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Probability and Random Processes

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Definition
Let $X : \Omega \to \mathbb{R}$ be a random variable. The *probability distribution function* of X is the function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(t) = \mathbb{P}\left[X \leq t\right].$$

$X : \Omega \longrightarrow \mathbb{R}$ values of X are real numbers

define events in terms of

$A_t = \{ X \leq t \}$ $B = \{ s \leq X \leq t \}$

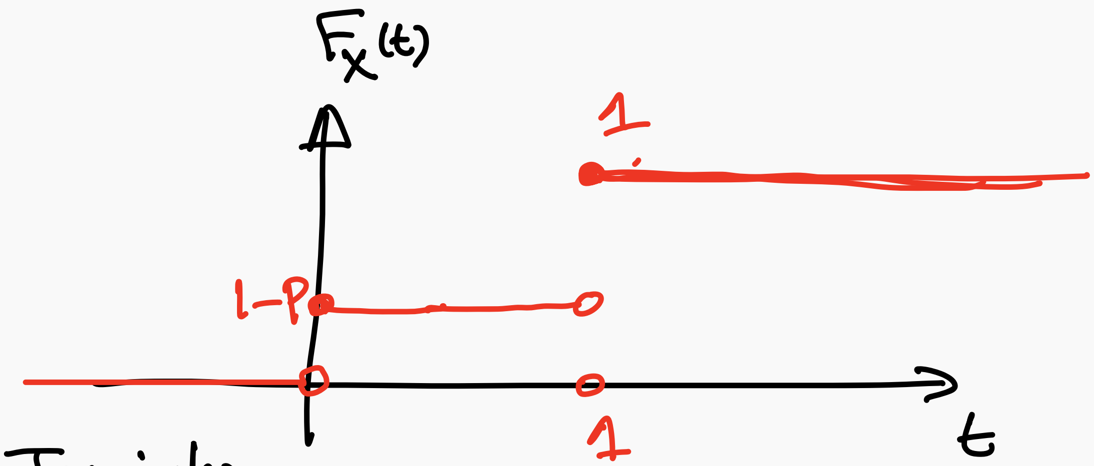
$\mathbb{P}(A) = \mathbb{P}(X \leq t)$ $\mathbb{P}(B) = \mathbb{P}(s \leq X \leq t)$

$\mathbb{R}$

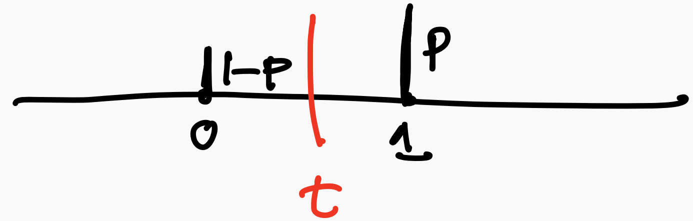
t

Examples: the distribution function of a Bernoulli random variable

Plot $F_X(t)$



x	0	1
$\mathbb{P}(X=x)$	$1-p$	p

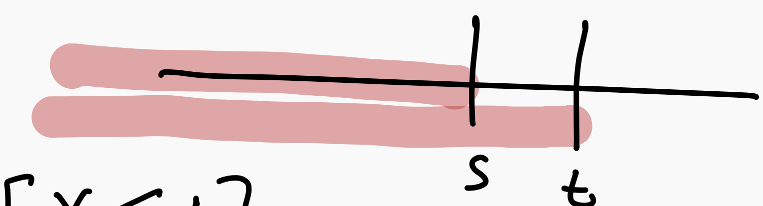


Terminology

increasing if $x < y \Rightarrow f(x) \leq f(y)$

$s \leq t$ $\qquad F_X(s) \leq F_X(t)$

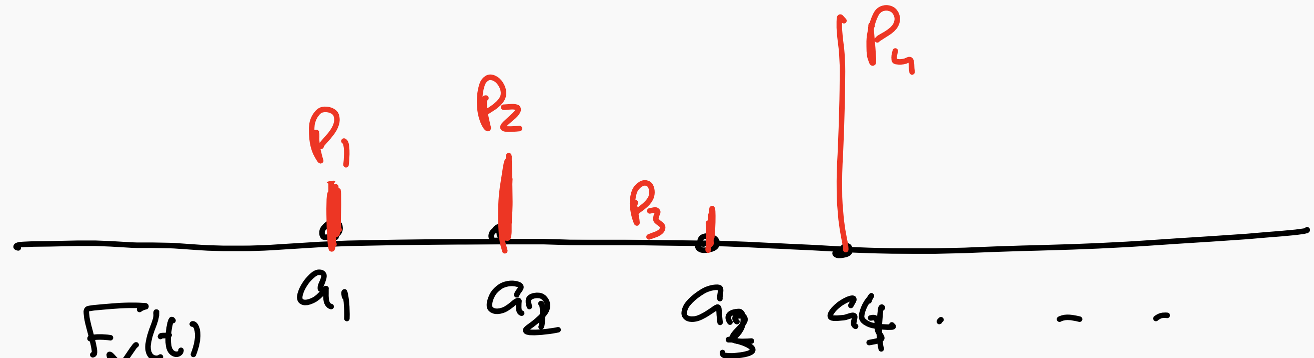
$$\mathbb{P}[X \leq s] \leq \mathbb{P}[X \leq t]$$



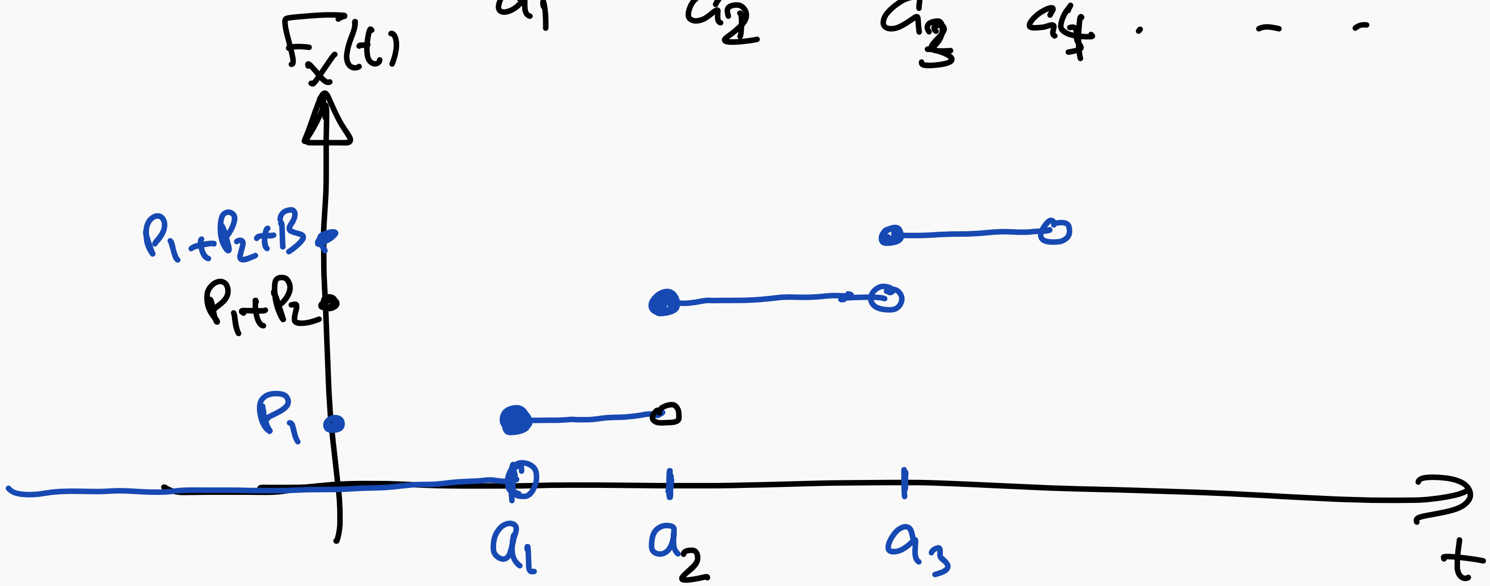
$$A_s = \{X \leq s\} \subseteq \{X \leq t\} = A_t$$

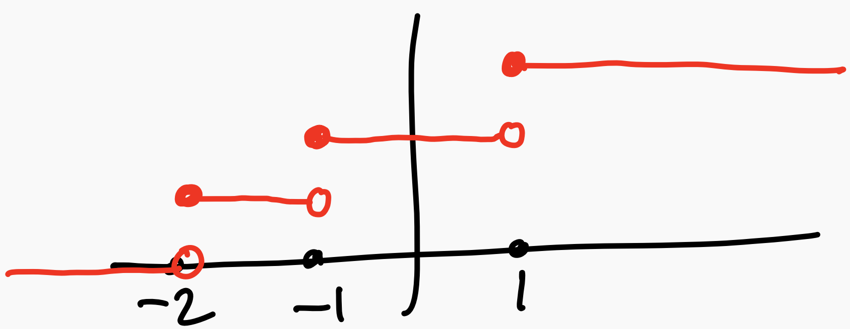
The distribution function of a general discrete random variable

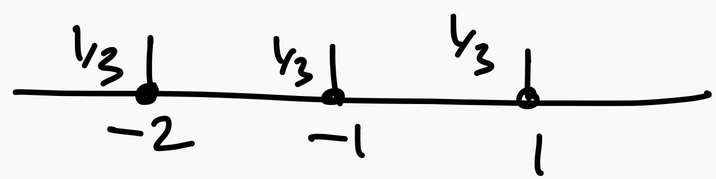
X



$F_X(t)$







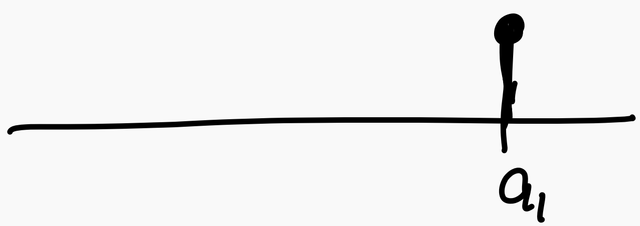
Properties of the distribution function

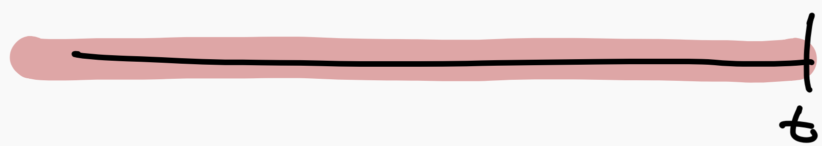
Theorem
The probability distribution function enjoys the following properties:

1. *F_X is (non-strictly) increasing: if $t_1 \leq t_2$, then $F_X(t_1) \leq F_X(t_2)$.*
2. *F_X is right-continuous, that is, for every $t \in \mathbb{R}$:*

$$\lim_{s \to t+} F_X(s) = F_X(t).$$

3. *F_X has limits at $\pm\infty$, namely, $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.*





$X \rightsquigarrow F_X(t)$

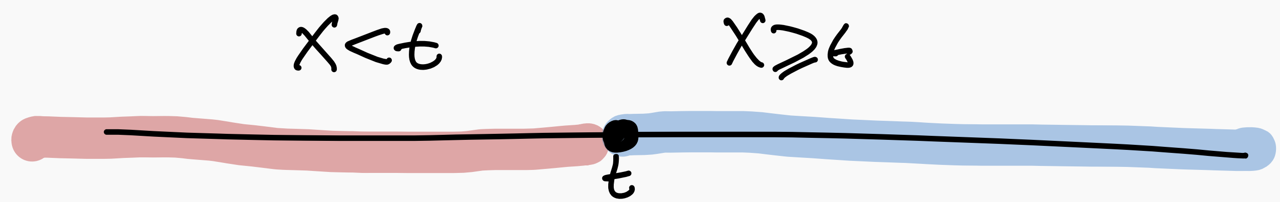
Theorem

For a random variables X and $t \in \mathbb{R}$, we have

1. $\mathbb{P}[X < t] = F_X(t-) := \lim_{s \to t-} F_X(s)$.
2. $\mathbb{P}[X \geq t] = 1 - F_X(t-)$.
3. $\mathbb{P}[X = t] = F_X(t) - F_X(t-)$.

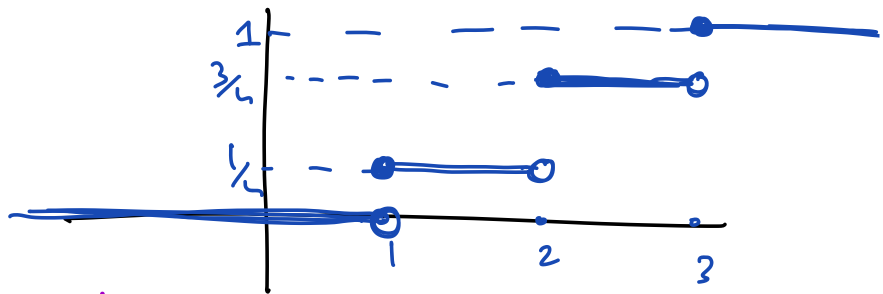
$\mathbb{P}(X \geq t) = 1 - \mathbb{P}(X<t)$

$\mathbb{P}(X \leq t) = F_X(t)$



$\mathbb{P}(X=t) = \mathbb{P}(X \leq t) - \mathbb{P}(X<t)$

X RV



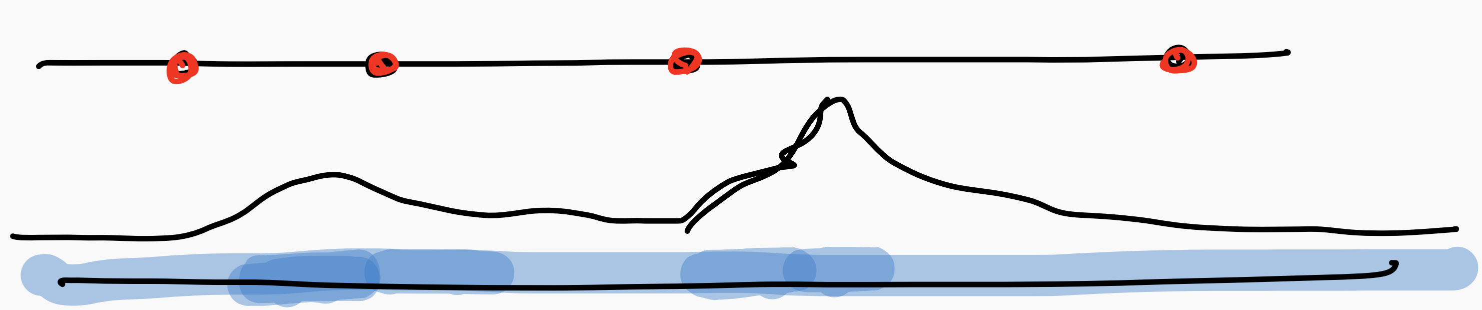
$$\mathbb{P}(X \leq 1.5) = 1/4$$

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\mathbb{P}(1 \leq X \leq 2) = \mathbb{P}(X \leq 2) - \mathbb{P}(X < 1)$$

$$= \frac{3}{4} - 0 = \frac{3}{4}.$$

Point mass

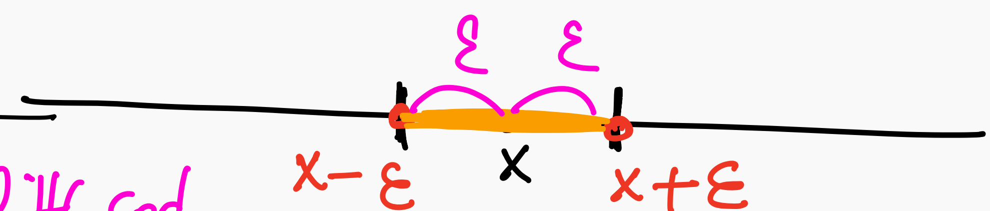


density

density function

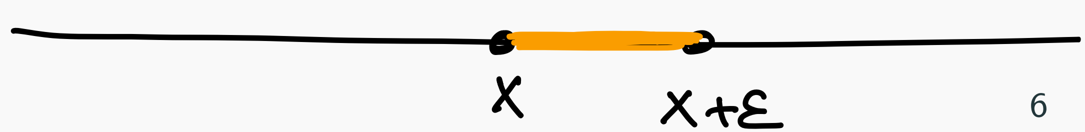
$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\text{mass between } (x-\varepsilon) \text{ and } (x+}{2\varepsilon}$$

↖ length of little rod



ε)

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\text{mass between } x \text{ and } x+\varepsilon}{\varepsilon}$$



Third version

$F(x) =$ mass up to x

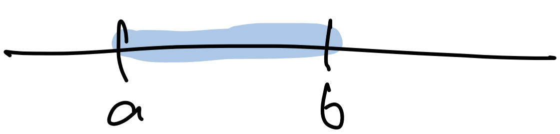
$F(x+\varepsilon) =$ mass up $x+\varepsilon$

mass between x and $x+\varepsilon$ $= F(x+\varepsilon) - F(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon) - F(x)}{\varepsilon} = F'(x).$$

If $\delta(x)$ is given

$$\boxed{F(t) = \int_{-\infty}^{t} \delta(x)\,dx}$$



mass between a & b $= \int_a^b \delta(x)\,dx$

Definition
A random variable $X : \Omega \to \mathbb{R}$ is called *continuous* if there exists a non-negative function $f_X : \mathbb{R} \to \mathbb{R}$, called the *probability density function* of X, such that for all $t \in \mathbb{R}$, we have

$$\mathbb{P}(X \leq t) = F_X(t) = \int_{-\infty}^{t} f_X(x)dx.$$

t

Definition (version 2)

Definition
A random variable $X : \Omega \to \mathbb{R}$ is called *continuous* if there exists a non-negative function $f_X : \mathbb{R} \to \mathbb{R}$, called the *probability density function* of X, such that for all $s \leq t$, we have

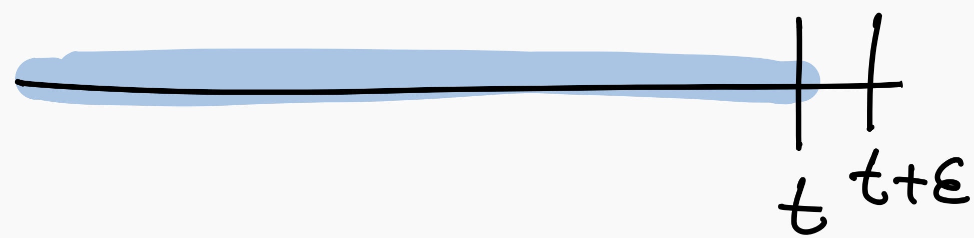
$$\mathbb{P}[s \leq X \leq t] = \int_s^t f_X(x)dx.$$

Relation between the density and the distribution functions

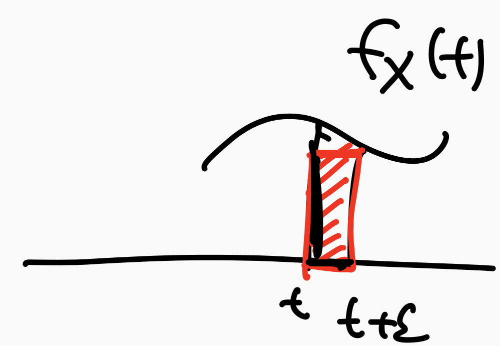
Theorem
If F_X is differentiable, then

$$f_X(t) = \frac{d}{dt} F_X(t).$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx$$



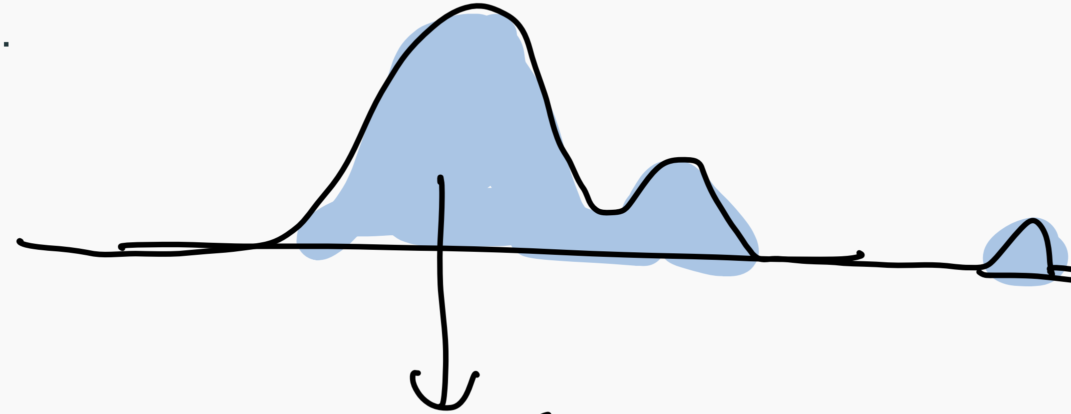
$$F_X(t+\varepsilon) = \int_{-\infty}^{t+\varepsilon} f_X(x)\,dx$$



$$F_X(t+\varepsilon) - F_X(t) = \int_{t}^{t+\varepsilon} f_X(t) \approx \varepsilon \cdot f_X(t)$$

$$\lim_{\varepsilon \to 0} \frac{F_X(t+\varepsilon) - F_X(t)}{\varepsilon} = f_X(t).$$

- (Non-negativity: $f_X \geq 0$.
- Total mass one: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.



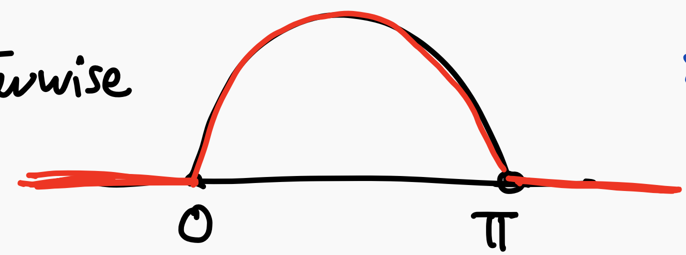
area = 1

$\int_{-\infty}^{\infty} \delta(x) = 1$

distribution.

example

$$f_X(t) = \begin{cases} \frac{1}{2}\sin t & 0 \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases}$$



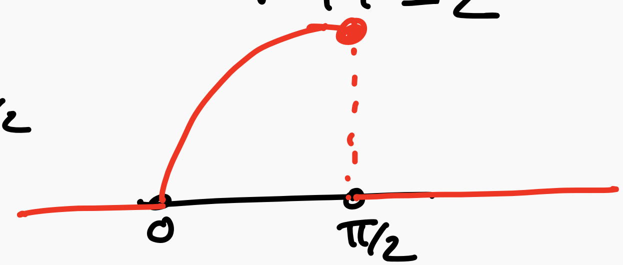
$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^{\pi} \sin x\,dx$$

$$= -\cos x \Big|_0^{\pi} = -\cos\pi + \cos 0$$

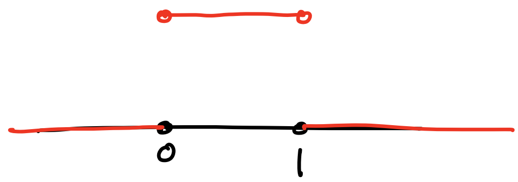
$$= 1 + 1 = 2$$

$$\int_0^{\pi/2} \sin x\,dx = 1$$

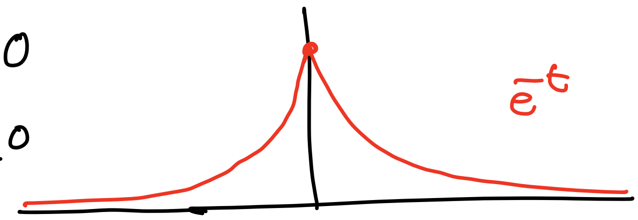
$$f_X(t) = \begin{cases} \sin t & 0 \leq t \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



$$f_X(t) = \begin{cases} \frac{e^{-t}}{2} & t \geq 0 \\ \frac{e^{t}}{2} & t \leq 0 \end{cases}$$



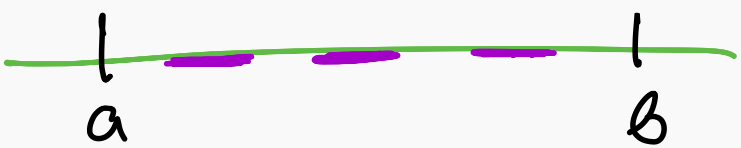
$$= e^{-|t|}$$

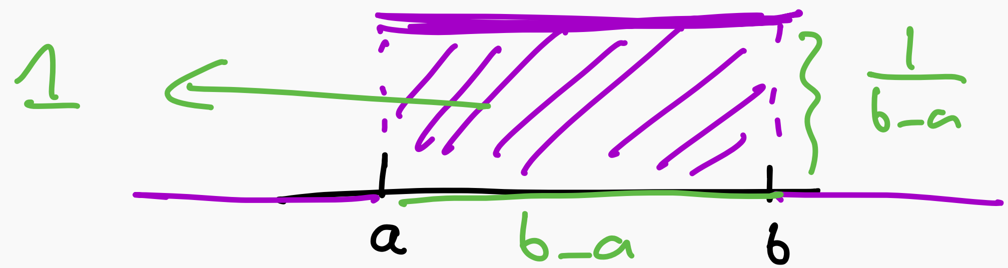
$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 2\int_0^{\infty} e^{-x}\,dx = -2 e^{-x}\Big|_0^{\infty}$$

$$= -2(0-1) = 2$$

Definition
A random variable X has uniform distribution over the interval $[a, b]$, if its probability density function is given by

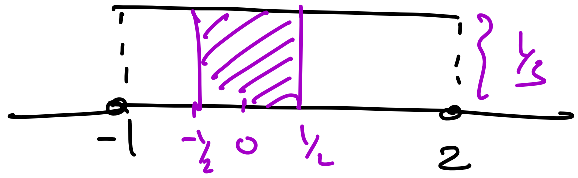
$$f_X(t) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}$$





X uniform RV on the interval $[-1, 2]$

Compute $P(\frac{-1}{2} \leq X \leq \frac{1}{2})$



$$\mathbb{P}\Big(\underbrace{\tfrac{-1}{2} \le X \le \tfrac{1}{2}}\Big) = \int_{\text{interval}} f_X(x)\,dx$$

$$= \int_{-1/2}^{1/2} \frac{1}{3}\,dx = \frac{1}{3} \, x \Big|_{-1/2}^{+1/2} = \frac{1}{3}$$

$$\mathbb{P}\left(-2 \le X \le \tfrac{1}{2}\right) = \int_{\text{interval}} f_X(x)\,dx$$

$$= \int_{-2}^{1/2} \frac{1}{3}\,dx = \frac{1}{3}x \Big|_{-2}^{1/2} = \frac{1}{3}\left(\frac{1}{2}+2\right)$$

-5

$$\frac{2.5}{3}$$

$$\frac{1}{3}\left(5+\frac{1}{2}\right) = \frac{5.5}{3}$$

$$\int_{-1}^{1/2} \frac{1}{3}\,dx = \frac{1}{3}\left(\frac{3}{2}\right) = \frac{1}{2}.$$