Introduction to Computer Science
Jacobs University Bremen
Dr. Jürgen Schönwälder

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ICS 2021 Problem Sheet #3

Problem 3.1: *cartesian products* (1+1 = 2 points)

Prove or disprove the following two propositions:

a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

Solution:

a) The proposition is true. We prove the equivalence by a chain of equivalences.

$$\begin{aligned}
(x, y) \in (A \cap B) \times (C \cap D) &\Leftrightarrow x \in (A \cap B) \wedge y \in (C \cap D) \\
&\Leftrightarrow x \in A \wedge x \in B \wedge y \in C \wedge y \in D \\
&\Leftrightarrow (x, y) \in (A \times C) \wedge (x, y) \in (B \times D) \\
&\Leftrightarrow (x, y) \in (A \times C) \cap (B \times D)
\end{aligned}$$

b) The proposition is not true. We prove this by providing a counter example:

$$\begin{array}{llll}
A = \{1\} & B = \{2\} & A \cup B = \{1, 2\} & A \times C = \{(1, a)\} \\
C = \{a\} & D = \{b\} & C \cup D = \{a, b\} & B \times D = \{(2, b)\}
\end{array}$$

$$(A \cup B) \times (C \cup D) = \{(1, a), (1, b), (2, a), (2, b)\} \neq \{(1, a), (2, b)\} = (A \times C) \cup (B \times D)$$

Marking:

a) 1pt for a correct proof

b) 1pt for a correct disproof

Problem 3.2: *reflexive, symmetric, transitive* (3 points)

For each of the following relations, determine whether they are reflexive, symmetric, or transitive. Provide a reasoning.

a) The absolute difference of the integer numbers a and b is less than or equal to 3.

$$R = \{(a, b) | a, b \in \mathbb{Z} \wedge |a - b| \leq 3\}$$

b) The last digit of the decimal representation of the integer numbers a and b is the same.

$$R = \{(a, b) | a, b \in \mathbb{Z} \wedge (a \bmod 10) = (b \bmod 10)\}$$

Solution:

a) $R = \{(a, b) | a, b \in \mathbb{Z} \wedge |a - b| \leq 3\}$

- reflexive since $|a - a| = 0 \leq 3$ for all $a \in \mathbb{Z}$
- symmetric since $|a - b| = |b - a|$ and hence $|a - b| \leq 3$ implies $|b - a| \leq 3$

- not transitive since $|2-5| \leq 3$ and $|5-8| \leq 3$ does not imply that $|2-8| \leq 3$

b) $R = \{(a,b) | a, b \in \mathbb{Z} \land (a \bmod 10) = (b \bmod 10)\}$

- reflexive since $(a \bmod 10) = (a \bmod 10)$ for all $a \in \mathbb{Z}$
- symmetric since $(a \bmod 10) = (b \bmod 10) \Leftrightarrow (b \bmod 10) = (a \bmod 10)$
- transitive since $(a \bmod 10) = (b \bmod 10)$ and $(b \bmod 10) = (c \bmod 10)$ implies $(a \bmod 10) = (c \bmod 10)$ (if a and b have the same last digit and b and c have the same last digit, then a and c have the same last digit as well)

Marking:

- *0.5pt for each correctly reasoned property*

Problem 3.3: *total, injective, surjective, bijective functions* (1+1 = 2 points)

Are the following functions total, injective, surjective, or bijective? Expain why or why not.

a) $f : \mathbb{N} \mapsto \mathbb{N}$ with $f(x) = 2x^2$

b) $f : \mathbb{R} \mapsto \mathbb{R}$ with $f(x) = x^2 + 6$

Solution:

a) The function f is defined for all $x \in \mathbb{N}$ and hence it is total. The function f is injective since every number in the domain maps to a distinct number in the codomain. The functions f is not surjective since it only maps to even numbers. Since the function is not surjective, it is not bijective.

b) The function f is is defined for all $x \in \mathbb{R}$ and hence it is total. The function f is not injective since $f(1) = f(-1)$. The function is not surjective since the function maps only to the numbers $\{x \in \mathbb{R} | x \geq 6\}$, which is a proper subset of $\mathbb{R}$. Since the function is not injective nor surjective, it is not bijective.

Marking:

a) 0.25pt total, 0.25pt injective, 0.25 not surjective, 0.25pt not bijective

b) 0.25pt total, 0.25pt not injective, 0.25 not surjective, 0.25pt not bijective

Problem 3.4: *function composition* (1 point)

Given the functions $f(x) = x + 1$. $g(x) = 2x$, and $h(x) = x^2$, determine an expression for the following function compositions:

a) $f \circ g$

b) $f \circ h$

c) $g \circ f$

d) $g \circ h$

e) $h \circ f$

f) $h \circ g$

g) $f \circ (g \circ h)$

h) $h \circ (g \circ f)$

Solution:

a) $f \circ g = f(g(x)) = f(2x) = 2x + 1$

b) $f \circ h = f(h(x)) = f(x^2) = x^2 + 1$

c) $g \circ f = g(f(x)) = g(x + 1) = 2(x + 1) = 2x + 2$

d) $g \circ h = g(h(x)) = g(x^2) = 2x^2$

e) $h \circ f = h(f(x)) = h(x + 1) = (x + 1)^2 = x^2 + 2x + 1$

f) $h \circ g = h(g(x)) = h(2x) = (2x)^2 = 4x^2$

g) $f \circ (g \circ h) = f(2x^2) = 2x^2 + 1$

h) $h \circ (g \circ f) = h(2x + 2) = (2x + 2)^2 = 4x^2 + 8x + 4$

Marking:

- *-0.2pt for each incorrect expression, not negative*

Problem 3.5: *list comprehensions (haskell)* (1+1 = 2 points)

Your list comprehensions should be correct, they do not have to be efficient. You are not getting points for a list comprehension simply returning a hard coded solution list. In other words, your list comprehensions should continue to function correctly if parameters are changed.

a) Write a list comprehension that returns all positive factors of the number 210. Try to write the list comprehension in such a way that 210 can easily be replaced by a different number.

b) Write a list comprehension that returns a list of Pythagorean triads (a, b, c), where a, b, c are positive integers in the range $1..100$ and the Pythagorean triad is defined as $a^2 + b^2 = c^2$. The list should not contain any “duplicates” where a and b are swapped. If the list contains $(3, 4, 5)$ (since $3^2 + 4^2 = 25 = 5^2$), then is should not also include $(4, 3, 5)$.

Solution:

a) A possible solution (not requiring language features not introduced yet):

```
[x | n <- [210], x <- [1..n], n `mod` x == 0]
```

b) A first not yet quite correct solution:

```
[(a,b,c) | a <- [1..100], b <- [1..100], c <- [1..100], c^2 == a^2 + b^2]
```

Note that this contains “duplicates” where a and b are reversed. This can be prevented by requiring that b does not range over values that are less than a.

```
[(a,b,c) | a <- [1..100], b <- [a..100], c <- [1..100], c^2 == a^2 + b^2]
```

This is still not very efficient. Since b is the longer “edge” and c must be longer the b, we can change this to:

```
[(a,b,c) | a <- [1..100], b <- [a..100], c <- [b..100], c^2 == a^2 + b^2]
```

We can do even better by restricting the upper bound for c to the minimum of 100 and $a + b$:

```
[(a,b,c) | a <- [1..100], b <- [a..100], c <- [b..min 100 (a+b)], c^2 == a^2 + b^2]
```

Marking:

a) - *1pt for a list comprehension returning the correct result (there can be many different correct solutions)*

b) - *1pt for a list comprehension returning the correct result (the solution does not have to be “efficient”)*