4. Use implicit differentiation to find an equation for the tangent line to the graph of $\sin(x+y) = y^2\cos(x)$ at point $(0,0)$. (5)

5. Compute the following integrals:

   (a) $\displaystyle\int_0^1 \ln x \, dx$

   (b) $\displaystyle\int \frac{x^2+1}{x^2-1}\, dx$

(5+10)

6. Is the following improper integral convergent? There is no need to compute the answer, but you should give detailed reasoning.

$$\int_0^\infty \frac{\ln x + e^{-x}}{1+x^2}\, dx$$

(10)

7. Consider the differential equation

$$\frac{dy}{dt} = t^3 y^3 .$$

   (a) Solve the initial value problem with $y(0) = 2$.

   (b) Does this equation have equilibrium points? Are they stable or unstable?

(10+5)

8. Show that, for $u, v \in \mathbb{R}^3$,

$$\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 .$$

(5)

9. Find the general solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ -3 \end{pmatrix} .$$

(10)

10. Let $L: \mathbb{R}^4 \to \mathbb{R}^4$ be the "shift mapping" defined as follows:

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} .$$

(a) Show that L is a linear transformation on $\mathbb{R}^4$.

(b) Write out the matrix S which represents L with respect to the standard basis.

(c) Find a basis for Range S and Ker S.

(d) State the "rank-nullity theorem" and verify explicitly that the result obtained in part (c) matches the statement of the theorem.

(5+5+5+5)