HW I Solution

①

a) $p(x) = 2x^2 + 12x + 26 \overset{!}{=} 0$

$$x_{1/2} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 26 \cdot 2}}{2 \cdot 2} = \frac{-\cancel{4} \cdot 3 \pm \sqrt{\cancel{4} \cdot 3 \cdot \cancel{4} \cdot 3 - \cancel{4} \cdot \cancel{4} \cdot 13}}{\cancel{2} \cdot \cancel{2}}$$

$$= -3 \pm 2i$$

b) $bx^2 - bx + 2 = 0$

Find $\Delta < 0$ (determinant Δ)

$$\Delta = (-b)^2 - 4 \cdot b \cdot 2 = b^2 - 8b$$

$\Rightarrow$ $b^2 - 8b < 0$ $\Rightarrow$ $b(b-8) < 0$

$\Rightarrow$ $\underbrace{b < 0 \ \& \ \underbrace{b - 8 > 0}_{b > 8}}_{\text{both conditions at the same time not possible!}}$ or $\underbrace{b > 0 \ \& \ \underbrace{b - 8 < 0}_{b < 8}}_{b \in (0,8)}$

Solution: $b \in (0,8)$

Remark: Actually, $b = 0$ has no solution at all.
(and therefore no real roots), as $0 \cdot x^2 - 0x + 2$
$= \underline{2 \neq 0}$

So, depending on how you interpret the question,
$b \in [0,8)$ is also correct.

I would except both answers for b)

c) $x^6 - x^5 - 3x^4 - 3x^3 - 22x^2 + 4x + 24 \overset{!}{=} 0$

long division by $(x-3)$ (as one root is $x_1 = 3$)

$$(x^6 - x^5 - 3x^4 - 3x^3 - 22x^2 + 4x + 24) : (x-3) = \underbrace{x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8}_{(*)}$$

$$\begin{aligned}
&-(x^6 - 3x^5) \\
&\quad 2x^5 - 3x^4 \\
&\quad -(2x^5 - 6x^4) \\
&\qquad 3x^4 - 3x^3 \\
&\qquad -(3x^4 - 9x^3) \\
&\qquad\quad 6x^3 - 22x^2 \\
&\qquad\quad -(6x^3 - 18x^2) \\
&\qquad\qquad -4x^2 + 4x \\
&\qquad\qquad -(-4x^2 + 12x) \\
&\qquad\qquad\quad -8x + 24 \\
&\qquad\qquad\quad -(-8x + 24) \\
&\qquad\qquad\qquad 0
\end{aligned}$$

Next guess: $x_2 = 1$ $\leadsto$ (*) divide by $(x-1)$

$$(x^5 + 2x^4 + 3x^3 + 6x^2 - 4x - 8) : (x-1) = \underbrace{x^4 + 3x^3 + 6x^2 + 12x + 8}_{(**)}$$

$$\begin{aligned}
&-(x^5 - x^4) \\
&\quad 3x^4 + 3x^3 \\
&\quad -(3x^4 - 3x^3) \\
&\qquad 6x^3 + 6x^2 \\
&\qquad -(6x^3 - 6x^2) \\
&\qquad\quad 12x^2 - 4x \\
&\qquad\quad -(12x^2 - 12x) \\
&\qquad\qquad 8x - 8 \\
&\qquad\qquad -(8x - 8) \\
&\qquad\qquad\quad 0
\end{aligned}$$

Next guess: $x_3 = -1$ $\rightsquigarrow$ divide (**) by $(x+1)$

$$(x^4+3x^3+6x^2+12x+8):(x+1) = \underbrace{x^3+2x^2+4x+8}_{(***)}$$

$$\begin{array}{l}
-(x^4+x^3) \\
\hline
\quad 2x^3+6x^2 \\
\quad -(2x^3+2x^2) \\
\hline
\qquad 4x^2+12x \\
\qquad -(4x^2+4x) \\
\hline
\qquad\quad 8x+8 \\
\qquad\quad -(8x+8) \\
\hline
\qquad\qquad 0
\end{array}$$

Next guess: $x_4 = -2$ $\rightsquigarrow$ divide (***) by $(x+2)$

$$(x^3+2x^2+4x+8):(x+2) = \underline{x^2+4}$$

$$\begin{array}{l}
-(x^3+2x^2) \\
\hline
\quad 0+4x+8 \\
\quad -(\quad 4x+8) \\
\hline
\qquad\qquad 0
\end{array}$$

Finally, solve $\underbrace{x^2+4}_{\substack{\text{needs to have} \\ \text{complex } x\text{, otherwise in } \mathbb{R} \text{ always} > 0}} = (x-2i)(x+2i) = 0$

So: $x_1 = 3$, $x_2 = 1$, $x_3 = -1$, $x_4 = -2$, $x_5 = 2i$, $x_6 = -2i$

②

a) $$\frac{1}{(z^*)^2} = \frac{1}{(a-ib)(a-ib)} = \frac{1}{a^2-b^2-2bai}$$

$z = a+ib$

$z^* = a-ib$

$$= \frac{1\cdot\left((a^2-b^2)+2bai\right)}{\left((a^2-b^2)-2bai\right)\left((a^2-b^2)+2bai\right)}$$

$$= \frac{a^2-b^2+2bai}{(a^2-b^2)^2+4b^2a^2}$$

$$= \frac{a^2-b^2}{a^4+2b^2a^2+b^4} + \frac{2ba}{a^4+2b^2a^2+b^4}\,i$$

$$= \frac{a^2-b^2}{(a^2+b^2)^2} + \frac{2ba}{(a^2+b^2)^2}\,i$$

b) $$\frac{2+z}{2z+2} = \frac{2+a+ib}{2a+2ib+2} = \frac{2+a+ib}{2((a+1)+ib)}$$

$$= \frac{((2+a)+ib)((a+1)-ib)}{2((a+1)+ib)((a+1)-ib)}$$

$$= \frac{(2+a)(a+1)+(2+a)\cdot(-ib)+ib(a+1)+b^2}{2\left((a+1)^2+b^2\right)}$$

$$= \frac{2a+2+a^2+a-2ib\cancel{-iab}\cancel{+iab}+ib+b^2}{2(a^2+2a+1+b^2)}$$

$$= \frac{a^2+3a+2+b^2}{2(a^2+2a+1+b^2)} - i\,\frac{b}{2(a^2+2a+1+b^2)}$$

c) $(z^*)^2 z = z^* \cdot (z^* \cdot z)$

$= (a - ib)(a^2 + b^2)$

$= a(a^2+b^2) - i\, b(a^2+b^2)$

Bonus:

d) $\left|\frac{1-i}{2+i}\right| = \left|\frac{(1-i)(2-i)}{(2+i)(2-i)}\right| = \left|\frac{2-3i \cancel{+1} -1}{4+1}\right|$

$= \left|\frac{1}{5} - \frac{3}{5}i\right| = \sqrt{\left(\frac{1}{5}-\frac{3}{5}i\right)\left(\frac{1}{5}+\frac{3}{5}i\right)}$

$= \sqrt{\frac{1}{25} + \frac{9}{25}} = \frac{\sqrt{10}}{5}$

e) $|4x+2| \leq |2x-3| \quad |^2$ (when squared, $|\cdot|$ can be dropped, as it also makes everything positive)

$(4x+2)^2 \leq (2x-3)^2$

$16x^2 + 16x + 4 \leq 4x^2 - 12x + 9$

$12x^2 + 28x - 5 \leq 0$

$x_{1/2} = \frac{-28 \pm \sqrt{(28)^2 - 4\cdot 12\cdot(-5)}}{2\cdot 12} = \frac{-28 \pm \sqrt{4^2\cdot 7^2 - 4^2\cdot 3\cdot(-5)}}{24}$

$= \frac{-\cancel{(4)}\cdot 7 \pm \cancel{4}\sqrt{49+15}}{\cancel{4}\cdot 6} = \frac{-7 \pm 8}{6}$

$\Rightarrow \left(x - \left(\frac{-7+8}{6}\right)\right)\left(x - \left(\frac{-7-8}{6}\right)\right) \leq 0$

$$\Rightarrow \left(x-\frac{1}{6}\right)\left(x+\frac{15}{6}\right) \leq 0$$

$$\Rightarrow \underbrace{x \leq \frac{1}{6} \quad \& \quad x \geq -\frac{15}{6}}_{x \in \left[-\frac{15}{6}, \frac{1}{6}\right]} \quad \text{or} \quad \underbrace{x \geq \frac{1}{6} \quad \& \quad x \leq -\frac{15}{6}}_{\text{not possible at same time}}$$

$$\Rightarrow \quad x \in \left[-\frac{15}{6}, \frac{1}{6}\right]$$

end values included
"closed interval"

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③ a)

$z^* = a - ib$
$w^* = c - id$
$z = a + ib$
$w^* = c + id$

$$\frac{z^*}{w^*} = \frac{a \mp ib}{c - id} = \frac{(a-ib)(c+id)}{c^2+d^2}$$

$$= \frac{(ac+bd)}{c^2+d^2} + i\,\frac{ad-bc}{c^2+d^2} \qquad (*)$$

$$\left(\frac{z}{w}\right)^* = \left(\frac{a+ib}{c+id}\right)^* = \left(\frac{(a+ib)(c-id)}{c^2+d^2}\right)^*$$

$$= \left(\frac{ac+bd}{c^2+d^2} - i\,\frac{ad \mp bc}{c^2+d^2}\right)^*$$

$$= \frac{ac+bd}{c^2+d^2} + i\,\frac{ad-bc}{c^2+d^2} \qquad (**)$$

$$(*) = (**) \quad \text{i.e.} \quad \frac{z^*}{w^*} = \left(\frac{z}{w}\right)^*$$

③ b) $z = a + ib$

$$Re(z) = a \overset{?}{=} \frac{a+ib+a-ib}{2} = \frac{2a}{2} = a \quad ✓$$

c)

$$Im(z) = b \overset{?}{=} \frac{a+ib-(a-ib)}{2i} = \frac{a-a+ib+ib}{2i} = \frac{2ib}{2i} = b \; ✓$$