28 sept 2021

Homework #3

Calculus & Linear Algebra

Problem 1

a) $\lim\limits_{\theta \to 0} \dfrac{1 - \cos^2\theta}{\theta^2}$ $\sin^2\theta + \cos^2\theta = 1$

$= \lim\limits_{\theta \to 0} \dfrac{\sin^2\theta}{\theta^2}$

$\Rightarrow$ Using L'Hopital Rule.

$= \lim\limits_{\theta \to 0} \dfrac{2\sin\theta\cos\theta}{2\theta}$

$= \lim\limits_{\theta \to 0} \dfrac{\cancel{2}\sin 2\theta}{2\theta}$ apply L'Hopital Rule again.

$\lim\limits_{\theta \to 0} \dfrac{2\cos 2\theta}{2}$

$= \dfrac{2\cos(2\cdot 0)}{2} = \dfrac{\cancel{2}\times 1}{\cancel{2}} = \boxed{1}$

b) Finding Asymptotes of graph.

$$\frac{3x^2-12x+9}{x^2-5}$$

For Horizontal Asyptotes, $x \to \infty$.

$$\cancel{3x^2} \lim_{x\to\infty} \frac{3x^2-12x+9}{x^2-5} \quad \cancel{x^2} \quad \frac{}{\cancel{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2-12x+9}{x^2}}{\frac{x^2-5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{12x}{x^2} + \frac{9}{x^2}}{\frac{x^2}{x^2} - \frac{5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3-12x^{-1}+9x^{-2}}{1-5x^{-2}}$$

$\frac{n}{\infty} = 0$ ~~for HAs~~.

$$= \frac{3 - 12/\infty + 9/\infty}{1 - 5/\infty}$$

$$= \frac{3-0+0}{1-0} = 3$$

So horizontal Asymptote will be at $\boxed{y=3}$

~~Verticle~~

Vertical Asymptotes:

$$\frac{3x^2 - 12x + 9}{x^2 - 5}$$ denominator $= 0$.

so $x^2 - 5 = 0$

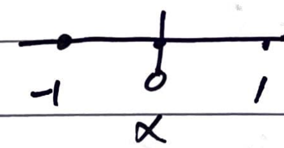
$x^2 = 5$

$x = \pm\sqrt{5}$

$$\boxed{\begin{array}{l} \text{Horizontal Asymptotes} \Rightarrow y = 3 \\ \text{Vertical Asymptotes } x = +\sqrt{5},\ x = -\sqrt{5} \end{array}}$$

Problem 2 (using Squeeze Law)

a) $\lim\limits_{x \to 0} x^n \cos\left(\frac{1}{x^n}\right)$ $\quad n \in \mathbb{N} \setminus \{0\}$



$$-1 \leq \cos\frac{1}{x^n} \leq 1$$

$$\lim_{x \to 0} -x^n \leq \lim_{x \to 0} \cos\frac{1}{x^n} \leq \lim_{x \to} x^n$$

$$-0^n \leq \lim_{x \to 0} \cos\frac{1}{x^n} \leq 0^n$$

means

$$\lim_{x \to 0} x^n \cos\frac{1}{x^n} = 0.$$

b) $\lim_{x \to 0} x^2 e^{\sin 1/x}$

$-1 \leq \sin 1/x \leq 1$

$e^{-1} \leq e^{\sin 1/x} \leq e^{1} \quad \times x^2$

$x^2 e^{-1} \leq x^2 e^{\sin 1/x} \leq x^2 e$

$\lim_{x \to 0} x^2 e = 0$

$\lim_{x \to 0} \frac{x^2}{e} = 0$

means $0 \leq x^2 e^{\sin 1/x} \leq 0$

$$\boxed{\lim_{x \to 0} x^2 e^{\sin 1/x} = 0}$$

Problem 3

a) $\cos(x) = e^{x} + x + 2$

applying intermediate Value Theorem

$y = e^{x} + x - \cos(x) + 2$

$f(x) = e^{x} + x - \cos x + 2.$

$f(1) = e + 1 + 3 - \cos(1)$
$= \boxed{5.18}$

$f(-1) = 1/e + (-1) + 2 - \cos(-1)$

$= 0.83$

Value of $f(x)$ exists at $x = -1$ and $x = 1$
so it is proven with IVT that that
$\cos(x) = e^{x} + x + 2$ is having atleast
one solution.

Bonus

$f(x) = e^x + x + 2$

$f'(x) = \cancel{x}e^x + 1 + 0$

Applying Intermediate Value Theorum

$f'(-1) = 1/e + 1 \quad = 1.37$

$f'(0) = e^0 + 1 \quad = 2$

$f(-1) = 1/e - 1 + 2 = 1.37.$

$f(-2) = 1/e^2 + 2 - 2 = 0.135.$ change of sign.

$f(-3) = 1/e^3 + 2 - 3 = -0.95$

$f(4) = 1/e^4 + 2 - 4 = -1.98$

hence the root lies between $x = -1, x = -3$