ICS 2021 Problem Sheet #2

Problem 2.1: *proof by contrapositive* (4 points)

Let x and y be real numbers, i.e., $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Solution:

Proof. We prove the contrapositive, if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$.

We know that $(x^2 + y^2)$ is positive. Assume $y > x$. Then the following deviation holds:

$$\begin{aligned} y &> x && \mid \cdot (x^2 + y^2) \\ y(x^2 + y^2) &> x(x^2 + y^2) \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

We have shown that the contrapositive is true and hence that the original statement is true. □

Marking:

- *1pt proper formulation of the contrapositive*
- *1pt proper idea to multiply with $(x^2 + y^2)$ which is positive*
- *1pt proper formulation of the proof step*
- *1pt proper final statement that the proof of the contrapositive implies the statement*

Problem 2.2: *proof by induction* (4 points)

Let n be a natural number with $n \geq 1$. Prove that the following holds:

$$1^2 + 3^2 + 5^2 + \ldots (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

Solution:

Proof. We prove the above statement by induction.

- Base case: We show that the equation is true for $n = 1$. On the left side of the equation, we get $1^2 = 1$. On the right side, we get for $n = 1$:

  $$\frac{2(2-1)(2+1)}{6} = \frac{2 \cdot 1 \cdot 3}{6} = 1$$

  Hence, the statement is true for the base case.

- Induction step: Assume that the equation is true for some n. Lets consider the case $n+1$:

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1)^2 &= \left(\sum_{k=1}^{n}(2k-1)^2\right) + (2(n+1)-1)^2 \\
&= \frac{2n(2n-1)(2n+1)}{6} + (2n+1)^2 \\
&= \frac{2n(2n-1)(2n+1) + 6(2n+1)^2}{6} \\
&= \frac{(2n+1)(2n(2n-1) + 6(2n+1))}{6} \\
&= \frac{(2n+1)(4n^2-2n+12n+6))}{6} \\
&= \frac{(2n+1)(4n^2+10n+6)}{6} \\
&= \frac{(2n+1)(2n+2)(2n+3)}{6} \\
&= \frac{2(n+1)(2(n+1)-1)(2(n+1)+1)}{6}
\end{aligned}$$

This shows that the equation holds for $n+1$.

$\square$

Marking:

- *1pt proper handling of the base case*
- *1pt proper start of the induction step*
- *1pt proper usage of the induction assumption*
- *1pt proper derivation steps*

Problem 2.3: *operator precedence and associativity (haskell)* (1+1 = 2 points)

Haskell operators have associativity and precedence. The associativity defines in which order operators with the same precedence are evaluated while the precedence defines in which order operators with different precedence levels are evaluated (higher precedence level first).

a) Some operators are neither left nor right associative. What happens if such operators appear multiple times in an expression (without additional parenthesis defining the evaluation order)? Provide an example and an explanation.

b) Haskell has a very special operator `$`. What is the precedence and associativity of this operator? Write the following prefix expression

```
(^) 2 $ (*) 5 $ (+) 2 3
```

in infix notation without the `$` operator, using parenthesis where necessary.

Solution:

a) The comparison operators have no associativity. Hence, an expression like `True == True == True` has no defined evaluation order. A Haskell compiler/interpreter generates a 'precedence parsing error'.

b) The `$` operator is right associative and has precedence level 0 (the lowest precedence level possible). The infix notation of the expression is:

```
2 ^ (5 * (2 + 3))
```

Marking:

a) - *0.5pt for an example*
- *0.5pt for a suitable explanation*

b) - *0.5pt for the correct precedence and associativity*
- *0.5ot for a correct infix expression*