Probability and Random Processes

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Jacobs University

Announcements

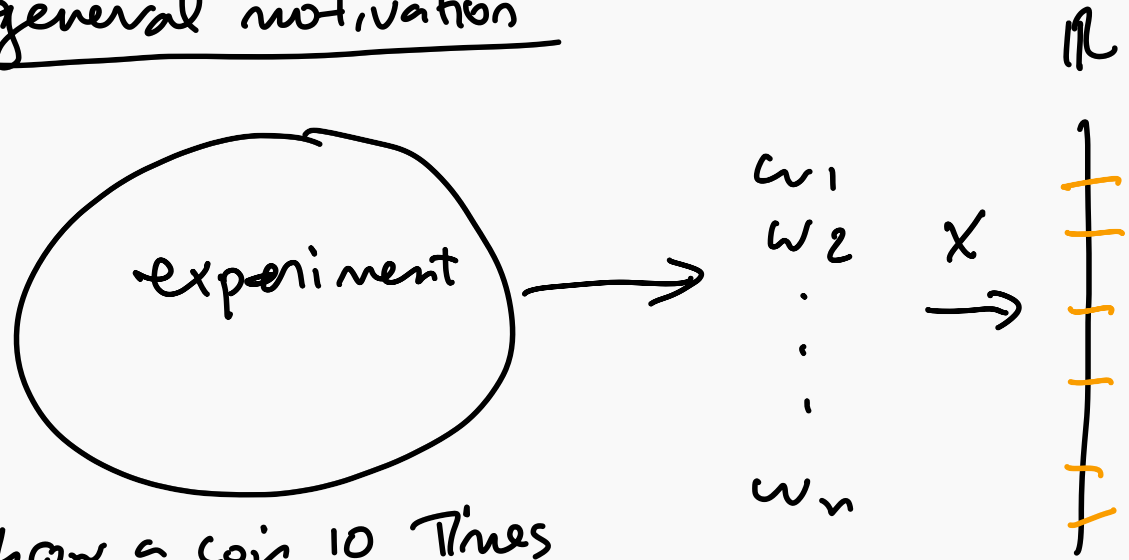
1. Problem Set 4 is due next Wednesday!

Agenda

1. Joint distribution of random variables
2. Joint probability mass function and marginals
3. Random walks

Studying several random variables at the same time

general motivation



Throw a coin 10 Times

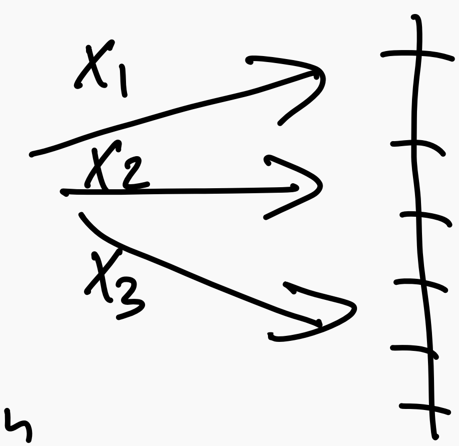
X_1 0 1

X_2 0 1

$\vdots$

X_n 0 1

$X_1 + \dots + X_n$ can takes $0, \dots, n$



Definition

For discrete random variables X and Y the *joint probability mass function* of X and Y is defined by

$$p(x, y) = \mathbb{P}[X = x, Y = y].$$

More generally, for n discrete random variable $X_1, X_2, \ldots, X_n$, the joint probability mass function of $X_1, \ldots, X_n$ the function defined by

$$p(x_1, x_2, \ldots, x_n) = \mathbb{P}[X_1 = x_1, \ldots, X_n = x_n].$$

probability mass function of X $\quad p_X(x) = \mathbb{P}[X = x]$

Example 1

A random integer N is chosen from the set $\{11, 12, \ldots, 35\}$. Let X denote the right digit of N and Y denote the left digit of N. Find the joint probability mass function of X and Y.

ordinary pmf

x	x_1	$\cdots$	x_n
	p_1	$\cdots$	p_n

$Y \backslash X$	0	1	2	3	4	5	6	7	8	9
1	0	1/25	1/25	1/25	·	-				1/25
2	1/25	-	-	— -	1/25	-		-		1/25
3	1/25	·	- -	-		1/25	0	0	0	0

$Y \backslash X$	x_1	$\cdots$	x_i	$\cdots$	x_m
y_1					
$\vdots$					
y_j			p_{ij}		
y_n					

$p_{ij} = P(x_i, y_j) = \mathbb{P}[X = x_i, Y = y_j]$

$N = \underline{Y}\ \underline{X}$

$\ 1\ \ 0$

$P(0,1) = \mathbb{P}(X=0, Y=1) =$

$P(2,4) = \mathbb{P}(X=4, Y=2)$
$= \mathbb{P}(N = 24) = 1/25$

Example 2

A random real number T is chosen from the interval $[0, 1]$. Let X_1 and X_2 denote the first and second digit in the binary expansion of T. Find the joint probability mass function of X_1 and X_2.

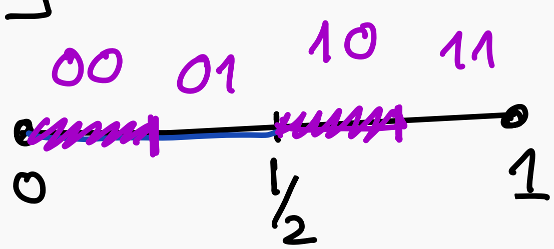
T random $[0,1]$ → X_1 first binary digit $0, 1$

→ X_2 second binary digit $0, 1$

$$p(0,0) = \mathbb{P}[X_1 = 0, X_2 = 0]$$

$0.\underline{0}\;\underline{0}$

$0.\underline{1}\;\underline{}$



$X_1 \backslash X_2$	0	1
0	$1/4$	$1/4$
1	$1/4$	$1/4$

$$p(0,0) = 1/4$$
$$p(0,1) = 1/4$$
$$p(1,0) = 1/4 \qquad p(1,1) = 1/4$$

Recovering individual probability mass functions

Example

Let X and Y be chosen randomly from the set $\{-1, 0, 1\}$ such that the joint probability mass function of X and Y is given by

	$Y=-1$	$Y=0$	$Y=1$	
$X=-1$	$1/10$	$1/10$	$1/10$	$3/10$
$X=0$	$1/10$	$2/10$	$1/10$	$4/10$
$X=1$	$1/10$	$1/10$	$1/10$	$3/10$
	$3/10$	$4/10$	$3/10$	

Find the probability mass functions of X and Y.

$$p_X(1) = \mathbb{P}(X=1) = \mathbb{P}(X=1, Y=-1) + \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=1, Y=1)$$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$$

Marginal probability mass functions

Theorem

Let $p(x, y) := p_{X,Y}(x, y)$ denote the probability mass function of discrete random variables X and Y. Then we have

1. $\sum_{x,y} p(x, y) = 1$.
2. $\mathbb{P}[Y = y] = \sum_x p(x, y)$.
3. $\mathbb{P}[X = x] = \sum_y p(x, y)$.

Recovering individual probability mass functions

Example
Let X and Y be numbers chosen randomly from the set $\{-1, 0, 1\}$ and $Z = XY$. Suppose that the joint probability mass function of X and Y is given by

	$Y = -1$	$Y = 0$	$Y = 1$
$X = -1$	$1/10$	$1/10$	$1/10$
$X = 0$	$1/10$	$2/10$	$1/10$
$X = 1$	$1/10$	$1/10$	$1/10$

Find the probability mass function of $Z = XY$.

what are the possible values of Z?

$-1, 0, 1$

$$\mathbb{P}(Z=-1) = \mathbb{P}(X=1, Y=-1) + \mathbb{P}(X=-1, Y=1) =$$

Z	-1	0	1
$\mathbb{P}[Z=z] = P_Z(z)$	$2/10$	$6/10$	$2/10$

Independence

Definition

Discrete random variables X and Y are called independent if for every x and y we have

$$p_{X,Y}(x,y) = \mathbb{P}[\overbrace{X = x}^{A}, \overbrace{Y = y}^{B}] = \mathbb{P}[\overbrace{X = x}^{A}]\,\mathbb{P}[\overbrace{Y = y}^{B}] = p_X(x)p_Y(y).$$

independence of events

A, B events independent $\quad \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B) = \mathbb{P}(A)$$

$$P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$$

$$\mathbb{P}[X = x \mid Y = y] = \mathbb{P}[X = x]$$

Example

Suppose that the joint probability mass function of X and Y is given by

	$Y=-1$	$Y=0$	$Y=1$	
$X=-1$	$1/10$	$1/10$	$1/10$	$3/10$
$X=0$	$1/10$	$2/10$	$1/10$	$4/10$
$X=1$	$1/10$	$1/10$	$1/10$	$3/10$
	$3/10$	$4/10$	$3/10$	

Are X and Y independent?

$9/100$	$12/100$	$9/100$	$3/10$
$12/100$	$16/100$	$12/100$	$4/10$
$9/100$	$12/100$	$9/100$	$3/10$
$3/10$	$4/10$	$3/10$	

Example

Suppose X_1 and X_2 denote the random variables from Example 2. Are X_1 and X_2 independent?

$X_1 \backslash X_2$	0	1	
0	$1/4$	$1/4$	$1/2$
1	$1/4$	$1/4$	$1/2$
	$1/2$	$1/2$	

Joint probability mass functions of independent random variables

Suppose X and Y are independent random variables with probability mass functions

$$\mathbb{P}[X=1]=\frac{1}{3},\quad \mathbb{P}[X=2]=\frac{2}{3}$$

$$\mathbb{P}[Y=0]=\frac{1}{4},\mathbb{P}[Y=1]=\frac{1}{4},\quad \mathbb{P}[Y=2]=\frac{1}{2}.$$

Find the joint probability mass function of X and Y and determine the probability mass function of $Z=XY$ and $T=X+Y$.

Z takes: $0, 1, 2, 4$

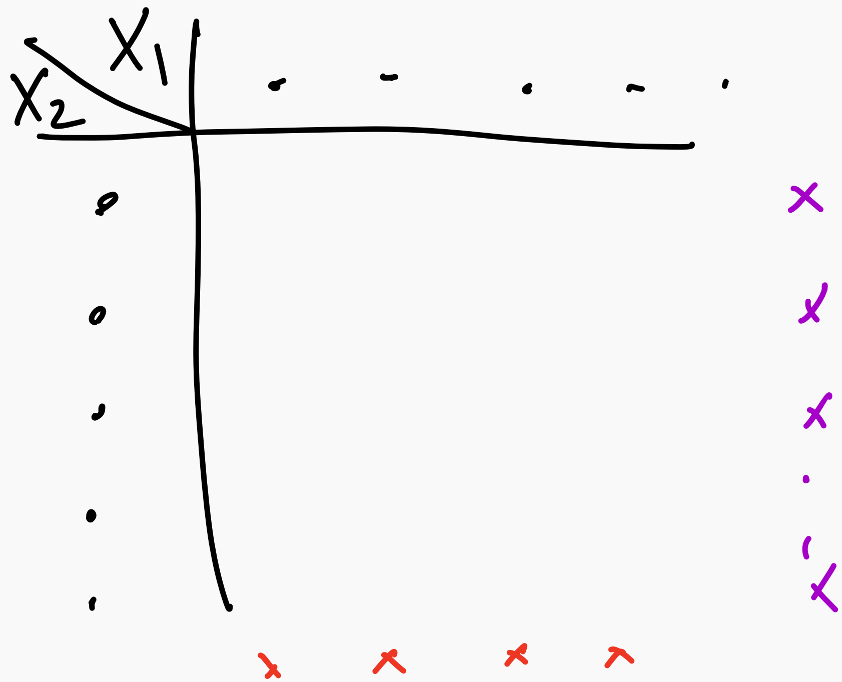
	0	1	2	4
	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{3}$

$X \backslash Y$	0	1	2	
1	$1/12$	$1/12$	$1/6$	$1/3$
2	$1/6$	$1/6$	$1/3$	$2/3$
	$1/4$	$1/4$	$1/2$	

	1	2	3	4
	$1/12$	$1/4$	$1/3$	$1/3$

Definition

Let X_1 and X_2 denote two random variables. A *coupling* of X_1 and X_2 is a random variable X with marginals given by X_1 and X_2.



Example

Describe all couplings of two Bernoulli random variables with $p = 1/2$.

$x=0$ $\qquad$ $x=1/4$ $\qquad$ $x=1/2$

$X_1 \backslash X_2$	0	1
0	0	$1/2$
1	$1/2$	0

$X_1 \backslash X_2$	0	1
0	$1/2$	0
1	0	$1/2$

$X_1 \backslash X_2$		
	$1/4$	$1/4$
	$1/4$	$1/4$

X_1, X_2 independent

$X_1 \backslash X_2$	0	1	
0	x	$\frac{1}{2} - x$	$1/2$
1	$\frac{1}{2} - x$	x	$1/2$
	$1/2$	$1/2$	

Transportation polytope

Problem

X_1	1 2 - - -	n	Prob	$1/n$
X_2	1 2 - - -	m		$1/m$

$x_1 \backslash x_2$	1	2	3	$\cdots$	
	p_1q_1	p_2q_1	-	-	q_1
	$\cdot$	p_2q_2	$\cdot$	$\cdot$	$\vdots$
	$\cdot$		$\cdot$	$\cdot$	q_m
	p_1 -	-	-	p_n	

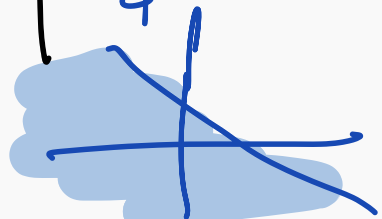
$n=3$
$m=4$

$x_1 \backslash x_2$	1	2	3	4	
1	p_1	p_2	p_3	$\cdot$ ✓	$1/3$
2	p_4	p_5	p_6	$\cdot$ ✓	$1/3$
3	$\cdot$ ✓	$\cdot$ ✓	$\cdot$ ✓	$\cdot$ ✓	$1/3$
	$1/4$	$1/4$	$1/4$	$1/4$	

$$p_1+p_2+p_3 \le 1/3$$
$$p_4+p_5+p_6 \le 1/3$$

$$p_1+p_4 \le 1/4$$
$$p_2+p_5 \le 1/4$$
$$p_3+p_6 \le 1/4$$

$x+y \le 1/4$



Independence of discrete random variables

Definition

Discrete random variables $X_1, \ldots, X_n$ are *independent* if for all values of $x_1, \ldots, x_n$ we have

$\mathbb{P}(X_1 = x_1, \ldots X_n = x_n)$

$$p_{X_1,\ldots,X_n}(x_1 \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n).$$

Here $p_{X_1,\ldots,X_n}$ is the joint probability mass function of $X_1, \ldots, X_n$ and p_{X_i} is the marginal density function of X_i, for $1 \leq i \leq n$.

Sums of independent random variables

Suppose $X_1, \ldots, X_n$ are independent random variables, each with Bernoulli distribution with parameter p. Let

$$S_n = X_1 + \cdots + X_n.$$

What is the probability mass function of S_n?

lazy random walk