# Homework 9 Solutions

## Problem 1

a) $\dfrac{dy}{dx} = y^2x^2 + y^2x \quad , \quad y(0) = 2$

$$\therefore \frac{dy}{dx} = y^2(x^2+x)$$

$$\therefore \int \frac{1}{y^2} \cdot dy = \int (x^2+x)\, dx$$

$$-\frac{1}{y} = \frac{x^3}{3} + \frac{x^2}{2} + c$$

when $y(0) = 2$ : $-\frac{1}{2} = 0 + 0 + c \quad \therefore \quad c = -\frac{1}{2}$

$$\therefore \quad -\frac{1}{y} = \frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{2}$$

$$\therefore \quad y = \frac{6}{3 - 2x^3 - 3x^2} \quad //$$

b) $e^x . e^y \dfrac{dy}{dx} + e^x . e^y = 2x^3 \quad , \quad y(0) = 0$

Through observation, $\dfrac{d}{dx}(e^x . e^y) = e^x . e^y \dfrac{dy}{dx} + e^x . e^y$

$$\therefore \int \frac{d}{dx}(e^x . e^y) = \int 2x^3$$

$$\therefore \quad e^x . e^y = \frac{2x^4}{4} + c$$

at $y(0) = 0$,

$e^0 . e^0 = 0 + C \quad \therefore \quad C = 1$

$\therefore \quad e^x . e^y = \frac{x^4}{2} + 1 \quad \Rightarrow \quad e^{(x+y)} = \frac{x^4}{2} + 1$

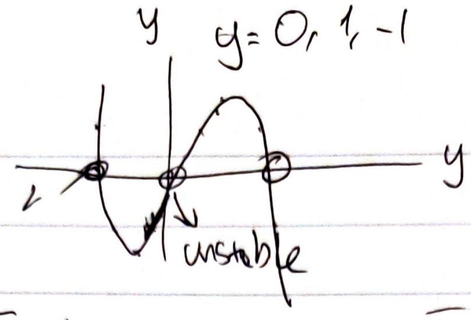
$\therefore \quad (x+y) \underbrace{\ln(e)}_{1} = \ln\left(\frac{x^4}{2} + 1\right)$

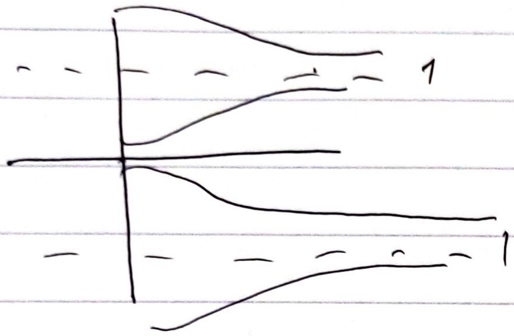
$\therefore \quad y = \ln\left(\frac{x^4}{2} + 1\right) - x$ //

2

$y' = y - y^3$

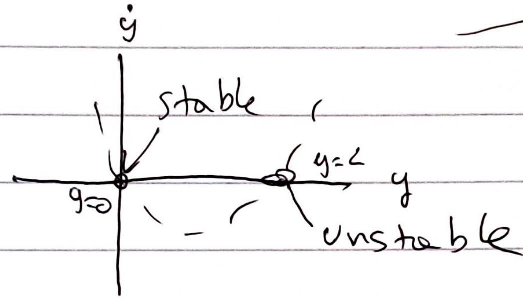
$y = 0, 1, -1$

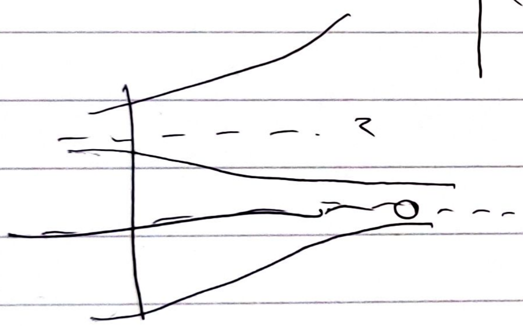


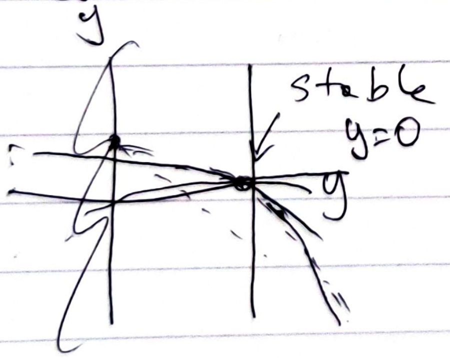

$y' = y(y-2)$ $y = 0, 2$

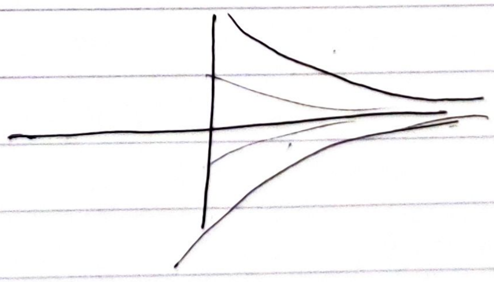
$y' = y^2 - 2y$




$\dot{y} = 2e^y - 2 = 2(1 - e^y)$ (one solution)



# Problem 3

$t_h = 5730$ years | carbon ratio $= 69\%$

Let $N(t) = N_0 \cdot e^{-\lambda t}$ denote no. of $^{14}C$ atoms in example.

$\frac{1}{2} N_0 = N_0 \cdot e^{-\lambda t_h}$ where $t_h = 5730$ is half life of $^{14}C$

$\therefore \frac{1}{2} = e^{-\lambda \cdot 5730}$ $\therefore \lambda = 1.2097 \times 10^{-4}$

$\therefore 0.69 \cancel{N_0} = \cancel{N_0} \cdot e^{-(1.2097 \times 10^{-4}) t}$

$\therefore t = 3067$ years //

Bonus: $\displaystyle\int \frac{\frac{1}{r^2}}{\sqrt{\left(E+\frac{1}{r}-\frac{1}{2r^2}\right)\cdot 2}}\,dr$ Let $u=\frac{1}{r}$ $du=-\frac{1}{r^2}\,dr$

$$= -\int \frac{du}{\sqrt{\left(E+u-\frac{u^2}{2}\right)2}} = \int \frac{du}{\sqrt{\left(\left(\frac{u}{2}-1\right)^2+E-1\right)\cdot 2}} = \frac{1}{\sqrt{E-1}}\int \frac{du}{\sqrt{\left(\frac{\left(\frac{u}{2}-1\right)^2}{E-1}+1\right)\cdot 2}}$$

~~Let $t = \frac{\frac{u}{2}-1}{\sqrt{E-1}}$ $dt = \frac{1}{2\sqrt{E-1}}$~~

$$\int \frac{\frac{1}{r^2}}{\sqrt{2\left(E+\frac{1}{r}-\frac{1}{2r^2}\right)}}\,dr \quad \text{Let } u=\frac{1}{r} \quad du=-\frac{1}{r^2} \quad = -\int \frac{du}{\sqrt{2\left(E+u-\frac{u^2}{2}\right)}} = -\int \frac{du}{\sqrt{E+2u-u^2}}$$

Complete the square $-\left[\left(-2E-2u+u^2+1-1\right)\right] = -\left[(u-1)^2-(1+2E)\right]$

$$= -\int \frac{du}{\sqrt{\left((u-1)^2-(1+2E)\right)\cdot -1}} \qquad t=\frac{u-1}{\sqrt{1+2E}} \quad dt=\frac{1}{\sqrt{1+2E}} \quad \rightarrow -\int \frac{dt}{\sqrt{-1(t^2-1)}} = -\int \frac{dt}{\sqrt{1-t^2}}$$

Let $t = \cos\theta$ then $-\int \frac{dt}{\sqrt{1-t^2}} = -\int \frac{-\sin\theta}{\sqrt{1-\cos^2\theta}} d\theta$

$dt = -\sin\theta \, d\theta$

$$= \int \frac{\sin\theta \, d\theta}{\sqrt{\sin^2\theta}} = \int 1 \, d\theta = \theta$$

Therefore $\varphi = \theta = \arccos(t) = \arccos\left(\frac{u-1}{\sqrt{2E-1}}\right)$

$$= \arccos\left(\frac{\frac{1}{r} - 1}{\sqrt{2E-1}}\right)$$