Introduction

Numerical Methods (NM) are algorithmic approaches to mathematical problems/equations which are hard to solve algebraic/analytically.

Goal of this:

Study efficient numerical methods and understand:

- How they work?
- When do they work? Limitation?
- What is the error introduced?

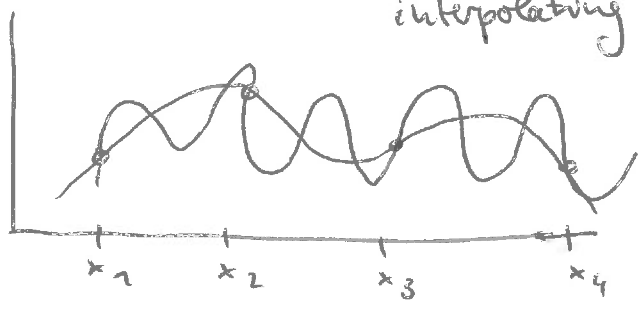
Types of NM

- Iteration: Solve $f(x) = y = \sin(x) + \ln(x)$ for roots x_i

  Procedure

  $$\underbrace{x_n}_{\text{iteration step}} = \overbrace{F}^{\text{iteration method}}(x_{n-1})$$

- Interpolation

  interpolating $f(x)$ with $g(x)$ s.t.

  $f(x_i) = g(x_i)$

  $f(x) \approx g(x)$

- Integration: $\frac{dy}{dx} = f(x)$, $y = \int f(x)\,dx$

  Approx. derivative and integral of functions.

Taylor Series:

Given a function $f: \mathbb{R} \to \mathbb{R}$ that is hard to evaluate for some $x \in \mathbb{R}$. But f and f^{n} are known for a value c which is close to x. (nth derivative of f)

Can we use this information to approx. $f(x)$?

How accurate is this approx.?

For example: $f(x) = \cos(x)$, $x = 0.1$

We know the values of $\cos^{(k)}(0)$, i.e. $c = 0$

$$\left.\begin{aligned} f^{0}(c) &= \cos(0) = 1 \\ f'(c) &= -\sin(0) = 0 \\ f''(c) &= -\cos(0) = -1 \end{aligned}\right\} \text{ for } c = 0$$

Can we get $\cos(0.1)$ from these values?

<u>Def.:</u> (Taylor series) Let $f: \mathbb{R} \to \mathbb{R}$ be (infinitely many times) differentiable at $c \in \mathbb{R}$. So we have $f^{(k)}(c)$ for $k = 0, 1, 2, \ldots$

The Taylor series of f at c is

$$f(x) \underset{\text{approx.}}{\approx} f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

This is a power series!

For $c = 0$ this is also known as Maclaurin series

Remember: A power series has a radius / interval of convergence. If $x \in$ interval of convergence, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

Ex.: Taylor series for $f(x) = e^x$ at $c = 0$

We have $f^{(k)}(x) = e^x$, so $f^{(k)}(c) = e^0 = 1$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

and the radius of convergence is ∞,

i.e. for any $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

For a numerical algorithm we need to stop the summation after a finite number of terms.

Eg.: $e^x \approx \frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2 + \ldots$

$$= \underbrace{1 + x + \frac{1}{2} x^2}_{\text{is a polynomial}}$$

Ex.: $f(x) = 4x^2 + 5x + 7$, $c = 2$

Taylor series of f at c?

$f(2) = 33$, $f'(x) = 8x + 5$, $f''(x) = 8$, $f'''(x) = 0$

$f'(2) = 21$, $f''(2) = 8$

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Taylor series of f at c:

$$33 + 21(x-2) + \frac{8}{2}(x-2)^2 = 4x^2 + 5x + 7 = f(x)$$

The Taylor series of a polynomial is the polynomial
<u>itself!</u>

<u>Theorem</u>: (Taylor theorem)

Let $f \in C^{n+1}([a,b])$, i.e. f is $(n+1)$-times continuously differentiable over $[a,b]$.

Then for many $c, x \in [a,b]$ we have that

$$f(x) = \boxed{\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k} \leftarrow \text{truncated Taylor series}$$

$$+ \boxed{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}} \leftarrow \text{remainder term}$$

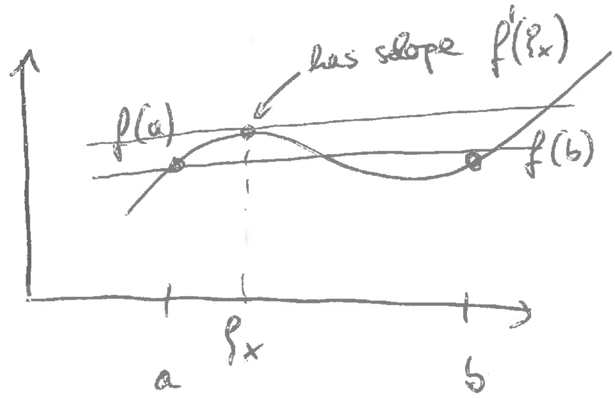
where ξ_x is a point that depends on x and which lies between c and x, i.e. $\xi_x \in (c, x)$ (for $c < x$) or $\xi_x \in (x, c)$ (for $x < c$)

For $n = 0$: $\quad f(x) = f(c) + f'(\xi_x)(x-c)$

choose $\quad c = a, \; x = b$

$$f(b) = f(a) - f'(\xi_x)(b-a)$$

$$\Leftrightarrow \frac{f(b) - f(a)}{b-a} = f'(\xi_x) \quad \left(\text{mean value theorem}\right)$$

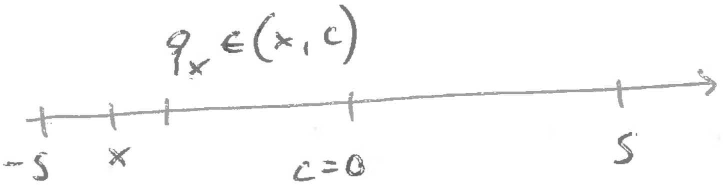


We say that a Taylor series <u>represents</u> the function f at x iff the Taylor series converges at that point, i.e. the remainder tends to zero as $n \to \infty$.

Back to e^x: $f(x) = e^x$, $c = 0$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \quad , \text{ with } \xi_x \in (c, x) \text{ or } (x, c)$$

For any $x \in \mathbb{R}$ we find $s \in \mathbb{R}_0^+$ (positive real numbers including 0) so that $|x| \leq s$, and $|\xi_x| \leq s$ because ξ_x is between c and x



Because e^x is monotone increasing, we have $e^{\xi_x} \leq e^s$, thus:

$$\lim_{n \to \infty} \left| \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \right| \leq \lim_{n \to \infty} \left| \frac{e^s}{(n+1)!} \right| s^{n+1}$$

$$= e^s \lim_{n \to \infty} \frac{s^{n+1}}{(n+1)!} = 0$$

because $(n+1)!$ will grow faster than any power of s. $\Rightarrow \lim_{n\to\infty} \left| \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \right| = 0$

Thus e^x is represented by its Taylor series.

Ex.: $f(x) = \ln(1+x)$, $c = 0$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$$

$\vdots$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\,\frac{1}{(1+x)^k}$$ so $f^{(k)}(0) = (-1)^{k-1}(k-1$

for $k \geq 1$

$$f(0) = \ln(1) = 0$$

Taylor series is then:

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k + \underbrace{\frac{(-1)^n}{n+1} \frac{1}{(1+\xi_x)^{n+1}} x^{n+1}}_{\text{remainder} = E_n(x)} = \ln(x+1)$$

Question: ~~U~~ For which x does $\lim E_n(x) \to 0$ as $n \to \infty$?

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} \underbrace{\frac{(-1)^n}{n+1}}_{\to 0} \underbrace{\left(\frac{x}{\xi_x+1}\right)^{n+1}}_{\to 0?} \overset{?}{=} 0 \qquad \xi_x \in (\overset{0}{\cancel{c}}, x)$$

$\Rightarrow$ $0 < \frac{x}{\xi_x+1} < 1$ $\Rightarrow$ $x - \xi_x < 1$ with $\xi_x \in (0, x)$

$\Rightarrow$ $x \leq 1$

se: $\lim_{n\to\infty} E_n(x) = 0$ iff $0 < x \leq 1$

This means that the Taylor series represents $\ln(1+x)$ for $x \in [0, 1]$

We can extend this to show $x \in (-1,1]$
(not shown here)

Putting it into practice: Compute $\cos(0.1)$

Taylor series approx. at $c=0$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \dots + \text{remainder}$$

Thus:

$$\left| \cos(x) - \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k!)} \right| = \underbrace{\left| (-1)^{n+1} \cos(\xi_k) \frac{x^{2(n+1)}}{(2(n+1))!} \right|}_{\text{remainder}}$$

$$\leq \frac{0.1^{2(n+1)}}{(2(n+1))!} \to 0 \text{ for } n \to \infty$$

n	Taylor poly.	$\lvert\text{error}\rvert \leq$
0	1	$\frac{(0.1)^2}{2!} = 0.005$
1	0.995	$\frac{(0.1)^4}{4!} = \frac{0.0001}{24}$
2	0.99500416	$\frac{0.000001}{6!}$

Error depends on choice of $|x-c|$ and n

Ex.: Compute $\ln(2)$

Taylor series for $\ln(x+1)$ at $c=0$
and evaluate at $x=1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

keeping 8 terms (until $k=8$) we get

$$\ln 2 \approx 0.63452$$

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We can use the Taylor series of $\ln((1+x)/(1-x))$ insted

Since $\ln((1+x)/(1-x)) = \ln(1+x) - \ln(1-x)$

we choose $x = \frac{1}{3}$ ~~instead of~~ instead of $x = 1$

$\left(\ln((1+\tfrac{1}{3})/(1-\tfrac{1}{3})) = \ln(2) \right)$

We then get $\ln(2) = 2 \cdot \left(\frac{1}{3} + \frac{1}{3^3 \cdot 3} + \frac{1}{3^5 \cdot 5} + \dots \right)$

So we only need 4 terms to get

$\ln 2 \simeq 0.69313$ (use calculator to get $\ln(2) \approx 0.69314$)

<u>Theorem:</u> Reformulation of Taylor's theorem

$f \in C^{n+1}([a,b])$. We change c to x and the old x to $x+h$ from previous version to get for $x, x+h \in [a,b]$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n+1)}(\xi_h)}{(n+1)!} h^{n+1}$$

$\xi_h \in (x, x+h)$

We can write the error term as

$$f(x+h) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k = \underbrace{\mathcal{O}(h^{n+1})}_{\text{Landau Symbol}}$$

<u>Recall:</u> $a(h) = \mathcal{O}(b(h))$ iff $\exists\, c > 0$ s.th. $\frac{a(h)}{b(h)} \leq c$ as $h \to 0$

So for $n=1$ the error decreases with h^2 (quadratic convergence). For $n=2$ error decreases cubically, i.e. h^3 etc.