Homework 8

Problem 1

(10 points)

Find

$$\int_{-1}^{0} \ln(x+1)\,\mathrm{d}x\,.$$

Problem 2

(10 points)

a) Show that the volume enclosed when revolving the curve $y = f(x)$ - where $f\colon [a,b] \to [0,\infty)$ - about the x-axis in three-dimensional x-y-z space is given by

$$V = \pi \int_{a}^{b} f^2(x)\,\mathrm{d}x\,.$$

Hint: Think about the cross-sectional areas and the perfect symmetry when revolving the function around the x-axis. **(5 points)**

b) Compute the volume of the solid obtained by revolving the graph of of $y = \dfrac{1+x^2}{2}$ on $[0,1]$ about the x-axis. **(5 points)**

Problem 3

(10 points)

a) Hook's law states that the force exerted by an ideal spring when extended from its equilibrium position at $x = 0$ to length x is given by

$$F(x) = -k\,x\,,$$

where k is a positive constant characterizing the stiffness of the spring. Compute the work required to expand the spring from its equilibrium position to length ℓ. **(5 points)**

b) Show that

$$\int_{1}^{\infty} \frac{1}{\sqrt{1+x^4}}\,\mathrm{d}x$$

is convergent. **(5 points)**

Hint: There is no elementary way to evaluate this integral. However, to only *test* convergence, you can bound the integrand by a simpler function and use the following fact without proof: Let $f\colon [a,\infty) \to \mathbb{R}$ be a bounded and increasing function. Then $\lim_{x\to\infty} f(x)$ exists. (The whole integral corresponds to the function f here.)