Tutorial 1 notes:
14-Sept

Variation of Birthday Problem:

Consider group of n people whose birthday equally likely to fall on any day.

Sample space: $\Omega = \{(x_1, x_2, \cdots, x_n) \mid 1 \le x_i \le 365\} \Rightarrow |\Omega| = 365^n$

Consider Events:

A = Someone shares your birthday (You're not part of group)

B = Any two people share a birthday

C = Any three people share a birthday

How to calculate P(A), P(B), P(C)?

First mathematically formulate the events A, B, C:

Suppose your birthday is at day 'd'. Then

Event $A \Leftrightarrow \{(x_1, x_2, \cdots, x_n) \mid x_k = d \text{ for some } k \le n\}$

Easier to calculate A^c instead.

$$A^c = \{(x_1, \cdots, x_n) \mid x_k \ne d \ \forall k \le n\}$$

$$|A^c| = 364^n \Rightarrow \boxed{P(A) = 1 - \left(\frac{364}{365}\right)^n}$$

Event $B \Leftrightarrow \{(x_1, \cdots, x_n) \mid x_i = x_j \text{ for some } 1 \le i, j \le 365\}$

Here also calculate $P(B^c)$ instead.

$$B^c = \{(x_1, x_2, \dots, x_n) \mid x_i \neq x_j \ \forall i,j\}$$

$$|B^c| = 365 \cdot (365-1) \cdot (365-2) \cdot \dots \cdot (365-(n-1))$$

$$P(B) = 1 - \frac{365 \cdot 364 \cdot \dots \cdot (366-n)}{365^n}$$

$$\boxed{P(B) = 1 - \frac{365!}{(365-n)! \, 365^n}}$$

Campare $P(A)$ and $P(B)$ to see $P(B)$ is much more likely than $P(A)$.

For $P(A) \geq 0.5 \Rightarrow n > 252$

$P(B) \geq 0.5 \Rightarrow n \geq 23$

Calculating $P(C)$ is much more complicated:

Event $C \Leftrightarrow \{(x_1, \dots, x_n) \mid x_i = x_j = x_k$ for some $i,j,k\}$

$$C^c = \{\underbrace{(x_1, \dots, x_n) \mid x_i \neq x_j \ \forall i,j}_{\text{all distinct birthdays}}\} \cup \{\underbrace{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i,j \text{ and } x_k \neq x_i \ \forall k \neq i,j}_{\text{one pair and rest distinct}}\} \cup \underbrace{\dots\dots}_{\text{two pairs and rest distinct}} \cup \dots$$

Finding explicit formula for $|C^c|$ is much more complicated. Can use numerical approx. for small n instead.

Pascal's triangle:

- Encodes combinatorial properties
- Defined by recursion: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$
\begin{array}{lccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$\binom{n}{0}$ $\quad \binom{n}{1} \; \dots \quad \dots \binom{n}{n-1} \quad \binom{n}{n}$