Guiding question:

Choice from a finite set $\rightsquigarrow$ Choice from a an infinite set such as an interval.

- Ω set of all outcomes sample space
- $A \subseteq \Omega$: set of favorable outcomes

Pascal's equiprobable model: $\mathbb{P}(A) = \dfrac{\#A}{\#\Omega}$

Monte Carlo method

$\Omega = [0, 1]$

$A = \{x \geq 1/2\} = [1/2, 1]$ 0.1235156...

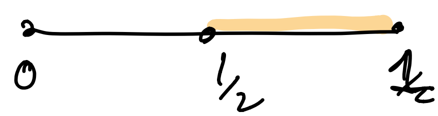
0 $B = \{\underline{x \text{ rational}}\}$

0 $C = \{$ decimal exp of x is period $\}$

0 $D = \{$ " " " x has no zeros in it $\}$

$$\mathbb{P}(A) = \frac{\# A}{\# \Omega} \to \infty$$

length $[c, d] = d - c$



$$\mathbb{P}(A) = \frac{\text{length}(A)}{\text{length}(\Omega)}$$

Geometric probability

Guiding question:

Choice from a finite set $\rightsquigarrow$ Choice from a an infinite set such as an interval.

A *random* number is chosen from the interval $[-2, 2]$ what is the probability that $x \in [1/2, 3/2]$?

Geometric probability in one dimension

- Suppose that the sample space is given by $\Omega = [a, b]$ and $A \subseteq \Omega$ is an event. Then we define

Geometric probability in one dimension

- Suppose that the sample space is given by $\Omega = [a, b]$ and $A \subseteq \Omega$ is an event. Then we define
- $$\mathbb{P}[A] = \frac{\mathcal{L}(A)}{b-a},$$
  where $\mathcal{L}$ denotes the length.

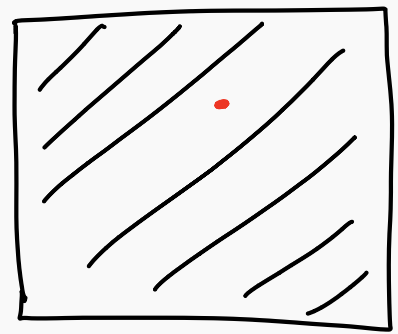
$\mathcal{L}(A)$ is well defined for a very large class of subsets of $[0,1]$

<u>measurable subsets</u>

Geometric probability in higher dimensions

$A \subseteq \Omega$

$$\mathbb{P}(A) = \frac{\text{Area}(A)}{\text{Area}(\Omega)}$$



Ω

higher dimensions.

$A \subseteq \mathbb{R}^n$

$$\mathbb{P}(A) = \frac{\text{volume}(A) \text{ in } \mathbb{R}^n}{\text{volume}(\Omega) \text{ in } \mathbb{R}^n}$$

Example

Alex and Anna are meeting between noon and 1 pm. Each of them picks a random time in the time interval to show up, wait for 15 minutes and leave. We also assume that they make their decision independently. What is the probability that they meet?

T_1 time that Alex shows up
T_2 time that Anna " "

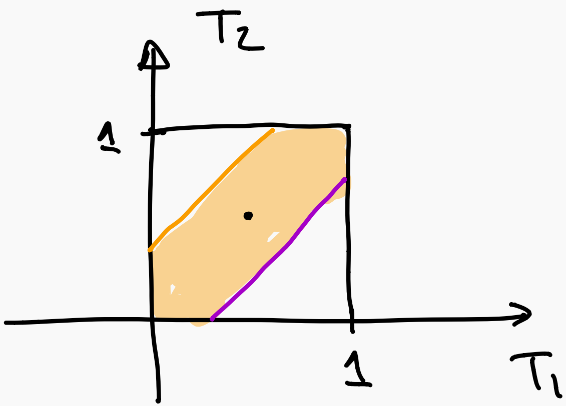
number to keep track of:

$0 \leq T_1 \leq 1$

$0 \leq T_1 \leq 1$
$0 \leq T_2 \leq 1$

$-\frac{1}{4} \leq T_1 - T_2 \leq \frac{1}{4}$

$T_1 - T_2 = -1/4$
$T_1 - T_2 = +1/4$



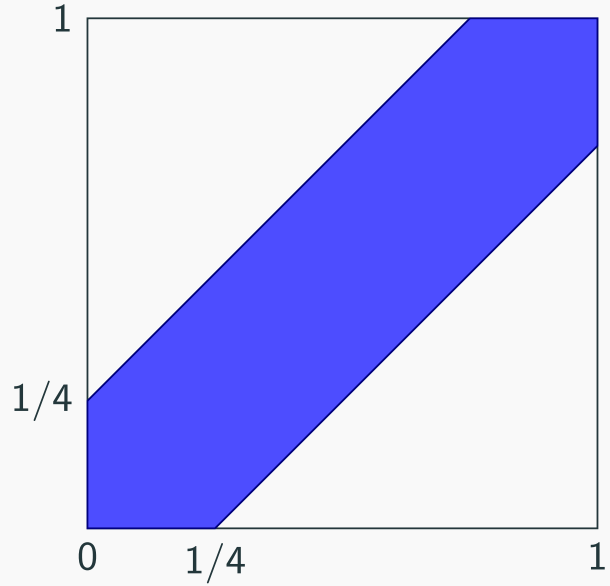
Geometric representation of the sample space

$$\mathbb{P}(\text{Meet}) = \frac{\text{area of blue region}}{\text{area of } \Omega}$$

$$= \frac{1 - \left(3/4\right)^2}{1}$$

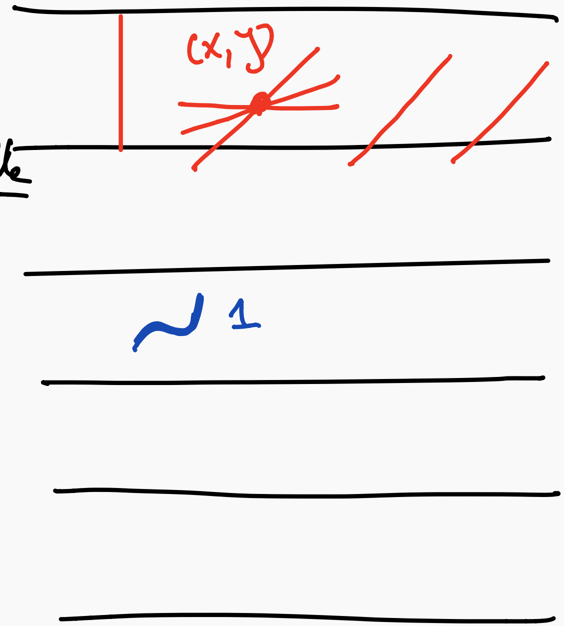
$$= \frac{1 - \frac{9}{16}}{1}$$

$$= \frac{7}{16} \approx 0.456 \text{ —}$$



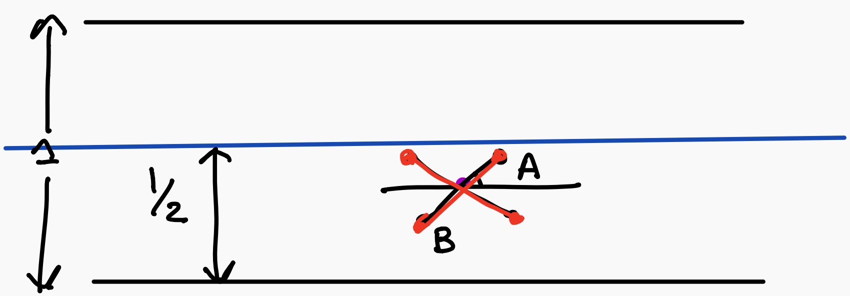
Buffon's needle

A needle of length 1 is randomly dropped on a plane which is ruled by parallel lines with distance 1 between any two consecutive ones. Compute the probability that it hits one of the lines.

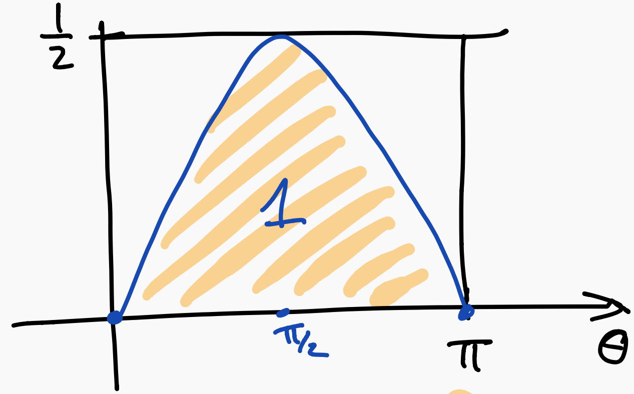


Need to keep track of:

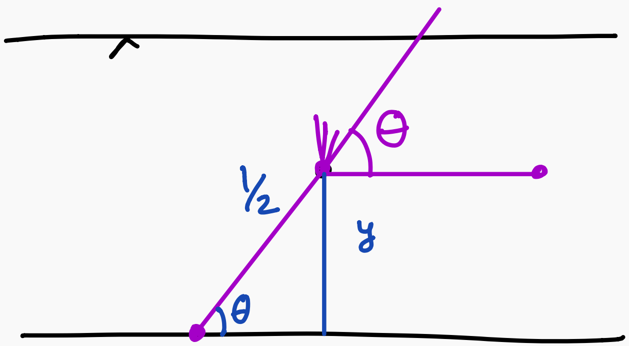
- y-coordinate of the center of the needle
- angle that the needle forms with the horizontal axis



$0 \le y \le \frac{1}{2} \qquad 0 \leq \theta \leq \pi$



$y < \frac{1}{2}\sin\theta$



$$\sin\theta = \frac{y}{1/2} \Rightarrow y = \frac{1}{2}\sin\theta$$

$y > \frac{1}{2}\sin\theta$ miss

$y < \frac{1}{2}\sin\theta$ hit

$$\int_0^{\pi} \frac{1}{2}\sin\theta \, d\theta$$

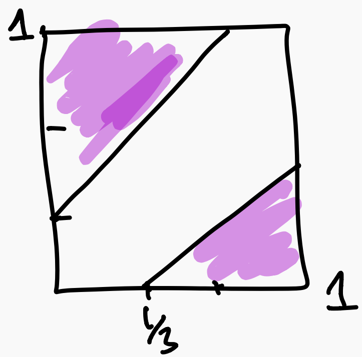
$$= -\frac{1}{2}\cos\theta \Big]_0^{\pi} = \frac{1+1}{2} = 1$$

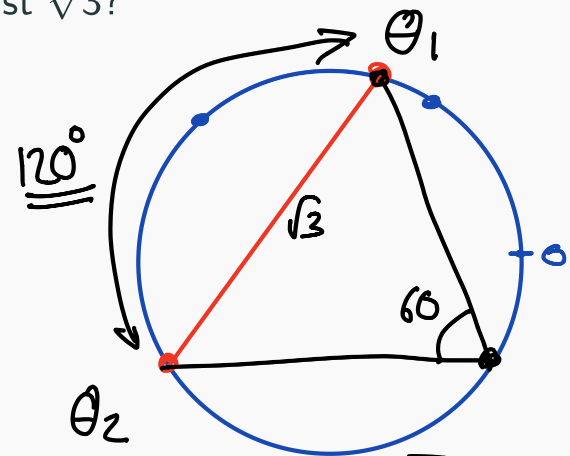
area of sample space $= \frac{\pi}{2}$

$$P(\text{hit}) = \frac{1}{\pi/2} = \frac{2}{\pi}$$

Betrand's paradox

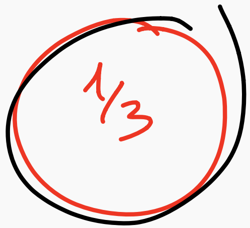
A chord of a circle of radius 1 is chosen randomly. What is the probability of the event E that the length of the cord is at least $\sqrt{3}$?

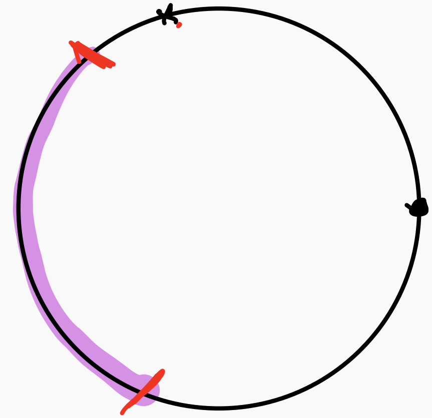


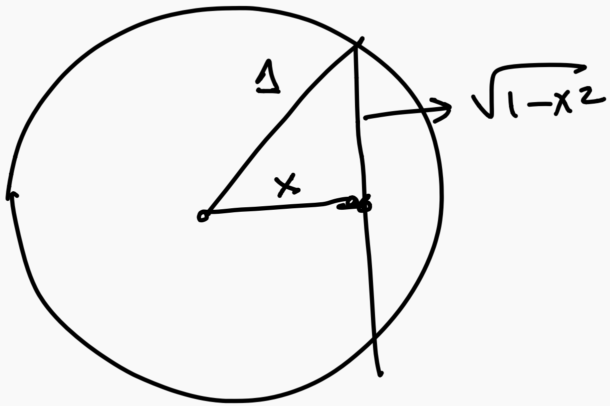


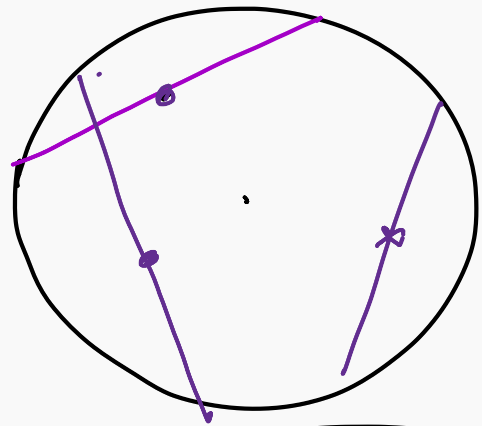
$$|\theta_1 - \theta_2| > \frac{1}{3}$$

$$\text{area} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$









$$2\sqrt{1-x^2} \geq \sqrt{3}$$

$$4(1-x^2) \geq 3$$

$$1 - x^2 \geq \tfrac{3}{4} \qquad x^2 \leq \tfrac{1}{4} \implies \boxed{x \leq \tfrac{1}{2}}$$

1/4