CH-231-A

# Algorithms and Data Structures

ADS

## Lecture 28

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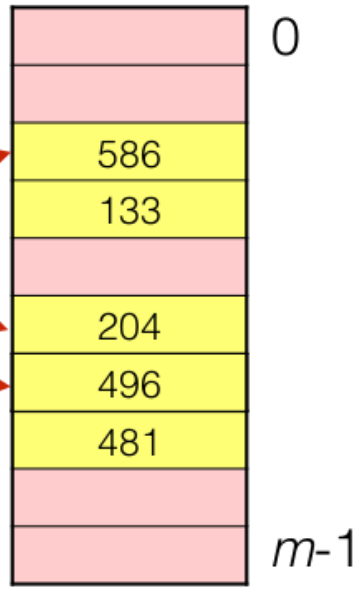
# Resolving Collisions by Open Addressing

- No additional storage is used.
- Only store one element per slot.
- Insertion probes the table systematically until an empty slot is found.
- The hash function depends on the key and the probe number, i.e., $h : U \times \{0, 1, ..., m-1\} \rightarrow \{0, 1, ..., m-1\}$.
- The probe sequence $< h(k, 0), h(k, 1), ..., h(k, m-1) >$ should be a permutation of $\{0, 1, ..., m-1\}$.

# Insert Example

- Insert key k = 496:

**0.** Probe $h(496,0)$
**1.** Probe $h(496,1)$
**2.** Probe $h(496,2)$

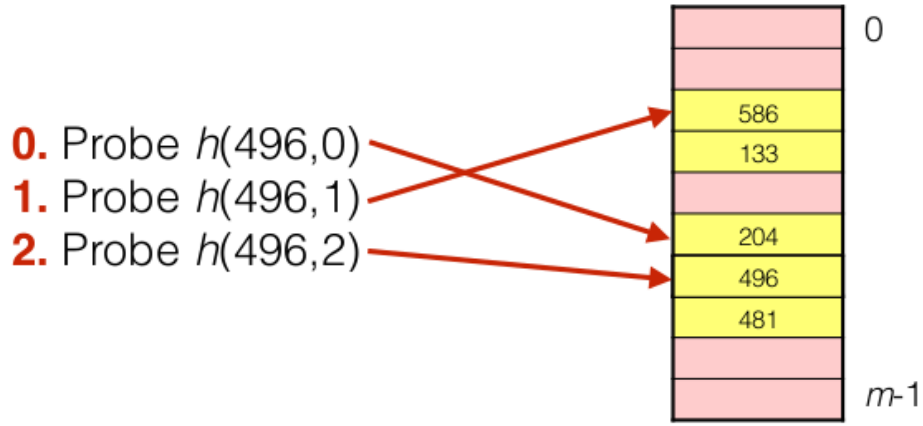



HASH-INSERT$(T, k)$

```
1  i = 0
2  repeat
3      j = h(k, i)
4      if T[j] == NIL
5          T[j] = k
6          return j
7      else i = i + 1
8  until i == m
9  error "hash table overflow"
```

# Search Example

```
HASH-SEARCH(T, k)
1  i = 0
2  repeat
3      j = h(k, i)
4      if T[j] == k
5          return j
6      i = i + 1
7  until T[j] == NIL or i == m
8  return NIL
```




- Search key $k = 496$
  - Search uses the same probe sequence, terminating successfully if it finds the key and unsuccessfully if it encounters an empty slot (or made it all the way through the list)
  - Search times no longer depend on load factor $\alpha$
- What about delete?
  - Have a special node type: DELETED
  - Chaining more commonly used when keys must also be deleted

# Probing Strategies (1)

Linear probing:

- Given an ordinary hash function $h'(k)$, linear probing uses the hash function $h(k, i) = (h'(k) + i) \bmod m$.
- This is a simple computation.
- However, it may suffer from primary clustering, where long runs of occupied slots build up and tend to get longer.
  - empty slot preceded by $i$ full slots gets filled next with probability $(i + 1)/m$

# Probing Strategies (2)

Quadratic probing:

- Quadratic probing uses the hash function $h(k, i) = (h'(k) + c_1 \cdot i + c_2 \cdot i^2) \bmod m$.
- Offset by amount that depends on quadratic manner, works much better than linear probing
- But, it may still suffer from secondary clustering: If two keys have initially the same value, then they also have the same probe sequence
- In addition $c_1$, $c_2$, and $m$ need to be constrained to make full use of the hash table

# Probing Strategies (3)

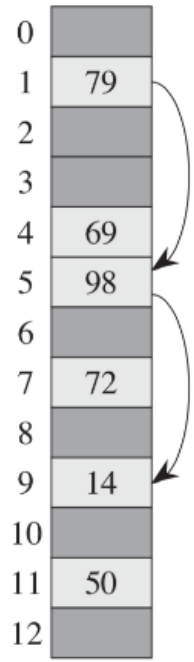
Double hashing:

- Given two ordinary hash functions $h_1(k)$ and $h_2(k)$, double hashing uses the hash function $h(k, i) = (h_1(k) + i \cdot h_2(k)) \mod m$.
- The initial probe goes to position $T[h_1(k)]$; successive probe positions are offset by $h_2(k) \rightarrow$ the initial probe position, the offset, or both, may vary
- This method generates excellent results, if $h_2(k)$ is relatively prime to the hash-table size $m$,

# Probing Strategies (4)

Double hashing (continue):

- e.g., by making $m$ a power of 2 and design $h_2(k)$ to only produce odd numbers.
- or let $m$ be prime and design $h_2$ such that it always returns a positive integer less than $m$, e.g. let $m'$ be slightly less than $m$:
  $h_1(k) = k \bmod m$
  $h_2(k) = 1 + (k \bmod m')$




$h_1(k) = k \bmod 13$
$h_2(k) = 1 + (k \bmod 11)$

—> k=14; $h_1(k)=1$, $h_2(k)=4$

—> k=27; $h_1(k)=1$, $h_2(k)=6$

# Analysis of Open Addressing (1)

Theorem:

- Assume uniform hashing, i.e., each key is likely to have any one of the $m!$ permutations as its probe sequence.
- Given an open-addressed hash table with load factor $\alpha = n/m < 1$.
- The expected number of probes in an unsuccessful search is, at most $\frac{1}{1-\alpha}$.

# Analysis of Open Addressing (2)

Proof:

- At least, one probe is always necessary.
- With probability $n/m$, the first probe hits an occupied slot, i.e., a second probe is necessary.
- With probability $(n-1)/(m-1)$, the second probe hits an occupied slot, i.e., a third probe is necessary.
- With probability $(n-2)/(m-2)$, the third probe hits an occupied slot, i.e., a fourth probe is necessary.
- ...

# Analysis of Open Addressing (3)

Given that $\frac{n-i}{m-i} < \frac{n}{m} = \alpha$ for $i = 1, 2, \ldots, n$.

$$1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(1 + \frac{n-2}{m-2}\left(\cdots\left(1 + \frac{1}{m-n+1}\right)\cdots\right)\right)\right)$$

$$\leq 1 + \alpha(1 + \alpha(1 + \alpha(\cdots(1+\alpha)\cdots)))$$

$$\leq 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$$

$$= \sum_{i=0}^{\infty} \alpha^i$$

$$= \frac{1}{1-\alpha}.$$

# Analysis of Open Addressing (4)

- For example, if the table is half full, the expected number of probes is $1/(1-0.5) = 2$.
- Or, if the table is 90% full, the expected number of probes is $1/(1-0.9) = 10$.
- The successful search takes less number of probes $\left[\text{expected number is at most } \frac{1}{\alpha} \ln \frac{1}{1-\alpha}\right]$.
- We conclude that if $\alpha$ is constant, then accessing an open-addressed hash table takes constant time.

# Summary

- Dynamic sets with queries and modifying operations.
- Array: Random access, search in $O(\lg n)$, but modifying operations $O(n)$.
- Stack: LIFO only. Operations in $O(1)$.
- Queue: FIFO only. Operations in $O(1)$.
- Linked list: Modifying operations in $O(1)$, but search $O(n)$.
- BST: All operations in $O(h)$.
- Red-black trees: All operations in $O(\lg n)$.
- Heap: All operations in $O(\lg n)$.
- Hash tables: Operations in $O(1)$, but additional storage space.