Homework 11 Solutions

Problem 1

(a) Rewrite the equations in a matrix.

$$\left(\begin{array}{ccc|c} -1 & 2 & -1 & 8 \\ 2 & 3 & 9 & 5 \\ -4 & -5 & -17 & -7 \end{array}\right)$$

Solve using Gaussian Elimination:

$$\left(\begin{array}{ccc|c} -1 & 2 & -1 & 8 \\ 2 & 3 & 9 & 5 \\ -4 & -5 & -17 & -7 \end{array}\right) \xrightarrow{\frac{r_1}{-1} \to r_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & -8 \\ 2 & 3 & 9 & 5 \\ -4 & -5 & -17 & -7 \end{array}\right)$$

This step must be read as "the new row 1 (r_1) is the old r_1 divided by -1"

$$\xrightarrow[\substack{r_2 - 2r_1 \to r_2 \\ \text{and} \\ r_3 + 4r_1 \to r_3}]{} \left(\begin{array}{ccc|c} 1 & -2 & 1 & -8 \\ 0 & 7 & 7 & 21 \\ 0 & -13 & -13 & -39 \end{array}\right) \xrightarrow{\frac{r_2}{7} \to r_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & -8 \\ 0 & 1 & 1 & 3 \\ 0 & -13 & -13 & -39 \end{array}\right)$$

$$\xrightarrow[\substack{r_1 + 2r_2 \to r_1 \\ r_3 + 13r_2 \to r_3}]{} \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, the solutions are:

$$x_1 + 3x_3 = -2$$
$$x_2 + x_3 = 3$$

$$\Rightarrow \quad x = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

1b) $a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}^{A^{-1}} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix},$$

Find the inverse: $\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right)$

$$\hookrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 5/2 \\ 1/2 \end{pmatrix}$$

Can also be done by solving $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$, although here you have the intuition of first finding the inverse matrix and solving the equation

Problem 2

Get the matrix form and peform a Gaussian Elimination.

$$\left(\begin{array}{ccc|c} 2 & -2 & \alpha & -2 \\ 4 & -4 & 12 & -4 \\ 2 & \alpha & 0 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & \alpha & -2 \\ 0 & 0 & 12-2\alpha & 0 \\ 2 & \alpha & 0 & 2 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & \alpha & -2 \\ 0 & 0 & 12-2\alpha & 0 \\ 0 & \alpha+2 & -\alpha & 4 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & \alpha & -2 \\ 0 & \alpha+2 & -\alpha & 4 \\ 0 & 0 & 12-2\alpha & 0 \end{array}\right)$$

$$2x_1 - 2x_2 + \alpha x_3 = -2$$
$$(\alpha+2)x_2 - \alpha x_3 = 4$$
$$(12-2\alpha)x_3 = 0$$

For $\alpha = 6$, we have x_3 to be a free variable (any value will satisfy its equation), so we have infinitely many solutions.

$\alpha = -2$ leads to a contradiction between equations, so it has no solutions. Everything else has a unique solution. (1 solution)

3a)

$$V_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} \quad V_4 = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

They are $\underbrace{\text{L.I.}}_{\text{linearly independent}}$ if $A := \left(V_1 \middle| V_2 \middle| V_3 \middle| V_4 \right)$ is invertible or $\det(A) \neq 0$

$$\begin{pmatrix} 2 & -1 & 0 & -2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & -1 & 2 \end{pmatrix} \xrightarrow[R_2/2]{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & -3 & 2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 1/2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 7/6 \end{pmatrix}$$

so it is invertible $\Rightarrow$ it is a basis

③ (b) By way of contradiction, assume A, B are non-singular.
This means that there does not exist a $\vec{v}$ such that $A\vec{v} = 0$ or $B\vec{v} = 0$ aside from $\vec{v} \neq 0$.

$A B \vec{v} = 0 \Rightarrow A\vec{\omega}\ 0$.
Since A is non-singular for $\vec{v} \neq 0$,
if $\vec{\omega} = B\vec{v}$
and if $\vec{\omega} = 0$, $B\vec{v} = 0$ $\boxed{\text{but } v \neq 0}$

Hence, contradiction.
$\Rightarrow$ Therefore, either A or B is singular.