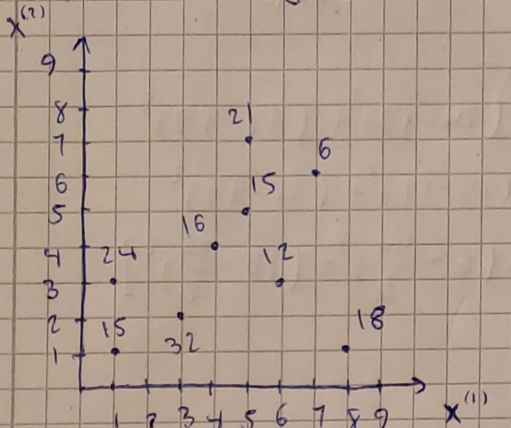
Exercise 1.

$$T = \{((1,1)^T, 15), ((1,3)^T, 24), ((3,2)^T, 32), ((4,4)^T, 16), ((5,5)^T, 15), ((5,7)^T, 21), ((6,3)^T, 12), ((7,6)^T, 6), ((8,1)^T, 18)\}$$

a) KNN regression $k=6$ training error



$x_1 = (1,1)^T$
$x_2 = (1,3)^T$
$x_3 = (3,2)^T$
$x_4 = (4,4)^T$
$x_5 = (5,5)^T$
$x_6 = (5,7)^T$
$x_7 = (6,3)^T$
$x_8 = (7,6)^T$
$x_9 = (8,1)^T$

$$f(x_1) = \frac{15 + 24 + 32 + 16 + 12 + 15}{6} = 19$$

$$f(x_2) = \frac{24 + 15 + 32 + 16 + 15 + 12}{6} = 19$$

$$f(x_3) = \frac{32 + 15 + 24 + 16 + 12 + 15}{6} = 19$$

$$f(x_4) = \frac{16 + 15 + 12 + 32 + 24 + 21}{6} = 20$$

$$f(x_5) = \frac{15 + 16 + 6 + 12 + 21 + 32}{6} = 17$$

$$f(x_6) = \frac{21 + 15 + 6 + 16 + 12 + 32}{6} = 17$$

$$f(x_7) = \frac{12 + 16 + 15 + 18 + 32 + 6}{6} = \frac{33}{2}$$

$$f(x_8) = \frac{6+21+15+12+16+18}{6} = \frac{44}{3}$$

$$f(x_9) = \frac{18+12+16+15+6+32}{6} = \frac{33}{2}$$

The training error TE is defined as:

$$TE(f,T) = \frac{1}{N}\sum_{i=1}^{N} L_2(y_i, f(x_i)) = \frac{1}{N}\sum_{i=1}^{N}(y_i - f(x_i))^2$$

$$TE(f,T) = \frac{1}{9}\Big[(15-19)^2 + (24-19)^2 + (32-19)^2$$
$$+(16-20)^2 + (15-17)^2 + (21-17)^2$$
$$+(12-\tfrac{33}{2})^2 + (6-\tfrac{44}{3})^2 + (18-\tfrac{33}{2})^2\Big]$$

$$TE(f,T) \approx 38.18$$

b) linear regression training error

The linear model's least squares estimator is:

$$\hat{\beta} = \begin{bmatrix}\hat{\beta}_0\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = (X^TX)^{-1}X^Ty \quad \text{where}$$

$$X = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 3\\ 1 & 3 & 2\\ 1 & 4 & 4\\ 1 & 5 & 5\\ 1 & 5 & 7\\ 1 & 6 & 3\\ 1 & 7 & 6\\ 1 & 8 & 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix}15\\24\\32\\16\\15\\21\\12\\6\\18\end{bmatrix}$$

$$X^TX = \begin{bmatrix} 9 & 40 & 32\\ 40 & 226 & 154\\ 32 & 154 & 150 \end{bmatrix} \qquad X^Ty = \begin{bmatrix}159\\637\\527\end{bmatrix}$$

Solve for $\hat{\beta}$

$\hat{\beta}_0 = \frac{77}{3}$ $\quad \hat{\beta}_1 = -\frac{259}{201}$ $\quad \hat{\beta}_2 = -\frac{257}{402}$

The predictor function is:

$$g(x) = \begin{bmatrix} 1 & x^T \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

$$g(x) = \hat{\beta}_0 + \hat{\beta}_1 x^{(1)} + \hat{\beta}_2 x^{(2)}$$

$$= \frac{77}{3} - \frac{259}{201} x^{(1)} - \frac{257}{402} x^{(2)}$$

$g(x_1) = \frac{3181}{134}$ $\quad g(x_4) = \frac{1203}{67}$ $\quad g(x_7) = \frac{6439}{402}$

$g(x_2) = \frac{9029}{402}$ $\quad g(x_5) = \frac{6443}{402}$ $\quad g(x_8) = \frac{2575}{201}$

$g(x_3) = \frac{1375}{67}$ $\quad g(x_6) = \frac{5929}{402}$ $\quad g(x_9) = \frac{5917}{402}$

$$TE(g, T) = \frac{1}{N} \sum_{i=1}^{N} (y_i - g(x_i))^2$$

$$TE(g,T) = \frac{1}{9}\Big[\left(15 - \frac{3181}{134}\right)^2 + \left(24 - \frac{9029}{402}\right)^2 + \left(32 - \frac{1375}{67}\right)^2 + \left(16 - \frac{1203}{67}\right)^2 + \left(15 - \frac{6443}{402}\right)^2 + \left(21 - \frac{5929}{402}\right)^2 + \left(12 - \frac{6439}{402}\right)^2 + \left(6 - \frac{2575}{201}\right)^2 + \left(18 - \frac{5917}{402}\right)^2\Big]$$

$$TE(g,T) \approx 36.41$$