Homework 4 solutions

1. a) $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{(x+h)\cdot(x^2+2hx+h^2) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} = \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3hx + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3hx + h^2 = 3x^2$$

b) $+\sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

conjugate pair

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

c) $f(x) = x$

$$f'(x) = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

independent of h

d) $f(x) = c$

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

independent of h

$f(x+h) = c$ as it is constant.

Problem 2

(a) $f(x) = \dfrac{x^2}{b-3x^2}$

Using quotient rule,

$$f'(x) = \frac{(b-3x^2).(2x) - (x^2)(-6x)}{(b-3x^2)^2}$$

$$= \frac{2bx - 6x^3 + 6x^3}{(b-3x^2)^2} = \frac{2bx}{(b-3x^2)^2} \quad //$$

(b) $g(t) = \cos(\omega t+\phi) + \sin(\omega t+\phi)$

$$g'(t) = \omega.(-\sin(\omega t+\phi)) + \omega(\cos(\omega t+\phi))$$

$$= \omega\left[\cos(\omega t+\phi) - \sin(\omega t+\phi)\right] \quad //$$

(c) $h(s) = \cos(s^2+s) + \sin(s/2)$

$$h'(s) = (2s+1).(-\sin(s^2+s)) + (1/2)\cos(s/2)$$

$$= (1/2)\cos(s/2) - \left[(2s+1).\sin(s^2+s)\right] \quad //$$

(d) $j(x) = \ln\left(x^{a^2} + x^{-a^2}\right)$

$$j'(x) = \frac{a^2 . x^{(a^2-1)} - a^2 . x^{-(a^2+1)}}{x^{a^2} + x^{-a^2}}$$

(e) $k(x) = \ln(x^a + b^x)$

$\therefore k'(x) = \dfrac{a.x^{a-1} + \ln(b).e^{x.\ln(b)}}{x^a + b^x}$

$\therefore k'(x) = \dfrac{a.x^{a-1} + \ln(b).b^x}{x^a + b^x}$ //

Here, for b^x, using logs,

$b = e^{\ln(b)}$

$\therefore b^x = e^{x\ln(b)}$

$\therefore$ derivative of b^x $y = \ln(b).e^{x.\ln(b)}$

(f) $l(x) = x^2.e^{-x^2}$

Using product rule,

$$l'(x) = x^2.\left((-2x).e^{-x^2}\right) + e^{-x^2}.(2x)$$

$$= 2x.e^{-x^2}\left(1 - x^2\right) \text{ //}$$

(g) $m(x) = x^{x^2}$

$\therefore m(x) = e^{x^2.\ln(x)}$

Using the same transformation,

$x = e^{\ln(x)}$

$x^{x^2} = e^{x^2.\ln(x)}$

First, computing the derivative of $x^2.\ln(x)$:

Using product rule,

$$(x^2).\left(\frac{1}{x}\right) + (\ln(x)).(2x) = x + 2x.\ln(x) = x(1 + 2\ln(x))$$

$\therefore m'(x) = x(1+2\ln(x)).e^{x^2.\ln(x)}$

$\therefore m'(x) = x(1+2\ln(x)).x^{x^2}$ //

3. $f(x) = |x|$ $f'(0)$ at 0 depends on where the limit is coming from (positive or negative)

$f'(0)^+$ (h is approaching from the positive reals)

$$f'(0)^+ = \lim_{h \to 0^+} \frac{|x+h| - |x|}{h}\Big|_0 = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

This means evaluated at $x = 0$

Since $h > 0$

$$f'(0)^- = \lim_{h \to 0^-} \frac{|x+h| - |x|}{h}\Big|_0 = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} -\frac{h}{h} = \lim_{h \to 0^-} -1 = 0$$

Since $h < 0$, $|h| = -h$

so $f'(0)^+ \neq f'(0)^-$, so it is not differentiable at $x = 0$