Homework 10

Problem 1

(10 points)

Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$.

1. The *"BAC–CAB-identity"*
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})\,. \tag{1}$$
2. The *Jacobi identity* in three dimensions
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0\,.$$

Problem 2

(10 points)

Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$.

1. The *Cauchy–Binet formula* in three dimensions
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c})\,.$$
Hint: Use the identity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u})$.
2. The identity
$$\|\boldsymbol{a} \times \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2\,.$$

Problem 3

(10 points)

1. Find the minimum distance between the point $\boldsymbol{p} = (2, 4, 6)$ and the line
$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\,.$$
2. Express the equation for the plane that contains the point $\boldsymbol{p}$ and the line $\boldsymbol{x}$ in parametric form. Then proceed to find the vector normal to this plane.

Bonus

(10 points)
Prove the following statement: Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be linearly independent. If a vector $\boldsymbol{w}$ can be written

$$\boldsymbol{w} = \sum_{k=1}^{n} \alpha_k \, \boldsymbol{v}_k \, ,$$

then the choice of the coefficients $\alpha_1, \ldots, \alpha_n$ is unique.

Hint: Recall that a set of vectors is said to be linearly independent if $\boldsymbol{w} = 0$ implies that all of the coefficients $\alpha_k = 0$.