9 Sept 2021

Thursday.

Calculus and linear Algebra
Homework 1

Q-1

a) $p(x) = 2x^2 + 12x + 26 \qquad \div 2$

$= x^2 + 6x + 13$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 - 52}}{2}$$

$$= -6/2 \pm \frac{\sqrt{-16}}{2}$$

$$\boxed{x = -3 \pm 2i}$$

b) $bx^2 - b + 2 = 0$ $\Rightarrow$ No real roots.

$b^2 - 4ac < 0$

$(-b)^2 - 4(b)(2) < 0$

$b^2 - 8b < 0$

$b(b-8) < 0$

	$b < 0$	$0 < b < 8$	$b > 8.$
$b=-1$	x	x	x
$b=1$	x	✓	x

$$\boxed{0 < b < 8}$$

c) $P(x) = x^6 - x^5 - 3x^4 - 3x^3 - 22x^2 + 4x + 24.$

$\rightarrow (x-3)$ is a root.

Using synthetic division:

$$\begin{array}{r|rrrrrrr} & 1 & -1 & -3 & -3 & -22 & +4 & +24 \\ 3\times & & 3 & 6 & 9 & 18 & -12 & -24 \\ \hline & 1 & 2 & 3 & 6 & -4 & -8 & \underline{0} \end{array}$$

$x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8$ $\quad x = +1$

$1 + 2 + 3 + 6 - 4 - 8$

$= 0$

$\rightarrow (x-1)$

$$\begin{array}{r|rrrrrr} & 1 & 2 & 3 & 6 & -4 & -8 \\ 1 & & 1 & 3 & 6 & 12 & 8 \\ \hline & 1 & 3 & 6 & 12 & +8 & \underline{0} \end{array}$$

$x^4 + 3x^3 + 6x^2 + 12x + 8.$ $\quad x = -1$

$1 - 3 + 6 - 12 + 8$

$= 0$

$\rightarrow (x+1)$

$$\begin{array}{r|rrrrr} & 1 & 3 & 6 & 12 & 8 \\ -1 & & -1 & -2 & -4 & -8 \\ \hline & 1 & 2 & 4 & 8 & \underline{0} \end{array}$$

$x^3 + 2x^2 + 4x + 8.$

$x = -2$

$-8 + 8 - 8 + 8.$

$= 0.$

$\rightarrow (x+2)$

$$
\begin{array}{r|rrrr}
 & 1 & 2 & 4 & 8 \\
-2 & & -2 & 0 & -8 \\
\hline
 & 1 & 0 & 4 & 0.
\end{array}
$$

$$x^2 + 4 = 0$$

$$x = \sqrt{-4}$$

$$x = \pm 2i$$

$$P(x) = (x-1)(x+1)(x-3)(x+2)(x+2i)(x-2i)$$

Q-2 $z = a + ib$

a) $\dfrac{1}{(z^*)^2}$

$$= \frac{1}{(a - ib)^2} = \frac{1}{a^2 + i^2b^2 - 2abi}$$

$$= \frac{1}{a^2 - 2abi - b^2} \times \frac{a^2 - b^2 + 2abi}{a^2 - b^2 + 2abi}$$

$$= \frac{a^2 - b^2 + 2abi}{(a^2 - b^2)^2 - 4a^2b^2i^2} = \frac{a^2 - b^2 - 2abi}{(a^2 - b^2)^2 + 4a^2b^2}$$

$$= \frac{a^2 - b^2 + 2abi}{(a^2 + b^2)^2}$$

$$Re\left(\frac{1}{(z^*)^2}\right) = \frac{a^2 - b^2}{(a^2 + b^2)^2}$$

$$Im\left(\frac{1}{(z^*)^2}\right) = \frac{2ab}{(a^2 + b^2)^2}$$

b) $\frac{2+z}{2z+2}$

let $z = a+ib$

$\frac{2+a+ib}{2a+2ib+2}$

$= \frac{(2+a)+ib}{(2a+2)+2ib} \times \frac{(2a+2)-2ib}{(2a+2)-2ib}.$

$\frac{((2+a)+ib) \times (2a+2-2ib)}{(2a+2)^2-(2bi)^2}$

$= \frac{2a^2+2a+4a+4-4bi+2bi+2b^2}{4a^2+8a+4+4b^2}$

$= \frac{a^2+3a+2-bi+b^2}{2a^2+4a+2+2b^2}$

$\frac{a^2+3a+2+b^2}{2a^2+4a+2b^2+2} - \frac{bi}{2a^2+4a+2+2b^2}$

$Re\left(\frac{z+2}{2z+2}\right) = \frac{a^2+3a+2+b^2}{2a^2+4a+2b^2+2}$

$Im\left(\frac{z+2}{2z+2}\right) = \frac{b}{2a^2+4a+2+2b^2}$

c) $(z^*)^2 z$

let $z = a + ib$

$z\ (a - ib)^2 (a + ib).$

$(a^2 + i^2 b^2 - 2abi)(a + ib).$

$(a^2 - 2abi - b^2)(a + ib)$

$a^3 + a^2bi - 2a^2bi - 2ab^2i^2 - ab^2 - b^3i$

$a^3 - a^2bi + 2ab^2 - ab^2 - b^3i$

$a^3 + ab^2 - a^2bi - b^3i$

$(a^3 + ab^2) + (-a^2b - b^3)i$

$\mathrm{Re}\left((z^*)^2 z\right) = a^3 + ab^2$ $\quad \mathrm{Im}\left((z^*)^2 z\right) = -a^2b - b^3$

d) $\left| \dfrac{1-i}{2+i} \right|$

$= \sqrt{\dfrac{1^2 + (-1)^2}{2^2 + 1^2}}$

$= \dfrac{\sqrt{2}}{\sqrt{5}} \quad = \dfrac{\sqrt{10}}{5} \quad \approx 0.632$

e) $|4x+2| \leq |2x-3|$ taking square on both sides removes $| \, |$

$(4x+2)^2 \leq (2x-3)^2$

$16x^2 + 16x + 4 \leq 4x^2 - 12x + 9$

$12x^2 + 28x - 5 \leq 0.$

$12x^2 + 30x - 2x - 5 \leq 0$

$(2x+5)(6x-1) \leq 0.$

$\begin{cases} 2x+5 \leq 0 \\ 6x-1 \geq 0 \end{cases}$

$\begin{cases} 2x+5 \geq 0 \checkmark \\ 6x-1 \leq 0 \checkmark \end{cases}$

$x \in \emptyset \qquad x \in \left[-\frac{5}{2}, \frac{1}{6}\right]$

Q-3

a) $\frac{z^*}{w^*} = \left(\frac{z}{w}\right)^*$

L.H.S	R.H.S
$\frac{a-ib}{c-id} \times \frac{c+id}{c+id}$	$\left(\frac{a+ib}{c+id} \times \frac{c-id}{c-id}\right)^*$
$\frac{ac+adi-cbi-bdi^2}{c^2-i^2d^2}$	$\left(\frac{ac-adi+bci-bdi^2}{c^2-i^2d^2}\right)^*$
$= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$	$\left(\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right)^*$
	$= \left(\frac{(ac+bd)-i(ad-bc)}{c^2+d^2}\right)^*$
	$\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$

L.H.S = R.H.S Hence Proved.

b) $Re(z) = \frac{z + z^*}{2}.$

let $z = a + ib$

$z = a - ib$

$Re(z) = a.$

$\frac{z+z^*}{2} = \frac{a + i\cancel{b} + a - i\cancel{b}}{2.}$

$= \frac{\cancel{2}a}{\cancel{2}}$

$= \cancel{2}\ a$

hence $\frac{z+z^*}{2} = Re(z) = a$

c) $Im(z) = \frac{z - z^*}{2i}$

let $z = a + ib$ so $Im(z) = b$.

$z^* = a - ib$

$$\frac{a + ib - (a - ib)}{2i}$$

$$= \frac{\cancel{a} + ib \cancel{- a} + ib}{2i}$$

$$= \frac{\cancel{2i}b}{\cancel{2i}}$$

$= b$ hence $Im(z) = b = \frac{z - z^*}{2i}$