Probability and Random Processes

7.12.2022

Recap

- Transition probabilities of a Markov chain

$$p_{ij} = \mathbb{P}\left[X_n = s_j | X_{n-1} = s_i\right].$$

- If the distribution of X_k is given by the row vector π, then the distribution of X_{k+1} is given by the row vector πP. More generally, the distribution of X_{k+n} is given by πP^n.
- The transition probabilities after n steps:

$$p_{ij}^{(n)} = (P^n)_{ij}.$$

Stationary distributions

Definition

A distribution π is called stationary if

$$\pi P = \pi.$$

Markov chain N states P $N \times N$ matrix

$\pi = (p_1 \ldots p_N)$ $N \times 1$ matrix

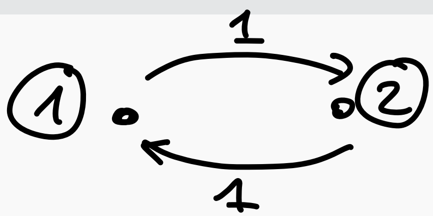
$\pi P = \pi$

Two definitions

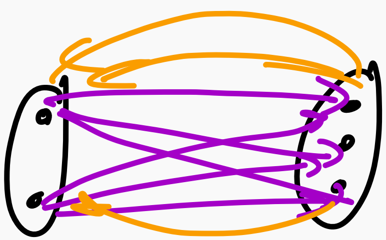
Definition

A Markov chain with transition matrix P is called

- irreducible if for every i and j there exists n such that $P^n_{ij} > 0$.
- ergodic if for every i and j there exists n such that $P^n_{ij} > 0$.



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Example

Determine whether the Markov chain with the matrix P below is irreducible/ergodic.

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

need to check that some power of P has only positive ~~integer~~ entries.

$$P^2 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$

ergodic $\Rightarrow$ irreducible.

Example

A Markov chain with the transition matrix

$$P = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/3 & 0 & 2/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

is ergodic.

Convergence to the stationary distribution

Theorem

Let P be the transition matrix of an ergodic Markov chain. Then

- There exists a unique stationary distribution π.
- When $n \to \infty$,

$$p_{ij}^{(n)} \to \pi_j.$$

  In other words, the matrix P^n converges to the matrix whose all rows are equal to π.

$$\pi = (\pi_1 \cdots \pi_N)$$

$$P^n = \begin{array}{c} j \\ \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \\ \pi_1 & \pi_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & & \pi_N \end{bmatrix} \end{array} \begin{array}{c} \leftarrow i \\ \\ \\ \\ \end{array}$$

$$p_{ij}^{(n)} \approx \pi_j$$

$$= \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

Example

Compute the stationary distribution for the Markov chain with the transition matrix

$$P = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/3 & 0 & 2/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

$$\pi = (x \quad y \quad z) \qquad x + y + z = 1$$

$$\pi P = \pi \qquad (x \quad y \quad z)\begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/3 & 0 & 2/3 \\ 0 & 1/2 & 1/2 \end{pmatrix} = (x \quad y \quad z)$$

$$\begin{cases} \frac{1}{6}x + \frac{1}{3}y = x \\ \frac{1}{3}x + \frac{1}{2}z = y \\ x + y + z = 1 \end{cases}$$

$$\Rightarrow \quad \frac{1}{3}y = \frac{5}{6}x \Rightarrow \boxed{y = \frac{5}{2}x}$$

$$\frac{1}{2}z = y - \frac{1}{3}x = \frac{5}{2}x - \frac{1}{3}x$$

$$= \frac{15-2}{6} x = \frac{13x}{6}$$

$$\boxed{z = \frac{13x}{3}}$$

$$x + y + z = 1 \implies x + \frac{5}{2} x + \frac{13}{3} x = 1$$

$$x \left(1 + \frac{5}{2} + \frac{13}{3} \right) = 1$$

$$x \cdot \frac{6 + 15 + 26}{6} = 1 \implies \boxed{x = \frac{6}{47}}$$

$$\boxed{y = \frac{15}{47}}$$

$$\boxed{z = \frac{26}{47}}$$

An important special cases

Theorem

Suppose that the transition matrix of an ergodic Markov chain with N states is doubly stochastic. Then the stationary measure of this Markov chain is the uniform measure

$$\pi = (\frac{1}{N}, \ldots, \frac{1}{N})$$

$P = (p_{ij})$

$\sum_{j=1}^{N} p_{ij} = 1$ always

$\sum_{i=1}^{N} p_{ij} = 1$ doubly stochastic

$(\frac{1}{N} \; \frac{1}{N} \cdots \frac{1}{N}) P = (\frac{1}{N} \cdots \frac{1}{N})$ ✓

uniquem $\longrightarrow$ $\boxed{\pi = (\frac{1}{N} \cdots \frac{1}{N})}$

Theorem of Perron-Frobenius

Theorem

Suppose P is the transition matrix of an ergodic Markov chain. Then there exists a vector π with positive entries such that

1. $\pi P = \pi$.
2. If v is a vector with $vP = v$ then v is a multiple of π.
3. For any other eigenvalue $\lambda \neq 1$ of P we have $\underline{\underline{|\lambda| < 1}}$.

<u>why do we care?</u>

Suppose: we can find a basis for $\mathbb{R}^n$ consisting of eigenvectors of P.

$\boxed{\pi = v_1} \longrightarrow$ eigenvalue $1 = \lambda_1$

$v_2 \longrightarrow \lambda_2$ $\quad |\lambda_i| < 1$ $\quad v_i P = \lambda_i v_i$

$\vdots \qquad \vdots$

$v_N \longrightarrow \lambda_n$

Suppose we want to compute

$$\lim_{n\to\infty} P_{ij}^{(n)}$$

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & i & \\ & & \downarrow & \\ [0 & \cdots & 1 & 0 \cdots 0] \end{array} P^n = [P_{i1}^{(n)} - \cdots - P_{iN}^{(n)}]$$

$$[0 \cdots 1 0 \cdots 0] = c_1 v_1 + c_2 v_2 + \cdots + c_N v_N$$

$$[0 \quad 1 \cdots 0] P^n = c_1 v_1 P^n + c_2 v_2 P^n + \cdots + c_N v_n P^n$$

$$= c_1 v_1 + c_2 \lambda_2^n v_2 + \cdots + c_N \lambda_N^n v_n$$

$v_1 = \pi$
$\lambda_1 = 1$

$n \to \infty \quad \lambda_i^n \to 0$ since $|\lambda_i| < 1$

as $n \to \infty$

$$[0 - - 1 0 - - 0] P^n \longrightarrow c_1 \pi$$

clearly: $c_1 = \cancel{\pi}\ 1 \Longrightarrow [0 \cdot\cdot 1 0 \cdot\cdot 0] P^n \longrightarrow \pi$

$$0.999^n \longrightarrow 0$$

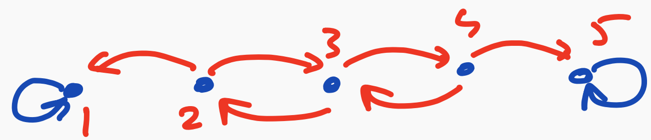
$0.999^n < 0.01$ $\underline{n \text{ large}}$

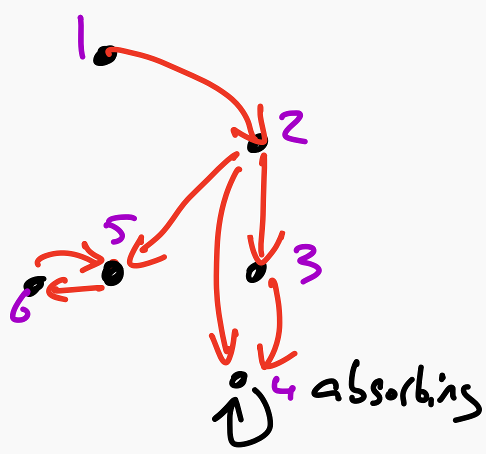
$\boxed{\text{Spectral gap}}$

$$Pv = \underline{\lambda} v$$

An example of a non-ergodic Markov chain

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/\cancel{2}3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & \cancel{0}\,2/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$





not an absorbing Markov chain.

Classification of states

Definition

A state of a Markov chain s_i is called absorbing if $p_{ii} = 1$. Otherwise, it is called transient.

Definition

A Markov chain is called absorbing if for every transient state s_i there exists an absorbing state s_j such that there exists a path from s_i to s_j.

Example

If $s_1, \ldots, s_k$ are the absorbing states and $s_{k+1}, \ldots, s_n$ are the transient states of an absorbing Markov chain, then the transition matrix is of the form

$$P = \begin{array}{cl} \begin{array}{cc} \text{abs} & \text{tr.} \end{array} & \\ \left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right) & \begin{array}{l} \text{abs} \\ \text{trans.} \end{array} \end{array}$$

Examples of absorbing Markov chains:penalty shootouts

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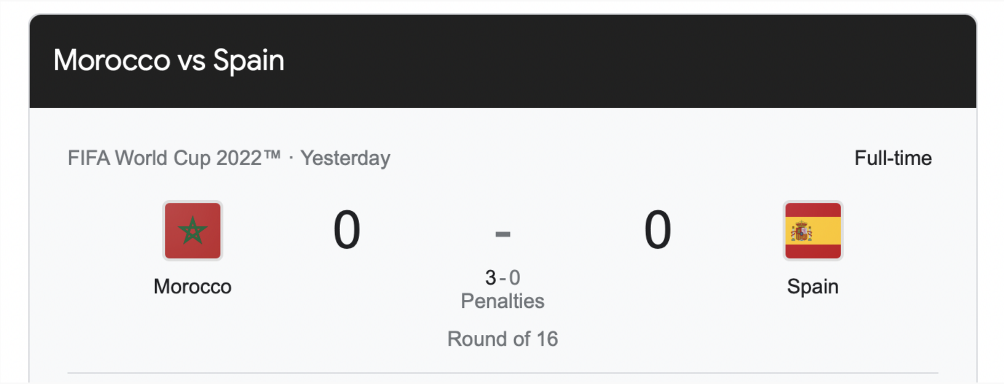
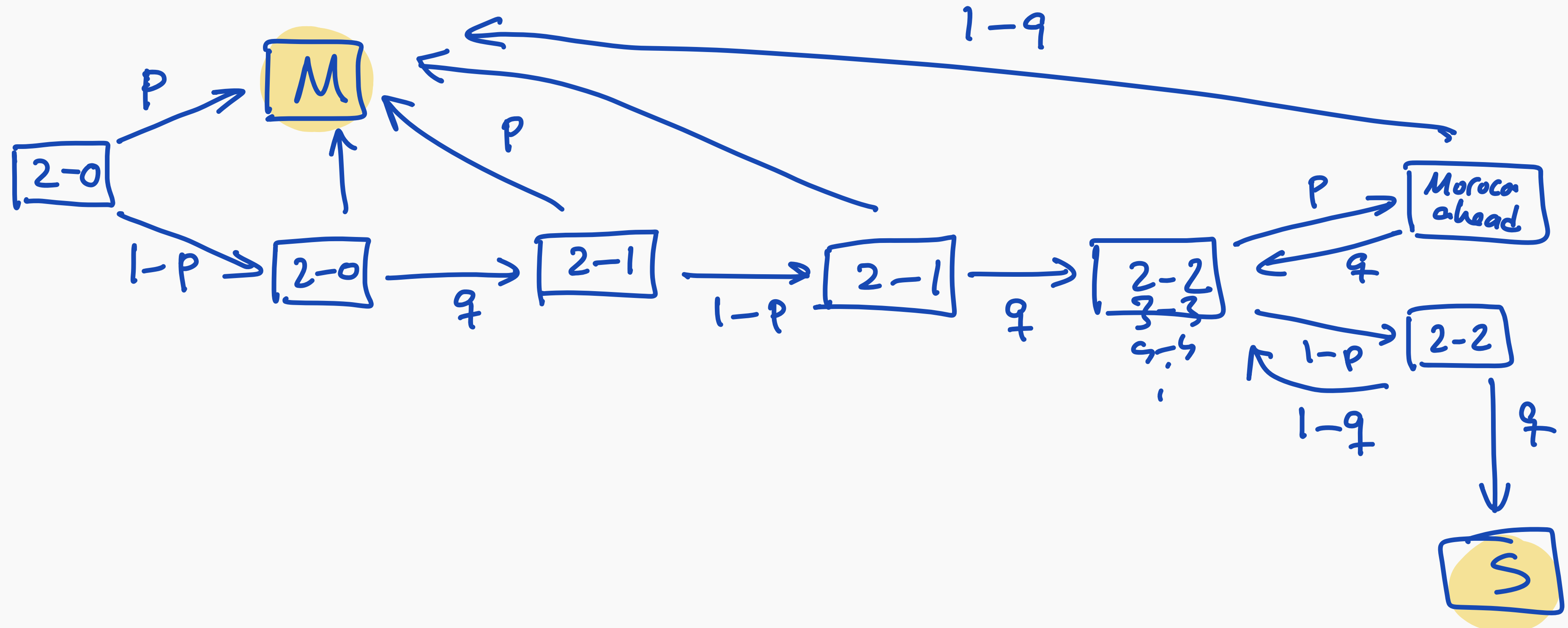


World Cup predictions

A football game between teams Morocco and Spain has gone into penalty kicks. At the end of the 3rd sound the result is 2-0 for Morocco, which will also start the fourth round. Suppose that in the remaining round players of Morocco score with probability p and misses with probability $1-p$. The corresponding probabilities for Spain players are q and $1-q$. Model the penalty kicks as a Markov chain.

5 round after 3 round 2-0



Iterations of an absorbing

$$P^2 = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ R+QR & Q^2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ R+QR & Q^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ R+QR+Q^2R & Q^3 \end{pmatrix}$$

$$P^n = \begin{pmatrix} I & 0 \\ R(I+Q+\cdots+Q^{n-1}) & Q^n \end{pmatrix}$$

$$I+Q+Q^2+Q^3+\cdots$$

$$= (I-Q)^{-1}.$$

$$n \to \infty \qquad \begin{pmatrix} I & 0 \\ R(I+Q+Q^2+\cdots) & 0 \end{pmatrix}$$

Theorem

Suppose that Q and R are as above. Then the (i, j) entry of the matrix $(I - Q)^{-1}R$ gives the probability of absorption in s_j if the chain starts at s_i.

i transient state

j absorbing state

Prob that the chain starting from i gets absorbed at j $= \left((I-Q)^{-1} R\right)_{ij}$

Example: gambler's ruin

A gabler has 3 Euros. In each round of gambling they win 1 euro with probability 0.4 and lose with probability 0.6. The gambler plays until reaching either 5 euros or going bankrupt. Find the probability for each one of these two possibilities.

Predicting the probability of winning for each team