Probability and Random Processes

Keivan Mallahi-Karai

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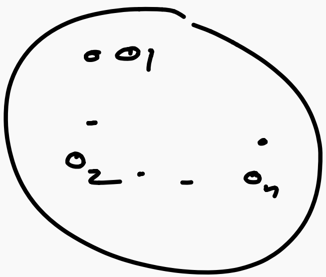
Jacobs University

Conditional probability: updating probabilities

Consider a probabilistic experiment with sample space Ω.

Probability $\longleftrightarrow$ degree of belief in some outcome



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Conditional probability: getting a peek

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2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

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$$\frac{4}{40} = \frac{1}{10}$$

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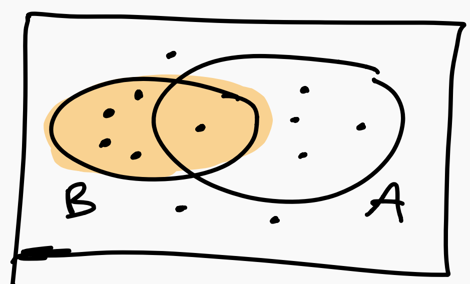
$$\mathbb{P}\left[A \text{ given } C\right] = \frac{4}{40} = \frac{1}{10}.$$

Definition
Suppose that A, B are two events and that $\mathbb{P}(B) \neq 0$. The conditional probability $\mathbb{P}(A|B)$ (read as A given B) is defined by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Simplicity: equiprobable model of Pascal

Ω



$$\mathbb{P}(A) = \frac{\#A}{\#\Omega}$$

$$\mathbb{P}(A \text{ given } B) = \frac{\# A \cap B}{\# B} = \frac{\frac{\# A \cap B}{\#\Omega}}{\frac{\# B}{\#\Omega}} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition of conditional probability

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Remark:

Note that for this definition to make sense, one needs to assume that $\mathbb{P}[B] \neq 0$.

Conditional probability: exmaples

Example

Two fair dice are rolled.

1. What is the probability that the sum of the resulted numbers is 7?
2. If both of the numbers obtained are at least 3, what is the probability that the sum of the resulted numbers is 7?

Conditional probability: exmaples

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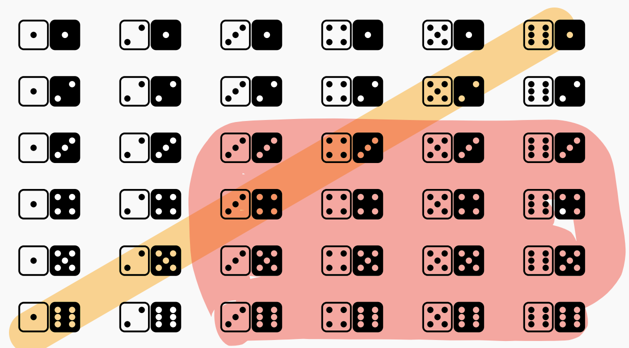
2. If both of the numbers obtained are at least 3, what is the probability that the sum of the resulted numbers is 7?

A

$$\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$$

$$\mathbb{P}(A/B) = \frac{2}{16} = \frac{1}{8}$$

Sample space:



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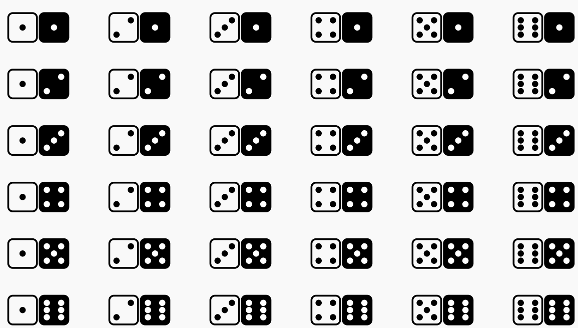
Sample space:

$$x + y = 9$$

$$1 \leq x, y \leq 6$$

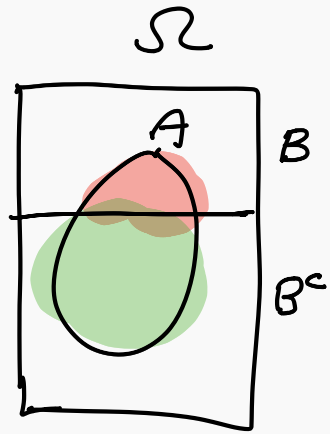
$$y = 7 - x$$



$$\mathbb{P}(\text{sum of outcome} = 7)$$

$$= \frac{}{36}$$

$$\mathbb{P}[A] = \underline{\mathbb{P}[A \cap B]} + \mathbb{P}[A \cap B^c]$$
$$= \underline{\mathbb{P}[A|B]\,\mathbb{P}[B]} + \mathbb{P}[A|B^c]\,\mathbb{P}[B^c]$$



$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\Rightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(B) \cdot \mathbb{P}(A|B),$$
$$\mathbb{P}(A \cap B^c) = \mathbb{P}(B^c) \cdot \mathbb{P}(A|B^c).$$

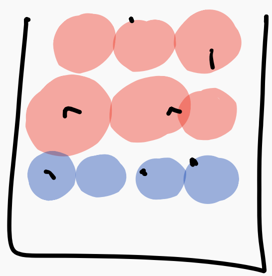
Example

An urn contains 6 red and 4 blue balls. A ball is drawn from the urn and discarded.

1. What is the probability that the discarded ball is blue?
2. Without knowing the color of the first ~~color~~ ball, what is the probability that a second ball drawn is blue?

B_2: second ball is blue

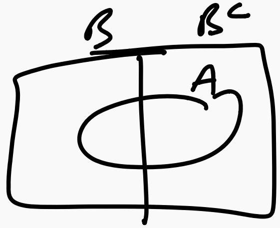
B_1 = first ball is blue



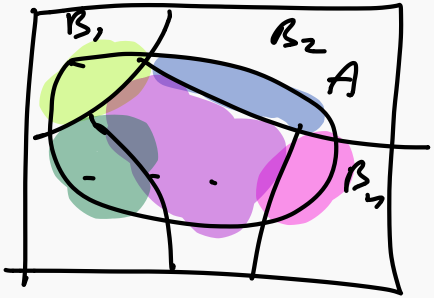
$$\mathbb{P}(B_1) = \frac{4}{10}$$

$$\begin{aligned}\mathbb{P}(B_2) &= \mathbb{P}(B_2 \mid B_1)\,\mathbb{P}(B_1) \\ &\quad + \mathbb{P}(B_2 \mid B_1^c)\,\mathbb{P}(B_1^c) \\ &= \frac{3}{9}\cdot\frac{4}{10} + \frac{4}{9}\cdot\frac{6}{10} = \frac{12}{90} + \frac{24}{90} \\ &= \frac{36}{90} = \frac{4}{10}.\end{aligned}$$

Conditioning formula; general case



$$\mathbb{P}[A] = \mathbb{P}[\cup_{i=1}^{n}(A \cap B_i)]$$
$$= \sum_{i=1}^{n} \mathbb{P}[A \cap B_i]$$
$$= \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i].$$



Theorem (Conditioning)

Let $\Omega = B_1 \cup B_2 \cdots \cup B_n$ be a partitioning of the sample space and A be an event. Then

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i].$$

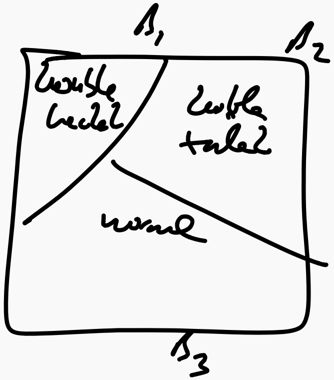
sample space is $\Omega = B_1 \cup B_2 \cup B_3$

Example

Alex has 5 coins in his pocket. Two are double-headed. one is double-tailed and the other two are normal. One of the coins is randomly chosen and flipped.

1. What is the probability that the outcome is heads?
2. He opens his eyes and sees that the outcome is heads. What is the probability that the flipped coin is double-headed?

$$\mathbb{P}(H) = \mathbb{P}(H|B_1)\,\mathbb{P}(B_1) + \mathbb{P}(H|B_2)\,\mathbb{P}(B_2) + \mathbb{P}(H|B_3)\,\mathbb{P}(B_3)$$



$$= \frac{1}{1} \cdot \frac{2}{5} + \frac{0}{1} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5}$$

$$= \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

Independence

Recall from the previous section that for events A and B, the conditional probability of A given B is defined by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

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$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}. = \mathbb{P}(A) \implies \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Definition
Events A and B are called independent when

$$\mathbb{P}[A|B] = \mathbb{P}[A],$$

Equivalently when

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B].$$

Examples of independence

A pair of dice are rolled. Consider the events

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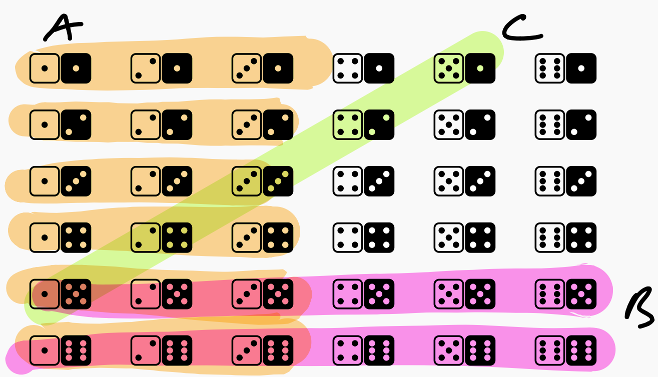
A pair of dice are rolled. Consider the events

1. A: The first die's score is at most 3.
2. B: The second die's score is at least 5.
3. C: Sum of the scores of the two dice is equal to 6.



A,B indept ?

A,C " ?

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$$\mathbb{P}(A) = \frac{18}{36} = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{12}{36} = \frac{1}{3}$$

$$\mathbb{P}(C) = \frac{5}{36}$$

$$\mathbb{P}(A \cap B) = \frac{6}{36} = \frac{1}{6} = \overset{1/2}{\mathbb{P}(A)}\,\overset{1/3}{\mathbb{P}(B)}$$

dependent $\mathbb{P}(A \cap C) = \frac{3}{36} = \frac{1}{12} \neq \overset{1/2}{\mathbb{P}(A)}\,\overset{5/36}{\mathbb{P}(C)}$

$$\mathbb{P}(B \cap C) = \frac{1}{36} \neq \underset{1/3}{\mathbb{P}(B)} \cdot \underset{5/36}{\mathbb{P}(C)}$$

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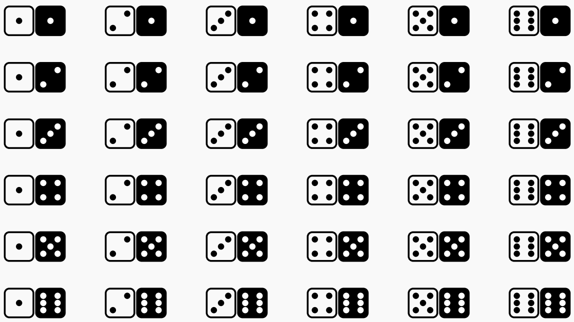

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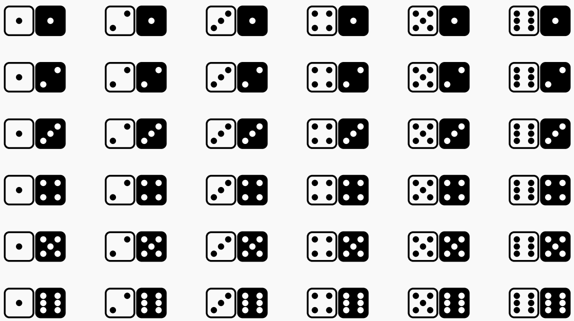

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$$\mathbb{P}[C] = \frac{5}{36}.$$

$$\mathbb{P}[A \cap B] = \frac{6}{36} = \frac{1}{6} = \mathbb{P}[A]\,\mathbb{P}[B].$$
$$\mathbb{P}[A \cap C] = \frac{3}{36} \neq \frac{5}{72} = \mathbb{P}[A]\,\mathbb{P}[C].$$
$$\mathbb{P}[B \cap C] = \frac{1}{36} \neq \frac{5}{108} = \mathbb{P}[B]\,\mathbb{P}[C].$$

Def Let $A_1, A_2, \dots, A_n$ be events in a sample space Ω. We say $A_1, A_2, \dots, A_n$ are independent:

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\,\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$$

$1 \le k \le n$ $\quad i_1, \dots, i_k$ are all distinct

$n=2$ $\quad \mathbb{P}(A_{i_1}) = \mathbb{P}(A_{i_1})$ ✓

$k=1,2$

$k=2$ $\quad \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)$.

$n=3$ $\quad \mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j)$

$k=2$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)$$
$$\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_3)$$
$$\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\,\mathbb{P}(A_3)$$

A_1, A_2, A_3 are pairwise indepent

$k=3$ $\quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$.

Verifying independence

Example
Let us unpack this definition to see what it entails for small values of n. For two events A, B, independence is equivalent to $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. For three events A, B, C, one requires the following four equalities:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B], \quad \mathbb{P}[A \cap C] = \mathbb{P}[A]\mathbb{P}[C], \quad \mathbb{P}[B \cap C] = \mathbb{P}[B]\mathbb{P}[C],$$

$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C].$$

Independence and pairwise independence

Suppose that three coins are thrown. Consider the events:

1. A: the outcome of the first and the second coin are the same.
2. B: the outcome of the first and the third coin are the same.
3. C: the outcome of the second and the third coin are the same.

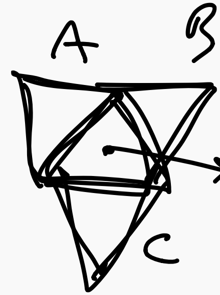
Are the events A, B, C independent?

$\Omega = \quad HHH$

$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/2$$

$\boxed{*}\ \boxed{1}\ \boxed{1}$

$$\mathbb{P}(A \cap B) = \frac{2}{8} = \frac{1}{4}$$



$A \cap B =$

$A \cap C =$

$A \cap C =$

$A \cap B \cap C$

$$\mathbb{P}(A \cap C) = 1/4$$

$$\mathbb{P}(B \cap C) = 1/4$$

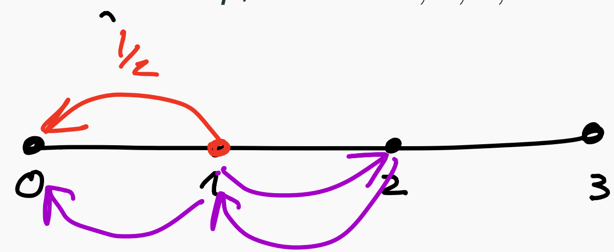
$$\mathbb{P}(A \cap B \cap C) = \frac{1}{4}$$

$$\neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C).$$

Random walks and absorbing probabilities

A particle starts at the point $0 \leq i \leq 3$ on the x-axis and starts a random walk as follows: at each step of the walk, the particle moves with probability $1/2$ one step to the right and with probability $1/2$ one step to the left. If the particle arrived at 0 or 3 it stops moving. Let p_i denote the probability that the particle eventually stops at 0. Determine the value of p_i for $i = 0, 1, 2, 3$.

P_i = Prob of eventually stopping at 0 if we start at i



$P_0 = 1$, $P_3 = 0$

E = eventually stop at 0

$$\mathbb{P}(E) = \mathbb{P}(E|\rightarrow)\,\mathbb{P}(\rightarrow) + \mathbb{P}(E|\leftarrow)\,\underline{\underline{\mathbb{P}(\leftarrow)}}$$

$$p_1 = p_2 \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{p_2+1}{2}$$

$$p_2 = 0 \cdot \frac{1}{2} + p_1 \cdot \frac{1}{2}$$

$$\begin{cases} 2p_1 = p_2 + 1 \\ p_2 = \frac{p_1}{2} \end{cases}$$

$$p_1 = \frac{2}{3} \qquad p_2 = \frac{1}{3}.$$

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