

Solutions HW 7

1. a) $\int \frac{\sin(\pi/x^2)}{x^3} dx$ Let $\eta = \frac{\pi}{x^2}$ $\frac{d\eta}{dx} = -2\pi \cdot \frac{1}{x^3} \Rightarrow \frac{dx}{x^3} = -\frac{1}{2\pi} d\eta$

$$= \int \sin(\eta) \left(-\frac{1}{2\pi} d\eta\right) = -\frac{1}{2\pi} \int \sin(\eta) d\eta = -\frac{1}{2\pi} \cos(\eta) + C = \frac{1}{2\pi} \cos\left(\frac{\pi}{x^2}\right) + C$$

b) $\int \frac{2 \ln(x)}{x} dx$ Let $u = \ln(x)$ $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$

$$= \int 2u du = \frac{2u^2}{2} + C = u^2 + C = \ln(x)^2 + C$$

c) $\int \cos(x) \ln(\sin(x)) dx = \sin(x) \ln(\sin(x)) - \int \sin(x) \frac{d \ln(\sin(x))}{dx} dx$

$$= \sin(x) \ln(\sin(x)) - \int \sin(x) \cdot \frac{1}{\sin(x)} \cdot \cos(x) dx = \sin(x) \ln(\sin(x)) - \int \cos(x) dx$$

$$= \sin(x) \ln(\sin(x)) - \sin(x) + C = \sin(x) (\ln(\sin(x)) - 1) + C$$

d) $\int_0^{\pi/2} x \cos x \sin x dx$ Since: $\sin(2x) = 2 \sin(x) \cos(x)$

$$= \int_0^{\pi/2} x \frac{\sin(2x)}{2} = -\frac{x \cos 2x}{4} - \frac{1}{2} \int \frac{\cos(2x)}{2} dx = -\frac{x \cos(2x)}{4} + \frac{\sin(2x)}{8} \Big|_0^{\pi/2}$$

$$= -\frac{\pi \cos(\pi)}{8} + 0 = \frac{\pi}{8}$$

all other terms evaluate to 0

$$2. \int \cos^n(x) dx = \int \cos^{n-1}(x) \cos(x) dx \quad \text{Now use integration by parts}$$

$$\sin(x) \cos^{n-1}(x) - \int \sin(x) \frac{d(\cos^{n-1}(x))}{dx} dx$$

$$= (n-1) \int \sin(x) \cos^{n-2}(x) \sin(x) dx = - \int \sin^2(x) \cos^{n-2}(x) (n-1) dx$$

$$= -(n-1) \int (1 - \cos^2(x)) \cos^{n-2}(x) dx = - \left(\int \cos^{n-2}(x) dx - \int \cos^n(x) dx \right) (n-1) dx$$

$$\Rightarrow \int \cos^n(x) = \sin(x) \cos^{n-1}(x) + (n-1) \int \cos^{n-2}(x) dx - (n-1) \int \cos^n(x) dx$$

$$\Rightarrow n \int \cos^n(x) = \sin(x) \cos^{n-1}(x) + (n-1) \int \cos^{n-2}(x) dx$$

$$\int \cos^n(x) dx = \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{n-1}{n} \int \cos^{n-2}(x) dx$$

Other ways are also possible

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$$

$$\int_{-a}^0 f(x) dx = \int_0^a f(-x) dx$$

↑ ↓
since $f(-x) = -f(x)$ as
 f is odd

$$\text{let } u = -x$$

$$du = -dx$$

$$(\text{also boundaries } \Rightarrow -a \mapsto a, 0 \mapsto 0)$$

$$\Rightarrow \int_0^a f(-x) dx = - \int_0^a f(u) du \quad \left(\text{same as } - \int_0^a f(x) dx \text{ as 'u' and 'x' are just "labels"} \right)$$

$$\therefore \int_{-a}^a f(x) dx = - \int_0^a f(u) du + \int_0^a f(x) dx = 0 //$$

$$3. a) \int \cos^2(x) dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 dx = \frac{\sin(2x)}{4} + \frac{1}{2} x + C$$

$$b) \int \cos^3(x) dx = \frac{1}{3} \cos^2 x \sin(x) + \frac{2}{3} \int \cos(x) = \frac{\cos^2 x \sin(x)}{3} + \frac{2 \sin(x)}{3}$$

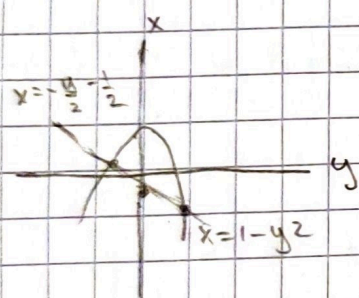
$$c) \int_0^{2\pi} \cos^5(x) dx = \frac{1}{5} \cos^3(x) \sin(x) + \frac{4}{5} \int_0^{2\pi} \cos^3(x) dx$$

$$= \frac{1}{5} \cos^3(x) \sin(x) + \frac{4}{5} \left(\frac{1}{3} \cos^2 x \sin x + \frac{2 \sin(x)}{3} \right) \Big|_0^{2\pi}$$

evaluates to zero

$$= \frac{8}{15} \sin(x) \Big|_0^{2\pi} = 0$$

Bonus



$$y = -2x - 1 \Rightarrow x = -\frac{y}{2} - \frac{1}{2}$$

When do they intersect?

$$1 - y^2 = -\frac{y}{2} - \frac{1}{2} \Rightarrow y^2 - \frac{y}{2} - \frac{3}{2}$$

$$y_{1,2} = \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} \right) / 2$$

$$= \left(\frac{1}{2} \pm \sqrt{\frac{25}{4}} \right) / 2 = \left(\frac{1}{2} \pm \frac{5}{2} \right) / 2$$

$$= \frac{3}{2}, -1$$

$$\int_{-1}^{3/2} (1 - y^2 - (-\frac{y}{2} - \frac{1}{2})) dy = \left[\frac{3y}{2} - \frac{y^3}{3} + \frac{y^2}{4} \right]_{-1}^{3/2}$$

$$= \frac{9}{4} - \frac{9}{8} + \frac{9}{16} - \left(-\frac{3}{2} + \frac{1}{3} + \frac{1}{4} \right)$$

$$= \frac{27}{16} - \left(-\frac{18}{12} + \frac{4}{12} + \frac{3}{12} \right) = \frac{27}{16} + \frac{11}{12} = \frac{125}{48}$$