Midterm Solutions

1) a) $$\lim_{s \to -1} \frac{\frac{1}{s} - 1}{s^3 - 1} = \frac{-2}{-2} = 1$$

(Other example:

$$\lim_{s \to 1} \frac{\frac{1}{s} - 1}{s^3 - 1} = \lim_{s \to 1} \frac{1}{s} \frac{\overset{-1}{\cancel{1-s}}}{\cancel{(s-1)}(s^2+s+1)}$$

$$= \lim_{s \to 1} \frac{-1}{s(s^2+s+1)} = -\frac{1}{3}$$)

b) $$\lim_{x \to \infty} \frac{e^{2x} + x^3 + \ln x}{3e^{2x} - x^3 + \cos x} = \lim_{x \to \infty} \frac{1 + \overset{\to 0}{\frac{x^3}{e^{2x}}} + \overset{\to 0}{\frac{\ln x}{e^{2x}}}}{3 - \underset{\to 0}{\frac{x^3}{e^{2x}}} + \underset{\to 0}{\frac{\cos x}{e^{2x}}}} = \frac{1}{3}$$

c) $$\lim_{r \nearrow 1} \frac{|r-1|}{r^2+1} = \lim_{r \nearrow 1} \frac{|r-1|}{(r+1)(r-1)} = \lim_{r \nearrow 1} \frac{\overset{-1}{\cancel{1-r}}}{(r+1)\cancel{(r-1)}} = -\frac{1}{2}$$

likewise, $$\lim_{r \searrow 1} \frac{|r-1|}{r^2+1} = \lim_{r \searrow 1} \frac{\cancel{r-1}}{(r+1)\cancel{(r-1)}} = \frac{1}{2}$$

so $\displaystyle\lim_{r \to 1} \frac{|r+1|}{r^2+1}$ does not exist.

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2) $y = 4 - x$ and $y = x^2 + 2$ touch at $x = 1$

($4 - x = x^2 + 2$ for $x = 1$) and are continuous.

Thus, by squeeze law, $\displaystyle\lim_{x \to 1} f(x) = 3 = f(1)$

so f is continuous at $x = 1$

3) $f(x) = x^7 - 3x - 1$ is continuous,

$f(-1) = -1 + 3 - 1 > 0$, and

$f(1) = 1 - 3 - 1 = -3 < 0$

$\Rightarrow$ by intermediate value theorem, f must have at least one root in the interval $(-1, 1)$.

4) a) $y = \arctan x \rightarrow x = \tan y$

Diff. w.r.t. x: $1 = \tan' y \frac{dy}{dx}$ (implicit diff.)

Moreover, $\tan' y = \left(\frac{\sin y}{\cos y}\right)' = \frac{\cos^2 y - (-\sin y)\sin y}{\cos^2 y}$

$$= \frac{1}{\cos^2 y}$$

and $x^2 = \tan^2 y = \frac{\sin^2 y}{\cos^2 y} = \frac{1 - \cos^2 y}{\cos^2 y} \Rightarrow \cos^2(y) \cdot x^2 = 1 - \cos^2 y$

$$\Rightarrow \cos^2 y = \frac{1}{1+x^2}$$

In total $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{\tan' y} = \cos^2 y = \underline{\underline{\frac{1}{1+x^2}}}$

b) • $D(f) = \mathbb{R}$, so no vertical asymptotes

• $\lim_{x \to \infty} (2 \arctan x - x) = -\infty$

$\lim_{x \to -\infty} (2 \arctan x - x) = \infty$ (as arctan is bounded)

$\Rightarrow$ no horizontal asymptotes

• $f'(x) = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2} = \frac{(1-x)(1+x)}{\underbrace{1+x^2}_{>0}}$

Sign change only due to numerator

$\Rightarrow$ for $x < -1$, $f'(x) < 0$ $\Rightarrow$ f is decreasing

for $x \in (-1, 1)$, $f'(x) > 0$ $\Rightarrow$ f is increasing

for $x > 1$, $f'(x) < 0$ $\Rightarrow$ f is decreasing

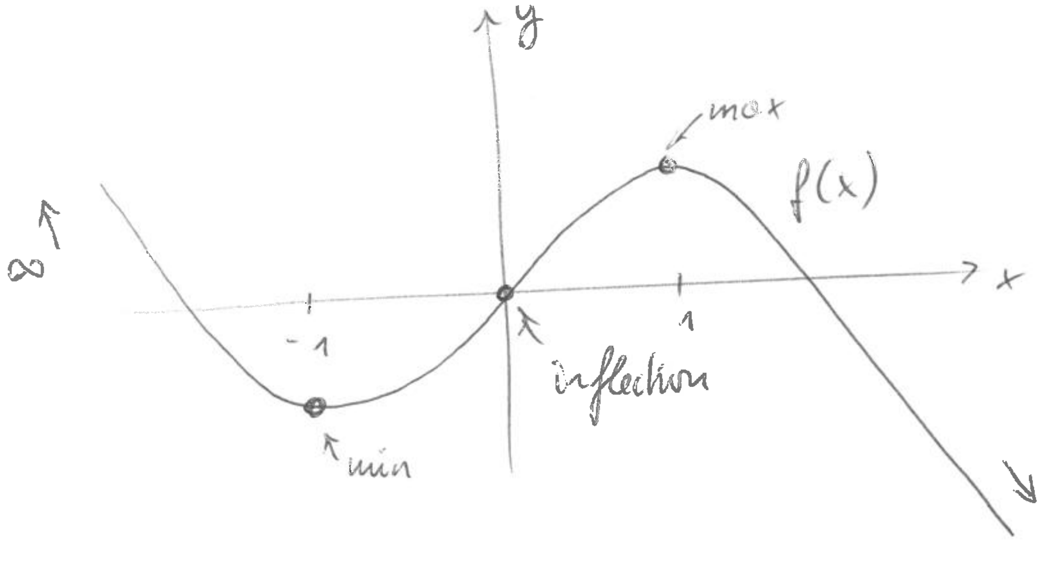
So f has ~~two~~ a local min at $(-1, f(-1))$ and a local max at $(1, f(1))$

• $f''(x) = \frac{-4x}{\underbrace{(1+x^2)^2}_{>0}}$

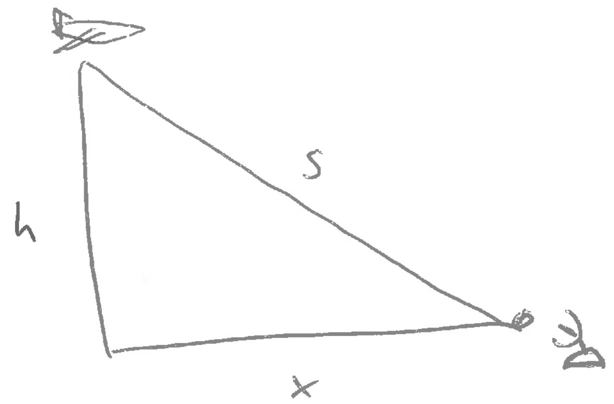
$\Rightarrow$ for $x < 0$, $f''(x) > 0$, so f is concave up

for $x > 0$, $f''(x) < 0$, so f is concave down

So f has a point of inflection at $x = 0$



5)



h constant

$s^2 = h^2 + x^2$, $x = x(t)$, $s = s(t)$

$\Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$

Here: $s = 10\,km$, $h = 6\,km$, $\frac{ds}{dt} = 400\,\frac{km}{h}$

$\Rightarrow x^2 = 100\,km^2 - 36\,km^2 = 64\,km^2$

$\Rightarrow x = 8\,km$

$\Rightarrow \frac{dx}{dt} = \frac{10\,km}{8\,km} \cdot 400\,\frac{km}{h} = 500\,\frac{km}{h}$

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6) a) Let $u = \frac{1}{x}$ $\Rightarrow du = -\frac{1}{x^2}dx$

$\Rightarrow \int \frac{1}{x^3} e^{\frac{1}{x}} dx = -\int u e^u du \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{=} -ue^u + \int e^u du$

$= -ue^u + e^u + c = -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} + c$

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b) Need partial fractions

$\frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+1)}$

$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$

Need to solve

$$x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

Collect different powers of x:

x^0: $1 = B$

x^1: $x = Ax \Rightarrow 1 = A$

x^2: $0x^2 = (B+D)x^2 \Rightarrow B+D=0 \Rightarrow D = -B = -1$

x^3: $0x^3 = (A+C)x^3 \Rightarrow A+C=0 \Rightarrow C = -A = -1$

$$\Rightarrow \int \frac{x+1}{x^2(x^2+1)}\,dx = \underbrace{\int \frac{\overset{=A}{1}}{x}\,dx}_{=\ln|x|} + \underbrace{\int \frac{\overset{=B}{1}}{x^2}\,dx}_{=-\frac{1}{x}} - \underbrace{\int \frac{\overset{=C}{x}}{x^2+1}\,dx}_{=\frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1|} - \underbrace{\int \frac{\overset{=D}{1}}{x^2+1}\,dx}_{=*}$$

$u = x^2+1 \Rightarrow du = 2x\,dx$

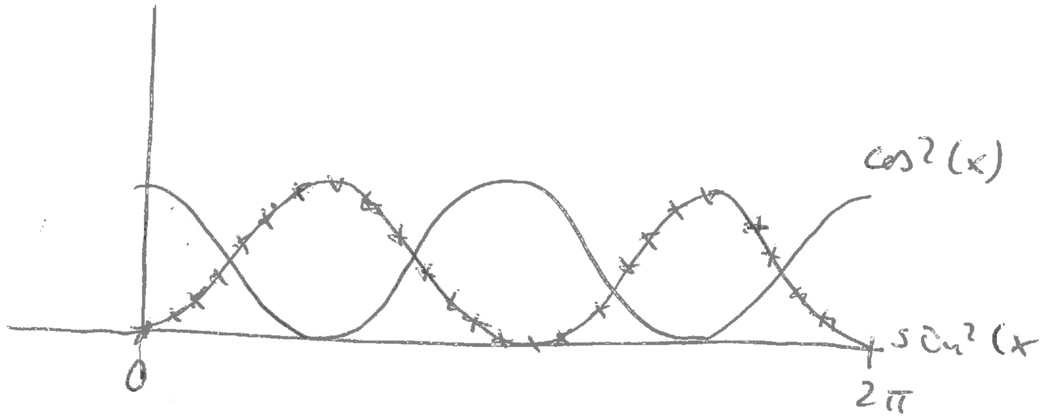
$$\left[* = \int \frac{1}{x^2+1}\,dx = \arctan x\right]$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2+1| - \arctan x + C$$

c) The integral is zero as both terms are periodic with period π and shifted copies of each other, so they cancel out.

The integral is actually $\frac{\sin(2x)}{2} + C = \cos(x)\sin(x) + C$

(as $\cos(2x) = \cos^2(x) - \sin^2(x)$)



areas under curves cancel (2 humps for each)

7) let $G(x) = \int_0^x \frac{e^t}{t}\,dt$ (via def.)

$$\Rightarrow G'(x) = \frac{e^x}{x}$$

$$\Rightarrow F(x) = G(x) - G(\sqrt{x})$$

$$\Rightarrow F'(x) = G'(x) - \underbrace{G'(\sqrt{x})\,\tfrac{1}{2}x^{-\frac{1}{2}}}_{\text{chain rule!}}$$

$$= \frac{e^x}{x} - \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}$$

$$= \frac{e^x - \frac{1}{2}e^{\sqrt{x}}}{x}$$