

# Robotics

## PS07 Solution

# 1 Problem

Given 4 beacons  $b_i$  at 4 known positions  $(x_i, y_i)$  in the plane as follows:

i=	$x_i$	$y_i$
1	35	40
2	12	23
3	7	18
4	9	9

A robot  $r$  at an unknown position  $p_r = (x_r, y_r)$  has a sensor to measure the distances to the beacons. Us multilateration to determine  $p_r$  when the ranges  $D(b_i, r)$  between all beacons and the robot are as follows:

	$b_1$	$b_2$	$b_3$	$b_4$
$D(b_i, r)$	37	9	2	10

# Problem 1

Multilateration as Linear Least Squares:

$$A = \begin{pmatrix} 2(x_1 - x_n) & 2(y_1 - y_n) \\ 2(x_2 - x_n) & 2(y_2 - y_n) \\ \dots & \dots \\ 2(x_{n-1} - x_n) & 2(y_{n-1} - y_n) \end{pmatrix} \quad b = \begin{pmatrix} x_1^2 - x_n^2 + y_1^2 - y_n^2 - d_1^2 + d_n^2 \\ x_2^2 - x_n^2 + y_2^2 - y_n^2 - d_2^2 + d_n^2 \\ \vdots \\ x_{n-1}^2 - x_n^2 + y_{n-1}^2 - y_n^2 - d_{n-1}^2 + d_n^2 \end{pmatrix}$$

find LLS fit  $\mathbf{x}^* = (x, y)^T$

$$Ax = b$$

$$x^* = (A^T A)^{-1} A^T b = A^+ b$$

# Problem 1

$i=$	$x_i$	$y_i$
1	35	40
2	12	23
3	7	18
4	9	9

	$b_1$	$b_2$	$b_3$	$b_4$
$D(b_i, r)$	37	9	2	10

$$A = \begin{pmatrix} 2(35-9) & 2(40-9) \\ 2(12-9) & 2(23-9) \\ 2(7-9) & 2(18-9) \end{pmatrix} = \begin{pmatrix} 52 & 62 \\ 6 & 28 \\ -4 & 18 \end{pmatrix}$$

$$b = \begin{pmatrix} 35^2 - 9^2 + 40^2 - 9^2 - 37^2 + 10^2 \\ 12^2 - 9^2 + 23^2 - 9^2 - 9^2 + 10^2 \\ 7^2 - 9^2 + 18^2 - 9^2 - 2^2 + 10^2 \end{pmatrix} = \begin{pmatrix} 1394 \\ 530 \\ 307 \end{pmatrix}$$

# Problem 1

$$A = \begin{pmatrix} 52 & 62 \\ 6 & 28 \\ -4 & 18 \end{pmatrix} = USV^T$$

$$U = \begin{pmatrix} -0.941 & 0.309 \\ -0.306 & -0.610 \\ -0.143 & -0.729 \end{pmatrix}, S = \begin{pmatrix} 85.737 & 0.000 \\ 0.000 & 18.898 \end{pmatrix}, V^T = \begin{pmatrix} -0.586 & -0.811 \\ 0.811 & -0.586 \end{pmatrix}$$

$$A^+ = VS^+U^T = \begin{pmatrix} 0.01968 & -0.02409 & -0.03031 \\ -0.00067 & 0.02181 & 0.02395 \end{pmatrix}$$

## Problem 1

$$\begin{aligned} x^* = A^+ b &= \begin{pmatrix} 0.01968 & -0.02409 & -0.03031 \\ -0.00067 & 0.02181 & 0.02395 \end{pmatrix} \begin{pmatrix} 1394 \\ 530 \\ 307 \end{pmatrix} \\ &= \begin{pmatrix} 5.361 \\ 17.978 \end{pmatrix} \end{aligned}$$

## 2 Problem

Given a sensor network with 4 nodes  $p_i$  at 4 positions  $(x_i, y_i)$  in the plane. The ranges  $D(p_i, p_j)$  between the nodes are given as follows:

$D(p_i, p_j)$	$p_1$	$p_2$	$p_3$	$p_4$
$p_1$	0	50	45	32
$p_2$	50	0	61	22
$p_3$	45	61	0	58
$p_4$	32	22	58	0

Determine the 4 positions  $(x_i, y_i)$  of the nodes via MDS using the step by step algorithm from the lecture, i.e.,:

- compute a suited matrix A from the distance matrix D
- double center A (getting B)
- use SVD on B
- find a good rank approximation B'
- get the locations X

## Problem 2

### MDS algorithm

1. set matrix A
2. set matrix B  
(double-centering)
3. compute SVD of B
4. get X  
(sqrt of diagonal L)

$$A = -\frac{1}{2} D^{(2)}$$

$$B = CAC$$

$$B = VLV^T$$

$$X = VL^{(1/2)}$$



## Problem 2

$D(p_i, p_j)$	$p_1$	$p_2$	$p_3$	$p_4$
$p_1$	0	50	45	32
$p_2$	50	0	61	22
$p_3$	45	61	0	58
$p_4$	32	22	58	0

$$A = -\frac{1}{2}D^{(2)} = -\frac{1}{2} \begin{pmatrix} 0 & 50 & 45 & 32 \\ 50 & 0 & 61 & 22 \\ 45 & 61 & 0 & 58 \\ 32 & 22 & 58 & 0 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 & -1250 & -1000 & -500 \\ -1250 & 0 & -1850 & -250 \\ -1000 & -1850 & 0 & -1700 \\ -500 & -250 & -1700 & 0 \end{pmatrix}$$

## Problem 2

$$C_{(4)} = I - \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix}$$

$$B = C_{(4)} A C_{(4)} = \begin{pmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 0 & -1250 & -1000 & -500 \\ -1250 & 0 & -1850 & -250 \\ -1000 & -1850 & 0 & -1700 \\ -500 & -250 & -1700 & 0 \end{pmatrix} \begin{pmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix}$$

$$= \begin{pmatrix} 556.25 & -543.75 & 6.25 & -18.75 \\ -543.75 & 856.25 & -693.75 & 381.25 \\ 6.25 & -693.75 & 1456.25 & -768.75 \\ -18.75 & 381.25 & -768.75 & 406.25 \end{pmatrix}$$

## Problem 2

$$B = \begin{pmatrix} 556.25 & -543.75 & 6.25 & -18.75 \\ -543.75 & 856.25 & -693.75 & 381.25 \\ 6.25 & -693.75 & 1456.25 & -768.75 \\ -18.75 & 381.25 & -768.75 & 406.25 \end{pmatrix} = VL V^T$$

$$V = \begin{pmatrix} -0.162 & -0.728 & 0.120 & -0.655 \\ 0.509 & 0.516 & 0.149 & -0.672 \\ -0.746 & 0.407 & -0.402 & -0.342 \\ 0.398 & -0.195 & -0.895 & -0.046 \end{pmatrix}, L = \begin{pmatrix} 2342.061 & 0.000 & 0.000 & 0.000 \\ 0.000 & 932.939 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

here: no noise, 2D => 2 singular values

## Problem 2

$$V = \begin{pmatrix} -0.162 & -0.728 & 0.120 & -0.655 \\ 0.509 & 0.516 & 0.149 & -0.672 \\ -0.746 & 0.407 & -0.402 & -0.342 \\ 0.398 & -0.195 & -0.895 & -0.046 \end{pmatrix}, L^{(\frac{1}{2})} = \begin{pmatrix} 48.39484 & 0.000 & 0.000 & 0.000 \\ 0.000 & 30.54405 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$X = VL^{(\frac{1}{2})} = \begin{pmatrix} -7.840 & -22.236 & 0.000 & 0.000 \\ 24.633 & 15.761 & 0.000 & 0.000 \\ -36.103 & 12.431 & 0.000 & 0.000 \\ 19.261 & -5.956 & 0.000 & 0.000 \end{pmatrix}$$

## Problem 2

$$V = \begin{pmatrix} -0.162 & -0.728 \\ 0.509 & 0.516 \\ -0.746 & 0.407 \\ 0.398 & -0.195 \end{pmatrix}, L^{(\frac{1}{2})} = \begin{pmatrix} 48.39484 & 0.000 \\ 0.000 & 30.54405 \end{pmatrix}$$

$$X = VL^{(\frac{1}{2})} = \begin{pmatrix} -7.840 & -22.236 \\ 24.633 & 15.761 \\ -36.103 & 12.431 \\ 19.261 & -5.956 \end{pmatrix}$$

## Problem 2

example with noise

$$A = -\frac{1}{2}D^{(2)} = -\frac{1}{2}\begin{pmatrix} 0.00 & 51.60 & 44.20 & 31.62 \\ 51.60 & 0.00 & 63.83 & 19.36 \\ 44.20 & 63.83 & 0.00 & 59.31 \\ 31.62 & 19.36 & 59.31 & 0.00 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 & -1331.28 & -976.82 & -500 \\ -1331.28 & 0 & -2036.98 & -187.418 \\ -976.82 & -2036.98 & 0 & -1758.81 \\ -500 & -187.418 & -1758.81 & 0 \end{pmatrix}$$

$$B = C_{(4)}AC_{(4)} = \begin{pmatrix} 555.1362 & -589.249 & 69.4443 & -35.3319 \\ -589.249 & 928.9266 & -803.823 & 464.1453 \\ 69.4443 & -803.823 & 1537.392 & -803.013 \\ -35.3319 & 464.1453 & -803.013 & 374.1999 \end{pmatrix} = VL V^T$$

## Problem 2

example with noise:

2D problem => consider only 2 largest singular values in L  
and corresponding columns in V

$$V = \begin{pmatrix} -0.188 & -0.729 & -0.428 & 0.500 \\ 0.534 & 0.485 & -0.478 & 0.500 \\ -0.729 & 0.441 & 0.155 & 0.500 \\ 0.383 & -0.197 & 0.751 & 0.500 \end{pmatrix}, L = \begin{pmatrix} 2566.489 & 0.000 & 0.000 & 0.000 \\ 0.000 & 895.942 & 0.000 & 0.000 \\ 0.000 & 0.000 & 66.776 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.188 & -0.729 \\ 0.534 & 0.485 \\ -0.729 & 0.441 \\ 0.383 & -0.197 \end{pmatrix}, L^{(\frac{1}{2})} = \begin{pmatrix} 50.661 & 0.000 \\ 0.000 & 29.932 \end{pmatrix} \quad X = VL^{(\frac{1}{2})} = \begin{pmatrix} -9.524 & -21.821 \\ 27.053 & 14.517 \\ -36.932 & 13.200 \\ 19.403 & -5.897 \end{pmatrix}$$