Tutorial 3 notes

Sept 29

Multivariable integration:

↳ Integration in heigher dimensions (here mostly $\mathbb{R}^2$ and $\mathbb{R}^3$).

↳ Naturally comes in probability theory.

1. Examples in $\mathbb{R}^2$:

1.a Integrate $f = x^2+y^2$ over $[0,1] \times [0,1]$

$$\iint_R f \, dx dy = \int_0^1 \int_0^1 x^2+y^2 \, dx dy$$

↑ order of int. important

↑ Bounds for x and y (depends on order)

$$= \int_0^1 \left(\int_0^1 x^2+y^2 \, dx \right) dy = \int_0^1 \tfrac{1}{3} + y^2 \, dy = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

1.b Integrate $f = x^2+y^2$ over region R with $x, y \geq 0$ and $x \leq y$

overall range where y varies (0 to 1)

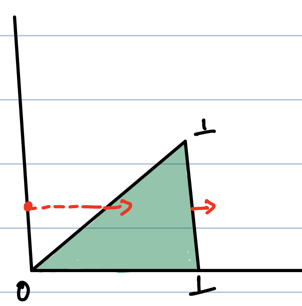
Fix some y and look for bounds of x. (x to 1)

$$\int_0^1 \int_0^y x^2+y^2 \, dx\, dy = \int_0^1 \left(\int_0^y x^2+y^2 \, dx \right) dy$$

$$= \int_0^1 \frac{y^3}{3} + y^3 \, dy = \int_0^1 \tfrac{4}{3} y^3 \, dy = \tfrac{1}{3}$$

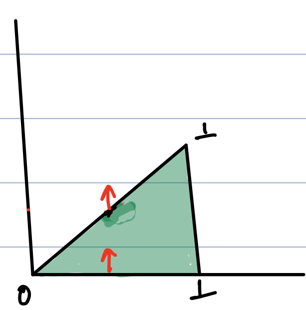
↑ note the order

dx.dy



what if we switch the order of int. to 'dy dx'?

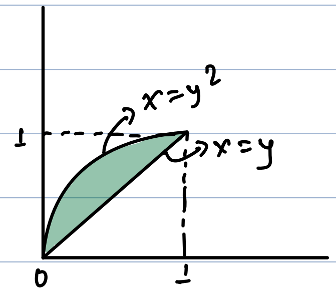
$$\int_0^1 \int_x^0 x^2+y^2 \, dy\, dx = \int_0^1 \left(\int_0^x x^2+y^2 \, dy \right) dx$$



$$= \int_0^1 x^3 + x^3/3 \, dx = \int_0^1 4/3 \, x^3 \, dx = 1/3$$

1.C more examples: Integrate $f(x,y)$ over domain bounded by parabola $x=y^2$ and $x=y$.

$$I = \int_0^1 \int_{y^2}^{y} f \, dx \, dy = \int_0^1 \int_x^{\sqrt{x}} f \, dy \, dx$$



Note: Some integrations are only possible in particular order.

Example: Take $f = \sin\left(\frac{\pi x}{y}\right)$ in above example. '$dx\,dy$' easier than '$dy\,dx$'.

Change of co-ordinates:

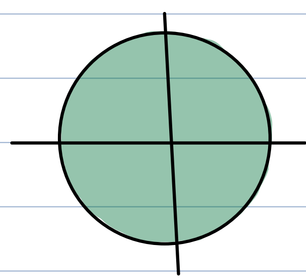
Suppose we have integral: $\int_{\alpha_2}^{\beta_2} \int_{\alpha_1}^{\beta_1} f \, dx \, dy$ and we perform transformation $x = f_1(r_1, r_2)$ and $y = f_2(r_1, r_2)$. How does integral changes?

$$\int_{\alpha_2}^{\beta_2} \int_{\alpha_1}^{\beta_1} f \, dx \, dy \xrightarrow[y = f_2(r_1, r_2)]{x = f_1(r_1, r_2)} \underbrace{\int_{\alpha_2'}^{\beta_2'} \int_{\alpha_1'}^{\beta_1'}}_{} f \cdot |\det J| \, dr_1 \, dr_2$$

new bounds in $r_1 - r_2$ plane.

$$J = \text{Jacobean matrix} = \begin{bmatrix} \frac{\partial f_1}{\partial r_1} & \frac{\partial f_1}{\partial r_2} \\ \frac{\partial f_2}{\partial r_1} & \frac{\partial f_2}{\partial r_2} \end{bmatrix}$$

$\rightsquigarrow$ Scales area from $x-y$ to r_1-r_2 plane.



Example: Area of circle:

$$A = \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1 \, dx \, dy = \int_0^r \int_0^{2\pi} \underbrace{r}_{} \, d\theta \, dr$$

Jacobean factor

Examples in probability theory:

1. Expectation of joint probability density function $f_{XY}(x,y)$.

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy \qquad E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y\, f_{XY}(x,y)\, dx\, dy$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xY f_{XY}(x,y)\, dx\, dy$$

Covariance of X and Y.

$$\mathrm{Cov}(X,Y) = \iint (x-\mu_x)(Y-\mu_Y)\, f_{XY}(x,y)\, dA$$

$\mu_x, \mu_y \rightsquigarrow$ Expectations of X and Y.

2. Other probability problems where we need to integrate PDF over a region.

Consider the following Example:

Suppose two numbers are picked uniformly from $[0,1]$. Call larger X and Smaller Y. Joint PDF is $f_{XY} = 2$ if $0 \le y \le x \le 1$ and 0 Else.
Find Average value of X and marginal distribution of X.

Solution: $E[X] = \int_0^1 \int_0^1 x\, f_{XY}(x,y)\, dy\, dx = \int_0^1 \int_0^x 2x\, dx\, dy = 2/3$

Marginal distribution: $f_X(x) = \int_0^1 f_{XY}(x,y)\, dy = \int_0^x 2\, dy = 2x$

⋮

And much more!