Homework 9

Problem 1

(10 points)

a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2x^2 + y^2x\,,$$
$$y(0) = 2\,.$$

(5 points)

b) Solve the differential equation

$$e^x\,e^y\,\frac{\mathrm{d}y}{\mathrm{d}x} + e^x\,e^y = 2x^3\,,$$
$$y(0) = 0\,.$$

Hint: Recognize the left hand side of the equation as the derivative of some expression, then integrate. **(5 points)**

Problem 2

(10 points)
For each of the following equations, determine the equilibrium points where $y'(x) = 0$ and classify each as stable (y' changes sign from positive to negative at x) or unstable (y' changes sign from negative to positive at x). Sketch a few solution curves (without trying to solve the equation; simple sketch is enough) in the x-y plane.

a) $y' = y - y^3$ **(4 points)**

b) $y' = y\,(y-2)$ **(3 points)**

c) $y' = 2e^y - 2$ **(3 points)**

Problem 3

(10 points)

Radiocarbon dating is a method to determine the age of objects composed of organic material. It is based on the fact that atmospheric carbon is composed of a fixed ratio of two stable carbon isotopes, ^{12}C and ^{13}C, and an unstable isotope, ^{14}C, which undergoes radioactive decay with a half-life of 5 730 years. (This ratio is maintained by the interaction of cosmic rays with atmospheric nitrogen.) Living organisms exchange carbon with the atmosphere sufficiently fast so that the carbon ratio in their tissue is close to the atmospheric ratio. When an organism dies, the amount of stable carbon in its organic matter remains constant while ^{14}C decays, so that the age can be estimated by measuring the "carbon ratio", the ratio of the number of ^{14}C to the number of stable carbon atoms.

An archaeological object is found to have a carbon ratio which is 69% of the atmospheric carbon ratio. Determine the age of the object.

Bonus Problem 4: Planetary Motion

(10 bonus points)
One step to deriving Kepler's Orbit is to derive the integral given by:

$$\varphi = \int \frac{\frac{1}{r^2}}{\sqrt{2\,(E - V_{\text{eff}}(r))}}\,\mathrm{d}r, \qquad V_{\text{eff}} = -\frac{1}{r} + \frac{1}{2r^2}$$

Here, most of the constants pertaining to the original equation have been set equal to 1 (except for 'E', which stands for the total Energy of the system and should be treated as a constant). Here, r represents the radius of the orbiting object, φ is the angle covered by the object in orbit, and V_{eff} is the effective potential energy we are considering for the system.

a) Complete the square in the denominator and find and use substitution to reformulate the integral and leave it in the form of

$$\varphi = -\int \frac{\mathrm{d}\mu}{\sqrt{1-\mu^2}}$$

b) Apply substitution once again and find an explicit expression for φ as a function of r