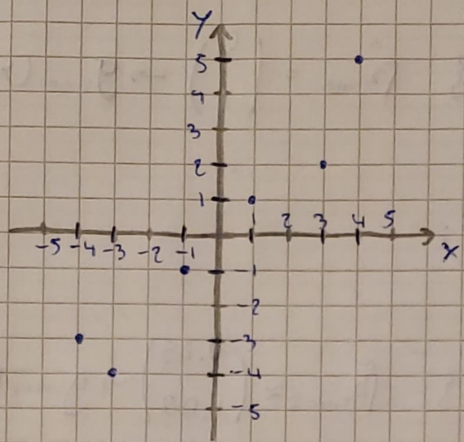
Exercise 2

$T = \{(1,1),(3,2),(4,5),(-4,-3),(-3,-4),(-1,-1)\}$

a) K-fold cross validation with $K=3$ for KNN regression $k=2$



$T_1 = \{(1,1),(3,2)\}$ $T_2 = \{(4,5),(-4,-3)\}$ $T_3 = \{(-3,-4),(-1,-1)\}$

1) $T_{train_1} = T \setminus T_1$ $\qquad T_{val_1} = T_1$

$f_{train_1}(1) = \frac{1}{2}(-1+5) = 2$ $\qquad f_{train_1}(3) = \frac{1}{2}(5-1) = 2$

$$\mathcal{E}_1 = \frac{1}{2}\left[(1-2)^2 + (2-2)^2\right] = \frac{1}{2}$$

2) $T_{train_2} = T \setminus T_2$ $\qquad T_{val_2} = T_2$

$f_{train_2}(4) = \frac{1}{2}(2+1) = \frac{3}{2}$ $\qquad f_{train_2}(-4) = \frac{1}{2}(-4-1) = -\frac{5}{2}$

$$\mathcal{E}_2 = \frac{1}{2}\left[\left(5-\frac{3}{2}\right)^2 + \left(-3+\frac{5}{2}\right)^2\right] = \frac{25}{4}$$

3) $T_{train_3} = T \setminus T_3$ $\qquad T_{val_3} = T_3$

$f_{train_3}(-3) = \frac{1}{2}(-3+1) = -1$ $\qquad f_{train_3}(-1) = \frac{1}{2}(1-3) = -1$

$$\mathcal{E}_3 = \frac{1}{2}\left[(-4+1)^2 + (-1+1)^2\right] = \frac{9}{2}$$

The generalization error of KNN:

$$\mathcal{E} = \frac{1}{K}\sum_{i=1}^{K} \mathcal{E}_i = \frac{1}{3}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) = \frac{1}{3}\left(\frac{1}{2} + \frac{25}{4} + \frac{9}{2}\right) = \frac{15}{4} = \underline{\underline{3.75}}$$

b) Leave-one-out cross validation for the linear model.

1) $T_{train_1} = T \setminus \{(1,1)\}$ $\quad \hat{\beta}_{train_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to g_{train_1}(x) = x$

$g_{train_1}(1) = 1$

$\mathcal{E}_1 = (1-1)^2 = 0$

2) $T_{train_2} = T \setminus \{(3,2)\}$ $\quad \hat{\beta}_{train_2} \approx \begin{pmatrix} 0.25 \\ 1.09 \end{pmatrix} \to g_{train_2}(x) = 0.25 + 1.09x$

$g_{train_2}(3) = 3.52$ $\qquad \mathcal{E}_2 = (2 - 3.52)^2 \approx 2.31$

3) $T_{train_3} = T \setminus \{(4,5)\}$ $\quad \hat{\beta}_{train_3} \approx \begin{pmatrix} -0.32 \\ 0.85 \end{pmatrix} \to g_{train_3}(x) = -0.32 + 0.85x$

$g_{train_3}(4) = 3.08$ $\qquad \mathcal{E}_3 = (5 - 3.08)^2 \approx 3.69$

4) $T_{train_4} = T \setminus \{(-4,-3)\}$ $\quad \hat{\beta}_{train_4} \approx \begin{pmatrix} -0.32 \\ 1.15 \end{pmatrix} \to g_{train_4}(x) = -0.32 + 1.15x$

$g_{train_4}(-4) = -4.92$ $\qquad \mathcal{E}_4 = (-3 + 4.92)^2 \approx 3.69$

5) $T_{train_5} = T \setminus \{(-3,-4)\}$ $\quad \hat{\beta}_{train_5} \approx \begin{pmatrix} 0.25 \\ 0.91 \end{pmatrix} \to g_{train_5}(x) = 0.25 + 0.91x$

$g_{train_5}(-3) = -2.48$ $\qquad \mathcal{E}_5 = (-4 + 2.48)^2 \approx 2.31$

6) $T_{train_6} = T \setminus \{(-1,-1)\}$ $\quad \hat{\beta}_{train_6} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to g_{train_6}(x) = x$

$g_{train_6}(-1) = -1$ $\qquad \mathcal{E}_6 = (-1+1)^2 = 0$

The generalization error of the linear model

$$\mathcal{E} = \frac{1}{N}\sum_{i=1}^{N} \mathcal{E}_i = \frac{1}{6}(0 + 2.31 + 3.69 + 3.69 + 2.31 + 0) = \underline{\underline{2}}$$