Example

$n \geq 2$ students are at a party. What is the probability that two of them are born on the same day of the year?

B = at least two students have the same birthday

$$\Omega = \{(x_1, x_2, \dots, x_n) \mid 1 \leq x_i \leq 365\}$$

$$\#\Omega = 365^n$$

$$\#B = ?$$

$$B = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$$

The birthday problem

$$\mathbb{P}(\underbrace{B^c}_{\text{not } B}) = 1 - \mathbb{P}(B)$$

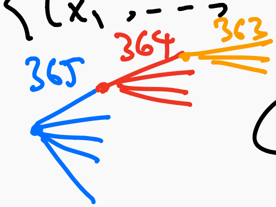
Example

$n \geq 2$ students are at a party. What is the probability that two of them are born on the same day of the year?

Simple observation: When $n > 365$ the probability is

$$\mathbb{P}(B^c) = \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{365^n} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}$$

$$B^c = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ are all different}\}$$



$$\rightarrow \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

$$P_n = \mathbb{P}(B) = 1 - \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

$$P_2 = 1 - \left(1-\frac{1}{365}\right) = \frac{1}{365} = 0.002\cancel{xx}$$

P_{25} $\quad P_{23} \simeq 0.51 \ldots \quad P_{80} = 0.99 \ldots$

Example

$n \geq 2$ students are at a party. What is the probability that two of them are born on the same day of the year?

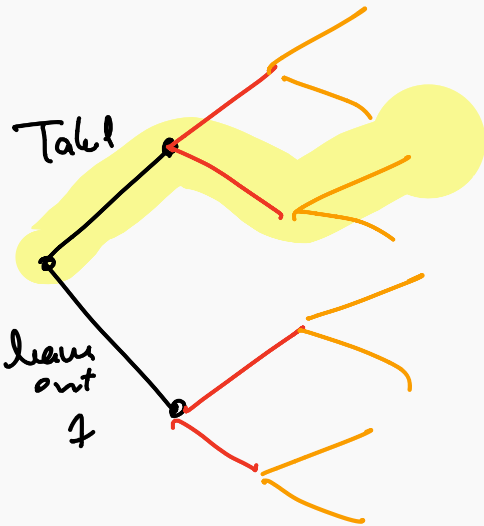
Simple observation: When $n > 365$ the probability is

$$p = 1.$$

Consider a set A with n elements.

$A = \{1, 2, \dots, n\}$

The total number of subsets of A is equal to 2^n:



$$\overbrace{2 \times 2 \times \dots \times 2}^{n \text{ times}}$$

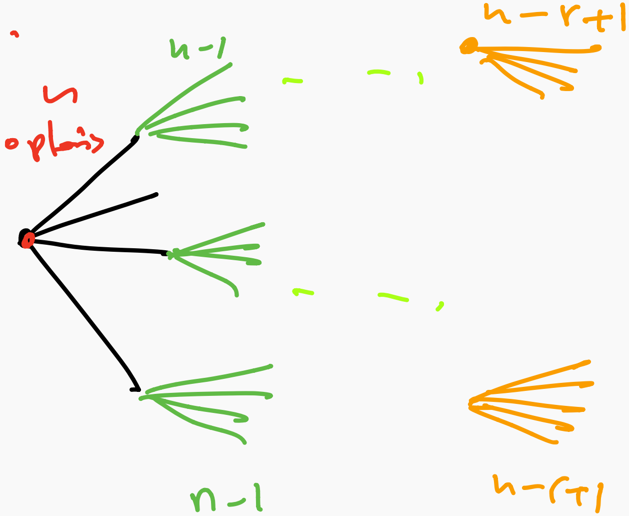
$$= 2^n$$

Counting the number of subsets of a given set

Consider a set A with n elements.

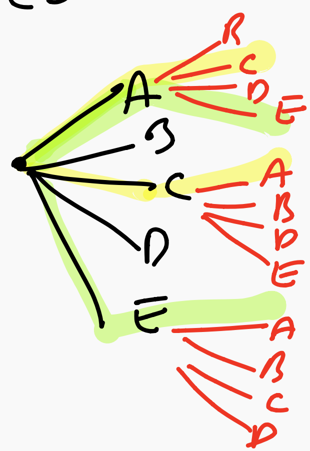
The total number of subsets of A is equal to 2^n:

<u>Question</u>: how many subsets of the set $\{1, \ldots, n\}$ are there with exactly r elements?



$$n(n-1)(n-2)\ldots(n-r+1)$$

Concrete A, B, C, D, E



$\#$ ways of forming a committee of r members out of n candidates

$\#$ r-element subsets of a set with n elements

$$= \frac{n(n-1)\cdots(n-r+1)}{r!}$$

$$= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 1}{r!\ \underbrace{(n-r)(n-r-1)\cdots 1}}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$\#$ r-element subsets of a set with n elements $= \dfrac{n!}{r!\,(n-r)!} = \dbinom{n}{r}$

C_n^r

n choose r

Counting the number of subsets of a given set

Consider a set A with n elements.

The total number of subsets of A is equal to 2^n:

The total number of subsets with k elements is given by

$$\begin{aligned}\binom{n}{r} &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\frac{(n-r)!}{(n-r)!} \\ &= \frac{n!}{r!(n-r)!}\end{aligned} \tag{1}$$

Counting questions

- From a group of 6 students how many different clubs consisting of 3 students can be formed?

$$\frac{6!}{3!\,(6-3)!} = \frac{\cancel{6} \times 5 \times 4 \times \cancel{3} \times \cancel{2}}{\cancel{3 \times 2} \times 1 \times \cancel{3} \times \cancel{2} \times 1} = 20$$

- From a group of 4 Math and 7 CS students how many clubs consisting of 2 math and 2 CS students can be formed?

$$\frac{4!}{2!\,2!} \times \frac{7!}{2! \cdot 5!} = 6 \times 21 = \underline{126}$$

Properties of binomial coefficients

Theorem

Binomial coefficients satisfy the following ~~tow~~ two properties:

- $\binom{n}{r} = \binom{n}{n-r}$ *for* $0 \leq r \leq n$.
- *For* $1 \leq r \leq n-1$ *we have*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

$$\frac{n!}{r!\,(n-r)!} = \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$1 \quad 2 \quad 3 \; . \; . \; . \qquad n$

$\binom{n-1}{r}$ = # teams not containing $\underline{\underline{n}}$

$\binom{n-1}{r-1}$ = # teams containing $\underline{\underline{n}}$

Pascal's triangle

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$$1\binom{0}{0}$$

$$1\binom{1}{0} \qquad \binom{1}{1}1$$

$$\overset{1}{\binom{2}{0}} \qquad \overset{2}{\binom{2}{1}} \qquad \binom{2}{2}1$$

$$\overset{1}{\binom{3}{0}} \qquad \overset{3}{\binom{3}{1}} \qquad \overset{3}{\binom{3}{2}} \qquad \binom{3}{3}1$$

$$\overset{1}{\binom{4}{0}} \qquad \overset{4}{\binom{4}{1}} \qquad \overset{6}{\binom{4}{2}} \qquad \overset{4}{\binom{4}{3}} \qquad \binom{4}{4}1$$

Example
A committee with an odd number of members (say, $N = 2n + 1$) is voting to choose one of the plans. Assume that the two plans are equally popular and each committee member votes for one plan with probability $1/2$. What is the probability that the last vote is a tie-breaker?

$2n =$ # voters excluding you

$$\mathbb{P}(\text{vote is tie breaking}) = \frac{\binom{2n}{n}}{2^{2n}}$$

$$= \frac{\frac{(2n)!}{n! \cdot n!}}{2^{2n}}$$

$$k! \sim \sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k$$

$$\frac{\dfrac{\sqrt{2\pi \cdot 2n}\,\left(\dfrac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\,\left(\dfrac{n}{e}\right)^{n}\right)^{2}}}{2^{2n}}$$

$$\approx \frac{\sqrt{4\pi n}\;\dfrac{\cancel{2^{2n}}\cdot\cancel{n^{2n}}}{\cancel{e^{2n}}}}{\dfrac{2\pi n\cdot\cancel{n^{2n}}}{\cancel{e^{2n}}}\cdot\cancel{2^{2n}}}$$

$$\overset{n\ \text{large}}{\approx} \frac{\cancel{2}\sqrt{\pi}\sqrt{n}}{\cancel{2}\,\pi n} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}$$

$$\frac{1}{1.7..}\cdot\frac{1}{\sqrt{n}}$$

Example

A committee with an odd number of members (say, $N = 2n + 1$) is voting to choose one of the plans. Assume that the two plans are equally popular and each committee member votes for one plan with probability $1/2$. What is the probability that the last vote is a tie-breaker?

For $N = 1001$ then the probability is approximately $p = 0.018$.

A B disjoint $\quad A \cap B = \emptyset$

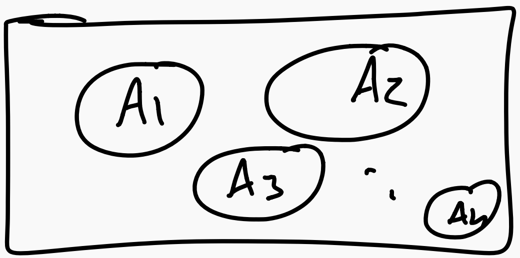
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Theorem

Let $A_1, \dots, A_n$ be n mutually disjoint events, i.e. $A_i \cap A_j = \emptyset$, when $i \neq j$. Then

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \mathbb{P}\left[A_i\right].$$

$$\mathbb{P}\left(\begin{array}{c}\text{one of the events}\\ A_1, \dots, A_n \text{ occurs}\end{array}\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$



From a group of 10 math and 10 CS students a committee with 6 memebers is formed. What is the probability that there are more ~~math~~ CS than ~~CS~~ math students in the committee?

A = There are more math than CS student

$A_1 = (4$ CS, 2 math

$A_2 = 5$ CS, 1 math

$A_3 = 6$ CS, 0 math

$$\mathbb{P}(A_1) = \frac{\binom{10}{4}\binom{10}{2}}{\binom{20}{6}}$$

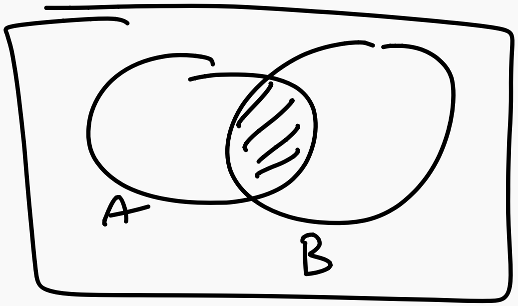
$$\mathbb{P}(A_2) = \frac{\binom{10}{5}\binom{10}{1}}{\binom{20}{6}}$$

$$\mathbb{P}(A_3) = \frac{\binom{10}{6}\binom{10}{0}}{\binom{20}{6}}$$

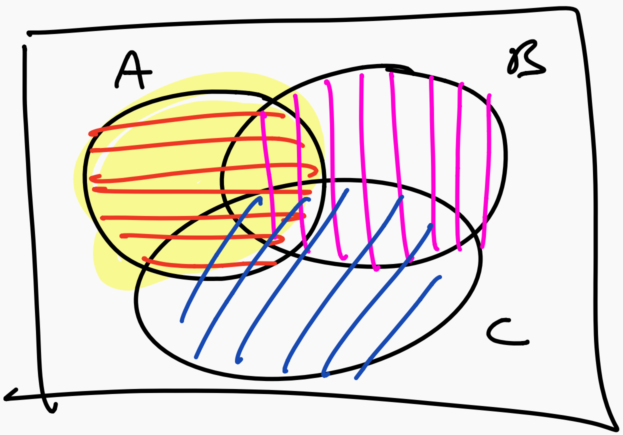
$$\mathbb{P}(A) = \frac{\binom{10}{4}\binom{10}{2}}{\binom{20}{6}} + \frac{\binom{10}{5}\binom{10}{1}}{\binom{20}{6}} + \frac{\binom{10}{6}\binom{10}{0}}{\binom{20}{6}}$$

The inclusion-excluson principle with three events

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$



$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$



Inclusion and exclusion

Theorem (Inclusion-Exclusion principle)

Let $A_1, A_2, ...A_n$ *be event and* $A = \bigcup_{i=1}^{n} A_i$*. Then*

$$\mathbb{P}[A] = \sum_{i} \mathbb{P}[A_i] - \sum_{i<j} \mathbb{P}[A_i \cap A_j] + \sum_{i<j<k} \mathbb{P}[A_i \cap A_j \cap A_k] - \cdots + (-1)^{n+1} \mathbb{P}[A_1 \cap A_2 \cap \cdots$$