CH-231-A

Algorithms and Data Structures

ADS

Lecture 31

Dr. Kinga Lipskoch

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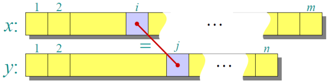
Strategy

- Look at length of longest-common subsequence.
- Let $|s|$ denote the length of a sequence s.
- To find $LCS(x, y)$, consider prefixes of x and y (i.e., we go from right to left)
- Definition: $c[i, j] = |LCS(x[1..i], y[1..j])|$.
  In particular, $c[m, n] = |LCS(x, y)|$.
- Theorem (recursive formulation):

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof (1)

Case $x[i] = y[j]$:



Let $z[1..k] = LCS(x[1..i], y[1..j])$ with $c[i,j] = k$.
Then, $z[k] = x[i] = y[j]$ (else z could be extended).
Thus, $z[1..k-1]$ is CS of $x[1..i-1]$ and $y[1..j-1]$.
Claim: $z[1..k-1] = LCS(x[1..i-1], y[1..j-1])$.

- Assume w is a longer CS of $x[1..i-1]$ and $y[1..j-1]$, i.e., $|w| > k-1$.
- Then the concatenation $w + z[k]$ is a CS of $x[1..i]$ and $y[1..j]$ with length $> k$.
- This contradicts $|LCS(x[1..i], y[1..j])| = k$.
- Hence, the assumption was wrong and the claim is proven.

Hence, $c[i-1, j-1] = k-1$, i.e., $c[i,j] = c[i-1,j-1] + 1$.

Proof (2)

Case $x[i] \neq y[j]$:

Then, $z[k] \neq x[i]$ or $z[k] \neq y[j]$.

- $z[k] \neq x[i]$:
  Then, $z[1..k] = LCS(x[1..i-1], y[1..j])$.
  Thus, $c[i-1, j] = k = c[i, j]$.
- $z[k] \neq y[j]$:
  Then, $z[1..k] = LCS(x[1..i], y[1..j-1])$.
  Thus, $c[i, j-1] = k = c[i, j]$.

In summary, $c[i, j] = \max\{c[i-1, j], c[i, j-1]\}$.

Dynamic Programming Concept (1)

Step 1: Optimal substructure.
An optimal solution to a problem contains optimal solutions to subproblems.

Example:
If $z = LCS(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

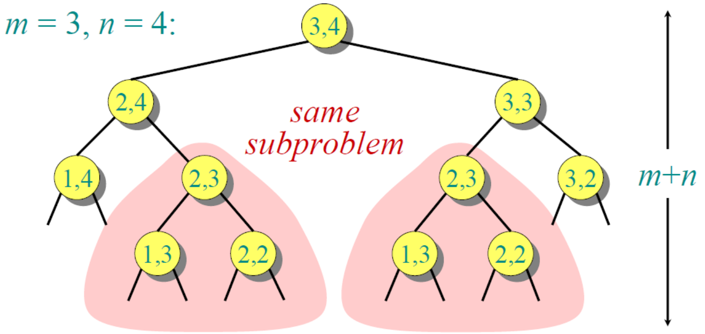
Recursive Algorithm

- Computation of the length of LCS:

```
LCSlength(x,y,i,j):
  if i == 0 or j == 0
    return 0
  else if x[i] == y[j]
    return LCSlength(x,y,i-1,j-1)+1
  else return max{LCSlength(x,y,i-1,j),
                  LCSlength(x,y,i,j-1)}
```

- Remark: if $x[i] \neq y[j]$, the algorithm evaluates two subproblems that are very similar.

Recursive Tree



Height $= m + n \Rightarrow$ work potentially exponential,
but we're solving subproblems already solved!

Dynamic Programming Concept (2)

Step 2: Overlapping subproblems.
A recursive solution contains a "small" number of distinct subproblems repeated many times.

Example:
The number of distinct LCS subproblems for two prefixes of lengths m and n is only $m \cdot n$.

Memoization Algorithm

Memoization:

- After computing a solution to a subproblem, store it in a table.
- Subsequent calls check the table to avoid repeating the same computation.

Recursive Algorithm with Memoization

Computation of the length of *LCS*:

```
LCSlength_(x,y,i,j):
  if c[i,j] == NIL
    then if i==0 or j==0
      c[i,j] = 0
    else if x[i] == y[j]
      c[i,j] = LCSlength_(x,y,i-1,j-1)+1
    else c[i,j] = max{LCSlength_(x,y,i-1,j),
                      LCSlength_(x,y,i,j-1)}
  return c[i,j]
```

Dynamic Programming (1)

Compute the table bottom-up:

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic Programming (2)

Compute the table bottom-up:

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

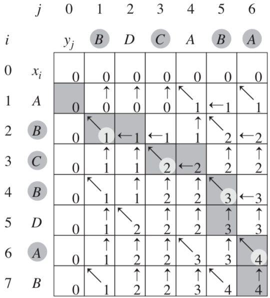
```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i - 1, j - 1] + 1
            b[i, j] = "↖"
        elseif c[i - 1, j] ≥ c[i, j - 1]
            c[i, j] = c[i - 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j - 1]
            b[i, j] = "←"
return c and b
```

Complexity

- Time complexity: $T(m, n) = \Theta(m \cdot n)$
- Space complexity: $S(m, n) = \Theta(m \cdot n)$

Reconstructing LCS

- Trace backwards:



```
PRINT-LCS(b, X, i, j)
if i == 0 or j == 0
    return
if b[i, j] == "↖"
    PRINT-LCS(b, X, i − 1, j − 1)
    print x_i
elseif b[i, j] == "↑"
    PRINT-LCS(b, X, i − 1, j)
else PRINT-LCS(b, X, i, j − 1)
```

- Time complexity: $O(m + n)$

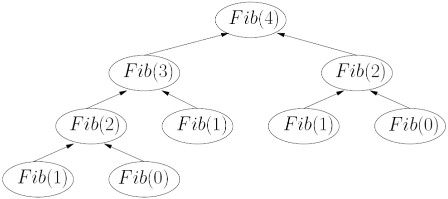
Fibonacci Numbers Revisited (1)

Recall:

- Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

- Recursion tree of brute-force implementation:



Fibonacci Numbers Revisited (2)

Dynamic programming solution:

- Avoid re-computations of same terms.
- Store results of subproblems in a table.
- Thus, $Fib(k)$ is computed exactly once for each k.
- This basically leads to the previously discussed bottom-up approach.
- Computation time is $T(n) = \Theta(n)$.