Exercise 1 - Polynomial Models

$T = \{(0,-5), (1,-2), (2,5)\}$

a) Linear Model $\quad f_{\hat{\alpha}}(x) = \hat{\alpha}_0 + \hat{\alpha}_1 x$

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad y = \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} \qquad \hat{\alpha} = \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \end{pmatrix}$$

$$X\hat{\alpha} = y$$

$$X^T X \hat{\alpha} = X^T y$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

$$X^T y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

Solve for $\hat{\alpha}$

$$\left(\begin{array}{cc|c} 3 & 3 & -2 \\ 3 & 5 & 8 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 3 & 3 & -2 \\ 0 & 2 & 10 \end{array}\right)$$

$$2\hat{\alpha}_1 = 10 \Rightarrow \boxed{\hat{\alpha}_1 = 5}$$

$$3\hat{\alpha}_0 + 3\hat{\alpha}_1 = -2$$

$$3\hat{\alpha}_0 + 3 \cdot 5 = -2$$

$$3\hat{\alpha}_0 = -17$$

$$\boxed{\hat{\alpha}_0 = -\frac{17}{3}}$$

$$\boxed{f_{\hat{\alpha}}(x) = -\frac{17}{3} + 5x}$$

b) Quadratic Model $\quad f_{\hat{\beta}}(x) = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \qquad y = \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} \qquad \hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$$

$$X\hat{\beta} = y$$

$$X^T X \hat{\beta} = X^T y$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix}$$

$$X^T y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ 18 \end{pmatrix}$$

Solve for $\hat{\beta}$

$$\left(\begin{array}{ccc|c} 3 & 3 & 5 & -2 \\ 3 & 5 & 9 & 8 \\ 5 & 9 & 17 & 18 \end{array}\right) \xrightarrow[R_3 - \frac{5}{3}R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 5 & -2 \\ 0 & 2 & 4 & 10 \\ 0 & 4 & \frac{26}{3} & \frac{64}{3} \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 3 & 3 & 5 & -2 \\ 0 & 2 & 4 & 10 \\ 0 & 0 & 2/3 & 4/3 \end{array}\right)$$

$$\frac{2}{3} \cancel{x} \hat{\beta}_2 = \frac{4}{3} \Rightarrow \boxed{\hat{\beta}_2 = 2}$$

$$2\hat{\beta}_1 + 4\hat{\beta}_2 = 10$$

$$2\hat{\beta}_1 + 4 \cdot 2 = 10$$

$$\boxed{\hat{\beta}_1 = 1}$$

$$3\hat{\beta}_0 + 3\hat{\beta}_1 + 5\hat{\beta}_2 = -2$$

$$3\hat{\beta}_0 + 3 \cdot 1 + 5 \cdot 2 = -2$$

$$3\hat{\beta}_0 = -15$$

$$\boxed{\hat{\beta}_0 = -5}$$

$$\boxed{f_{\hat{\beta}}(x) = -5 + x + 2x^2}$$

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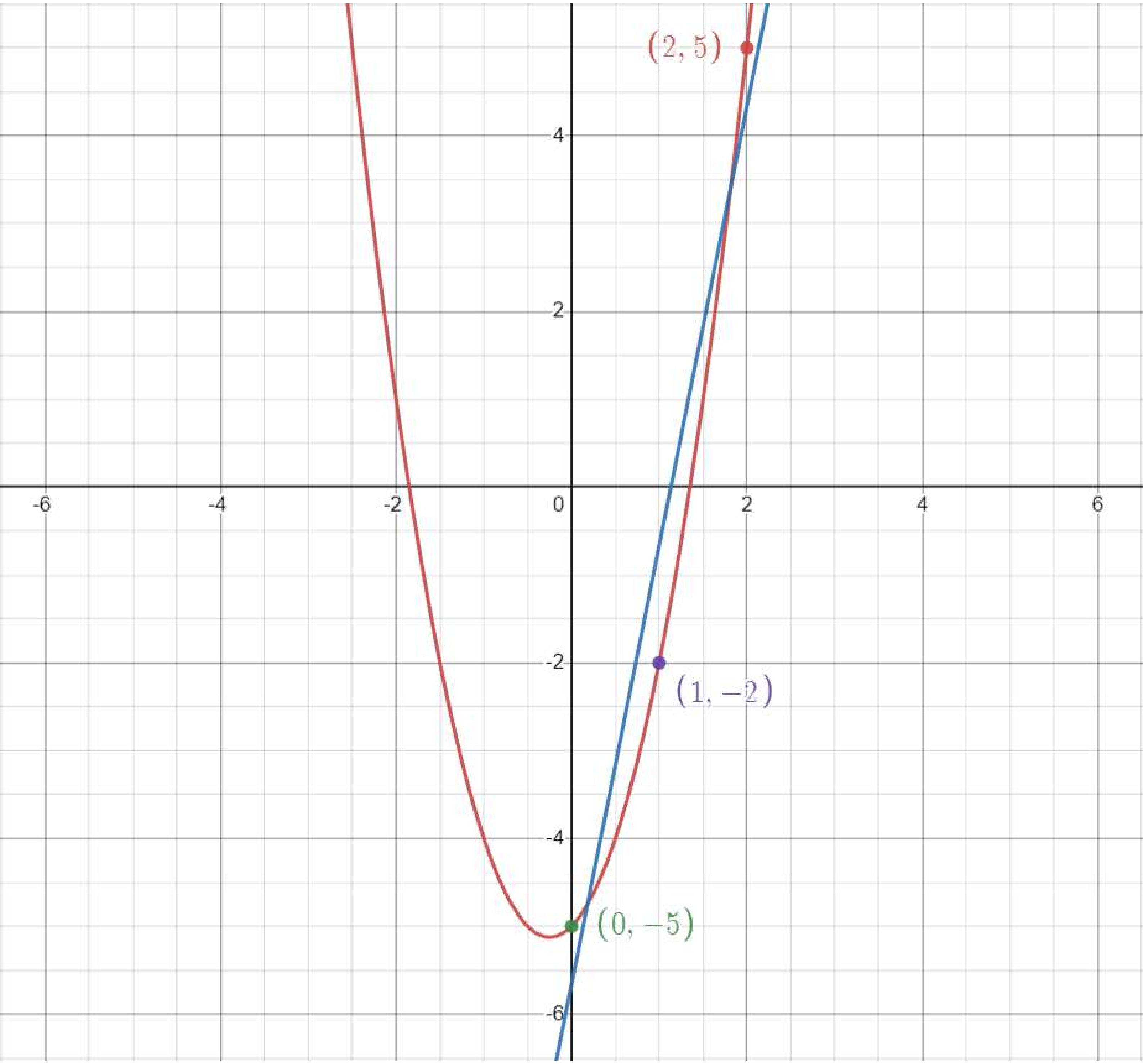

(2, 5)
(1, −2)
(0, −5)
-6
-4
-2
0
2
4
6
4
2
-2
-4
-6