Homework 9.

1.

a).

$$\frac{dy}{dx} = y^2x^2 + y^2x,$$

$$y(0) = 2$$

$$\frac{dy}{dx} = y^2(x^2+x)$$

$$\frac{dy}{y^2} = (x^2+x)\,dx$$

$$\int \frac{1}{y^2}\,dy = \int x^2+x\,dx$$

$$+\frac{1}{y} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

$$y = -\frac{6}{2x^3+3x^2+C}$$

Applying initial condition: $x=0$, $y=2$

$$2 = -\frac{6}{C}$$

$$C = -3$$

$$\Rightarrow y = -\frac{6}{2x^3+3x^2-3}$$

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b).
$$e^x e^y \frac{dy}{dx} + e^x e^y = 2x^3,$$
$$y(0) = 0$$

Let $u = e^x e^y$
$$\Rightarrow \frac{du}{dx} = e^x e^y \frac{dy}{dx} + e^x e^y$$

Then,
$$e^x e^y \frac{dy}{dx} + e^x e^y = 2x^3$$
$$\Rightarrow \frac{du}{dx} = 2x^3$$
$$du = 2x^3 dx$$
$$\int du = \int 2x^3 dx$$
$$u = \frac{x^4}{2} + C$$
$$e^x e^y = \frac{x^4}{2} + C$$
$$e^x e^y - \frac{x^4}{2} = C$$

Applying initial condition: $x = 0, y = 0$
$$e^0 e^0 - \frac{0}{2} = C$$
$$1 = C$$

Therefore: $e^x e^y = \frac{x^4}{2} + 1$
$$x + y = \ln\left(\frac{x^4}{2} + 1\right)$$
$$y = \ln\left(\frac{x^4}{2} + 1\right) - x$$

2.

a). $y' = y - y^3$

$0 = y - y^3$

$0 = y(1 - y^2)$

$y_1 = 0$

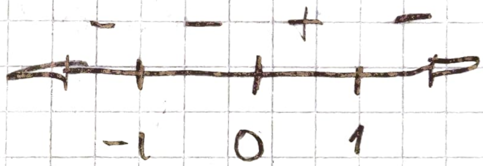
$1 - y^2 = 0 \Rightarrow y_2 = 1, \; y_3 = -1$

For $y = -2 \rightarrow y' = -10$

For $y = -0.5 \rightarrow y' = -0.625$

For $y = 0.5 \rightarrow y' = 0.375$

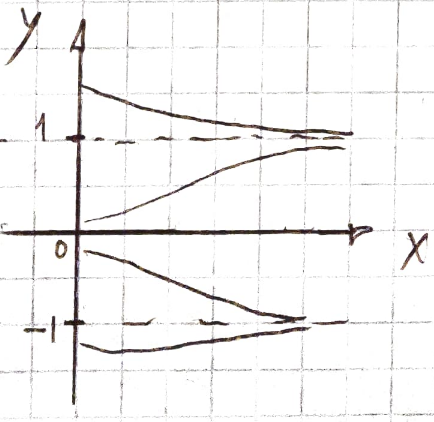
For $y = 2 \rightarrow y' = -6$



$y = -1$ is stable

$y = 0$ is unstable

$y = 1$ is stable.



b). $y' = y(y-2)$

$0 = y(y-2)$

$y_1 = 0$

$y_2 = 2$

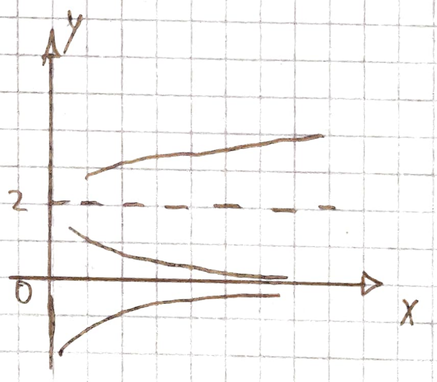
For $y = -1 \rightarrow y' = 3$

For $y = 1 \rightarrow y' = -1$

For $y = 3 \rightarrow y' = 3$

$y = 0$ is stable

$y = 2$ is unstable



c). $y' = 2e^y - 2$

$0 = 2e^y - 2$

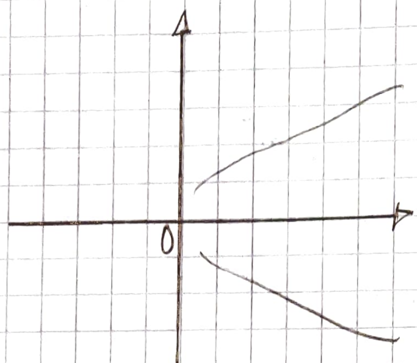
$1 = e^y$

$y = 0$

For $y = -1 \rightarrow y' = \frac{2}{e} - 2 = -1.26$

For $y = 1 \rightarrow y' = 2e - 2 = 3.44$

$y=0$ is unstable



3. The change of the amount of ^{14}C can be described as:

$$(1) \quad \frac{dC}{dt} = rC$$

amount of ^{14}C

decay rate

At t_0: $C(0) = C_0$ initial amount of carbon 14

Note that $r<0$ because the amount decreases.

From (1):

$$\frac{dC}{C} = r\,dt$$

$$\Rightarrow \int_{C_0}^{C(t)} \frac{dC}{C} = \int_{t_0}^{t} r\,dt$$

$$\Rightarrow \left[\ln C\right]_{C_0}^{C(t)} = rt$$

$$\ln C(t) - \ln(C_0) = rt$$

$$\ln\left(\frac{C(t)}{C_0}\right) = rt$$

$$\frac{C(t)}{C_0} = e^{rt}$$

$$\text{(2)} \qquad C(t) = C_0 e^{rt}$$

Now, to calculate r, the half-life time is used:

$$C_0 = 1 \quad , \quad C(t_{1/2}) = \frac{1}{2}$$

In (2) $\qquad \frac{1}{2} = 1 \cdot e^{r t_{1/2}}$

$$r = -\frac{\ln 2}{t_{1/2}}$$

We know that $t_{1/2} = 5730$ years, so:

$$r = -\frac{\ln 2}{5730}$$

$$r = -12{,}1 \cdot 10^{-5}$$

The amount of stable carbon remains constant but ^{14}C decays. The initial carbon ratio is close to the atmospheric ratio, so:

$$\overbrace{0.69}^{69\%} \underbrace{\frac{\overbrace{C_0}^{\text{Initial } ^{14}C}}{C_{stable}}}_{\text{atmospheric carbon ratio}} = \underbrace{\frac{\overbrace{C_t}^{^{14}C \text{ after time } t}}{C_{stable}}}_{\text{Object's carbon ratio}}$$

$\Longrightarrow C_t = 0.69\,C_0$

Then in (2):

$$0.69\,C_0 = C_0\,e^{rt}$$

$$t = \frac{\ln 0.69}{-12.1 \cdot 10^{-5}}$$

$t = 3066.64$ years //