CH-231-A

Algorithms and Data Structures

ADS

Lecture 42

Dr. Kinga Lipskoch

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General: Linear Programming[1]

- ▶ Important tool for optimal allocation of scarce resources, among a number of competing activities.
- ▶ Powerful and general problem-solving method.
- ▶ Applications:
    - ▶ Computer science: Compiler register allocation, data mining.
    - ▶ Electrical engineering: VLSI design, optimal clocking.
    - ▶ Economics: Equilibrium theory, two-person zero-sum games.
    - ▶ Environment: Water quality management.
    - ▶ Logistics: Supply-chain management, Berlin airlift.
    - ▶ Manufacturing: Production line balancing, cutting stock.
    - ▶ Telecommunication: Network design, Internet routing.

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[1]Source of slides: Kevin Wayne: Algorithms and Data Structures, Spring 2004, Princeton University

Brewery Problem: A Toy LP Example

Small brewery produces ale and beer.

- ▶ Production limited by scarce resources: corn, hops, malt.
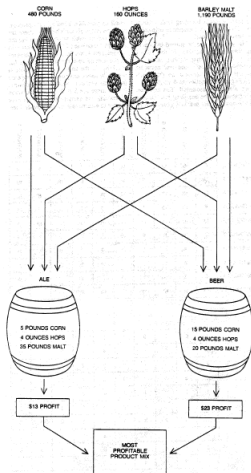- ▶ Recipes for ale and beer require different proportions of resources.

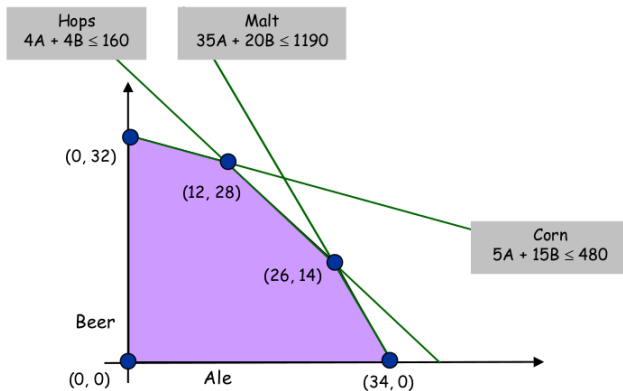| Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale	5	4	35	13
Beer	15	4	20	23
Quantity	480	160	1190	

How can brewer maximize profits?

- ▶ Devote all resources to ale: 34 barrels of ale → $442.
- ▶ Devote all resources to beer: 32 barrels of beer → $736.
- ▶ 7.5 barrels of ale, 29.5 barrels of beer → $776.
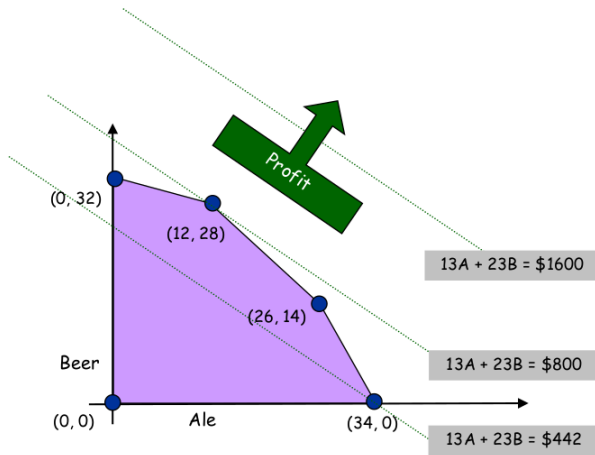- ▶ 12 barrels of ale, 28 barrels of beer → $800.

Brewery Problem

$$\begin{array}{rl}
 & \text{Ale} \quad \text{Beer} \\
\max & 13A + 23B \qquad \text{Profit} \\
\text{s.t.} & 5A + 15B \leq 480 \qquad \text{Corn} \\
& 4A + 4B \leq 160 \qquad \text{Hops} \\
& 35A + 20B \leq 1190 \qquad \text{Malt} \\
& A \quad , \quad B \geq 0
\end{array}$$

Brewery Problem: Feasible Region

Hops
$4A + 4B \leq 160$

Malt
$35A + 20B \leq 1190$

$(0, 32)$

$(12, 28)$

Corn
$5A + 15B \leq 480$

$(26, 14)$

Beer

$(0, 0)$

Ale

$(34, 0)$

Brewery Problem: Objective Function

(0, 32)

(12, 28)

(26, 14)

Beer

(0, 0)   Ale   (34, 0)

Profit

$13A + 23B = \$1600$

$13A + 23B = \$800$

$13A + 23B = \$442$
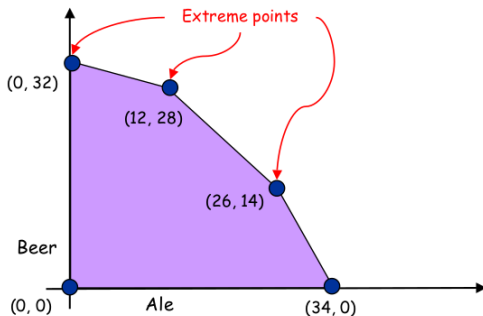
Brewery Problem: Geometry

Observation: Regardless of objective function coefficients, an optimal solution occurs at an extreme point.

Linear Programming: Standard Form

Standard form:

- ▶ Input: real numbers c_j, b_i, a_{ij}.
- ▶ Output: real numbers x_j.
- ▶ $n = \#$ nonnegative variables, $m = \#$ constraints.
- ▶ Maximize linear objective function subject to linear inequalities.

$$\text{(P)} \max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \le i \le m$$
$$x_j \ge 0 \quad 1 \le j \le n$$

$$\text{(P)} \max c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \ge 0$$

Linear: No x^2, xy, $\arccos(x)$, etc.

Programming: Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input:

$$
\begin{array}{rl}
\max & 13A + 23B \\
\text{s.t.} & 5A + 15B \le 480 \\
& 4A + 4B \le 160 \\
& 35A + 20B \le 1190 \\
& A, \quad B \ge 0
\end{array}
$$

Standard form:

- ▶ Add slack variable for each inequality.
- ▶ Now a 5-dimensional problem.

$$
\begin{array}{rl}
\max & 13A + 23B \\
\text{s.t.} & 5A + 15B + S_C \qquad\qquad = 480 \\
& 4A + 4B \qquad + S_H \qquad = 160 \\
& 35A + 20B \qquad\qquad + S_M = 1190 \\
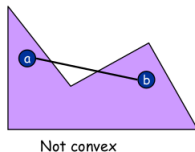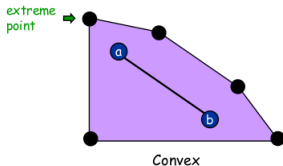& A, \quad B, \quad S_C, \quad S_H, \quad S_M \ge 0
\end{array}
$$

Geometry (1)

Geometry:

- ▶ Inequalities : halfplanes (2D), hyperplanes.
- ▶ Bounded feasible region: convex polygon (2D), (convex) polytope.

Convex: if a and b are feasible solutions, then so is $(a + b)/2$.

Extreme point: feasible solution x that cannot be written as $(a + b)/2$ for any two distinct feasible solutions a and b.

Convex   Not convex

Geometry (2)

Extreme point property: If there exists an optimal solution to (P), then there exists one that is an extreme point.
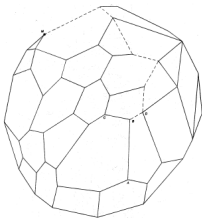
- Only need to consider finitely many possible solutions.

Challenge: Number of extreme points can be exponential.

- Consider n-dimensional hypercube.

Greedy: Local optima are global optima.

- Extreme point is optimal if no neighboring extreme point is better.
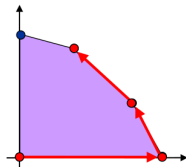
Simplex Algorithm

Simplex algorithm: (George Dantzig, 1947)

- ▶ Developed after WWII in response to logistical problems.
- ▶ Used for 1948 Berlin airlift.

Generic algorithm:

- ▶ Start at some extreme point.
- ▶ Pivot from one extreme point to a neighboring one. (never decrease objective function)
- ▶ Repeat until optimal.

How to implement? Linear algebra.

Simplex Algorithm: Basis

Basis: Subset of m of the n variables.

Basic feasible solution (BFS): Set $n - m$ nonbasic variables to 0, solve for remaining m variables.

- ▶ Solve m equations in m unknowns.
- ▶ If unique and feasible solution $\rightarrow$ BFS.
- ▶ BFS corresponds to extreme point.
- ▶ Simplex only considers BFS.

$$
\begin{aligned}
\max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B + S_C && = 480 \\
& 4A + 4B && + S_H && = 160 \\
& 35A + 20B && && + S_M && = 1190 \\
& A, \quad B, \quad S_C, \quad S_H, \quad S_M && \geq 0
\end{aligned}
$$
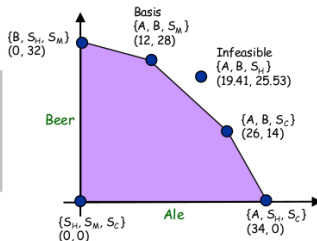
Simplex Algorithm: Pivot 1 (1)

max Z subject to

$$13A + 23B \qquad\qquad - Z = 0$$
$$5A + \boxed{15B} + S_C \qquad\qquad = 480$$
$$4A + 4B \qquad + S_H \qquad = 160$$
$$35A + 20B \qquad\qquad + S_M = 1190$$
$$A \ , \ B \ , \ S_C \ , \ S_H \ , \ S_M \ \geq \ 0$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Substitute: $B = 1/15 \ (480 - 5A - S_C)$

max Z subject to

$$\frac{16}{3}A \qquad - \frac{23}{15}S_C \qquad\qquad - Z = -736$$
$$\frac{1}{3}A + B + \frac{1}{15}S_C \qquad\qquad = 32$$
$$\frac{8}{3}A \qquad - \frac{4}{15}S_C + S_H \qquad = 32$$
$$\frac{85}{3}A \qquad - \frac{4}{3}S_C \qquad + S_M = 550$$
$$A \ , \ B \ , \qquad S_C \ , \qquad S_H \ , \ S_M \ \geq \ 0$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Simplex Algorithm: Pivot 1 (2)

$$\max Z \text{ subject to}$$

$$13A + 23B \qquad\qquad - Z = 0$$
$$5A + \boxed{15B} + S_C \qquad\qquad = 480$$
$$4A + 4B \qquad + S_H \qquad\qquad = 160$$
$$35A + 20B \qquad\qquad + S_M = 1190$$
$$A \quad, \quad B \quad, \quad S_C \quad, \quad S_H \quad, \quad S_M \geq 0$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Why pivot on column 2?

▶ Each unit increase in B increases objective value by \$23.

▶ Pivoting on column 1 also OK.

Why pivot on row 2?

▶ Preserves feasibility by ensuring $RHS \geq 0$.

▶ Minimum ratio rule: $\min\{480/15, 160/4, 1190/20\}$.

Simplex Algorithm: Pivot 2

max Z subject to

$$\frac{16}{3} A \quad - \frac{23}{15} S_C \qquad\qquad - Z = -736$$

$$\frac{1}{3} A + B + \frac{1}{15} S_C \qquad\qquad = 32$$

$$\frac{8}{3} A \quad - \frac{4}{15} S_C + S_H \qquad = 32$$

$$\frac{85}{3} A \quad - \frac{4}{3} S_C \qquad + S_M = 550$$

$$A \ , \ B \ , \quad S_C \ , \quad S_H \ , \ S_M \quad \geq \quad 0$$

Basis = {B, S_H, S_M}
A = S_C = 0
Z = 736
B = 32
S_H = 32
S_M = 550

Substitute: $A = 3/8 (32 + 4/15 \, S_C - S_H)$

max Z subject to

$$- \quad S_C - 2 S_H \qquad - Z = -800$$

$$B + \frac{1}{10} S_C + \frac{1}{8} S_H \qquad = 28$$

$$A \quad - \frac{1}{10} S_C + \frac{3}{8} S_H \qquad = 12$$

$$- \frac{25}{6} S_C - \frac{85}{8} S_H + S_M = 110$$

$$A \ , \ B \ , \quad S_C \ , \quad S_H \ , \ S_M \quad \geq \quad 0$$

Basis = {A, B, S_M}
S_C = S_H = 0
Z = 800
B = 28
A = 12
S_M = 110

Simplex Algorithm: Optimality

When to stop pivoting?

▶ If all coefficients in top row are non-positive.

Why is resulting solution optimal?

▶ Any feasible solution satisfies system of equations in tableaux.
   - in particular: $Z = 800 - S_C - 2S_H$

▶ Thus, optimal objective value $Z^* \leq 800$ since $S_C, S_H \geq 0$.

▶ Current BFS has value $800 \rightarrow$ optimal.

max Z subject to

$$
\begin{array}{rcl}
- S_C - 2 S_H - Z &=& -800 \\
B + \tfrac{1}{10} S_C + \tfrac{1}{8} S_H &=& 28 \\
A - \tfrac{1}{10} S_C + \tfrac{3}{8} S_H &=& 12 \\
- \tfrac{25}{6} S_C - \tfrac{85}{8} S_H + S_M &=& 110 \\
A, B, S_C, S_H, S_M &\geq& 0
\end{array}
$$

Basis = {A, B, S_M}
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Algorithm: Issues

Remarkable property: In practice, simplex algorithm typically terminates in at most $2(m + n)$ pivots.

- ▶ No polynomial pivot rule known.
- ▶ Most pivot rules known to be exponential in worst-case.

Issues: Which neighboring extreme point?

Degeneracy: New basis, same extreme point.

- ▶ "Stalling" is common in practice.

Cycling: Get stuck by cycling through different bases that all correspond to same extreme point.

- ▶ Does not occur in the wild.
- ▶ Bland's least index rule $\rightarrow$ finite # of pivots.