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Probability and Random Processes

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Random variables: definition

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Definition
Let $(\Omega, \mathbb{P})$ be a probability space. A function

$$X : \Omega \to \mathbb{R}$$

is called a real valued *random variable*. Similarly, a function $X : \Omega \to \mathbb{R}^n$ is called a vector-valued random variable.

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Discrete random variables

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Definition
The probability mass function of a random variable X with sets of values $x_1, \ldots$ is defined by

$$p(x) = \mathbb{P}\left[X = x\right].$$

Note that for every i, we have $p(x_i) = p_i > 0$.

discrete random variable

x_1	x_2	x_3	x_4	$\ldots$
p_1	p_2	p_3	p_4	

$$p_i = \mathbb{P}[X = x_i]$$

$$\boxed{\begin{gathered} 0 \leq p_i \leq 1 \\ \sum_i p_i = 1 \end{gathered}}$$

Bernoulli random variables

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A random variable X is called the *Bernoulli* random variable with parameter p if it only takes values 0 and 1, and

$$\mathbb{P}[X = 1] = p, \qquad \mathbb{P}[X = 0] = 1 - p.$$

	0	1	2
	p	q	r

x_i	0	1
$P(X=x_i)$	$1-p$	p

$$\boxed{0 \le p, q, r \qquad p+q+r=1}$$

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A Bernoulli random variable X tells us whether something happened or not. The probability of happening $\mathbb{P}[X = 1]$ is called the parameter of X.

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Example

A die is rolled. Let X be the random variable that tells us whether the outcome is larger than 4 or not. $\mathbb{P}(X>4) = \mathbb{P}(\{5,6\}) = 1/3$.

Binomial distribution

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A coin is thrown n times. Suppose that

- The coin shows H with probability p and T with probability $1 - p$.
- Outcomes of different rounds are independent.

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Suppose $n=2$: Then

$$\begin{aligned}
\mathbb{P}[HH] &= \mathbb{P}[\text{ first} H]\,\mathbb{P}[\text{ second} H] = p^2. \\
\mathbb{P}[HT] &= \mathbb{P}[\text{ first} H]\,\mathbb{P}[\text{ second} T] = p(1-p). \\
\mathbb{P}[TH] &= \mathbb{P}[\text{ first} T]\,\mathbb{P}[\text{ second} H] = (1-p)p. \\
\mathbb{P}[TT] &= \mathbb{P}[\text{ first} T]\,\mathbb{P}[\text{ second} T] = (1-p)^2.
\end{aligned} \tag{1}$$

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$X =$ # of Heads

$$\begin{array}{ccccc}
\text{HH} & \longrightarrow & 2 & p^2 \\
\text{HT TH} & \longrightarrow & 1 & 2p(1-p) \\
\text{TT} & \longrightarrow & 0 & (1-p)^2
\end{array}$$

0	1	2
$(1-p)^2$	$2p(1-p)$	p^2

Binomial distribution

Suppose $n = 3$. Then the number of heads could be $0, 1, 2, 3$

Bernoulli trial

$$\begin{array}{ccc} \text{HHH} & \longrightarrow & 3 & p^3 \\ \text{HHT HTH THH} & \longrightarrow & 2 & 3p^2(1-p) \\ \text{HTT THT TTH} & \longrightarrow & 1 & 3p(1-p)^2 \\ \text{TTT} & \longrightarrow & 0 & (1-p)^3 \end{array}$$

$n = 1, 2, 3, \ldots$

$0 \leq p \leq 1$

Definition
A random variable X has the Binomial distribution with parameters (n, p) if,

$$\mathbb{P}[X = k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(\underbrace{HH\ldots H}_{k}\underbrace{T\ldots T}_{n-k}) = p^k(1-p)^{n-k}$$

$\binom{n}{k}$ = # ways of picking k items out of n items.

$\underline{H}\ \underline{\ }\ \underline{H} \cdots \underline{H}\ \underline{H}$

Suppose X is a Binomial random variable with parameter $(n, p) = (5, 1/3)$.
Find $\mathbb{P}[X \geq 4]$ and $\mathbb{P}[X \leq 1]$.

$$\mathbb{P}(\underline{X \geq 4}) = \mathbb{P}(X=4 \text{ or } X=5) =$$

$$\mathbb{P}(X=5) = p^5 = (1/3)^5$$

$$\mathbb{P}(X=4) = \binom{5}{4} p^4 (1-p)^1$$

$$= 5p^4(1-p) = 5(1/3)^4 \cdot 2/3$$

X	0	1	2	3	4	5
					$5(1/3)^4 \cdot 2/3$	$(1/3)^5$

$$\mathbb{P}(X \geq 4) = (1/3)^5 + 5(1/3)^4 \cdot 2/3$$

$$\mathbb{P}(X \leq 1) = \mathbb{P}(X=0) + \mathbb{P}(X=1)$$

$$(2/3)^5 + \binom{5}{1}(2/3)^4(1/3)^1$$

for every k

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

<u>special case</u>

$p = 1/2$

$$\mathbb{P}(X=k) = \binom{n}{k} (1/2)^k \cdot (1/2)^{n-k}$$

$$= \binom{n}{k} \cdot (1/2)^n = \frac{\binom{n}{k}}{2^n}$$

$n=5$

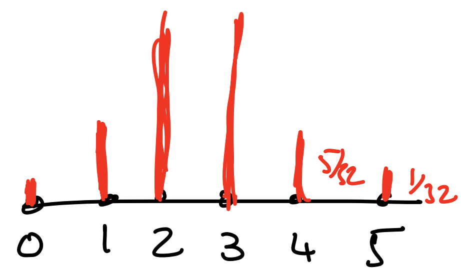
X	0	1	2	3	4	5
$P(X=x)$	$1/32$	$5/32$	$10/32$	$10/32$	$5/32$	$1/32$

$\binom{5}{0} = 1$ $\qquad$ $\binom{5}{4} = 5$

$\binom{5}{1} = 5$ $\qquad$ $\binom{5}{5} = 1$

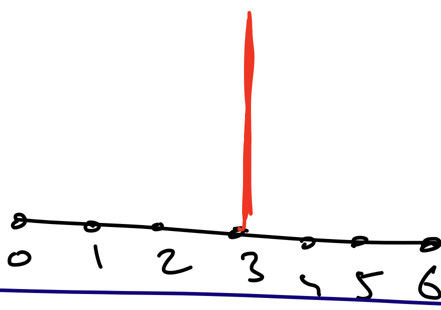
$\binom{5}{2} = 10$

$\binom{5}{3} = 10$

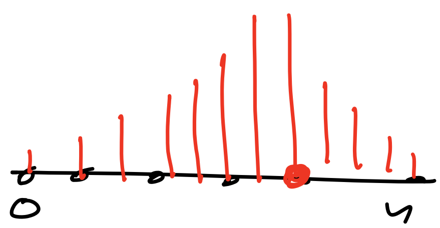


unimodal

$n=6$



$p \neq 1/2$



Geometric distribution

Consider a coin that turns up heads with probability p and tails with probability $1 - p$. The coin is flipped until the a heads shows ups. Let X be the number of the flips needed. Assuming that the outcome of the flips are independent, we have

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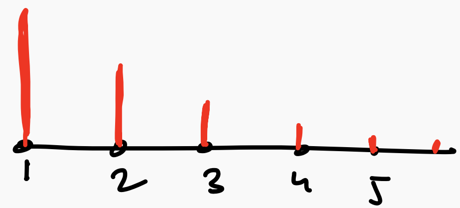
$X =$ first occurance of H

$$\mathbb{P}[X = n] = (1 - p)^{n-1} p.$$

$\mathbb{P}(X=1) = p$

$\mathbb{P}(X=2) = (1-p) \cdot p$

$\mathbb{P}(X=n) = (1-p)^{n-1} \cdot p$



$p = 1/3$

Consider a coin that turns up heads with probability p and tails with probability $1-p$. The coin is flipped until the a heads shows ups. Let X be the number of the flips needed. Assuming that the outcome of the flips are independent, we have

$$\mathbb{P}[X=n]=(1-p)^{n-1}p.$$

Definition
A discrete random variable X has Geometric distribution with parameter p if

$$\mathbb{P}[X=k]=\begin{cases} p(1-p)^{k-1} & \text{if } k\geq 1 \quad k=1,2,\ldots \\ 0 & \text{otherwise}\end{cases}$$

Example

Let X be a random variable with the geometric distribution with parameter p. Find $\mathbb{P}[X > k]$.

$$\mathbb{P}[X>k] = \mathbb{P}(X = k+1, k+2, \ldots)$$

$$= \sum_{i=k+1}^{\infty} \mathbb{P}(X=i) = \sum_{i=k+1}^{\infty} p(1-p)^{i-1}$$

$$= p\left[(1-p)^k + (1-p)^{k+1} + (1-p)^{k+2} + \cdots\right]$$

$0<q<1$

$$= p(1-p)^k \left[1 + (1-p) + (1-p)^2 + \cdots\right] = p(1-p)^k \cdot \frac{1}{1-(1-p)}$$

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}$$

$$= p(1-p)^k \cdot \frac{1}{p} = (1-p)^k.$$

$$\mathbb{P}(X > k) = \mathbb{P}\left(\begin{array}{c}\text{first } k \text{ trials were} \\ \text{failure}\end{array}\right)$$

$$= (1-p)^k.$$

Poisson random variables

There are $n = 200$ people in a seminar. Suppose that the mobile phone of each participant rings during the meeting with probability $p = 0.01$. Assuming the independence of the ringing events, find the probability of the event that k phones ring during the meeting.

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n participants

p probability of success for each participant

n large $\quad p$ small $\quad np = \lambda$

$X = \#$ success

distribution of X $\quad \underline{\mathbb{P}(X = k)}$

$$\mathbb{P}(X=k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

$n \longrightarrow \infty$

$p \longrightarrow 0$ $\quad np = \lambda \Longrightarrow p = \frac{\lambda}{n}$

$$\mathbb{P}(X=0) = \binom{n}{0} \cdot p^0 (1-p)^n$$

$$= \left(1 - \frac{\lambda}{n}\right)^n$$

$$\boxed{\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x} \qquad \underset{n\to\infty}{=} e^{-\lambda}.$$

$$\mathbb{P}(X=1) = \binom{n}{1} \cdot p^1 (1-p)^{n-1}$$

$$= n \cdot p \cdot \left(1-\frac{\lambda}{n}\right)^{n-1}$$

$$\frac{\lambda\left(1-\frac{\lambda}{n}\right)^n}{1-\frac{\lambda}{n}} \xrightarrow[n\to\infty]{} \frac{\lambda \cdot e^{-\lambda}}{1} = \lambda e^{-\lambda}.$$

exercice

$$\boxed{\mathbb{P}(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$$

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)\cdots 1}{(n-k)\cdots 1} \cdot$$

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$\frac{n^k - n^{k-1} + \cdots}{n^k} \to 1$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} \cdot \frac{1}{k!} \cdot \frac{\lambda^k \left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

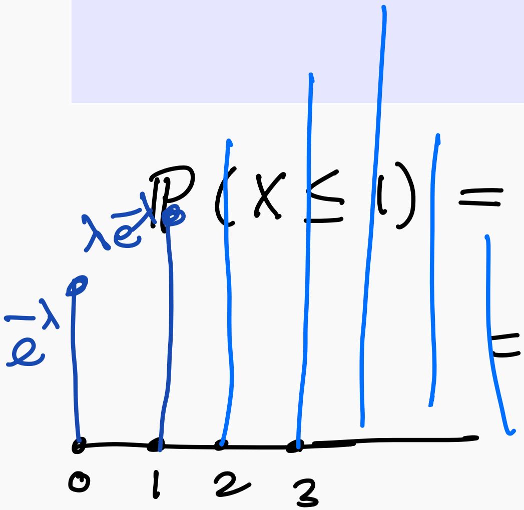
$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$

$\left(1-\frac{\lambda}{n}\right)^k \to 1$

Poisson distribution

Definition
A discrete random variable X has Poisson distribution with parameter λ if

$$\mathbb{P}[X=k] = \begin{cases} e^{-\lambda}\frac{\lambda^k}{k!} & \text{if } \cancel{k \geq 1}\ k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$



$$\mathbb{P}(X \leq 1) = \mathbb{P}(X=0) + \mathbb{P}(X=1)$$

$$= e^{-\lambda} \cdot \frac{\lambda^0}{0!} + e^{-\lambda} \cdot \frac{\lambda^1}{1!}$$

$$= e^{-\lambda}(1+\lambda).$$

$$\mathbb{P}(X=2) = e^{-\lambda} \cdot \frac{\lambda^2}{2} \qquad \mathbb{P}(X=3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!}$$

Distribution function of a random variable

Definition
Let $X : \Omega \to \mathbb{R}$ be a random variable. The *probability distribution function* of X is the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(t) = \mathbb{P}\left[X \le t\right].$$

Examples: the distribution function of a Bernoulli random variable

	0	1
	$1-p$	p

$$F_X(t) = \mathbb{P}(X \le t)$$

$t < 0$

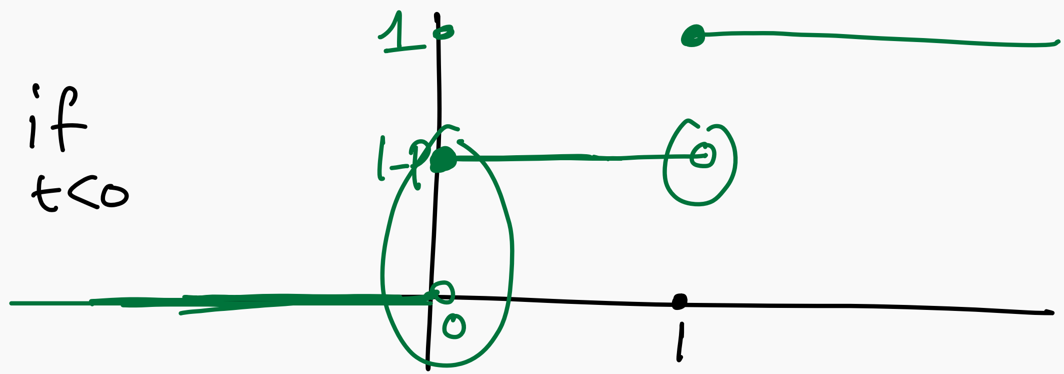
$\mathbb{P}(X \le t) = 0$ if $t<0$

$\mathbb{P}(X \le 0)$

$= \mathbb{P}(X = 0) = 1-p$

$0 < t < 1$

$\mathbb{P}(X \le t) = 1-p$



$\mathbb{P}(X \le 1) = 1$