Robotics
PS02

IMPORTANT NOTES

- **do not just watch and listen** to the presentation
- have a copy of the lecture slides at hand
- use **paper and pencil** to
  - **first try** to solve the problems **on your own**
  - and **then follow** the solution **step by step**
  - i.e., **pause** the video after each slide or even within the presentation of each slide
  - and make sure that you can **replicate** every step

Problem 1

Proof that when turning in circles you end up where you started. Or more concretely: given the motion `move(`α`, `d`)` (in 2D is sufficient) that turns with angle α and then makes a translation by a distance d, proof that the sequence of motions `move(`90`, `d`)`, `move(`90`, `d`)`, `move(`90`, `d`)`, `move(`90`, `d`)` executed in pose p_{start} gets you into pose p_{end} with $p_{start} = p_{end}$.

Problem 1: Notes - 2D homogeneous matrix

homogeneous matrix in 2D

angle α, translation (tx, ty)

$$H = \begin{pmatrix} c\alpha & -s\alpha & tx \\ s\alpha & c\alpha & ty \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1: first rotation, then translation

angle α, translation t = (tx, ty)

$$\mathrm{H}(\alpha, \mathrm{t}) = \begin{pmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\alpha & -s\alpha & tx \\ s\alpha & c\alpha & ty \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1

$$\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

w.l.o.g.
start in the origin

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d \\ d \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ d \\ 1 \end{pmatrix}$$

just chain the motions

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ q.e.d.

Problem 2

Suppose an object, e.g., the earth, has the pose P_e and a 2nd object, e.g., the moon, with pose P_m is rotating around it with angle θ around the z-axis of P_e.

What is the new pose of P'_m for

$$\theta = 90^o, \quad p_e = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad p_m = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 2: Notes – Consider 2D first

homogeneous matrix R’ for

rotating by α around point $(x_{1,}y_1)^T$ in frame *F*, i.e., ${}^F(x_{1,}y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

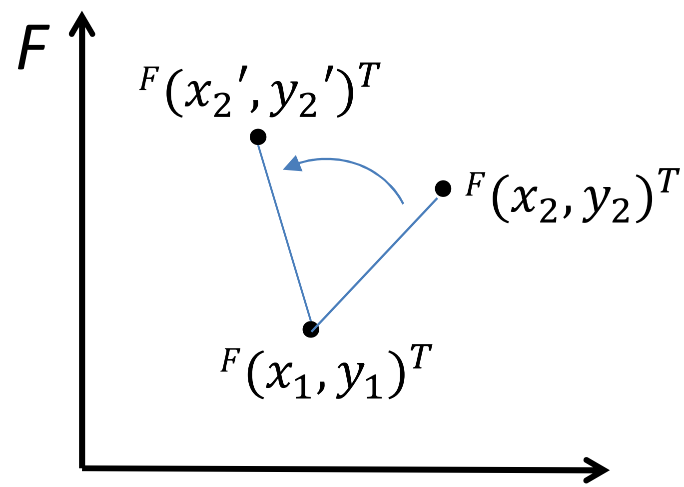
- shift to origin
- rotate
- shift back

Problem 2: Notes – Consider 2D first

rotate ${}^{F}(x_2, y_2)^T$ by α around point ${}^{F}(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} x_2{}' \\ y_2{}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$

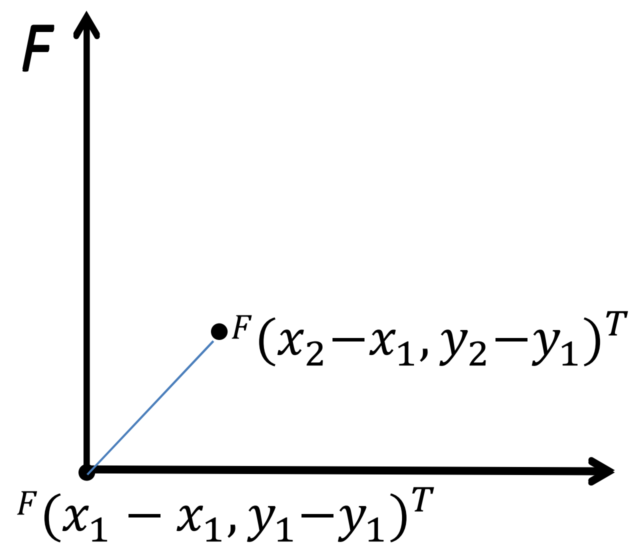


Problem 2: Notes – Consider 2D first

rotate ${}^{F}(x_2, y_2)^T$ by α around point ${}^{F}(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$p_2' = \begin{pmatrix} {x_2}' \\ {y_2}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$

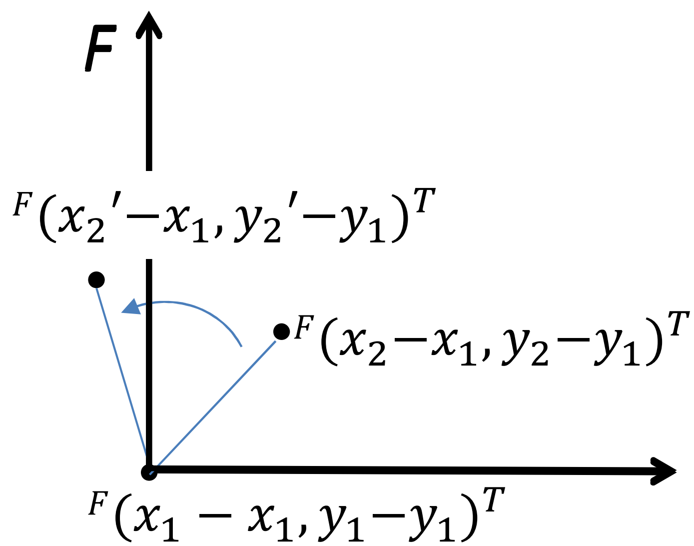


Problem 2: Notes – Consider 2D first

rotate ${}^{F}(x_2, y_2)^T$ by α around point ${}^{F}(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} {x_2}' \\ {y_2}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$

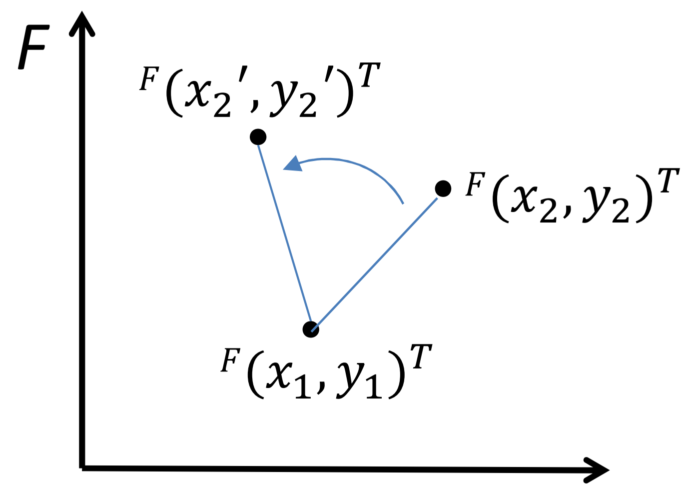


Problem 2: Notes – Consider 2D first

rotate ${}^F(x_2, y_2)^T$ by α around point ${}^F(x_1, y_1)^T$

$$R' = \boxed{\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} {x_2}' \\ {y_2}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$



Problem 2: Notes – Same Game in 3D

- reference frame ${}^{F}F_0$
- rotate a frame within ${}^{F}F_0$ by α with R(α)

$${}^{F}_{0}R' = {}^{F}F_0 \cdot {}^{F}R(\alpha) \cdot {}^{F}F_0{}^{-1}$$

- move to origin
- rotate
- move back

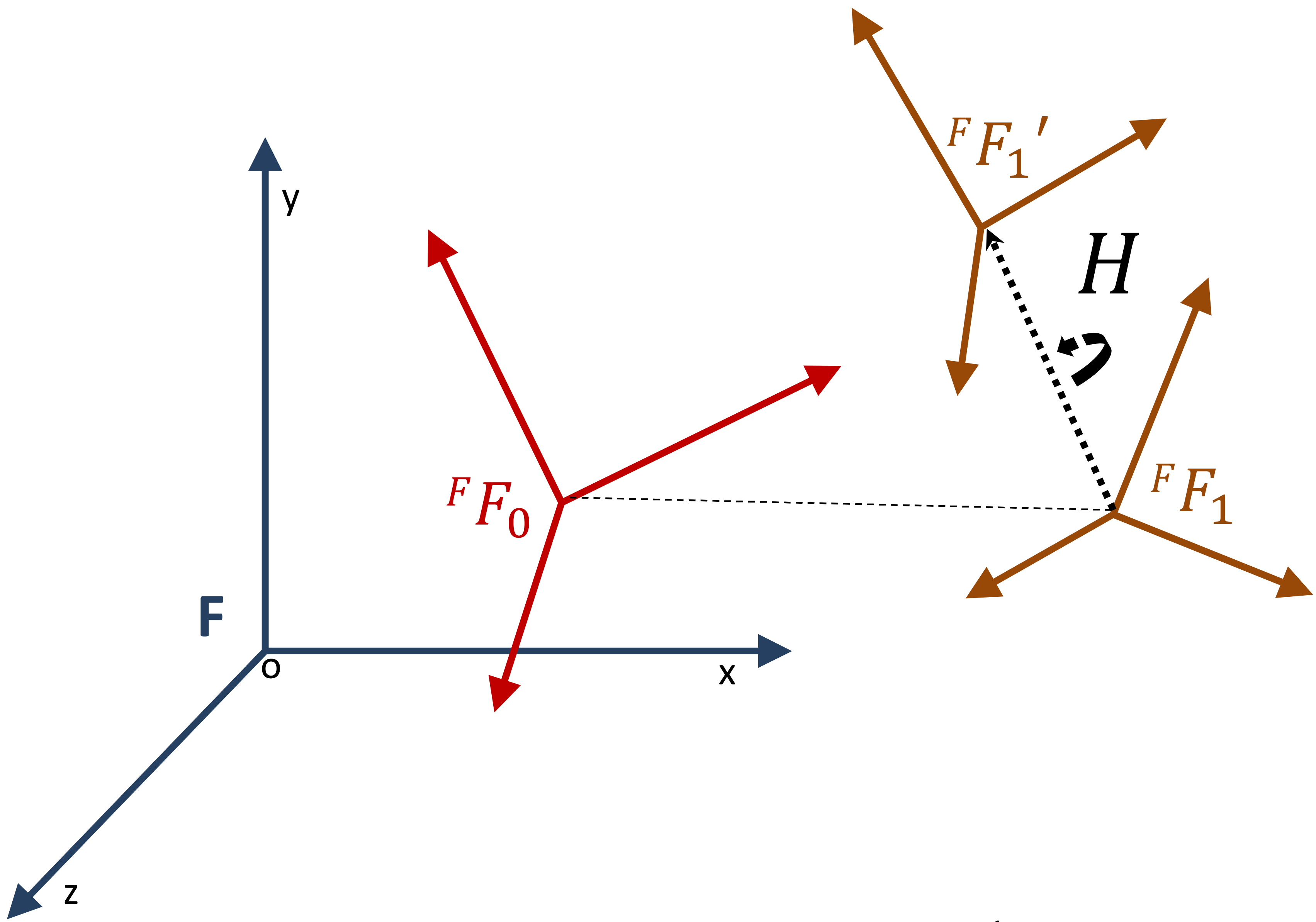
Problem 2: Notes – Same Game in 3D

- reference frame ${}^{F}F_0$
- "arbitrary" motion within ${}^{F}F_0$ (homogeneous transform H)

$$ {}^{F_0}H' = {}^{F}F_0 \cdot H \cdot {}^{F}F_0^{\,-1} $$

- move to origin
- transform with H
- move back

Problem 2: Notes



$$ {}^{F}F_1{}' = ({}^{F}F_0 \cdot H \cdot {}^{F}F_0{}^{-1}){}^{F}F_1 $$

Problem 2

$$p'_m = p_e \cdot R_z(90^o) \cdot p_e^{-1} \cdot p_m$$

$$p_e = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, p_m = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, p_e^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda = 90^o, s\lambda = 1, c\lambda = 0$$

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_z(90^o) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p'_m = p_e \cdot R_z(90^o) \cdot p_e^{-1} \cdot p_m$$

$$= \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 11 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 2

Problem 3

Given a world-frame F_w as identity matrix and an object with pose P_o with

$$p_o = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Suppose the object rotates by 90^o around the z-axis of F_w. What is the new pose P_o' of the object?
- Suppose world frame is an observer/sensor, who/which rotates by 90^o around its z-axis. What is the new pose P_o' of the object?

Problem 3

object rotates: $p'_o = R_z(90^o) \cdot p_o$

observer rotates: $p'_o = R_z^{-1}(90^o) \cdot p_o$

$$\lambda = 90^o, s\lambda = 1, c\lambda = 0$$

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_z(90^o) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R^{-1}{}_z(90^o) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 3

object rotates:

$$p'_o = R_z(90^o) \cdot p_o = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

observer rotates:

$$p'_o = R_z^{-1}(90^o) \cdot p_o = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

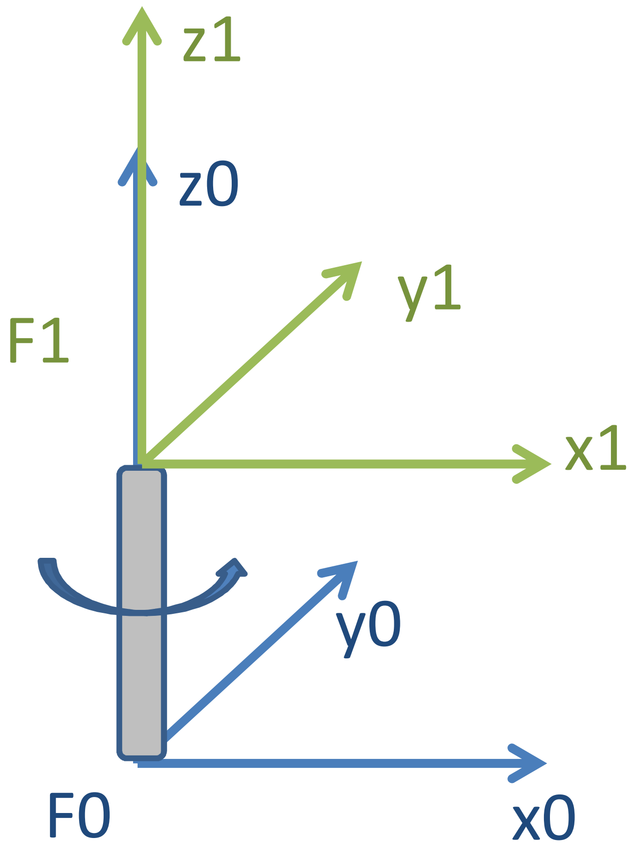
Problem 4

Given a simple robot arm with

- a base frame F_0 as identity matrix
- a rotational joint j_1 that can rotate along the z-axis of F_0 with angle θ_1
- a link l_1 of length 5 along the z-axis of F_0
- a rotational joint j_2 at the end of l_1 that can rotate with angle θ_2 around the y-axis and where its frame F_2 is co-aligned with the base frame for $\theta_1 = \theta_2 = 0$
- a link l_2 of length 3 along the z-axis of F_2
- an end-effector, e.g., a gripper, with pose $P_g = F_3$ at the end of link l_2

How can we express the pose of the end-effector with homogeneous matrices? What is the exact pose P_g for $\theta_1 = 90^o$ and $\theta_2 = 180^o$?

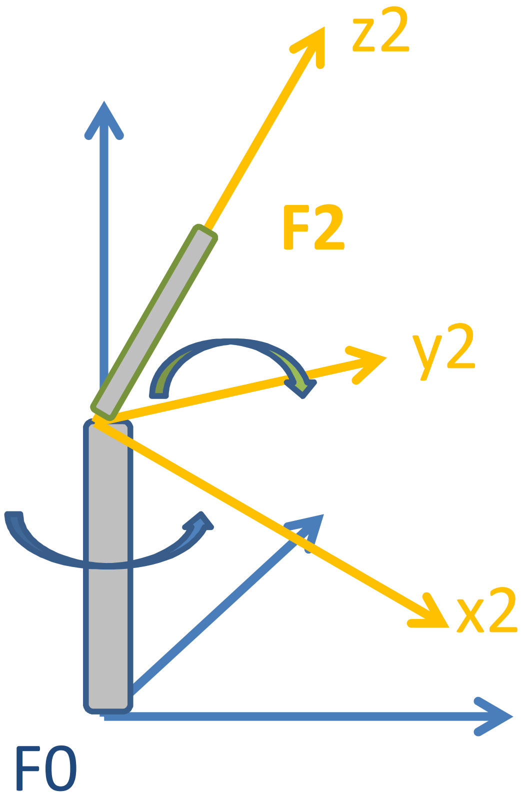
Problem 4



$$F_0 = I$$

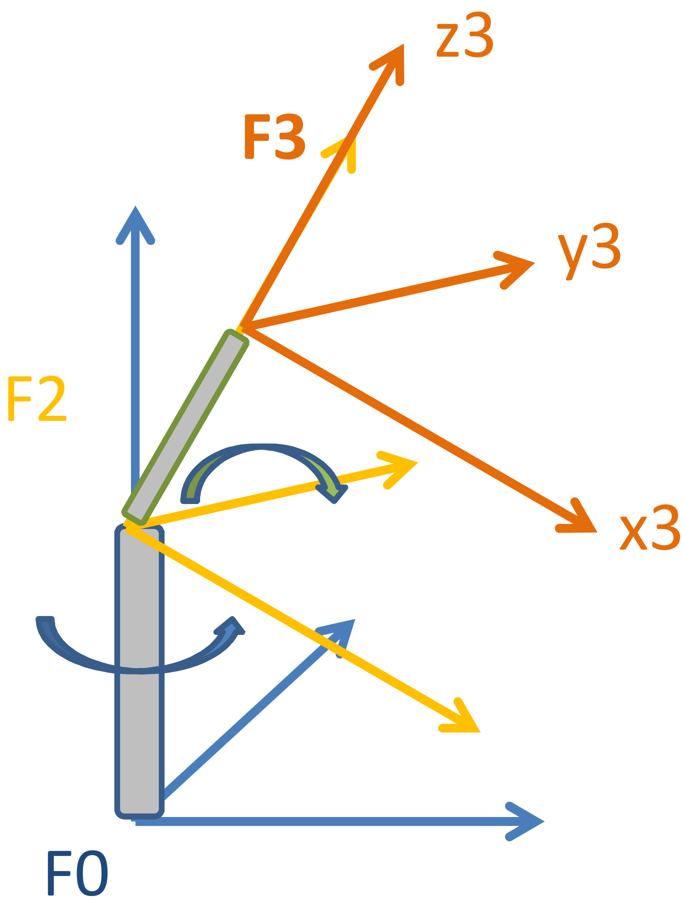
$$^{I}F_1 = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4



$$
{}^{F_1}F_2 = \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

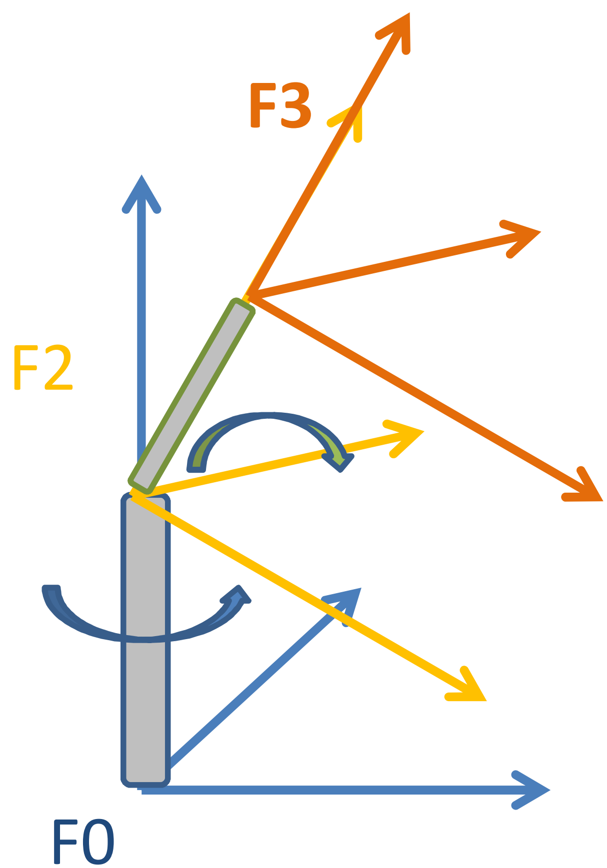
Problem 4



$$
{}^{F_2}F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Problem 4: Notes

$$F_0 = I, \, {}^{I}F_1 = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \, {}^{F_1}F_2 = \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \, {}^{F_2}F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



we want:

$${}^{I}p_g = {}^{I}F_3 \cdot {}^{I}F_2(\theta_2) \cdot {}^{I}F_1(\theta_1) \cdot I$$

but links moves

w.r.t. a reference frame ≠ I

Problem 4: Notes

in general (see also problem 2)

$$
\begin{aligned}
{}^{A}F &= {}^{A}_{C}T_2 \cdot {}^{A}_{B}T_1 \\
&= \left({}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2 \cdot {}^{A}_{B}T_1{}^{-1}\right) \cdot {}^{A}_{B}T_1 \\
&= {}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2 \cdot \left({}^{A}_{B}T_1{}^{-1} \cdot {}^{A}_{B}T_1\right) \\
&= {}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2
\end{aligned}
$$

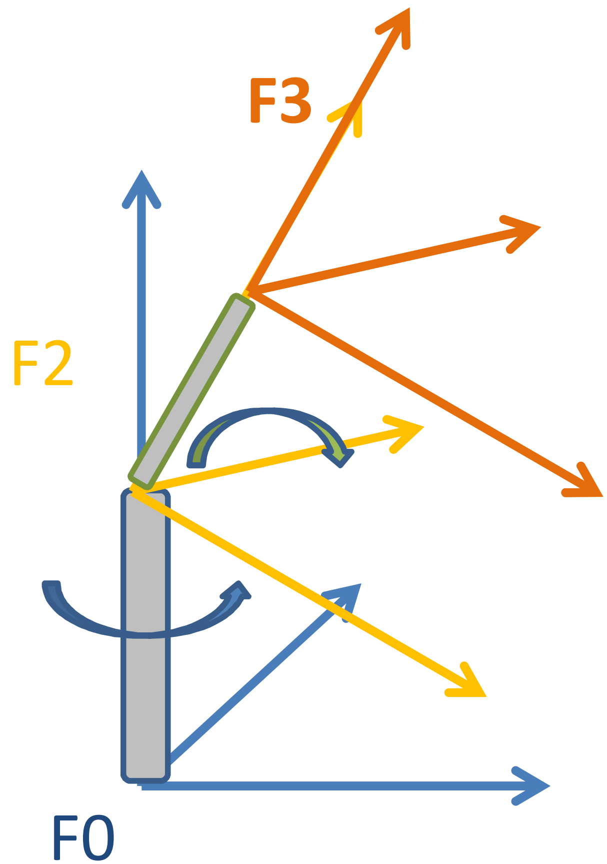
Problem 4: Notes

respectively:

$$
\begin{aligned}
{}^{I}F &= {}^{I}F_n \cdot \ldots \cdot {}^{I}F_2 \cdot {}^{I}F_1 \\
&= {}^{I}F_1 \cdot {}^{F_1}F_2 \cdot \ldots \cdot {}^{F_{n-1}}F_n
\end{aligned}
$$

Problem 4: Notes

$$F_0 = I,\ {}^{I}F_1 = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ {}^{F_1}F_2 = \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ {}^{F_2}F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



hence:

$$\begin{aligned} {}^{I}p_g &= {}^{I}F_3 \cdot {}^{I}F_2(\theta_2) \cdot {}^{I}F_1(\theta_1) \\ &= {}^{I}F_1(\theta_1) \cdot {}^{F_1}F_2(\theta_2) \cdot {}^{F_2}F_3 \end{aligned}$$

Problem 4

$$
{}^{I}p_g = {}^{I}F_1(\theta_1)\cdot {}^{F_1}F_2(\theta_2)\cdot {}^{F_2}F_3
$$

$$
= \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Problem 4

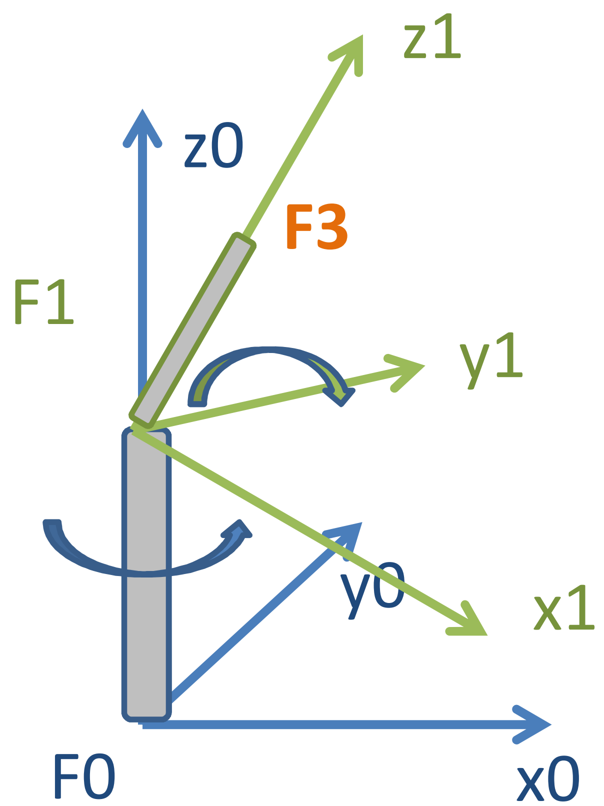
$$
{}^{I}p = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} \theta_1 = 90^o \\ \theta_2 = 180^o \end{matrix}
$$

$$
= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Problem 4: Notes

$$\begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ l_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ l_1 \\ 1 \end{pmatrix} \Rightarrow F_1 = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ l_2 \\ 1 \end{pmatrix} = \begin{pmatrix} s\theta_2 l_2 \\ 0 \\ c\theta_2 l_2 \\ 1 \end{pmatrix}$$

$$\Rightarrow {}^{F_1}F_3 = \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & s\theta_2 l_2 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & c\theta_2 l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

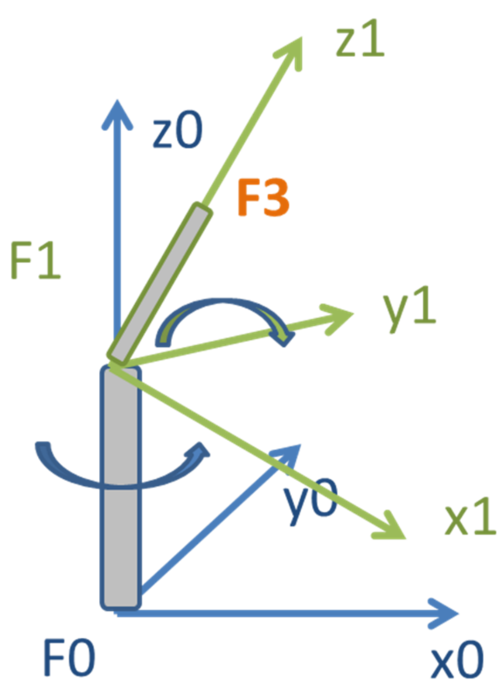
$${}^{F_1}F_3(\theta_2) = {}^{F_1}F_2(\theta_2) \cdot {}^{F_2}F_3$$

Problem 4: Notes

or starting with *I* and “move it” to Pg...

$$ {}^A_C T_3 = {}^A_C T_2 \cdot {}^A_B T_1 = {}^A_B T_1 \cdot {}^B_C T_2 $$

$$ F_0 = I, {}^I F_1 = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{F_1}F_2 = \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{F_2}F_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

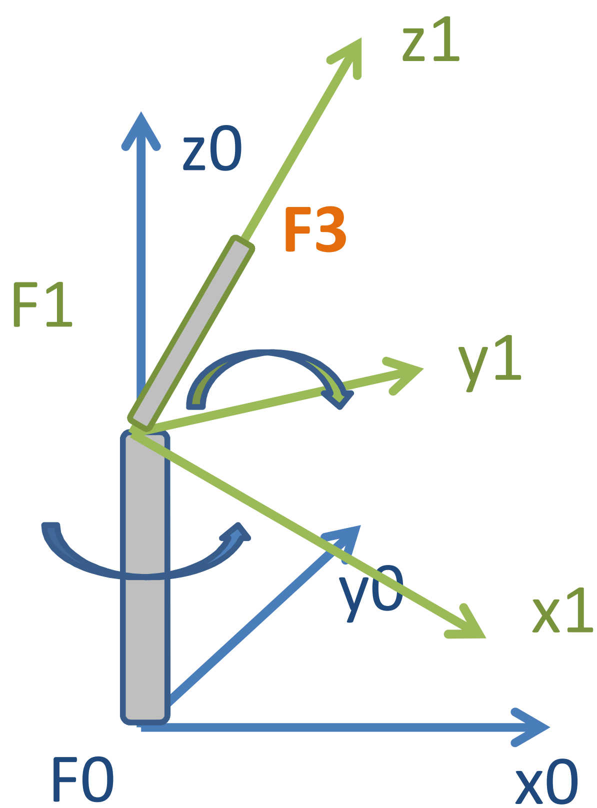


$$ \begin{aligned} {}^I p_g &= {}^I F_3 \cdot {}^I F_2(\theta_2) \cdot {}^I F_1(\theta_1) \cdot I \\ &= {}^I F_1(\theta_1) \cdot {}^{F_1} F_2(\theta_2) \cdot {}^{F_2} F_3 \\ &= {}^I F_1(\theta_1) \cdot {}^{F_1} F_3(\theta_2) \end{aligned} $$

Problem 4: Notes

2[nd] link moves
w.r.t. a reference frame ≠ I
(see also problem 2)

$$
\begin{aligned}
{}^{A}_{C}T_3 &= {}^{A}_{C}T_2 \cdot {}^{A}_{B}T_1 \\
&= ({}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2 \cdot {}^{A}_{B}T_1^{-1}) \cdot {}^{A}_{B}T_1 \\
&= {}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2 \cdot ({}^{A}_{B}T_1^{-1} \cdot {}^{A}_{B}T_1) \\
&= {}^{A}_{B}T_1 \cdot {}^{B}_{C}T_2
\end{aligned}
$$



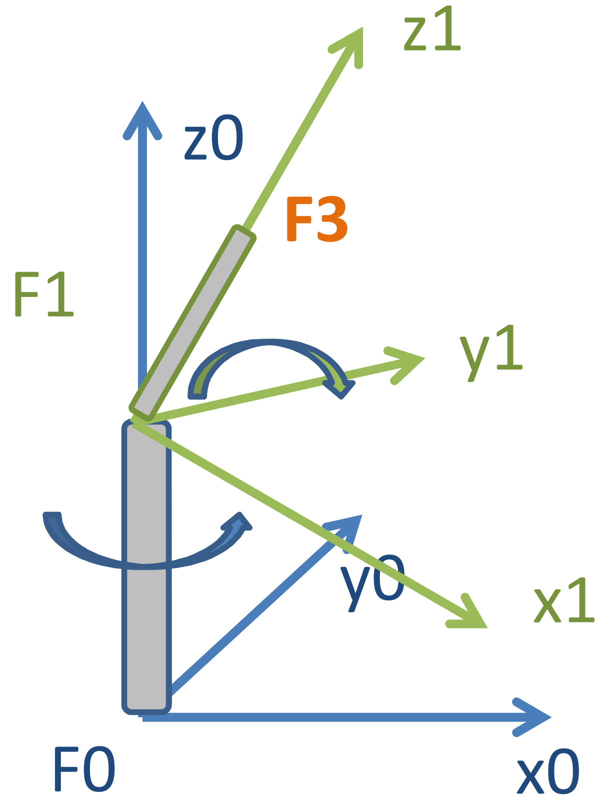
$$
{}^{I}F_3(\theta_2) \cdot F_1(\theta_1) = F_1(\theta_1) \cdot {}^{F_1}F_3(\theta_2)
$$

Problem 4

$$P_g = {}^{I}F_3(\theta_2)\cdot {}^{I}F_1(\theta_1)$$

$$= F_1(\theta_1)\cdot {}^{F_1}F_3(\theta_2)$$

$$= \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta_2 & 0 & s\theta_2 & 3\cdot s\theta_2 \\ 0 & 1 & 0 & 0 \\ -s\theta_2 & 0 & c\theta_2 & 3\cdot c\theta_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Problem 4

$$\theta_1 = 90^o, \theta_2 = 180^o, F_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{F_1}F_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$