Midterm Solutions

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a) $$\lim_{x \to -2} \frac{\frac{1}{x} - \frac{1}{2}}{x^3 - 8} = \frac{-\frac{1}{2} - \frac{1}{2}}{-8-8} = \frac{-1}{-16} = \frac{1}{16}$$

b) $$\lim_{y \to \infty} \frac{e^{-y} \sin(y) \cos(y)}{y}$$

Use squeeze law, as $|\sin(y) \cdot \cos(y)| \leq 1$

$$\frac{-e^{-y}}{y} \leq \frac{e^{-y} \sin(y) \cos(y)}{y} \leq \frac{e^{-y}}{y}$$

$$\frac{-e^{-y}}{y} \xrightarrow{y \to \infty} 0 \qquad \frac{e^{-y}}{y} \xrightarrow{y \to \infty} 0$$

Since all the above functions are continuous,

$$\frac{e^{-y} \sin(y) \cos(y)}{y} \xrightarrow[y \to \infty]{} 0$$ by squeeze law.

c) $$\lim_{r \to 1} \frac{|r-1|}{2r-2} = \lim_{r \to 1} \frac{1}{2} \frac{|r-1|}{r-1}$$

There are two different limits, as $r - 1 < 0$ for $r < 1$ and $r - 1 > 0$ for $r > 1$

We get

$$\lim_{r \searrow 1} \frac{1}{2} \frac{|r-1|}{r-1} = \frac{1}{2} \lim_{r \searrow 1} \underbrace{\frac{r-1}{r-1}}_{=1}$$

$$= \frac{1}{2}$$

and

$$\lim_{r \nearrow 1} \frac{1}{2} \frac{|r-1|}{r-1} = \frac{1}{2} \lim_{r \nearrow 1} \underbrace{\frac{-r+1}{r-1}}_{=-1}$$

$$= -\frac{1}{2}$$

So limit $\lim_{r \to 1}$ does not exist!

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(2) $x^6 - 5x - 5 = 0$

We look for roots of the function

$f(x) = x^6 - 5x - 5$ in $[-1, 0]$

f is continuous. We use intermediate value theorem

$$\left.\begin{array}{l} f(-1) = 1 + 5 - 5 = 1 > 0 \\ f(0) = -5 < 0 \end{array}\right\} \text{sign changes}$$

$\Rightarrow$ IVT says $\exists\, x$ s.t. $f(x) = 0$ for $x \in [-1, 0]$

b) $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x^2+2hx+h^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x^2 - (x^2+2hx+h^2)}{x^2(x^2+2hx+h^2)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}x + h^{\cancel{2}}}{\cancel{h}x^2(x^2+2hx+h^2)} = \lim_{h \to 0} \frac{-2x + \overset{\to 0}{h}}{x^4 + \underset{\to 0}{2hx^3} + \underset{\to 0}{x^2h^2}}$$

$$= -\frac{2x}{x^4} = -\frac{2}{x^3}$$

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③ $f(x) = \frac{x^2}{2-x^2}$, $2 - x^2 = 0 \iff x = \pm\sqrt{2}$

1) $D(f) = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$

2) Horizontal asymptote

$$\lim_{x \to \infty} \frac{x^2}{2-x^2} = \lim_{x \to \infty} \frac{1}{\underset{\to 0}{\frac{2}{x^2}} - 1} = -1$$

Same for $x \to -\infty$.

Horizontal asymptotes are -1 for $x \to \pm\infty$

3) Vertical asymptote:

Check points excluded from domain $\mathcal{D}(f)$

$x = \sqrt{2}$ and $x = -\sqrt{2}$.

In both cases, denominator goes to 0, but we get

$$\lim_{x \searrow \sqrt{2}} \frac{x^2}{(2-x^2)} = \lim_{x \searrow \sqrt{2}} \frac{1}{\underbrace{\frac{2}{x^2} - 1}_{<0 \text{ for } x>\sqrt{2}}} = -\infty$$

$$\lim_{x \nearrow \sqrt{2}} \frac{x^2}{2-x^2} = \lim_{x \nearrow \sqrt{2}} \frac{1}{\underbrace{\frac{2}{x^2} - 1}_{>0 \text{ for } x<\sqrt{2}}} = \infty$$

Similarly for $-\sqrt{2}$ (but other way around)

$$\lim_{x \searrow -\sqrt{2}} \frac{x^2}{(2-x^2)} = \infty$$

$$\lim_{x \nearrow -\sqrt{2}} \frac{x^2}{2-x^2} = -\infty$$

So we have two vertical asymptotes at $\sqrt{2}$ and $-\sqrt{2}$ but with opposite "limits" on each side

4) $$f'(x) = \frac{2x(2-x^2) - x^2(-2x)}{(2-x^2)^2}$$

$$= \frac{4x - 2x^3 + 2x^3}{(2-x^2)^2} = \frac{4x}{(2-x^2)^2} \overset{!}{=} 0$$

In this case $4x \overset{!}{=} 0 \iff x = 0$ critical point

We can look at sign of x, which again is only defined by numerator (as $(2-x^2)^2 > 0$)

$$\Rightarrow \left.\begin{array}{lll} f'(x) < 0 & \text{if} & x < 0 \\ f'(x) > 0 & \text{if} & x > 0 \end{array}\right\} f(0) = 0 \text{ is a local minimum}$$

5) $$f''(x) = \frac{4(2-x^2)^{\cancel{2}} - 4x\,\cancel{(2-x^2)} \cdot 2 \cdot (-2x)}{(2-x^2)^{\cancel{4}\,3}}$$

$$= \frac{8 - 4x^2 + 16x^2}{(2-x^2)^3} = \frac{8 + 12x^2}{(2-x^2)^3}$$

Sign change of $f''(x)$ only when denominator changes sign

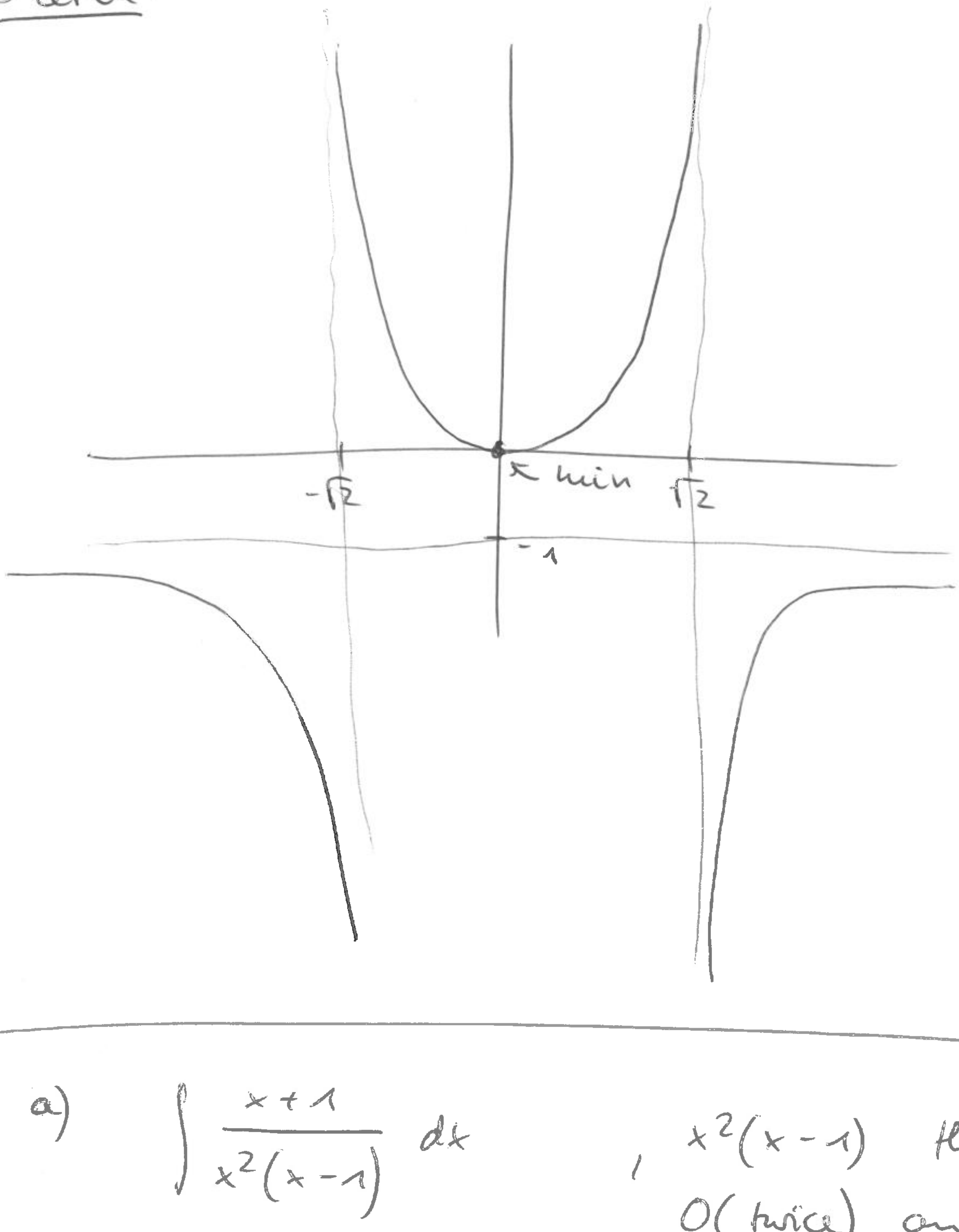
$f''(x) > 0$ for $(2-x^2) > 0$, i.e. $|x| < \sqrt{2}$
(<u>concave up</u>) $\Rightarrow x \in (-\sqrt{2}, \sqrt{2})$

$f''(x) < 0$ for $(2-x^2) < 0$, i.e. $|x| > \sqrt{2}$

$\Rightarrow$ for $x > \sqrt{2}$ or $x < -\sqrt{2}$

(concave down)

<u>Sketch:</u>



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(4) a) $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$, $x^2(x-1)$ three roots 0 (twice) and 1

Need to find A, B, C s.t.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} = \frac{x+1}{x^2(x-1)}$$

$$\frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)} \overset{!}{=} \frac{x+1}{x^2(x-1)}$$

$$\Rightarrow \quad Ax^2 - Ax + Bx - B + Cx^2 = x+1$$

$$\left.\begin{array}{lll} Ax^2 + Cx^2 = 0 & \Rightarrow & A = -C \\ -Ax + Bx = x & \Rightarrow & \left.\begin{array}{l} -A+B=1 \\ B=-1 \end{array}\right\} A=-2 \\ -B = 1 & \Rightarrow & \end{array}\right\} C=2$$

$$\int -\frac{2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{2}{x-1}\,dx$$

$$= -2\ln|x| - (-1)\frac{1}{x} + 2\ln|x-1| + c$$

$$= -2\ln|x| + 2\ln|x-1| + \frac{1}{x} + c$$

$$\left(= -\ln(x^2) + \ln\left((x-1)^2\right) + \frac{1}{x} + c = \ln\left(\frac{(x-1)^2}{x^2}\right) + \frac{1}{x} + c \right)$$

b) $\int x^{-3} e^{-\frac{1}{x^2}}\,dx$ substitute $u = -\frac{1}{x^2}$, $\frac{du}{dx} = 2\frac{1}{x^3}$

$\frac{du}{2} = \frac{1}{x^3}\,dx$

$$= \int \frac{e^u}{2}\,du = \frac{e^u}{2} + c = \frac{e^{-\frac{1}{x^2}}}{2} + c$$

c) $\int_0^{2\pi} \underset{u'}{\underset{\shortparallel}{e^x}} \underset{v}{\underset{\shortparallel}{\cos(x)}}\, dx = \left[e^x \cos(x)\right]_0^{2\pi} - \int_0^{2\pi} e^x (-\sin(x))\, dx$

$u = e^x \quad v' = -\sin(x)$

$+$

Now

$$\int_0^{2\pi} \underset{u'}{\underset{\shortparallel}{e^x}} \underset{v}{\underset{\shortparallel}{\sin(x)}}\, dx = \left[e^x \sin(x)\right]_0^{2\pi} - \int_0^{2\pi} e^x \cos(x)$$

$u = e^x \quad v' = \cos(x)$

$$\Rightarrow 2\int_0^{2\pi} e^x \cos x \, dx = \left[e^x \cos(x)\right]_0^{2\pi} + \underset{0}{\underset{\shortparallel}{\left[e^x \sin(x)\right]_0^{2\pi}}}$$

0, since $\sin(0) = \sin(2\pi) = 0$

$$\Rightarrow \int_0^{2\pi} e^x \cos x \, dx = \frac{1}{2} e^{2\pi} \cdot \overset{1}{\overset{\shortparallel}{\cos(2\pi)}} - \frac{1}{2} \underbrace{e^0 \cos(0)}_{=1} = \frac{1}{2}(e^{2\pi} - 1)$$

$(\hat{=} 267.25)$

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d) $f(t) = t^{t^3} = e^{\ln t \cdot t^3}$

$$f'(t) = \underbrace{e^{\ln t \cdot t^3}}_{\text{outer}} \cdot \underbrace{\left(\frac{1}{t} \cdot t^3 + \ln t \cdot 3t^2\right)}_{\text{inner}}$$

$$= t^{t^3}(t^2 + 3\ln t \cdot t^2) = t^{t^3+2}(1 + 3\ln t)$$

5

a) Find points of intersection

$x = y^2 \quad , \quad x = -y^2 + 2$

$\Rightarrow \quad y^2 = -y^2 + 2 \quad \Rightarrow \quad 2y^2 = 2 \quad \Rightarrow \quad y_{1/2} = \pm 1$

Now integrate w.r.t. y

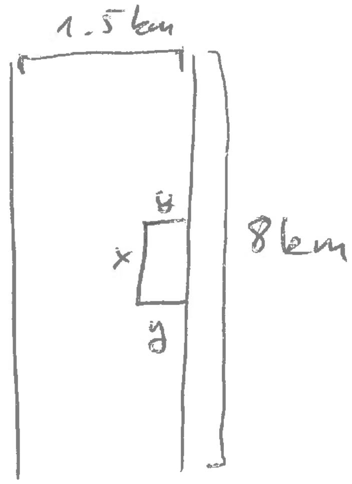
$$\int_{-1}^{1} y^2 - (-y^2 + 2)\, dy = \int_{-1}^{1} 2y^2 - 2\, dy$$

$$= \left[\frac{2}{3} y^3 - 2y\right]_{-1}^{1} = \left(\frac{2}{3} - 2\right) - \left(-\frac{2}{3} + 2\right)$$

$$= \frac{4}{3} - 4 = -\frac{8}{3}$$

In absolute terms $\left|-\frac{8}{3}\right| = \frac{8}{3}$ is <u>area</u>.

b)



(i) We have $A = x \cdot y$

$C = 2y + x = 1 (km)$ (fencing)

$\Rightarrow \quad x = 1 - 2y$

$A = (1 - 2y)\, y = y - 2y^2$

$$\frac{dA}{dy} = 1 - 4y \overset{!}{=} 0$$

$\Rightarrow \quad y = \frac{1}{4}$ (has to be max. as $-y^2$ is concave down)

$\Rightarrow \quad x = 1 - 2\frac{1}{4} = \frac{1}{2} \quad \Rightarrow \quad A = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \; (km^2)$

b) (ii) In this case, all of his land can be fenced off, as 2×1.5 km is only 3 km, but he has 4 km of fence

$\Rightarrow \quad A = 1.5 \cdot 8 = 12\,(km^2)$

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c) $\sin(2x+y) = y^3 \sin(x)$

Do $\frac{d}{dx}$ on both sides

$$\underbrace{\cos(2x+y)}_{\text{outer}}\underbrace{\left(2+\frac{dy}{dx}\right)}_{\text{inner}} = \underbrace{3y^2}_{\text{outer}}\underbrace{\frac{dy}{dx}}_{\text{inner}} \cdot \sin(x) + y^3\cos(x)$$

Now do this at $(0,0)$: $\sin(0)=0$, $\cos(0)=1$

$$\Rightarrow \quad 1\cdot\left(2+\frac{dy}{dx}\right) = \underbrace{3\cdot 0\cdot \frac{dy}{dx}\cdot 0 + 0\cdot 1}_{=0}$$

$$\Rightarrow \quad 2+\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -2$$

For tangent we use equation of line

$$y - y_1 = m\,(x - x_1) \quad , \text{ with } (x_1, y_1) = (0,0)$$

m: slope $\frac{dy}{dx}$

$\Rightarrow \quad \underline{y = -2x}$