Exercise 1

a) Compute, whether M and N are positive definite.

$$M = \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix} \qquad N = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

Determinant Test: All upper left determinants are positive.

$$M = \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}$$

$$|3| = 3 > 0$$

$$\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = 3 \cdot 7 - 5 \cdot 5 = 21 - 25 = -4 < 0$$

$\underline{M \text{ is not positive definite}}$

$$N = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

$$|3| = 3 > 0$$

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot 1 = 6 - 1 = 5 > 0$$

$$\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & -3 \\ -2 & -3 & 5 \end{vmatrix} = 3 \cdot \begin{vmatrix} 2 & -3 \\ -3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} =$$

$= 3(10-9) - 1(5-6) - 2(-3+4) =$

$= 3 \cdot 1 - 1 \cdot (-1) - 2 \cdot 1 =$

$= 3 + 1 - 2 = 2 > 0$

N is positive definite

b) $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 6x$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 6x^2 + 6y^2 - 6 \\ 12xy - 9y^2 \end{bmatrix}$$

Stationary points:

$\nabla f = \vec{0}$

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6x^2 + 6y^2 - 6 \\ 12xy - 9y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} ① \; 6x^2 + 6y^2 - 6 = 0 \\ ② \; 12xy - 9y^2 = 0 \end{cases} \iff \begin{cases} 6x^2 + 6y^2 = 6 \\ y(12x - 9y) = 0 \end{cases}$$

From ② $y = 0 \rightarrow$ Substitute in ①

$6x^2 = 6$

$\Downarrow$

$x = -1$ or $x = 1$

From ② $12x = 9y$ Substitute in ①

$x = \frac{3}{4}y \Rightarrow 6\left(\frac{3}{4}y\right)^2 + 6y^2 = 6$

$\frac{9}{16}y^2 + y^2 = 1$

$y^2 = \frac{16}{25}$

$\Downarrow$

$y = -\frac{4}{5}$ or $y = \frac{4}{5}$)

For $y = -\frac{4}{5} \Rightarrow 12x - 9 \cdot \left(-\frac{4}{5}\right) = 0$

$x = -\frac{3}{5}$

$y = \frac{4}{5} \Rightarrow 12x - 9 \cdot \frac{4}{5} = 0$

$x = \frac{3}{5}$

Stationary Points:

$$\left.\begin{array}{l} A_1 \ (-1, 0) \\ A_2 \ (1, 0) \\ A_3 \ \left(-\frac{3}{5}, -\frac{4}{5}\right) \end{array}\right\}$$

$A_4 \ \left(\frac{3}{5}, \frac{4}{5}\right)$

To see if they are local minima/maxima or ~~sta~~ saddle points: Compute Hessian

$$H_f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x & 12y \\ 12y & 12x - 18y \end{bmatrix}$$

$\underline{A_1(-1;0)}$

$$H_f(-1;0) = \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix} \text{ is negative definite}$$

because:

$$[x \ y] \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -12x^2 - 12y^2 < 0 \ \forall \vec{v} \in \mathbb{R}^2 \ (\vec{v} \neq \vec{0})$$

Hence $A_1(-1;0)$ is <u>local maxima</u>

$A_2(1,0)$

$$H_f(1,0) = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$ is positive definite

because

$$[x \quad y]\begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 12x^2 + 12y^2 > 0 \ \forall \vec{v} \in \mathbb{R}^2 \ (\vec{v} \neq \vec{0})$$

Hence $A_2(1,0)$ is <u>local minima</u>

$A_3(-\frac{3}{5}, -\frac{4}{5})$

$$H_f(-\frac{3}{5}, -\frac{4}{5}) = \begin{bmatrix} -\frac{36}{5} & -\frac{48}{5} \\ -\frac{48}{5} & \frac{36}{5} \end{bmatrix}$$ is indefinite

because

$$[x \quad y]\begin{bmatrix} -\frac{36}{5} & -\frac{48}{5} \\ -\frac{48}{5} & \frac{36}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-36x^2 - 96xy + 36y^2}{5}$$

may be positive or negative depending on the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Hence $A_3(-\frac{3}{5}, -\frac{4}{3})$ is a <u>saddle point</u>

$A_4(\frac{3}{5}, \frac{4}{5})$

$$H_f(\frac{3}{5}; \frac{4}{5}) = \begin{bmatrix} \frac{36}{5} & \frac{48}{5} \\ \frac{48}{5} & -\frac{36}{5} \end{bmatrix}$$ is indefinite

because

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{36}{5} & \frac{48}{5} \\ \frac{48}{5} & -\frac{36}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{36x^2 + 96xy - 36y^2}{5}$$

may be positive or negative depending on the chosen vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Hence $A_4\left(\frac{3}{5}, \frac{4}{5}\right)$ is a saddle point