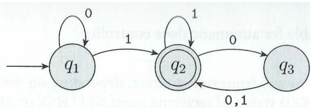
Finite Automata

Definition 1.1 (Finite automaton) A **finite automaton** (FA) M is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

1. Q is a finite set called the **states**,
2. Σ is a finite set called the **alphabet**,
3. $\delta : Q \times \Sigma \to Q$ is the **transition function**,
4. $q_0 \in Q$ is the **start state**, and
5. $F \subseteq Q$ is the **set of accept states** / **final states**.



state transition diagram (STD)

$$M_1 = (Q, \Sigma, \delta, q_1, F)$$

$\Sigma = \{0, 1\}$

$Q = \{q_1, q_2, q_3\}$

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

Definition 1.2 (Strings accepted by M) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and $w = w_1 w_2 \cdots w_n$ be a string over alphabet Σ.
M **accepts** w if there exists a sequence of states $r_0, r_1, \ldots, r_n$, such that all following three conditions hold:

1. $r_0 = q_0$ *(M starts in start state.)*
2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots n-1$ *(State change follows transition function.)*
3. $r_n \in F$ *(M ends up in accept state)*

If M does not accept w, it **rejects** it.

$\Rightarrow$ "computation" of M on w

$$\to \boxed{q_1} \xrightarrow{1} \boxed{q_2} \xrightarrow{1} \boxed{q_2} \xrightarrow{0} \boxed{q_3} \xrightarrow{1} \boxed{\boxed{q_2}}$$

Definition 1.4 (Language of machine M) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton. The **language of machine M** $L(M)$ is the set of all strings that are accepted by M. We say: M **recognizes** $L(M)$.

$L(M_1) = \{w | w$ contains at least one 1 and an even number of 0s follow the last 1$\}$.

Definition 1.5 (Regular language) A language is called a **regular language** if some finite automaton recognizes it.

$\leftarrow$ class of languages

Non-regular Language:

$$L = \{ 0^n 1^n \mid n \geq 1 \}$$

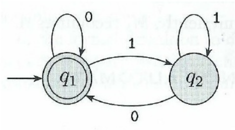
Σ alphabet $\qquad \Sigma = \{0, 1\}$

language $\hat{=}$ sets of strings over some alphabet

$$\Sigma^*$$

1 Reading and understanding STDs and FAs

You are given the following state transition diagram:



1. Give the formal description of the corresponding FA as five-tuple.
2. Let the automaton "run" on the input strings 101, 100.
3. Find the language recognized by this automaton.

1) $M = (Q, \Sigma, \delta, q_1, F)$ $Q = \{q_1, q_2\}$ $F = \{q_1\}$ $\Sigma = \{0, 1\}$

δ	0	1
q_1	q_1	q_2
q_2	q_1	q_2

2) On reading 101, the FA goes through states q_1, q_2, q_1, q_2
$\Rightarrow$ rejects

On reading 100, the FA goes through states q_1, q_2, q_1, q_1
$\Rightarrow$ accept

3) $L(M_3) = \{ w \in \Sigma^* \mid w$ is empty or ends with a $0 \}$

2 Constructing an FA

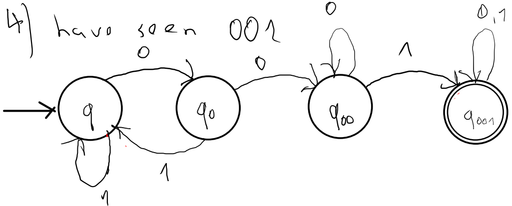
We consider the language

$$L = \{w | w \text{ contains } 001 \text{ as substring}\}$$

over the alphabet $\Sigma = \{0, 1\}$.
Find the finite automaton the recognizes L and describe it by an STD.

Idea: Four states

1) haven't seen any symbol of that pattern q
2) have just seen 0 q_0
3) have seen 00 q_{00}
4) have seen 001 q_{001}



1 1 1 0 0 1 0 1 0 1
1001

Closure of regular operations

Definition 1.6 (Regular operations) Let A and B be languages. We define the regular operations **union**, **concatenation** and **star** as follows:

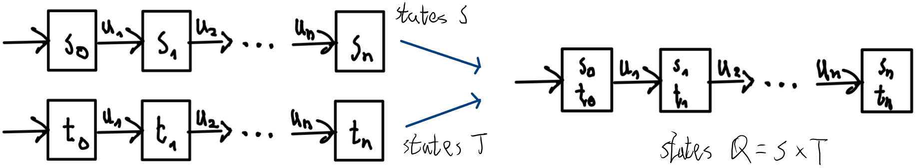
- **Union**: $A \cup B := \{x | x \in A \text{ or } x \in B\}$
- **Concatenation**: $A \circ B := \{xy | x \in A \text{ and } y \in B\}$.
- **Star**: $A^* := \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$.

Theorem 1.1 The class of regular languages is closed under the union operation. In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

$A_1 = \{good, bad\}$

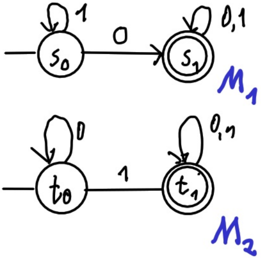
$A_2 = \{day, night\}$

$A \cup B = \{good, bad, day, night\}$



3 Example for proof of Theorem 1.1

Let the two FAs M_1 and M_2 be given via their STDs:

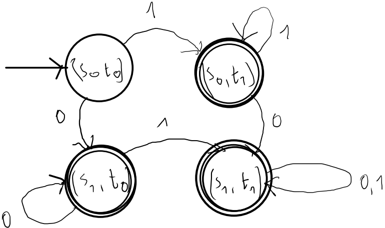


$S = \{s_0, s_1\}$

$T = \{t_0, t_1\}$

We would like to find a third FA M such that $L(M) = L(M_1) \cup L(M_2)$.

$Q = S \times T = \{(s_0, t_0), (s_0, t_1), (s_1, t_0), (s_1, t_1)\}$



2 Constructing an FA

We consider the language

How to extend the previous example?

$$L = \{w|w \text{ contains } 001\text{0} \text{ as substring}\}$$

over the alphabet $\Sigma = \{0, 1\}$.

Find the finite automaton the recognizes L and describe it by an STD.

1001

~~1011~~

111100

111000 0

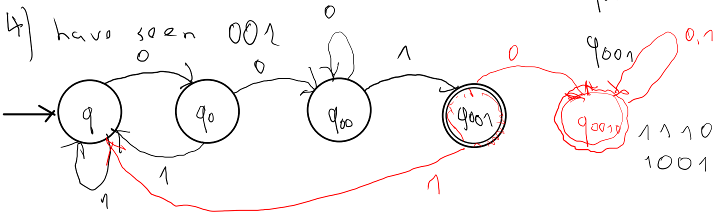
0011

0000

0011

Idea: Four states

1) haven't seen any symbol of that pattern — q
2) have just seen 0 — q_0
3) have seen 00 — q_{00}
4) have seen 001 — q_{001}



1110010101

1001

Re-iteration of the proof for the union of two regular languages

A_1, A_2 regular

show: $A_1 \cup A_2$ regular

$A_1 \quad M_1 = (S, \Sigma, \delta, s_0, F)$

$A_2 \quad M_2 = (T, \Sigma, \delta', t_0, F')$

$A_1 = L(M_1)$

$A_2 = L(M_2)$

$M = (Q, \Sigma, \delta'', q_0, F'')$

$L(M) = A_1 \cup A_2$

$A_1 \cup A_2 = \{ w \in \Sigma^* \mid w \in A_1 \text{ or } w \in A_2 \}$