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Probability and Random Processes

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30 September 2022

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Distribution function of a random variable (for continuous random variable)

Definition
Let $X : \Omega \to \mathbb{R}$ be a random variable. The *probability distribution function* of X is the function $F_X : \mathbb{R} \to [0,1]$ defined by

$$F_X(t) = \mathbb{P}[X \leq t] = \int_{-\infty}^{t} f_X(x)dx.$$

Know $\quad F_X(t) = \mathbb{P}(X \leq t) \quad \mathbb{P}(X > t) = 1 - F_X(t)$

$$\mathbb{P}(X < t) = \lim_{s \to t-} F_X(s)$$

$$\mathbb{P}(X \leq t) = F_X(t) = \lim_{s \to t+} F_X(s)$$

$$\mathbb{P}(X = t) = F_X(t) - \lim_{s \to t-} F_X(s).$$

Continuous X with density function f_X $\quad \mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx.$

Continuous random variables and the density function

Definition
Let $X : \Omega \to \mathbb{R}$ be a *continuous random variable* with the probability density function f_X. The for all real values of $a \leq b$ we have

$$= \int_{-\infty}^{t} f_X(x) dx.$$

Definition (version 2)

Definition
A random variable $X : \Omega \to \mathbb{R}$ is called *continuous* if there exists a non-negative function $f_X : \mathbb{R} \to \mathbb{R}$, called the *probability density function* of X, such that for all $s \leq t$, we have

$$\mathbb{P}[s \leq X \leq t] = \int_s^t f_X(x)dx.$$

Relation between the density and the distribution functions

Theorem
If F_X is differentiable, then

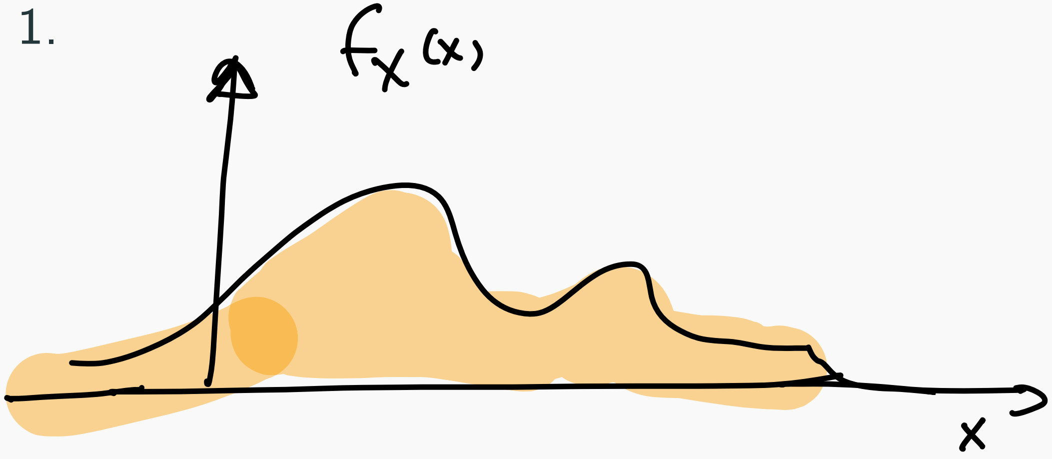
$$f_X(t) = \frac{d}{dt} F_X(t).$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\, dx$$

Fundamental Theorem of Calculus

Properties of the density function and comparison with discrete case

- (Non-negativity: $f_X \geq 0$.
- Total mass one: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.



Uniform random variables

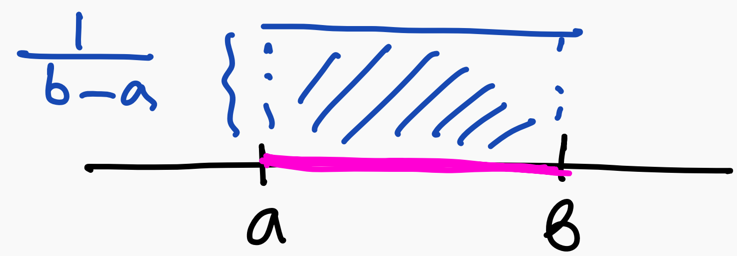
Definition
A random variable X has uniform distribution over the interval $[a, b]$, if its probability density function is given by

$$f_X(t) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}$$

warm-up

X uniform $[-1, 1]$

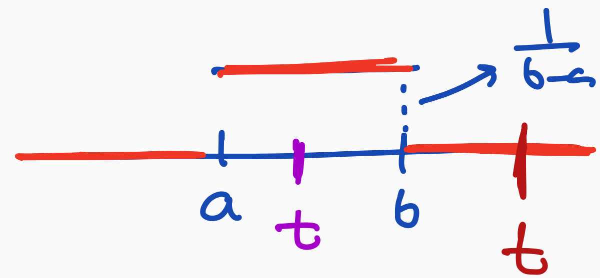
$$f_X(t) = \begin{cases} \frac{1}{2} & -1 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



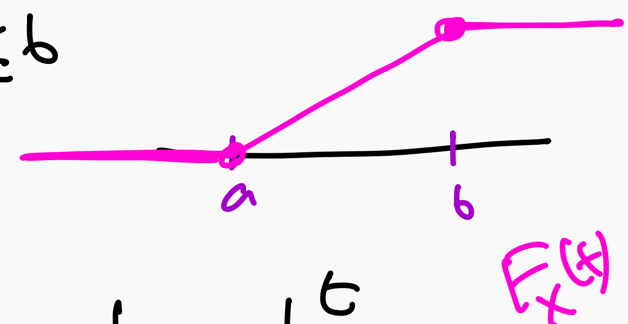
Distribution function of a uniform random variable

The distribution function of a random variable X with uniform distribution over the interval $[a, b]$ is given by

$$F_X(t) = \begin{cases} 0 & \text{if } t \leq a \\ \frac{t-a}{b-a} & \text{if } a \leq t \leq b \\ 1 & t \geq b. \end{cases}$$



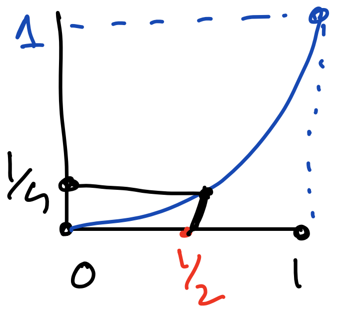
$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \begin{cases} 0 & t \leq a \\ \frac{t-a}{b-a} & a \leq t \leq b \\ 1 & t \geq b \end{cases}$$



$a \leq t \leq b$ $\quad \int_{-\infty}^{t} f_X(x)\,dx = \int_{a}^{t} \frac{1}{b-a}\,dx = \frac{1}{b-a}\, x \Big|_a^t$

$$= \frac{t-a}{b-a}$$

$$\int_{-\infty}^{t} f_X(x)\,dx = \int_a^b \frac{1}{b-a} + \int_b^t 0\,dx = \frac{b-a}{b-a} = 1$$



X random variable
Uniform distribution over $[0,1]$

$$Y = X^2$$

Y is a random variable.

$\mathbb{P}(0 \le Y \le \frac{1}{4}) = \frac{1}{3}$

$$\tfrac{1}{2} = \mathbb{P}(0 \le X \le \tfrac{1}{2}) \Longrightarrow \mathbb{P}(0 \le Y \le \tfrac{1}{4}) = \tfrac{1}{2}$$

$$\tfrac{1}{2} = \mathbb{P}(\tfrac{1}{2} \le X \le 1) \Longrightarrow \mathbb{P}(\tfrac{1}{4} \le Y \le 1) = \tfrac{1}{2}$$

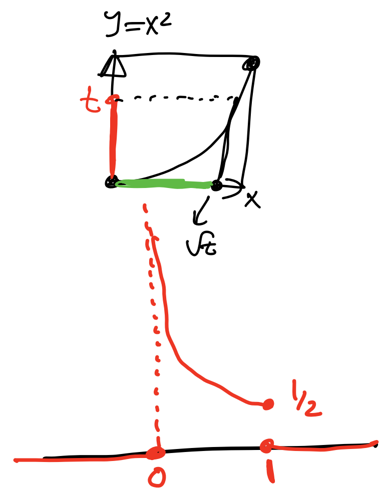
Y is not uniform

X continuous random variable with some density function $\rightsquigarrow Y = f(X)$ Q what is the density function?

Question Find the density function of Y?

$$F_Y(t) = \mathbb{P}(Y \le t) = \begin{cases} 0 & t \le 0 \\ ? & 0 \le t \le 1 \\ 1 & t \ge 1 \end{cases}$$

Y is in $[0,1]$



$$F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(X \le \sqrt{t})$$

$$= F_X(\sqrt{t}) = \sqrt{t}$$

$$f_Y(t) = \frac{d}{dt}\sqrt{t} = \frac{d}{dt}(t^{1/2}) = \frac{1}{2\sqrt{t}}$$

Example
Suppose θ has uniform distribution over the interval $[-\pi/2, \pi/2]$ and $X = \sin\theta$. Compute the probability density function of X.

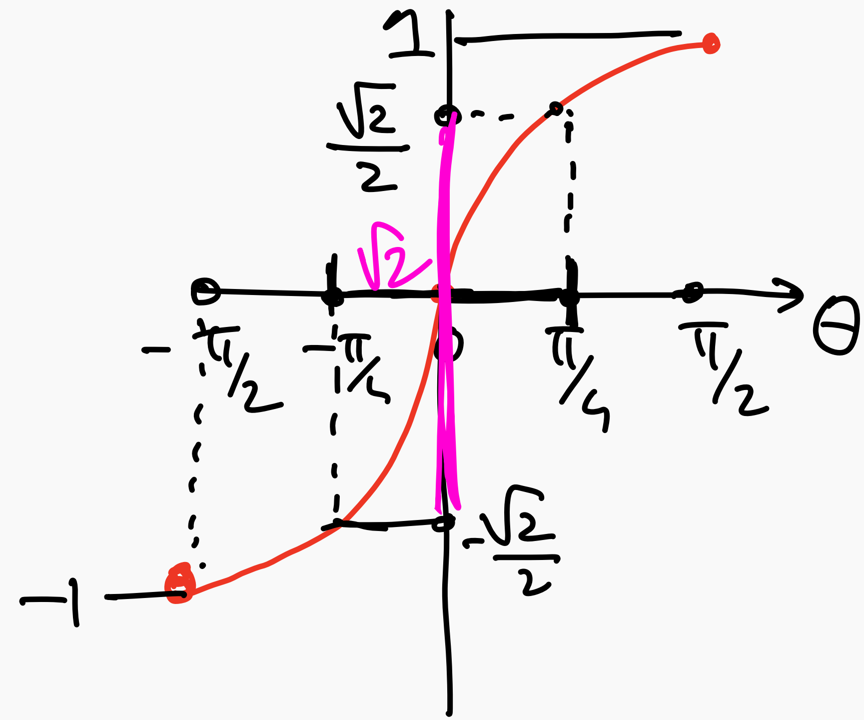
$Y = \sin\theta$

θ uniform $[-\pi/2, \pi/2]$

$-1 \leq Y \leq 1$

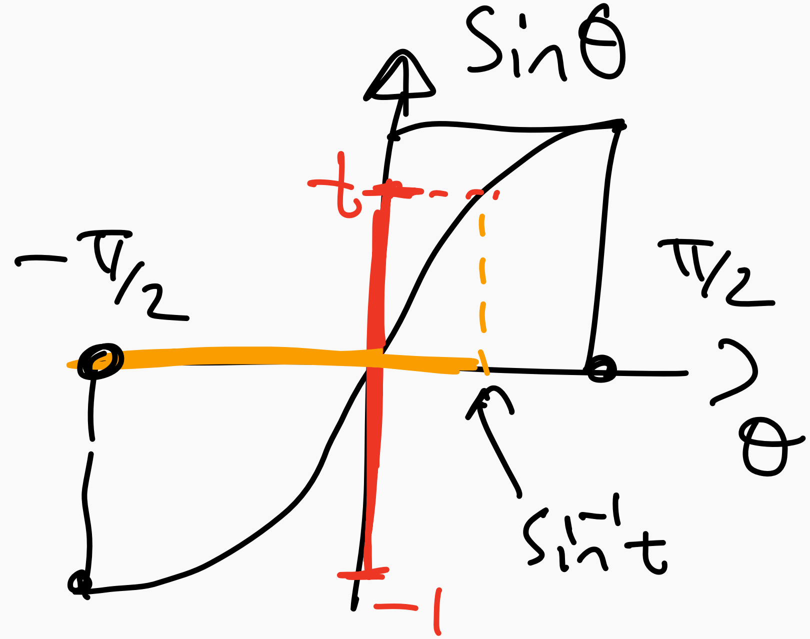
$\mathbb{P}(-\frac{\pi}{4} \leq \theta \leq \pi/4) = \frac{1}{2}$

$\mathbb{P}\underbrace{(-\frac{\sqrt{2}}{2} \leq Y \leq \frac{\sqrt{2}}{2})} = 1/2$



If Y were uniform $\quad \mathbb{P}(-\frac{\sqrt{2}}{2} \leq Y \leq \frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

density function



$$F_Y(t) = \mathbb{P}(Y \le t)$$

$$= \begin{cases} 0 & t \le -1 \\ & -1 \le t \le 1 \\ 1 & t \geq 1 \end{cases}$$

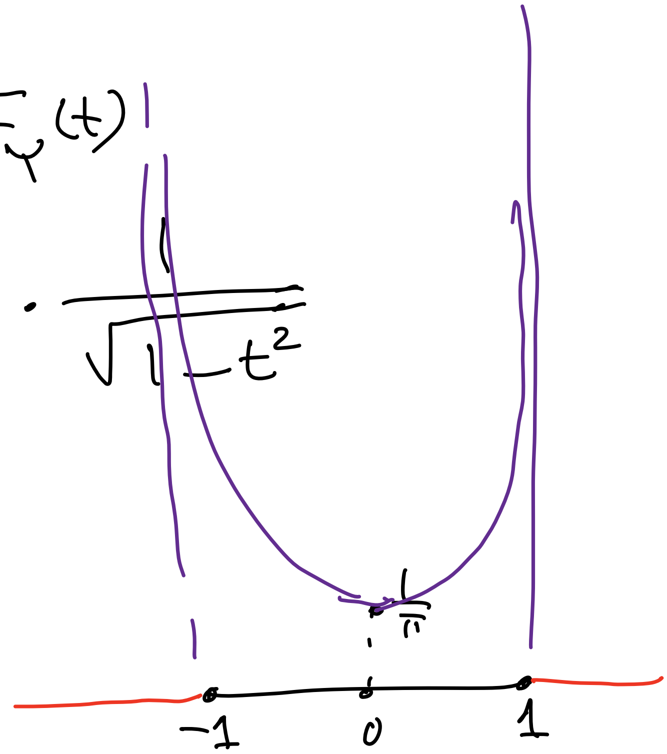
$$\mathbb{P}(Y \le t) = \mathbb{P}(\sin\theta \le t)$$

$$= \mathbb{P}(\theta \le \sin^{-1} t) = \frac{\sin^{-1} t + \pi/2}{\pi} = \frac{1}{2} + \frac{1}{\pi}\sin^{-1} t$$

$$F_Y(t) = \frac{1}{2} + \frac{1}{\pi}\sin^{-1} t$$

$$f_Y(t) = \frac{d}{dt} F_Y(t)$$

$$= \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-t^2}}$$



Example

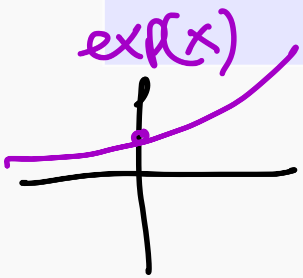
Example

Let X be uniformly distributed in $[-1, 1]$, and $Y = X^2$. Compute the distribution function of Y.

Definition
A continuous random variable X has an exponential distribution with parameter λ if

$\lambda > 0$

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(t)\,dt = \int_0^\infty \lambda e^{-\lambda t}\,dt$$

$$\lambda \cdot \frac{e^{-\lambda t}}{-\lambda}\Big|_0^\infty = -e^{-\lambda t}\Big|_0^\infty = 0-(-1) = 1$$

Distribution function of an exponential random variable

Theorem
The distribution function of a random variable X with geometric distribution with parameter λ is given by

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}. \quad t<0$$

$$F_X(t) = \mathbb{P}(X \leq t) = 0 \quad \text{if } t \leq 0$$

if $t>0$ $$\mathbb{P}(X \leq t) = \int_{-\infty}^{t} f_X(x)\,dx = \int_0^t \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_0^t = -e^{-\lambda t} + 1 = 1 - e^{-\lambda t}$$

$$\mathbb{P}(X > t) = \mathbb{P}(X \geq t) = 1 - F_X(t) = e^{-\lambda t}$$

Conditional Probability

X random variable with exp. distribution with parameter λ:

$$\mathbb{P}(X>t) = e^{-\lambda t} \qquad \mathbb{P}(X>1) = e^{-\lambda}$$

$t_2 > t_1$

$$\mathbb{P}(X>t_2 \mid X>t_1)$$

ex. $$\mathbb{P}(X>5 \mid X>1) = \frac{\mathbb{P}((X>5)\cap(X>1))}{\mathbb{P}(X>1)}$$

$$= \frac{\mathbb{P}(X>5)}{\mathbb{P}(X>1)}$$

$$= \frac{e^{-5\lambda}}{e^{-\lambda}} = e^{-4\lambda}$$

$$\mathbb{P}(X>t_1+t_2 \mid X>t_1) = \frac{\mathbb{P}(X>t_1+t_2 \cap X>t_1)}{\mathbb{P}(X>t_1)}$$

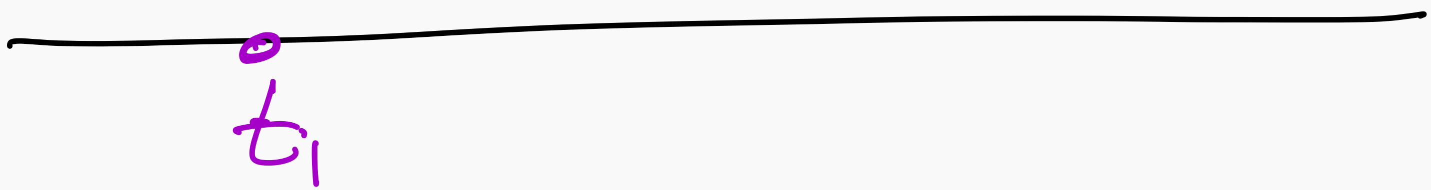
$$= \frac{\mathbb{P}(X>t_1+t_2)}{\mathbb{P}(X>t_1)} = \frac{e^{-\lambda(t_1+t_2)}}{e^{-\lambda t_1}} = e^{-\lambda t_2}$$

$$= \mathbb{P}(X>t_2)$$

Exponential random variables are Memoryless

$$\mathbb{P}(X > t_1 + t_2 \mid X > t_1) = \mathbb{P}(X > t_2).$$

X time wait until train arrives



Gaussian (Normal) random variables

Gaussian (Normal) random variables

Definition
A continuous random variable X is said to have ~~standard~~ Gaussian or ~~standard~~ normal distribution with parameters (μ, σ) if the probability density function of X is given by

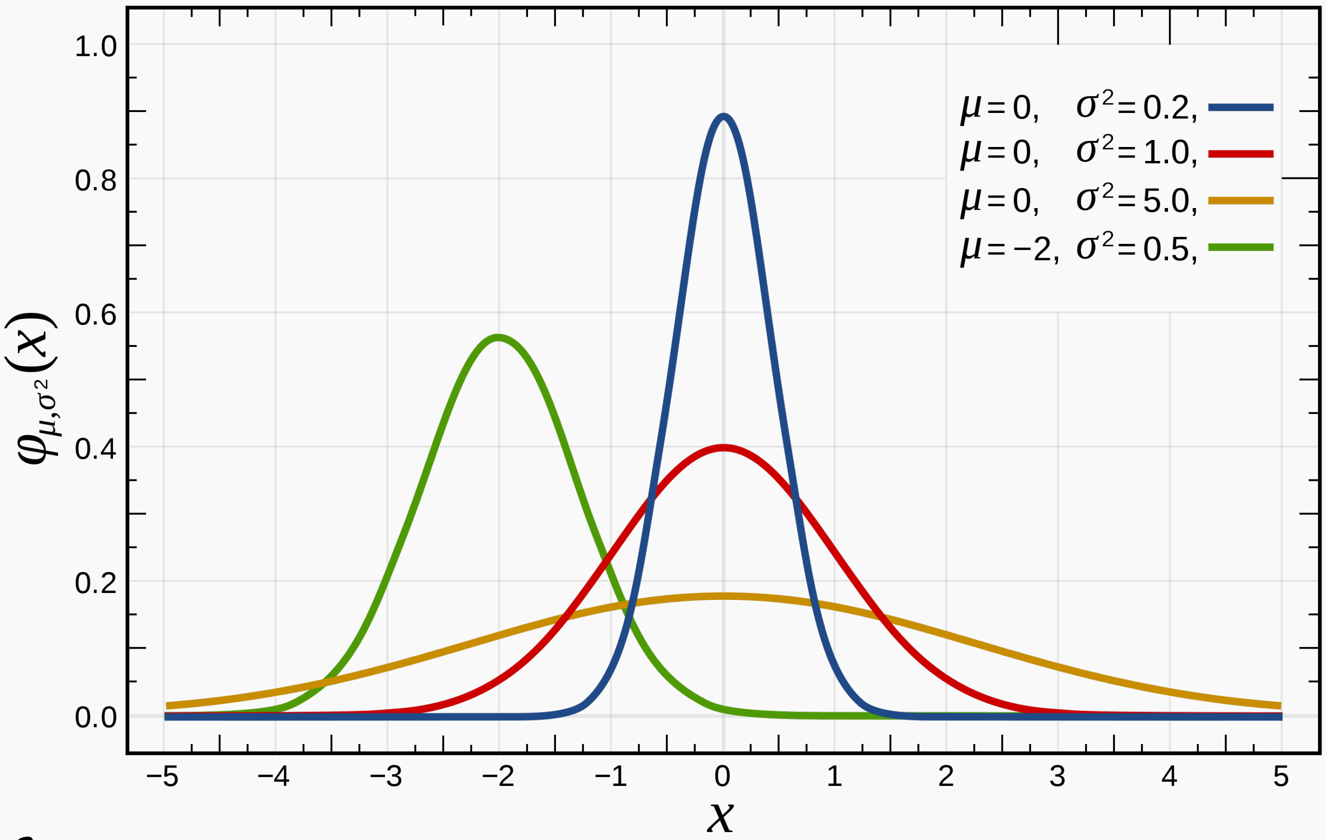
$$f_X(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}.$$

A random variable with normal distribution with parameters $\mu = 0$ and $\sigma = 1$ is called a *standard normal distribution*.

The density function of a standard normal random variable is given by

$$f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Density functions

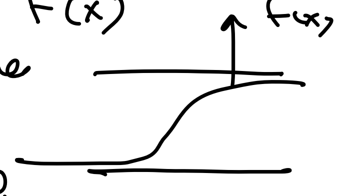


$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1.$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx$$

$$e^{-x^2/2} \xrightarrow[\text{antiderivative}]{} F(x)$$



F(x) cannot be expressed in a closed form using polynomial, trig, exp, log.

F(1)

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}}$$