Solutions HW 7

1. a) $\int \frac{\sin(\pi/x^2)}{x^3}\,dx$ Let $y = \frac{\pi}{x^2}$ $\frac{dy}{dx} = -2\pi \cdot \frac{1}{x^3} \Rightarrow \frac{dx}{x^3} = \frac{-1}{2\pi}\,dy$

$= \int \sin(y)\left(\frac{-1}{2\pi}\,dy\right) = \frac{-1}{2\pi}\int \sin(y)\,dy = \frac{1}{2\pi}\cos(y) + C = \frac{1}{2\pi}\cos\left(\frac{\pi}{x^2}\right) + C$

b) $\int \frac{2\ln(x)}{x}\,dx$ Let $u = \ln(x)$ $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$

$= \int 2u\,du = \frac{2u^2}{2} + c = u^2 + c = \ln(x)^2 + c$

c) $\int \cos(x)\ln(\sin(x))\,dx = \sin(x)\ln(\sin(x)) - \int \sin(x)\frac{d}{dx}\ln(\sin(x))\,dx$

$= \sin(x)\ln(\sin(x)) - \int \sin(x)\cdot\frac{1}{\sin(x)}\cdot\cos(x)\,dx = \sin(x)\ln(\sin(x)) - \int \cos(x)\,dx$

$= \sin(x)\ln(\sin(x))\,dx - \sin(x) + c = \sin(x)(\ln(\sin(x)) - 1)$

d) $\int_0^{\pi/2} x\cos x\sin x\,dx$ Since: $\sin(2x) = 2\sin(x)\cos(x)$

$= \int_0^{\pi/2} x\frac{\sin(2x)}{2} = \frac{-x\cos 2x}{4} - \frac{1}{2}\int \frac{\cos(2x)}{-2}\,dx = \frac{-x\cos(2x)}{4} + \frac{\sin(2x)}{8}\Bigg|_0^{\pi/2}$

$= -\frac{\pi}{8}\cos(\pi) + \underbrace{0}_{\text{all other terms evaluate to 0}} = \frac{\pi}{8}$

2. $\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\cos(x)\,dx$ Now use integration by parts

$\sin(x)\cos^{n-1}(x) - \int \sin(x)\frac{d}{dx}(\cos^{n-1}(x))\,dx$

$= -(n-1)\int \sin(x)\cos^{n-2}(x)\sin(x)\,dx = (-\int \sin^2(x)\cos^{n-2}(x))(n-1)dx$

$= -(n-1)\int (1-\cos^2(x))(\cos^{n-2}(x))dx = -(\int \cos^{n-2}(x) - \int \cos^n(x))(n-1)\,dx$

$\Rightarrow \int \cos^n(x) = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx$

$\Rightarrow n\int \cos^n(x) = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx$

$$\int \cos^n(x)\,dx = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

Other ways are also possible

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$-\int_{0}^{-a} f(x)\,dx = \int_{0}^{-a} f(-x)\,dx$$

since $f(-x) = -f(x)$ as f is odd

let $u = -x$

$du = -dx$

(also boundories $\Rightarrow$ $-a \mapsto a$, $0 \mapsto 0$)

$$\Rightarrow \int_{0}^{-a} f(-x)\,dx = -\int_{0}^{a} f(u)\,du$$

= (same as $-\int_{0}^{a} f(x)\,dx$ as 'u' and 'x' are just "labels")

$$\therefore \int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0$$

3. a) $\int \cos^2(x)\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx = \frac{\sin(2x)}{4} + \frac{1}{2}x + C$

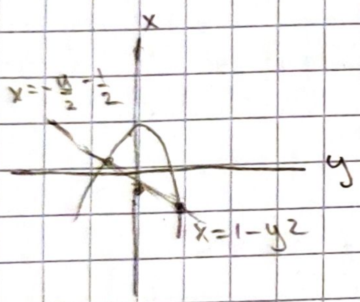
b) $\int \cos^3(x)\,dx = \frac{1}{3}\cos^2 x \sin(x) + \frac{2}{3}\int \cos(x) = \frac{\cos^2 x \sin(x)}{3} + \frac{2\sin(x)}{3}$

c) $\int_0^{2\pi} \cos^5(x)\,dx = \frac{1}{5}\cos^3(x)\sin(x)\Big|_0^{2\pi} + \frac{4}{5}\int_0^{2\pi}\cos^3(x)\,dx$

$= \underbrace{\frac{1}{5}\cos^3(x)\sin(x) + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x\right.}_{\text{evaluates to zero}} \left. + \frac{2}{3}\sin(x)\right)\Big|_0^{2\text{ pi}}$

$= \frac{8}{15}\sin(x)\Big|_0^{2\text{pi}} = 0$

Bonus



$y = -2x - 1 \Rightarrow x = -\frac{y}{2} - \frac{1}{2}$

When do they intersect?

$1 - y^2 = -\frac{y}{2} - \frac{1}{2} \Rightarrow y^2 - \frac{y}{2} - \frac{3}{2}$

$y_{1,2} = \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6}\right)/2$

$= \left(\frac{1}{2} \pm \sqrt{\frac{25}{4}}\right)/2 = \left(\frac{1}{2} \pm \frac{5}{2}\right)/2$

$= \frac{3}{2}, -1$

$\int_{-1}^{3/2} 1 - y^2 - \left(\frac{y}{2} - \frac{1}{2}\right) dy = \frac{3y}{2} - \frac{y^3}{3} - \frac{y^2}{4}\Big|_{-1}^{3/2} = \frac{9}{4} - \frac{9}{8} - \frac{9}{16} - \left(-\frac{3}{2} + \frac{1}{3} - \frac{1}{4}\right)$

$= \frac{9}{16} - \left(\frac{-18 + 4 - 3}{12}\right) = \frac{9}{16} + \frac{17}{12} = \frac{95}{48}$