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Due: December 7, 2022
Assignment 7

Probability and Random Processes

(7.1) Show that for a non-negative random variable X with mean $\mathbb{E}[X] = \mu$, we have

$$\mathbb{P}[X \geq 2\mu] \leq \frac{1}{2}.$$

Give an example of a non-negative random variable X with mean $\mathbb{E}[X] = \mu > 0$ such that

$$\mathbb{P}[X \geq 2\mu] = \frac{1}{2}.$$

Solution. The inequality

$$\mathbb{P}[X \geq 2\mu] \leq \frac{1}{2}.$$

follows from Markov inequality by setting $t = 2\mu$.

Let X be a random variable taking values 0 and 2 with probability 1/2. Then $\mu = 1$ and

$$\mathbb{P}[X \geq 2\mu] = \mathbb{P}[X \geq 2] = \mathbb{P}[X = 2] = \frac{1}{2}.$$

For (b), consider the random variable X which takes values 2μ and 0, each with probability 1/2. Then

$$\mathbb{E}[X] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (2\mu) = \mu.$$

At the same time,

$$\mathbb{P}[X \geq 2\mu] = \mathbb{P}[X = 2\mu] = \frac{1}{2}.$$

(7.2) Use the Central Limit theorem to compute the approximate value of the probability that the average of 1200 points chosen randomly according to the uniform distribution from the interval $(0, 1)$ is within 0.01 of the midpoint of the interval? The answer can be given in terms of the distribution function F of a standard normal random variable.

Solution. Note that if X_i has uniform distribution over $(0, 1)$ then $\mu = \mathbb{E}[X] = 1/2$ and $\mathrm{Var}[X] = \sigma^2 = 1/2$. Hence $\sigma = \sqrt{1/12}$. Hence $\sqrt{n}\sigma = \sqrt{1200}\sqrt{1/12} = 10$. Let us denote the sum of these 1200 random numbers by S_{1200}. Then we ahve

$$\mathbb{P}\left[|\frac{S_{1200}}{1200} - \frac{1}{2}| \leq 0.01\right] = \mathbb{P}[|S_{1200} - 600| \leq 12] = \mathbb{P}\left[\frac{|S_{1200} - 600|}{10} \leq 1.2\right] \approx \mathbb{P}[N < 1.2] = F(1.2).$$

(7.3) Suppose X has geometric distribution with parameter p. Show that the moment generating function of X is given by

$$M_X(t) = \frac{pe^t}{1 - (1 - p)e^t}.$$

Solution. Using the definition and the geometric series summation formula we have

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{j=1}^{\infty} e^{tj} p(1-p)^{j-1} = pe^t \sum_{k=0}^{\infty}((1-p)e^t)^k = \frac{pe^t}{1-(1-p)e^t}.$$

(7.4) Suppose X is a random variable whose moment generating function is given by

$$M_X(t) = \frac{1}{4}e^{2t} + \frac{1}{3}e^{-t} + \frac{5}{12}.$$

Find the probability $\mathbb{P}\left[|X| \le 1\right]$.

Hint: Try to guess a candidate for the random variable X and then use the uniqueness theorem.

Solution. Let Y be a random variable with the probability mass function given by

$$\mathbb{P}\left[Y=2\right] = \frac{1}{4}, \quad \mathbb{P}\left[Y=-1\right] = \frac{1}{3}, \quad \mathbb{P}\left[Y=0\right] = \frac{5}{12}.$$

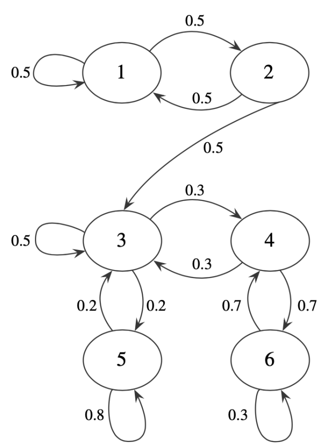
It follows from the definition of the moment generating function that

$$M_Y(t) = \frac{1}{4}e^{2t} + \frac{1}{3}e^{-t} + \frac{5}{12} = M_X(t).$$

Hence, X and Y have the same distribution. It follows that

$$\mathbb{P}\left[|X| \le 1\right] = \mathbb{P}\left[|Y| \le 1\right] = \mathbb{P}\left[Y=0\right] + \mathbb{P}\left[Y=-1\right] = \frac{5}{12} + \frac{1}{3} = \frac{3}{4}.$$

(7.5) Consider the following Markov chain on the state space



(a) Compute the transition matrix of this Markov chain.
(b) Compute the probability $p_{12}^{(2)}$.
(c) Determine the transient and absorbing states and compute absorbing probabilities.

Solution. It is easy to see from the transition probabilities that

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 3/10 & 2/10 & 0 \\ 0 & 0 & 3/10 & 0 & 0 & 7/10 \\ 0 & 0 & 2/10 & 0 & 8/10 & 0 \\ 0 & 0 & 0 & 7/10 & 0 & 3/10 \end{pmatrix}$$

We have

$$p_{12}^{(2)} = (P^2)_{12} = 1/4.$$

(c) It is clear from the matrix that for all states i we have $p_{ii} < 1$. This means that there are no absorbing states and all states are transient. In particular, there are no absorbing probabilities to compute!