Tutorial 6
Oct 20

Expectation of Random variables:

For X discrete taking values $\{x_1, x_2, \cdots\}$ with probability $\mathbb{P}[x_i]$, its expectation is defined as:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} x_k \, \mathbb{P}[X = x_k] = \sum_{k=1}^{\infty} x_k f_X(k)$$

$\uparrow$ pmf

For $\mathbb{E}[X]$ to be well defined, above infinite series has to be "absolutely" convergent.

i.e. we want $\displaystyle\sum_{k=1}^{\infty} |x_k| \, \mathbb{P}[X = x_k] < \infty$

what if the series is not absolutely convergent?

Take the following series:

$$\begin{aligned} S_n &= 1 - 1 + 1 - 1 + \cdots \\ &= (1-1) + (1-1) + \cdots \\ &= 0 + 0 + \cdots \\ &= 0 \end{aligned}$$

other way,

$$\begin{aligned} S_n &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1 \end{aligned}$$

So, if series is not absolutely convergent, can sum up to any value from series.

more Examples: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ not convergent.

$\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \longrightarrow$ Converges to $\frac{\pi^2}{6}$.

Expectations of standard distributions:

i) X = Bernoulli RV:

$$f_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$$

2) X = Poisson with parameter 'λ'.

i.e. $$\mathbb{P}[X = x] = \frac{e^{-\lambda} \lambda^{k}}{k!}$$

$$E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda^{k}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

Substitute $k-1 = j$

$$\Rightarrow E[X] = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j}}{j!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Taylor expansion of e^{λ}.

3) X = Binomial with paramater p.

$$\mathbb{P}[X = k] = \binom{n}{k} p^{k} (1-p)^{n-k}$$

$$E[X] = np$$

Theorem: X RV taking only positive integers, than

$$E[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$$

Proof:

$$E[X] = \sum_{k=0}^{\infty} k\,\mathbb{P}[X=k] = \sum_{k=1}^{\infty} k\,\mathbb{P}[X=k] \quad \text{---- (1)}$$

$$= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}[X=i] \quad \text{---- (2)}$$

$$= \sum_{k=1}^{\infty} \mathbb{P}[X \geq k] \qquad \square$$

Intution behind double sum:

Take (1): $\sum_{k=1}^{\infty} k\,\mathbb{P}[X=k] = \mathbb{P}(1) + 2\cdot\mathbb{P}(2) + 3\cdot\mathbb{P}(3) + \cdots + k\cdot\mathbb{P}(k) + \cdots$

$\mathbb{P}[k]$ repeated k times.

Take $k=1$ in the double sum:

$$S_1 := \sum_{i=1}^{\infty} \mathbb{P}[X=i] = \mathbb{P}[1] + \mathbb{P}[2] + \cdots$$

$$S_2 := \sum_{i=2}^{\infty} \mathbb{P}[X=i] = \mathbb{P}[2] + \mathbb{P}[3] + \cdots$$

$$\vdots$$

$$\sum_{k=1}^{\infty} S_k = S_1 + S_2 + \cdots = \left(\mathbb{P}(1) + \mathbb{P}(2) + \cdots\right) + \left(\mathbb{P}(2) + \mathbb{P}(3) + \cdots\right) + \cdots$$

$$= \mathbb{P}(1) + 2\cdot\mathbb{P}(2) + 3\cdot\mathbb{P}(3) + \cdots$$

In general $\mathbb{P}(k)$ is counted k times.

Application of above theorem:

Consider X = geometric dist.

i.e. $\mathbb{P}[X=k] = p(1-p)^{k-1}$ 1 success and k-1 failures

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\,\mathbb{P}[X=k] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$$

need k-1 failures

$$= \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{p}$$

Geometric sum

as $0 < 1-p < 1$