Homework 6 Solutions

Problem 1

$f(x) = \frac{x^2}{4-x^2}$ | Domain: All real numbers excluding 2 and -2.

Intercepts $\Rightarrow (0,0)$

Horizontal asymptotes $\Rightarrow \lim_{x\to\infty} \frac{f(x)}{x^2} = \lim_{x\to\infty} \frac{1}{\frac{4}{x^2}-1} = \frac{1}{-1} = -1$

$\therefore$ $y=-1$ is an asymptote.

Vertical asymptotes $\Rightarrow$ Check for where denominator becomes zero.

$4-x^2=0 \Rightarrow x = \pm 2$

$\therefore$ $x=2$ and $x=-2$ are vertical asymptotes.

First Derivatives

$f(x) = \frac{x^2}{4-x^2}$

$\therefore f'(x) = \frac{2x(4-x^2) - x^2(-2x)}{(4-x^2)^2} = \frac{2x^3 + 2x(4-x^2)}{(4-x^2)^2}$

$= \frac{8x}{(4-x^2)^2}$

At $\frac{dy}{dx} = 0$, $x = 0$ and $y = 0$

→ $f(x)$ is decreasing from $-2 < x < 0$
→ $f(x)$ is increasing from $0 < x < 2$
→ There is a minimum point at $x = 0$

$\Rightarrow (0,0)$ is a minimum point.

Second Derivatives

$$f'(x) = \frac{8x}{(4-x^2)^2}$$

$$\therefore f''(x) = \frac{8\left[(x^2-4)^2 - 2(x^2-4)(2x+0)x\right]}{(x^2-4)^4}$$

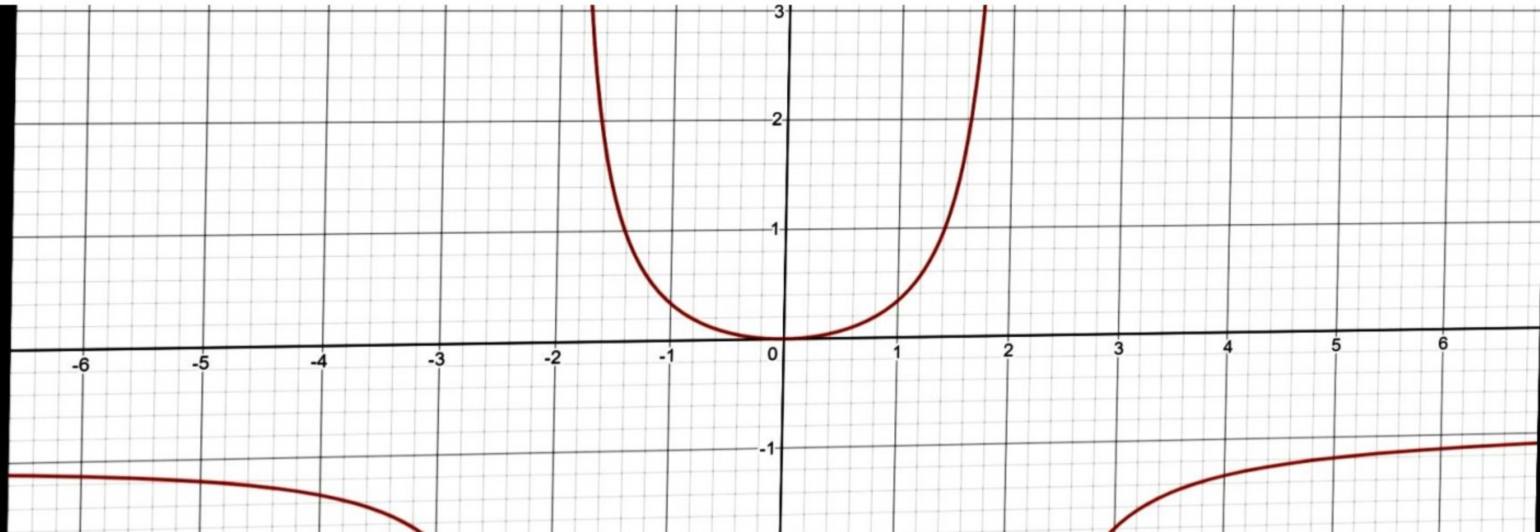
$$= \frac{8(3x^2+4)}{(x^2-4)^3}$$

→ $f''(x)$ is $(-)$ve

→ $f(x)$ is concave up from $-2 < x < 2$
→ $f(x)$ is concave down ~~from~~ everywhere else.

→ $f''(x)$ is $(+)$ve

→ No points of inflection since when $f''(x) = 0$, roots are imaginary.



-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
3
2
1
-1
-2
-3

Problem 2

$f(x) = -\ln(x) + \sqrt{x}$

Domain $\Rightarrow$ $x > 0$ for all real x.

Intercepts $\Rightarrow$ No intercepts since $\ln(0)$ is NOT defined.

Horizontal Asymptotes : NO horizontal asymptotes.

Vertical Asymptotes : $x = 0$ is a vertical asymptote.

First Derivative

$$f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{x} = \frac{1}{2\sqrt{x}} - \frac{1}{x}$$

At $f'(x) = 0$, $x = 4$

$\therefore$ $(4, 2 - \ln(4))$ is a minimum point.

→ $f(x)$ is decreasing from $0 < x < 4$

→ $f(x)$ is increasing ~~from~~ for $x > 4$

Second Derivative

$$f''(x) = \frac{\left(-\frac{1}{2}\right) \cdot x^{-3/2}}{2} + \frac{1}{x^2}$$

$$= \frac{1}{x^2} - \frac{1}{4x^{3/2}}$$

at $f''(x) = 0$, $x = 16$

$\therefore$ at $x = 16$, there is an inflection point.

→ $f(x)$ is concave up from $0 < x < 16$
→ $f(x)$ is concave down for $x > 16$

Problem 3

$f(x) = 2 \cdot e^{-4/x}$

Domain : All real x excluding 0.

Intercepts ⇒ NO intercepts

$$\lim_{x \nearrow 0} 2 \cdot e^{-4/x} = \infty$$

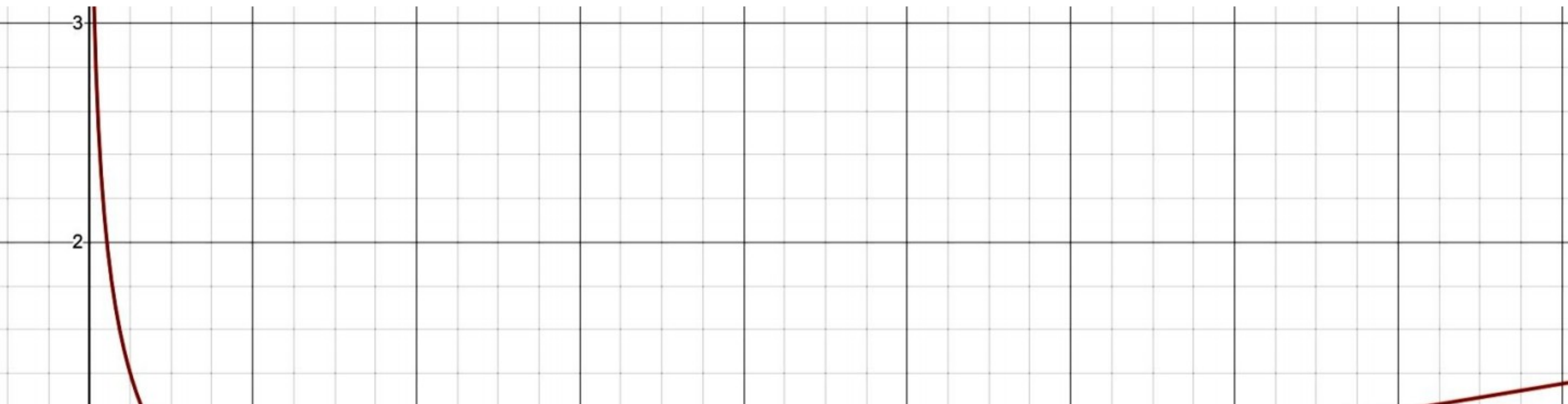
$$\lim_{x \searrow 0} 2 \cdot e^{-4/x} = 0$$

Horizontal Asymptotes

→ $y = 2$ is a horizontal asymptote.

Vertical Asymptotes

→ $x = 0$ is a vertical asymptote



3
2
1
0
2
4
6
8
10
12
14
16
18

First Derivative

$$f'(x) = 2 \cdot e^{-4/x} \cdot \frac{4}{x^2} = \frac{8 \cdot e^{-4/x}}{x^2}$$

At $f'(x) = 0$, no solution, hence no turning points. (maxima or minima)

Since $f'(x) > 0$ for all x, the function is always increasing

Second Derivative

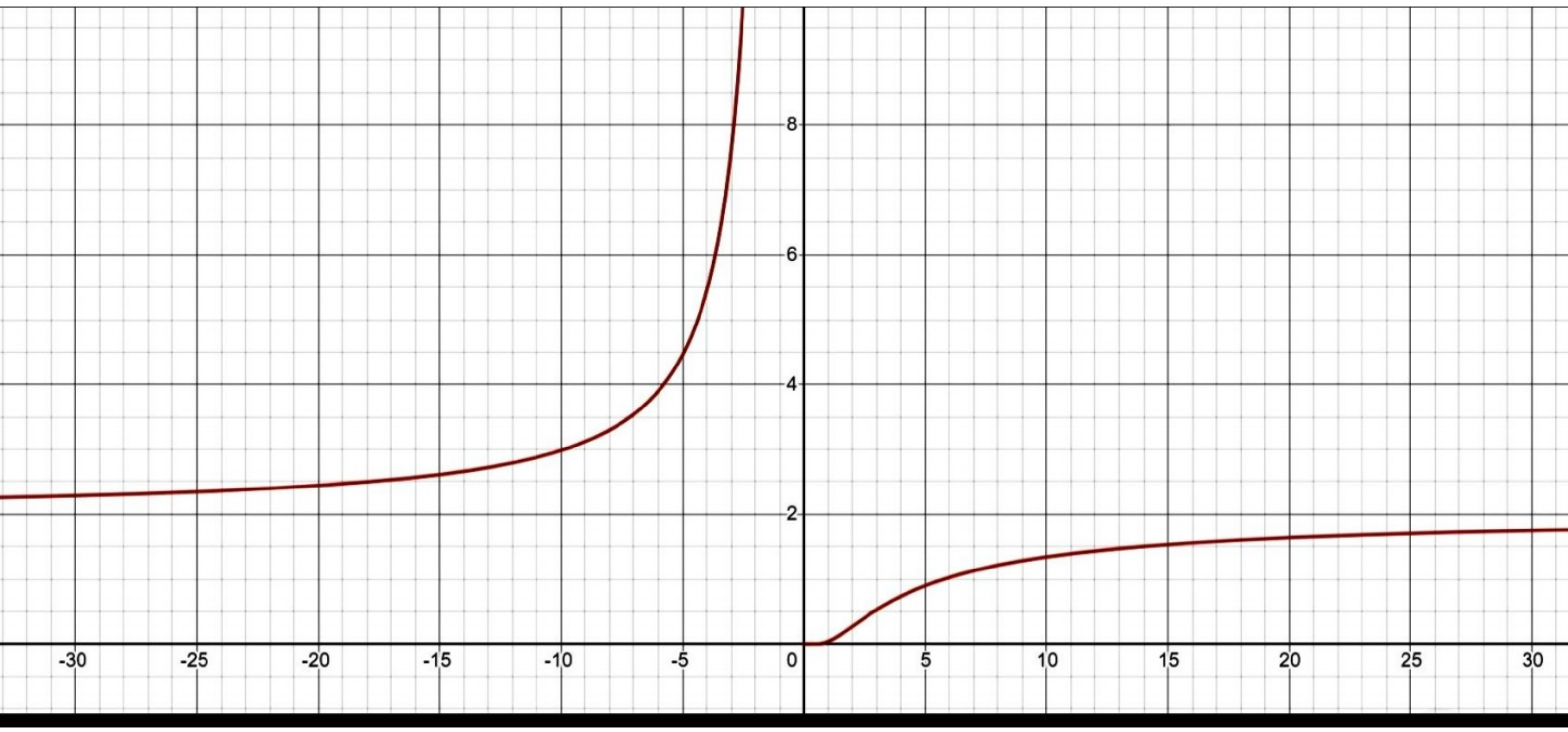
$$f''(x) = \frac{8\left(4 \cdot e^{-4/x} \cdot 1 - 2x \cdot e^{-4/x}\right)}{x^4}$$

$$= \frac{8\left(4 \cdot e^{-4/x} - 2x \cdot e^{-4/x}\right)}{x^4} = - \frac{(16x - 32) \cdot e^{-4/x}}{x^4}$$

At $f''(x) = 0 \Rightarrow x = 2$

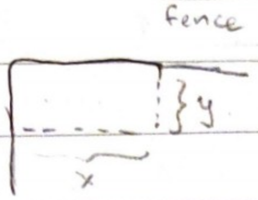
$\rightarrow \therefore$, there is an inflection point at $x = 2$

$\rightarrow f(x)$ is concave up for $x > 2$
$\rightarrow f(x)$ is concave down for $x < 2$



8
6
4
2
-30
-25
-20
-15
-10
-5
0
5
10
15
20
25
30

4)



$x + y = 50 \Rightarrow y = 50 - x$

$A = x \cdot y$ (Area of rectangle

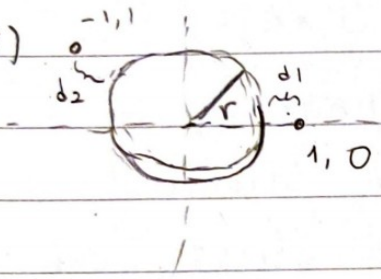
$A(x) = x(50 - x) = 50x - x^2$

$\frac{dA(x)}{dx} = 50 - 2x \Rightarrow x = \frac{50}{2} = 25$ maximizes area function

$y = 50 - x = 50 - 25 = 25$

Max Area = x×y = 25×25 = 625

5)



$D = d_1^2 + d_2^2$ (Sum of distance squared)

$D(r) = (\sqrt{1^2+0^2} - r)^2 + (\sqrt{(-1)^2+(1)^2} - r)^2$

$= (1 - r)^2 + (\sqrt{2} - r)^2$

$\frac{dD(r)}{dr} = 2(1 - r)(-1) + 2(\sqrt{2} - r)(-1)$

$= -2 + 2r - 2\sqrt{2} + 2r$

$\Rightarrow r = \frac{2(1 + \sqrt{2})}{4} = \frac{1 + \sqrt{2}}{2}$ minimizes D function

(also makes sense as it is average of the 2 distances)