I. Quaternions

1) $\left(\cos\frac{\theta}{2}, n \sin\frac{\theta}{2}\right)$

$$q = \left(\cos\frac{60}{2}, (0\ 1\ 0)\sin\frac{60}{2}\right) = \left(\frac{\sqrt{3}}{2}, 0\ \frac{1}{2}\ 0\right)$$

$$q^* = \left(\frac{\sqrt{3}}{2}, 0\ -\frac{1}{2}\ 0\right)$$

$$p = (0, 1\ 1\ 1)$$

$$\boxed{\text{Rotation} = qpq^* = S_1S_2 - \vec{V_1}\cdot\vec{V_2},\ S_1\vec{V_2} + S_2\vec{V_1} + \vec{V_1}\times\vec{V_2}}$$

$$pq^* = S_1S_2 - \vec{V_1}\cdot\vec{V_2},\ S_1\vec{V_2} + S_2\vec{V_1} + \vec{V_1}\times\vec{V_2}$$

$$= 0\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right),\ 0 + \frac{\sqrt{3}}{2}(1\ 1\ 1) + \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1/2 & 0 \end{bmatrix}$$

$$= \frac{1}{2}, \left(\frac{\sqrt{3}}{2}\ \frac{\sqrt{3}}{2}\ \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\ 0\ -\frac{1}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{\sqrt{3}+1}{2}\ \frac{\sqrt{3}}{2}\ \frac{\sqrt{3}-1}{2}\right)$$

$$q(pq^*) = S_1S_2 - \vec{V_1}\cdot\vec{V_2},\ S_1\vec{V_2} + S_2\vec{V_1} + \vec{V_1}\times\vec{V_2}$$

$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) - \frac{\sqrt{3}}{4},\ \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}+1}{2}\ \frac{\sqrt{3}}{2}\ \frac{\sqrt{3}-1}{2}\right) + \frac{1}{2}\left(0\ \frac{1}{2}\ 0\right) + \begin{bmatrix} 0 & 1/2 & 0 \\ \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}-1}{2} \end{bmatrix}$$

$$= 0, \left(\frac{\sqrt{3}(\sqrt{3}+1)}{4}\ 1\ \frac{\sqrt{3}(\sqrt{3}-1)}{4}\right) + \left(\frac{\sqrt{3}-1}{4}\ 0\ \frac{-\sqrt{3}+1}{4}\right)$$

$$\left(0, \frac{\sqrt{3}+1}{2}\ 1\ \frac{1-\sqrt{3}}{2}\right)$$

Hence $p' = \left(\frac{\sqrt{3}+1}{2}, 1, \frac{1-\sqrt{3}}{2}\right)$

2) $\left(\cos\frac{\theta}{2}, n \sin\frac{\theta}{2}\right)$

$$q = \cos\frac{30}{2}, (0\ 1\ 0)\sin\frac{30}{2}) = (0.97, 0\ 0.26\ 0)$$

$$q^* = (0.97, 0\ -0.26\ 0)$$

$p = (0, 1\ 1\ 1)$

$$\boxed{\text{Rotation} = qpq^* = s_1 s_2 - \vec{v_1} \cdot \vec{v_2},\ s_1 \vec{v_2} + s_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}}$$

$$pq^* = s_1 s_2 - \vec{v_1} \cdot \vec{v_2},\ s_1 \vec{v_2} + s_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}$$

$$= 0 - (-0.26),\ 0 + 0.96\,(1\ 1\ 1) + \begin{bmatrix} 1 & 1 & 1 \\ 0 & -0.26 & 0 \end{bmatrix}$$

$$= 0.26,\ (0.96\quad 0.96\quad 0.96) + (0.26\quad 0\quad -0.26)$$

$$= 0.26,\ (1.22\quad 0.96\quad 0.7)$$

$$q(pq^*) = s_1 s_2 - \vec{v_1} \cdot \vec{v_2},\ s_1 \vec{v_2} + s_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}$$

$$= (0.96 \times 0.26) - (0.96 \times 0.26),\ 0.96(1.22\quad 0.96\quad 0.7) + 0.26(0\quad 0.26\quad 0) + \begin{bmatrix} 0 & 0.26 & 0 \\ 1.22 & 0.96 & 0.7 \end{bmatrix}$$

$$= 0,\ (1.17\quad 0.92\quad 0.67) + (0\quad 0.068\quad 0) + (0.18\quad 0\quad -0.32)$$

$$= 0,\ (1.35\quad 0.98\quad 0.35)$$

Hence $p' = (1.35,\ 0.98,\ 0.35)$

3) Same as question 1 and 2

4) Same as question 1 and 2

5) $\left[0.5,\ \left(0, 0, \frac{1}{2}\sqrt{3}\right)\right]$

- The axis vector is $(0, 0, 1)$ about the z-axis
- The angle of rotation $= \cos\frac{\theta}{2} = 0.5$ or $\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}$

$\frac{\theta}{2} = 60$ or $\frac{\theta}{2} = 60$

$\theta = 120^\circ$

6) $q_1 = 4 + 3i + 2j - K \qquad q_2 = i - K$

$(4 + 3i + 2j - K)\,(i - K)$

$4i - 4K + 3i^2 - 3iK + 2ji - 2jK - Ki + K^2$

$\boxed{i \to j \to K \quad = +ve}$

$4i - 4K + 3i^2 - 3iK + 2ji - 2jK - Ki + K^2$

$$\boxed{\begin{array}{ll} i \rightarrow j \rightarrow K & = +ve \\ i \leftarrow j \leftarrow K & = -ve \end{array}}$$

$4i - 4K - 3 + 3j - 2K - 2i - j - 1$

$-4 + 2i + 2j - 6K$

7) $(4 + 4i + 2j + K)(j - K)$

$4j - 4K + 4ij - 4iK + 2j^2 - 2jK + Kj - K^2$

$4j - \cancel{4K + 4K} + 4j - 2 - 2i - i + 1$

$-1 - 3i + 8j$