Problem 1

a) $\vec{a}\times(\vec{b}\times\vec{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - c_2b_3 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = \begin{pmatrix} a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ a_3(b_2c_3 - c_2b_3) - a_1(b_1c_2 - b_2c_1) \\ a_1(b_3c_1 - c_3b_1) - a_2(b_2c_3 - c_2b_3) \end{pmatrix}$

$$= \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{pmatrix} \underbrace{\begin{matrix} +a_1b_1c_1 - a_1b_1c_1 \\ +a_2b_2c_2 - a_2b_2c_2 \\ +a_3b_3c_3 - a_3b_3c_3 \end{matrix}}_{\text{Ammount to adding } 0}$$

$$= \begin{pmatrix} a_2b_1c_2 + a_3b_1c_3 + a_1b_1c_1 - (a_2b_2c_1 + a_3b_3c_1 + a_1b_1c_1) \\ a_3b_2c_3 + a_1b_2c_1 + a_2b_2c_2 - (a_3b_3c_2 + a_1b_1c_2 + a_2b_2c_2) \\ a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - (a_1b_1c_3 + a_2b_2c_3 + a_3b_3c_3) \end{pmatrix}$$

$$= \begin{pmatrix} b_1(\vec{a}\cdot\vec{c}) - c_1(\vec{a}\cdot\vec{b}) \\ b_2(\vec{a}\cdot\vec{c}) - c_2(\vec{a}\cdot\vec{b}) \\ b_3(\vec{a}\cdot\vec{c}) - c_3(\vec{a}\cdot\vec{b}) \end{pmatrix} = (\vec{a}\cdot\vec{c})\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - (\vec{a}\cdot\vec{b})\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

b) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}(\vec{c}\times\vec{a}) + \vec{c}(\vec{a}\times\vec{b})$

$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) + \vec{c}(\vec{b}\cdot\vec{a}) - \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}(\vec{c}\cdot\vec{b}) - \vec{b}(\vec{c}\cdot\vec{a})$

$= 0$ (all terms cancel out)

Problem 2

a) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{c}\cdot(\vec{d}\times(\vec{a}\times\vec{b})) = \vec{c}\cdot[\vec{a}(\vec{d}\cdot\vec{b}) - \vec{b}(\vec{d}\cdot\vec{a})]$

by formula in hint   BAC-CAB

$= \vec{c}\cdot\vec{a}(\vec{d}\cdot\vec{b}) - \vec{c}\cdot\vec{b}(\vec{d}\cdot\vec{a}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

only scalars so can be factored out

Also, dot product is commutative

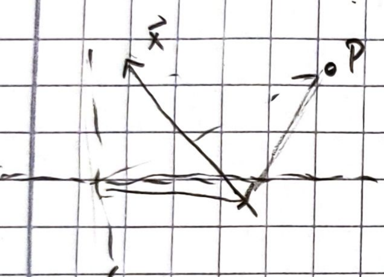
b) $\|\vec{a}\times\vec{b}\|^2 \stackrel{?}{=} \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$

$\|\vec{a}\times\vec{b}\|^2 = (\|\vec{a}\|\|\vec{b}\|\sin\theta)^2 = \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta = \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)$

$= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta = \|\vec{a}\|^2\|\vec{b}\|^2 - (a\cdot b)^2$

Problem 3

Let $D(\lambda)$ be the distance squared function from the point P to a point along $\vec{x}$



$$D(\lambda) = (P_1 - X_1)^2 + (P_2 - X_2)^2 + (P_3 - X_3)^2$$

$$= (2 - (-1+\lambda))^2 + (4 - (1-\lambda))^2 + (6-6)^2$$

$$= (3-\lambda)^2 + (3+\lambda)^2 = 9 - \cancel{6\lambda} + \lambda^2 + 9 + \cancel{6\lambda} + \lambda^2$$

$$= 18 + 2\lambda^2$$

$\frac{d}{d\lambda} D(\lambda) = 4\lambda$ $\qquad$ $\frac{d}{d\lambda} D(\lambda) = 0$ at $\lambda = 0$

$\Rightarrow D_{min} = 18 + 0 = 18$ $\quad$ so distance $= \sqrt{18} = 3\sqrt{2}$

Need a second vector for equation of the plane:

$$V_2 = \begin{pmatrix} 2 - -1 \\ 4 - 1 \\ 6 - 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \sim 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Parametric eq: $\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

In general many solutions are correct (infinitely many!), as long as the two vectors lie on a plane. $\vec{n}$ should be the same up to scalar multiplication

Problem 4

Suppose $\alpha_1, \dots, \alpha_n$ are not unique

then $\vec{\omega} = \sum_{k=1}^{n} \beta_k V_k$ and $\vec{\omega} = \sum_{k=1}^{n} \alpha_k V_k$

with $\beta_k \neq \alpha_k$ for all k's

Then $\sum_{k=1}^{n} \beta_k V_k - \sum_{k=1}^{n} \alpha_k V_k = \vec{\omega} - \vec{\omega} = 0$

$\Rightarrow \sum_{k=1}^{n} (\beta_k - \alpha_k) V_k = 0$

Since $V_1, \dots, V_k$ are linearly independent

$\sum_{k=1}^{n} (\beta_k - \alpha_k) V_k = 0$ only if all $(\beta_k - \alpha_k) = 0$

$\Rightarrow \beta_k = \alpha_k$ for all k which contradicts initial supposition

Therefore, $\vec{\omega}$ has a unique representation