Tutorial 2 notes:

22-Sept

Secretary Problem:

How to hire best candidate out of n candidate by interviewing one by One? Cannot make decision after interviewing all.

Strategy:

Reject first 'r' people and hire the first better candidate than first r.

How to find 'r' which maximizes probability?

$$P(r) = \sum_{i=1}^{n} P(\text{Person } i \text{ is selected} \cap \text{Person } i \text{ is best})$$

$$= \sum_{i=1}^{r-1} 0 + \sum_{i=r}^{n} P(i \text{ selected} \mid i \text{ best}) \cdot P(i \text{ best})$$

Reject first r with probability 1

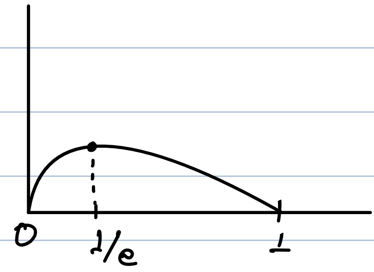
$$= \frac{1}{n} \sum_{i=r}^{n} P(\text{best of } i-1 \text{ is in first } r-1 \mid i \text{ is best})$$

$$= \frac{1}{n} \sum_{i=r}^{n} \frac{r-1}{i-1} = \frac{r-1}{n} \sum_{i=r}^{n} \frac{1}{i-1}$$ Find r to maximize the sum.

Substitute $x = \frac{r-1}{n}$ $\quad t = \frac{i-1}{n}$ $\quad dt = \frac{1}{n}$

Approx $P(x)$ by integral:

$$P(x) = x \int_x^1 \frac{dt}{t} = -x \ln x$$



Solve for $\frac{d}{dx} P(x) = 0$

local maxima at $x = 1/e$ $\Rightarrow$ optimal cutoff $(r) = n/e$

$\Rightarrow$ Can hire best candidate with probability $\approx 37\%$

\# Suppose $A = \{x_1, \ldots, x_m\}$, $B = \{y_1, \ldots, y_n\}$

F = set of all functions from A to B

Pick any $f \in F$, what is probability f is one-to-one?

First note: If $m > n$, no one-to-one functions are formed.

$\Rightarrow p(m,n) = 0$ for $m > n$

For $|F|$: Each $x_i \in A$ have n options to map $\Rightarrow |F| = \underbrace{n \cdot n \cdot \cdots \cdot n}_{m} = n^m$

For $m \leq n$: n options for $f(x_1)$

$n-1$ options for $f(x_2)$

$\vdots$

$n-m+1$ options for $f(x_m)$

$$\Rightarrow P(m,n) = \frac{n \cdot (n-1) \cdots (n-m+1)}{n^m} = \prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right)$$