Midterm (calculus)

Q-1

a) $$\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x^3 - 8}$$

$$\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4$$

$$\lim_{x \to 2} \frac{\frac{2-x}{2x}}{x^3 - 8}$$

$$\lim_{x \to 2} \frac{2-x}{2x(x^3 - 8)}$$

$$\lim_{x \to 2} \frac{-\cancel{(x-2)}}{2x\cancel{(x-2)}(x^2 + 2x + 4)}$$

$$\lim_{x \to 2} \frac{-1}{2x^3 + 4x^2 + 8x}$$

$$= \frac{-1}{2(2)^3 + 4(2)^2 + 8(2)}$$

$$= \frac{-1}{48}.$$

b) $\lim_{y \to \infty} \frac{e^{-y} \sin(y) \cos(y)}{y}$.

$\lim_{y \to \infty} \frac{\sin(y) \cos(y)}{ye^{y}}$

applying Squeeze Theorum

$\lim_{y \to \infty} \frac{\sin(y)}{ye^{y}} \cdot \cos(y)$

$\lim_{y \to \infty} - \frac{\sin y}{ye^{y}} \leq \lim_{y \to \infty} \cos(y) \leq \lim_{y \to \infty} \frac{\sin y}{ye^{y}}$

$= 0$

c) $\lim\limits_{r \to 1} \dfrac{|r-1|}{2r-2} = 0$

$$\lim_{r \to 1^-} \left(\frac{|r-1|}{2r-2} \right) = \frac{-(r-1)}{2r-2} = \frac{-\cancel{(r-1)}}{2\cancel{(r-1)}} = \frac{-1}{2}$$

$$\lim_{r \to 1^+} \left(\frac{|r-1|}{2r-2} \right) = \frac{(r-1)}{2(r-1)} = \frac{1}{2}$$

Since left hand and right hand limits are different, the limit does not exist for $r \to 1$

Q.2 $x^6 - 5x - 5 = 0$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ Mean Value Theorum

$f(0) = -5$
$f(-1) = 1$

$$f'(c) = 6x^5 - 5 = \frac{-5 - 1}{0 - (-1)}$$

$= 6x^5 - 5 = -6$
$$x^5 = \frac{-6 + 5}{6}$$

$$\boxed{x \approx -0.698}$$

b) $f(x) = \frac{1}{x^2}$

$$\lim_{h\to 0} \frac{(x+h)^{-2} - x^{-2}}{h}$$

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$\lim_{h\to 0} \frac{\frac{1}{x^2+h^2+2xh} - \frac{1}{x^2}}{h}$$

$$\lim_{h\to 0} \frac{x^2 - x^2 - h^2 - 2xh}{x^2 h (x+h)^2}$$

$$\frac{\cancel{h}(-2x-h)}{\cancel{h}x^2(x+h)^2}$$

$$\lim_{h\to 0} - \frac{2x-h}{(x^2+hx)^2}$$

$$- \frac{2x+0}{(x^2)^2}$$

$$= -\frac{2x}{x^4} = \frac{-2}{x^3}$$

Q-3 $f(x) = \frac{x^2}{2-x^2}$

$$\boxed{\text{domain} = x \in R \setminus \{-\sqrt{2}, \sqrt{2}\}}$$

~~Verticle~~ Vertical asymtotes

$$\lim_{x \to \pm\sqrt{2}^{-+}} \frac{x^2}{2-x^2} \quad +- \infty$$

left hand limit $\neq$ Right hand limit.

$\boxed{x = -\sqrt{2} \quad \text{and} \quad x = +\sqrt{2}}$
are vertical Asymptotes.

Horizontal Asyptotes

$$\lim_{x \to \pm\infty} \frac{x^2}{2-x^2} \Big/ \div x^2$$

$$= \frac{1}{\frac{2}{x^2} - 1} = -1$$

$\boxed{y = -1}$ ~~vertica~~ Horizontal Asymptote.

$$\frac{d}{dx}\left(\frac{x^2}{2-x^2}\right)$$

$$\frac{2x(2-x^2) - x^2(-2x)}{(2-x^2)^2}$$

$$= \frac{4x - \cancel{2x^3} + \cancel{2x^3}}{(2-x^2)^2} = 0$$

$$= 4x = 0$$

$$x = 0$$

$$\langle -\sqrt{2}, 0 \rangle, \langle 0, \sqrt{2} \rangle$$

$x_1 = -1$

$x_2 = 1$ intermediate.

$$\frac{4(-1)}{(2-(-1)^2)^2} = -4$$

$$\frac{4(1)}{(2-(1)^2)^2} = 4$$

local minima is at $x = 0$

No local ~~ext~~ maxima.

~~No inflection points.~~

$x = 0$ is ~~reflection~~ inflection point.

$$\frac{d}{dx}\frac{4x}{\cancel{x}(2-x^2)^2}$$

$$\frac{4(2-x^2)^2 - 4x\left(2(2-x^2)\times 2x\right)}{\left((2-x^2)^2\right)^2}$$

$$\frac{\cancel{(2-x^2)}\left(4\times(2-x^2) - 4x(2\times 2x)\right)}{(2-x^2)^{\cancel{4}\,3}}$$

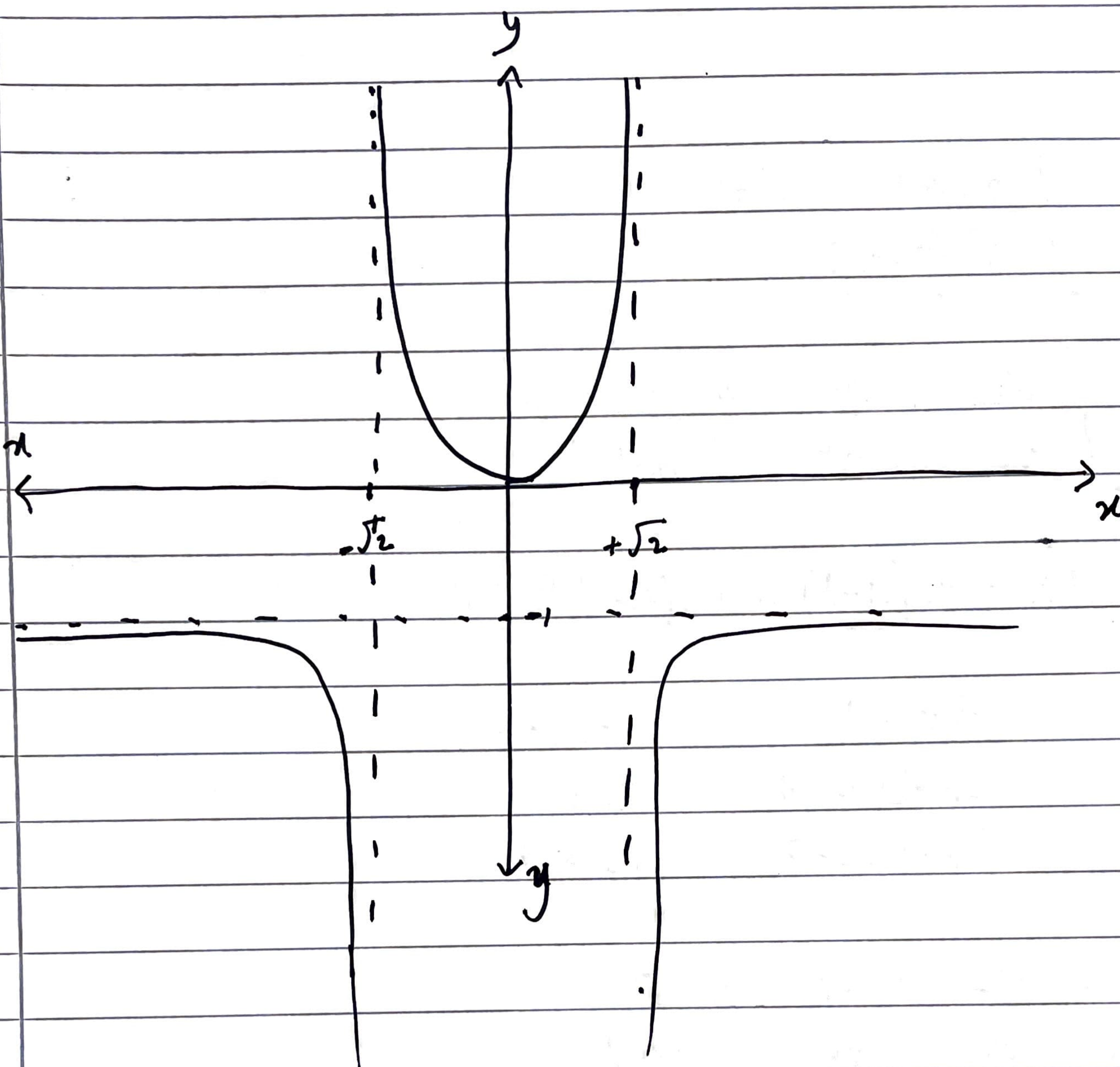
$$f'' = \frac{8+12x^2}{(2-x^2)^3}$$

$$\frac{8+12(\sqrt{2})^2}{(2+(\sqrt{2})^2)^3} > 0 \quad \text{concave up}$$

concave down ~~four~~ for $-\infty, -\sqrt{2}$, $\sqrt{2}, \infty$.

$(-\sqrt{2}, \sqrt{2})$ concave up.

$(-\infty, -\sqrt{2}), (\sqrt{2}, \infty)$ conc down


y
x
x
$-\sqrt{2}$
$+\sqrt{2}$
-1
y

Q-4

a) $\int \frac{x+1}{x^2(x-1)} dx$

$\int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} dx.$

$x+1 = x(x-1)A + (x-1)B + Cx^2$

$B = -1$ $\quad\quad$ $A = -2$

$A - B = -1$ $\quad\quad$ $B = -1$

$A + C = 0$ $\quad\quad$ $C = 2$

$\int \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}$

$-2\ln(x) + \frac{1}{x} + 2\ln(x-1) + C$

b) $\int x^{-3} e^{-1/x^2} \, dx.$

$u = -\frac{1}{x^2}$

$\frac{du}{dx} = \frac{2}{x^3}$

$\int \frac{e^{-1/x^2}}{x^3} \, dx$

$dx = \frac{x^3}{2} du$

$\int \frac{e^u}{\cancel{x^3}} \cdot \frac{\cancel{x^3}}{2} du$

$\frac{1}{2} \int e^u \, du$

$\frac{1}{2} e^u + C$

$= \frac{1}{2} e^{-1/x^2} + C$

c) $\displaystyle\int_0^{2\pi} e^x \cos(x)\,dx$

$\int fg' = fg - \int f'g$

$f = \cos x \qquad g' = e^x$

$f' = -\sin x \qquad g = e^x$

$e^x \cos x - \int -\overset{\sin x}{\cancel{\cos(x)}})\, e^x \, dx$

$e^x \cos x - \left(-e^x \sin x - \int -e^x \cos x \, dx\right)$

$e^x \cos x + e^x \sin x + \int \cancel{-}e^x \cos x \, dx = 2f(x).$

$\left. \dfrac{e^x \cos x + e^x \sin x}{2} \right|_0^{2\pi}$

$\dfrac{e^{2\pi}}{2} - \dfrac{1}{2}$

$\dfrac{e^{2\pi} - 1}{2} \approx 267.2.$

d) $\frac{d}{dt}\left(t^{t^3}\right)$

$\frac{d}{dt}\left(e^{\ln t \times t^3}\right)$

$e^{\ln t \times t^3} \times \frac{d}{dt}\left(\ln t \times t^3\right)$

$e^{\ln t \times t^3} \cdot \left(\frac{1}{t} \cdot t^3 + \ln t \cdot 3t^2\right)$

$t^{t^3} \cancel{\times} \left(t^2 + 3t^2 \ln t\right)$

$= \underline{\underline{t^{t^3+2} + 3t^{t^3+2} \ln(t)}}$

Q-5

a) $x = y^2$, $-x - y^2 + 2 = 0$

$$\downarrow$$

$$2 - y^2 = x$$

$y^2 = 2 - y^2$

$2y^2 = 2$

$\boxed{y = \pm 1 \ , \ x = 1}$

Integrate w.r.t y-axis.

$A = \int y^2 - (2 - y^2) \, dy.$

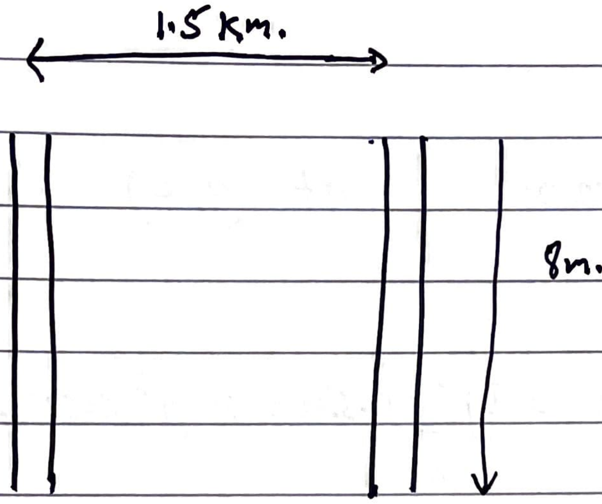
$= \int_{-1}^{1} \frac{y^3}{3} - 2y + \frac{y^3}{3} \Big|$

$= \left| \frac{2y^3}{3} - 2y \right|_{-1}^{1}$

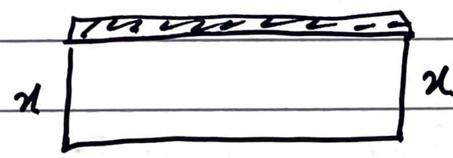
$\left(\frac{2}{3} - 2\right) - \left(-\frac{2}{3} + 2\right)$

$= -\frac{8}{3}$

b)

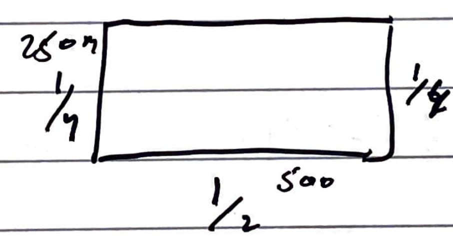


i) 1km fence.



$y = 1 - 2x.$

$2x + y = 1$



$A = x(1-2x).$

$= x - 2x^2$

$A' = 1 - 4x = 0$

$4x = 1$

$x = 1/4.$

$\frac{1}{4} \times \frac{1}{2} = 1/8 \text{ km}^2$

$= 0.125 \text{ km}^2$ or 125000 m^2

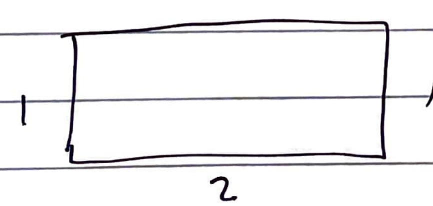
ii) <u>4km</u>

$A = x(4-2x)$

$= 4x - 2x^2$

$A' = 4 - 4x = 0$

$x = 1$



$A = 1 \times 2 = 2 \text{ km}^2$ or 2000000 m^2

c) $\frac{d}{dx}\sin(2x+y) = \frac{d}{dx}y^3 \sin x$ at $(0,0)$

$\cos(2x+y)\cdot\left(2+\frac{dy}{dx}\right) = 3y^2\frac{dy}{d}\cdot\sin x + y^3\cos x.$

$2\cos(2x+y) + \cos(2x+y)\frac{dy}{d} - 3y^2\frac{dy}{dx}\sin x = y^3\cos x.$

$\frac{dy}{dx}\left(\cos(2x+y) - 3y^2\sin x\right) = y^3\cos x - 2\cos(2x+y)$

$\frac{dy}{dy}(1) = -2(1)$ $\quad m = -2.$

$y - 0 = -2(x-0).$

$y = -2x.$