
Breadth-First Search (BFS)

Problem:

- Given (directed or undirected) graph $G = (V, E)$ and a starting vertex $s \in V$.
- Systematically explore all vertices reachable from s.

BFS strategy:

- First find all vertices of distance 1 from s, then of distance 2, then of distance 3, etc.

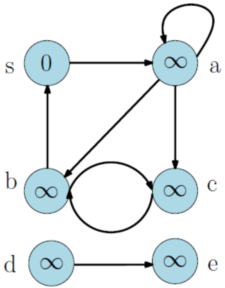


BFS Approach

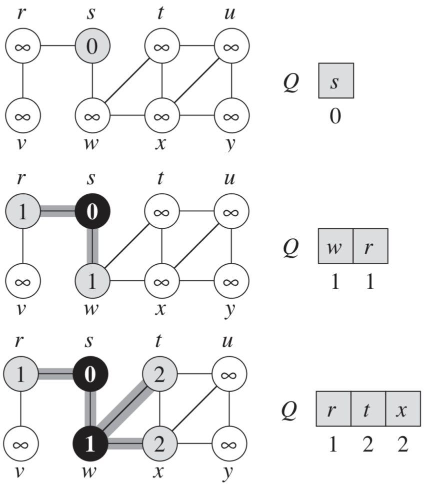
- Use adjacency-list representation.
- Use a color attribute for each *vertex* $\in \{\text{white}, \text{gray}, \text{black}\}$.
  - white: not detected yet
  - gray: just detected, waiting for us to explore their adjacency lists
  - black: done, all neighbors have been visited
- Store all gray vertices in a queue (FIFO principle).
- In addition, store for each vertex an attribute with the (topological) distance to starting vertex s.
- Finally, also store a pointer to the predecessor.

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```



BFS Example (1)

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for each vertex u ∈ G.V − {s}
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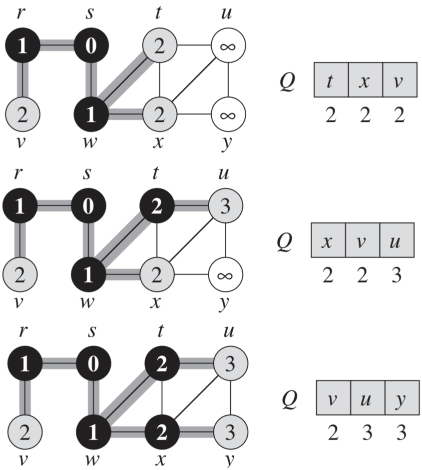
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

BFS Example (2)




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
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s.d = 0
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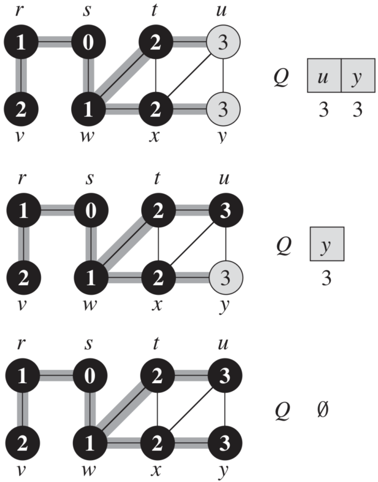
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

BFS Example (3)




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

BFS Analysis

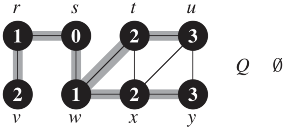
- Each vertex is enqueued and dequeued once.
- Each queue operation is $O(1)$.
- Total time for queue operations is $O(|V|)$.
- Loop over adjacency list of all vertices is in total $\Theta(|E|)$.
- Together, we get a time complexity of $O(|V| + |E|)$.

Breadth-First Tree

- When storing the predecessors, we can construct the predecessor subgraph $G_\pi = (V_\pi, E_\pi)$ of G with

$$V_\pi = \{v \in V \mid v.\pi \neq NIL\} \cup \{s\}$$
$$E_\pi = \{(v.\pi, v) \mid v \in V_\pi - \{s\}\}$$

- This subgraph represents a tree structure.
- It is called the breadth-first tree.
- It contains a unique path from s to every vertex in V_π.
- All these paths are shortest paths in G.



Depth-First Search (DFS)

DFS Strategy:

First follow one path all the way to its end, before we step back to follow the next path ($u.d$ and $u.f$ are start/finish time for vertex processing)

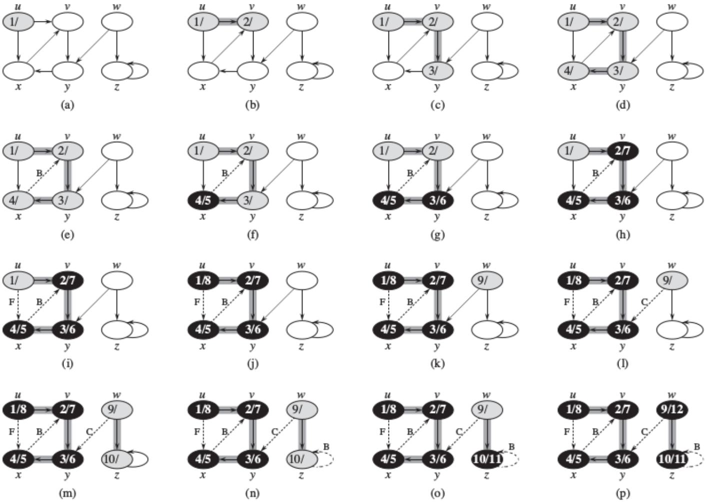
DFS(G)

```
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

DFS-VISIT(G, u)

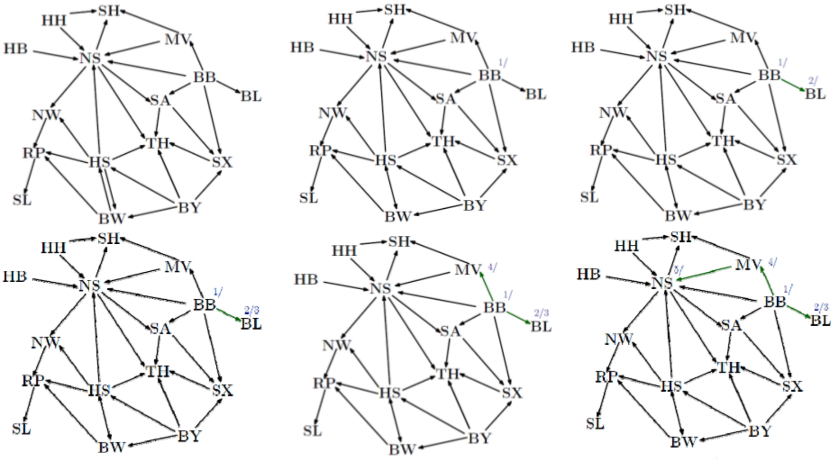
```
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

DFS Example

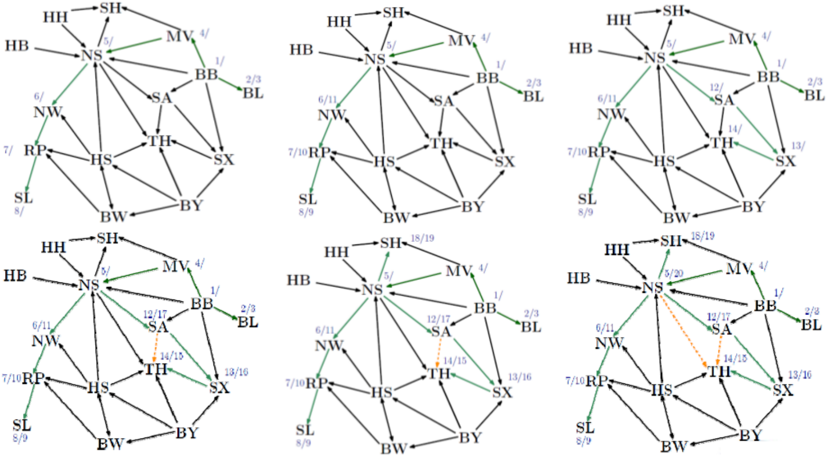


(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o) (p)

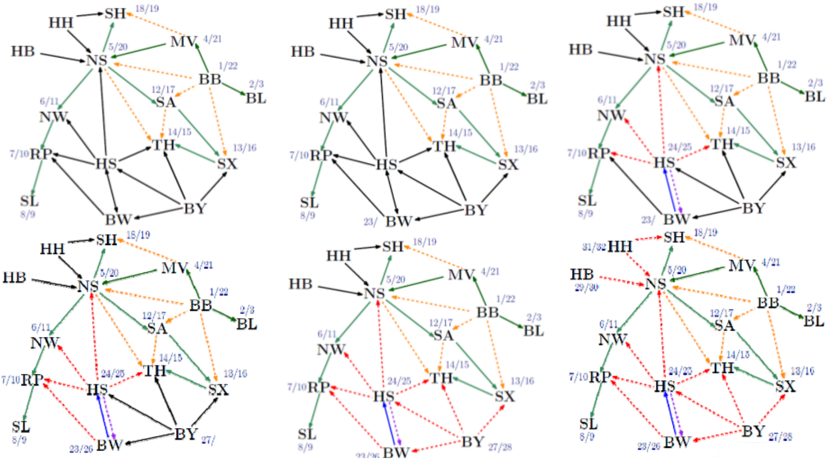
Another DFS Example (1)



Another DFS Example (2)



Another DFS Example (3)



DFS Analysis

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

Each vertex and each edge is processed once.
Hence, time complexity is $\Theta(|V| + |E|)$.

Edge Types

- Different edge types for (u, v):
  - Tree edges (solid): v is white.
  - Backward edges (purple): v is gray.
  - Forward edges (orange): v is black and $u.d < v.d$
  - Cross edges (red): v is black and $u.d > v.d$
- The tree edges form a forest.
- This is called the depth-first forest.
- In an undirected graph, we have no forward and cross edges.