Probability and Random Processes

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Ω: sample space. Every subset of Ω is called an event.

Definition
To every event A we associate a probability denoted by $\mathbb{P}[A]$ such that

1. $\mathbb{P}[\Omega] = 1$. $1 \geq P(A) \geq 0$
2. If A and B are events with $\cancel{AB = \emptyset}$ $A \cap B = \emptyset$ then

$$\underbrace{\mathbb{P}[A \cup B]}_{A \text{ or } B} = \mathbb{P}[A] \cup \mathbb{P}[B].$$

Comment

Example

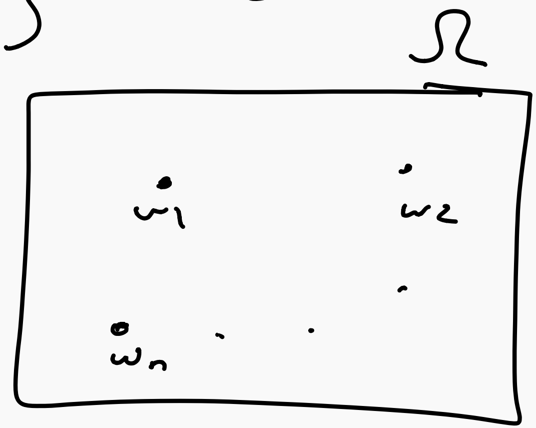
~~A coin is flipped. The sample space is~~

Probability according to Pascal

ω omega

$$\Omega = \{\omega_1, \dots, \omega_n\}$$

Ω



$$\mathbb{P}(\{\omega_i\}) = p$$

$$1 = \mathbb{P}(\{\omega_1\} \cup \dots \cup \{\omega_n\})$$

$$= \mathbb{P}(\Omega) =$$

$$\mathbb{P}(\{\omega_1\}) + \dots + \mathbb{P}(\{\omega_n\}) = np \Rightarrow p = 1/n$$

$$\mathbb{P}(\{\omega_1, \omega_2\}) = \frac{2}{n}, \quad \boxed{\mathbb{P}(A) = \frac{k}{n}}$$

$k = \#$ elements in A

Example
A coin is flipped. The sample space is

$$\Omega = \{H, T\}.$$

$$\mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(\{T\}) = 1/2.$$

The equiprobable model of Pascal: examples

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A coin is flipped. The sample space is

$$\Omega = \{H, T\}.$$

$$\mathbb{P}\left[\{H\}\right] = \frac{1}{2}.$$

Example
For two coins:

The equiprobable model of Pascal: examples

Example
A coin is flipped. The sample space is

$$\Omega = \{H, T\}.$$

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Example
For two coins:

$$\Omega = \{\underset{1/4}{HH}, \underset{1/4}{HT}, \underset{1/4}{TH}, \underset{1/4}{TT}\}$$

n coins

flip n coins

outcome : sequence $x_1 x_2 \dots x_n$

x_i is either H, T

x_i is either 0, 1

$00 \dots 0$

$11 \dots 1$

$101111 \dots 1$

Size of sample space 2^n

The equiprobable model of Pascal: examples

Example
A coin is flipped. The sample space is

$$\Omega = \{H, T\}.$$

$$\mathbb{P}[\{H\}] = \frac{1}{2}.$$

Example
For two coins:

$$\Omega = \{HH, HT, TH, TT\}$$

$$\mathbb{P}[\{HH, HT, TH\}] = \frac{3}{4}.$$

One can generalize this to more than two coins:

If the experiment consists of throwing n coins, then we consider sequences of length n as the sample space.

Examples of events

Example
A die is rolled. What is the probability that the outcome is an even number.

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$$\mathbb{P}[A] = \frac{3}{6} = 0.5$$

Let B be the event that the outcome is at most 4. Then

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$$B = \{⚀, ⚁, ⚂, ⚃\}.$$

Example

A die is rolled. What is the probability that the outcome is an even number.

$$\Omega = \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}.$$

The event A is defined by

$$A = \{\epsdice{2}, \epsdice{4}, \epsdice{6}\}.$$

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Let B be the event that the outcome is at most 4. Then

$$B = \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}\}.$$

$$\mathbb{P}[B] = \frac{4}{6} = 0.67.$$

Example

A random card is dealt from a well-shuffled deck of cards. What is the probability that the card is (a) an ace (b) red (c) an ace or red.

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2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

$$\mathbb{P}[A] = \frac{4}{52} = \frac{1}{13}.$$

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2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

$$\mathbb{P}[R] = \frac{26}{52} = \frac{1}{2}.$$

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$$\mathbb{P}[A \cup R] = \frac{4 + 26 - 2}{52} = \frac{28}{52}.$$

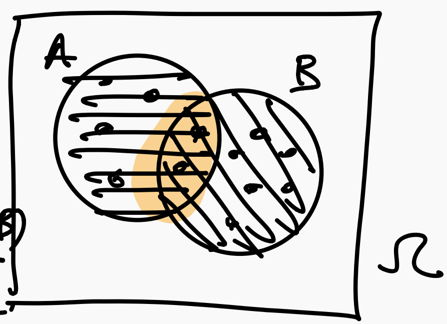
The union law

Theorem (The Union law)
Suppose A and B are two events. Then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B].$$

Proof.
Proof using Venn diagram:

Pascal's model

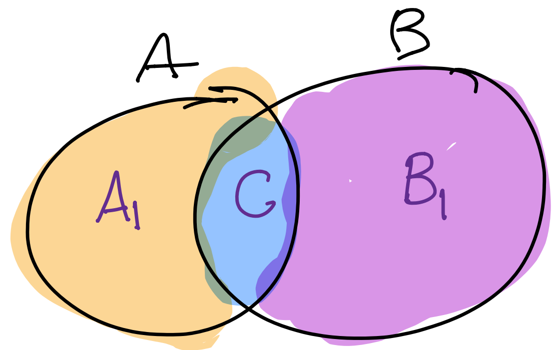


$$P(A) = \frac{\#A}{\#\Omega}, \quad P(B) = \frac{\#B}{\#\Omega}, \quad P(A \cup B) = \frac{\#(A \cup B)}{\#\Omega}$$

$$\#(A \cup B) = \#A + \#B - \#(A \cap B)$$

□

<u>Idea of a proof</u>



$$A = A_1 \cup C$$

$$A_1 \cap C = \emptyset$$

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(C)$$

$$\mathbb{P}(B) = \mathbb{P}(B_1) + \mathbb{P}(C)$$

$$\mathbb{P}(A) + \mathbb{P}(B) = \underbrace{\mathbb{P}(A_1) + \mathbb{P}(B_1) + \mathbb{P}(C)}_{\mathbb{P}(A\cup B)} + \underbrace{\mathbb{P}(C)}_{\mathbb{P}(A\cap B)}$$

$$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B).$$

Many problems in probability boil down to finding out how many elements are in a set. This turns out to be an art, but there are also methods.

Example

A 3-digit number x is chosen randomly. What is the probability that x is at least 200.

$$\Omega = \{100, 101, \ldots, 999\} \quad 900$$

$$A = \{200, 201, \ldots, 999\} \quad 800$$

$$\underbrace{a, a+1, \ldots, b}$$

$$\mathbb{P}(A) = \frac{800}{900} = \frac{8}{9}$$

Counting

Many problems in probability boil down to finding out how many elements are in a set. This turns out to be an art, but there are also methods.

Example

A 3-digit number x is chosen randomly. What is the probability that x is at least 200.

If $a < b$ are integers, then the number of integers in the list

$$a, a+1, \ldots, b$$

is $b - a + 1$.

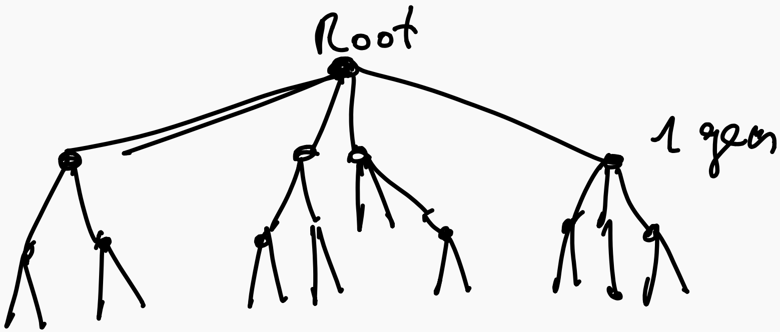
$110, \ldots, 255$

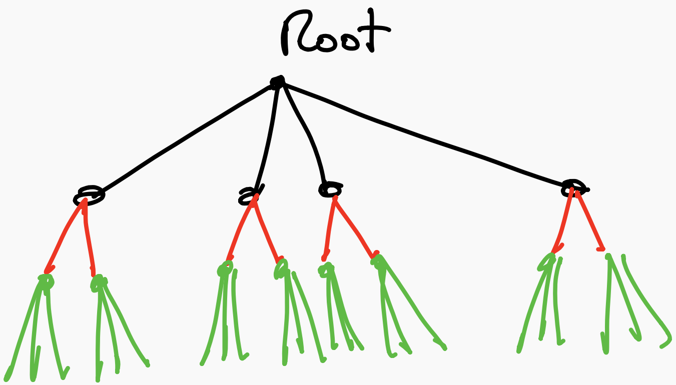
$a+0, a+1, \ldots, a+(b-a)$

$145+1 = 146$

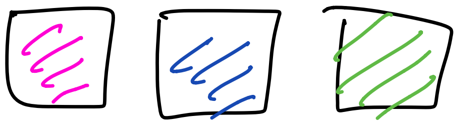
$b-a+1$

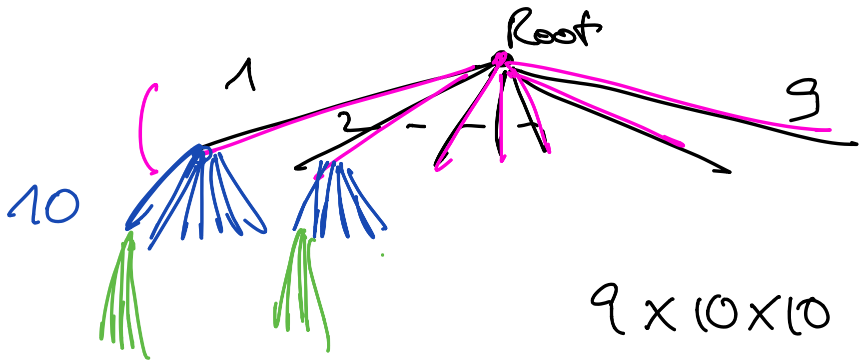
Counting using choice tree





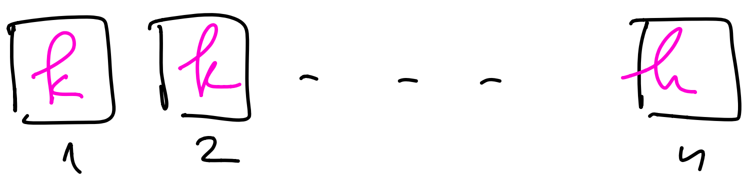
Count # 3 digit number



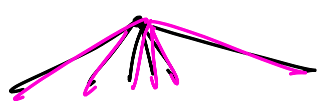


$9 \times 10 \times 10 = \underline{900}$

<u>Question</u> how many sequences ^are there^ of <u>length n</u> using digits $1, 2, \ldots, k$?



1 1 1 - - - 1



$\underline{\#\ \text{sequences}} = k^n$

Choice tree and unconstrained sequences

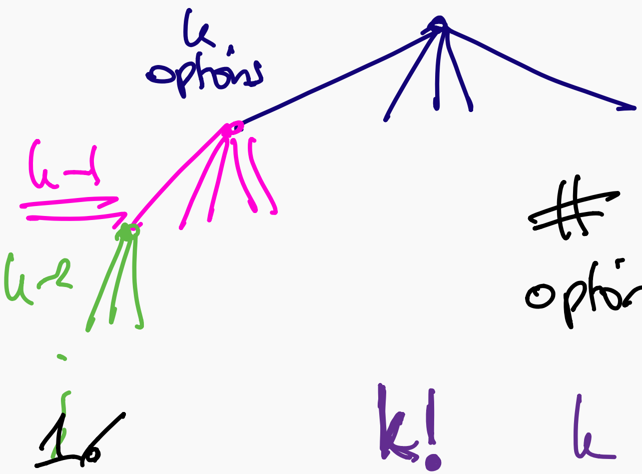
In how many ways can one form a sequence of length k using letters $1, 2, \underset{\ldots}{\cancel{dots}}, k$?

Choice tree and Factorials

In how many ways can one list $1, 2, \ldots, k$ such that every number appears exactly once?

$k = 2$ 12 21

$k = 3$ 123 132 213 231 312 321



$$\#\text{ options} = k \times (k-1) \times (k-2) \times \cdots \times 1$$

$$= 1 \times 2 \times 3 \times \cdots \times k.$$

$k!$ k factorial

Properties of $n!$

$k! = 1 \times 2 \times \cdots \times k$

$1! = 1, \quad 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

Convention $0! = 1$

how large is $n!$ as a function of n?

logarithmic $\log n$

Polynomial growth n^{10} α fixed

exponential growth 10^n α fixed $\alpha > 1$

double exponential 2^{2^n}

$n! \neq$ 2^n

100^n

n	2^n	$n!$	10^n
1	2	1	10
2	4	2	100
3	8	6	1000
4	16	24	10000
5	32	120	
6	64	720	

$n!$

Properties of $n!$

The sequence $n!$ grows very quickly. In fact it grows faster than any exponential function.

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Theorem (Stirling's formula)

For large values of n, one can use the following asymptotic formula to approximate $n!$:

$$n! \sim \underbrace{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}.$$

$n^n = n \cdot n \cdots n$

$n! = n \cdot (n-1) \cdots 1$

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to 1$$

as $n \to \infty$.

The complement trick

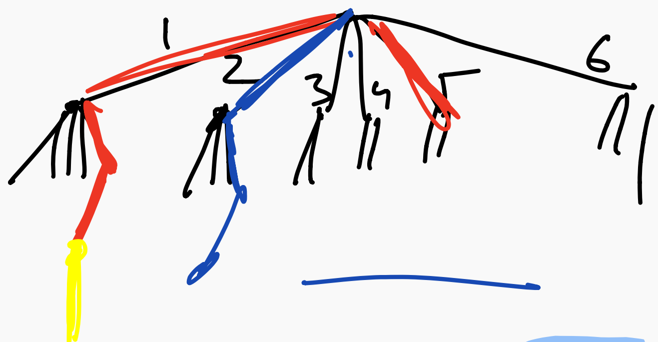
A fair die is thrown 4 times. What is the probability that the score 5 appears at least once.

$\Omega =$ • 4 throws of a fair die

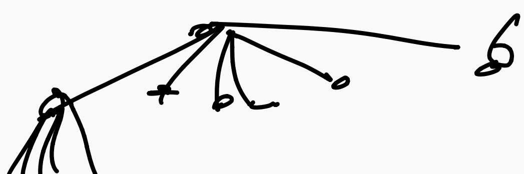
$A =$ at least one appearance of 5

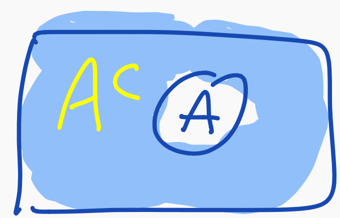
$\#\Omega = 6^4,$

$\#A = ?$



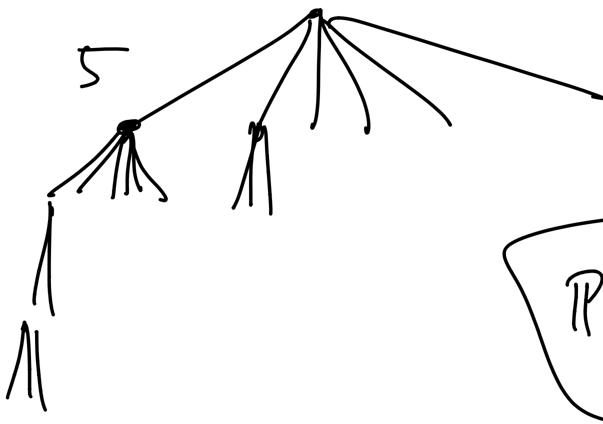
$A^c = \Omega$ with A removed





$$\mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

$A^c =$ 5 does not show up



favorable # outcome $= 5^4$

$$\mathbb{P}(A^c) = \frac{5^4}{6^4}$$

$$\boxed{\mathbb{P}(A) = 1 - \frac{5^4}{6^4}}$$

The complement trick

A fair die is thrown 4 times. What is the probability that the score 5 appears at least once.

The sample space is

$$\Omega = \{(x_1, x_2, x_3, x_4) : 1 \leq x_i \leq 6\}.$$

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The event A^c, indicating that A did not happen consists of those outcomes that consist only of $1, 2, 3, 4, 6$. So,

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$$\mathbb{P}[A^c] = \frac{5^4}{6^4} = 0.48.$$

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Hence

$$\mathbb{P}[A] = 1 - 0.48 = 0.52.$$