Probability and Random Processes

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Announcements

1. Problem Set 3 is due tomorrow at 23:59.
2. The assessment phase for PS3 will start on Sunday noon.
3. Problem Set 4 will be posted today

Agenda

1. Review: notion of expected value
2. Operations on random variables
3. linearity and its application
4. Variance
5. The expected value of continuous random variables
6. Examples
7. Variance of continuous random variables

Last class

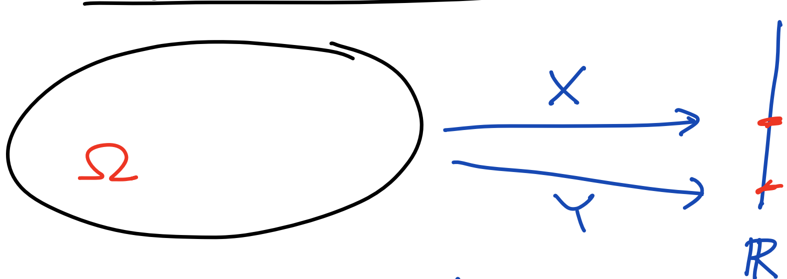
X discrete RV

X	x_1	x_2	x_3	$\cdots$
$\mathbb{P}(X=x)$	p_1	p_2	p_3	$\cdots$

$$\mathbb{E}[X] = \sum_i p_i x_i$$

	Expected Value
X Bernoulli (p) $\rightsquigarrow$	p
binomial (n,p) $\longrightarrow$	np
Poisson (λ) $\longrightarrow$	λ
Geometric (p) $\longrightarrow$	$1/p$

Algebraic operations with random variables



Example: Throw a die 5 times

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) \mid 1 \le x_i \le 6,\ x_i \text{ integer}\} \quad 6^5 \text{ elements.}$$

$X = \#\{1 \le i \le 5 \mid x_i = 6\}$

$Y = \max\{x_1, x_2, x_3, x_5, x_6\}$

$X : \Omega \longrightarrow \mathbb{R}$ function RV

$Y : \Omega \longrightarrow \mathbb{R}$ function RV

$$(X+Y)(\omega) = X(\omega) + Y(\omega)$$

$$(XY)(\omega) = X(\omega) \cdot Y(\omega)$$

Example: Through a die 5 times

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) \mid \underset{x_i \text{ integer}}{1 \le x_i \le 6}\} \quad 6^5 \text{ elements.}$$

$X = \#\{1 \le i \le 5 \mid x_i = 6\}$

$Y = \max\{x_1, x_2, x_3, x_4, x_5\}$

Possible values of X : $0, 1, 2, 3, 4, 5$

X	0	1	2	3	4	5
P	$\left(\frac{5}{6}\right)^5$					

Possib values of Y : $1, 2, 3, 4, 5, 6$

Y	1	2	3	4	5	6
P	$\left(\frac{1}{6}\right)^5$					

$$\mathbb{P}(X=0) = \mathbb{P}(\#\{1 \le i \le 5 \mid x_i = 6\} = 0) = \mathbb{P}(x_1 \ne 6, \dots, x_5 \ne 6)$$
$$= \left(\frac{5}{6}\right)^5$$

$$\mathbb{P}(Y=1) = \mathbb{P}(\max(x_1, x_2, x_3, x_4, x_5) = 1) = \left(\frac{1}{6}\right)^5$$

Random variable: $\underline{\underline{X+Y}}$

$Z = X+Y$ Possible values for Z

Z	1	2	3	$\dots$	11
$\mathbb{P}(Z=z)$	$\left(\frac{1}{6}\right)^5$				$\left(\frac{1}{6}\right)^5$

$$\mathbb{P}(Z=1) = \mathbb{P}(X+Y=1)$$

$$= \mathbb{P}(X=0, Y=1)$$

$$x_1 \neq 6, x_2 \neq 6, \ldots, x_5 \neq 6$$

AND

$$\max(x_1, x_2, \ldots, x_5) = 1$$

$$\mathbb{P}(Z=11) = \mathbb{P}(X+Y=11)$$

$$= \mathbb{P}(X=5, Y=6)$$

$$= \mathbb{P}\begin{pmatrix} x_1 = x_2 = x_3 = x_4 = x_5 = 6 \\ \max(x_1, x_2, x_3, x_4, x_5) = 6 \end{pmatrix} = \left(\frac{1}{6}\right)^5$$

$$\mathbb{P}(Z=6) = \mathbb{P}(X+Y=6)$$

$$\mathbb{P}\begin{pmatrix} X=0, Y=6 \\ X=1, Y=5 \\ X=2, Y=4 \\ X=3, Y=3 \\ X=4, Y=2 \\ X=5, Y=1 \end{pmatrix}$$

Properties of the expected value

Theorem
Let X, Y be random variables and c a constant. We have

- *Linearity:* $\mathbb{E}[cX + Y] = c\mathbb{E}[X] + \mathbb{E}[Y]$.
- *Comparison: if $X \leq Y$ ~~with probability 1~~, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$,*

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n].$$

important point

In general $\mathbb{E}[X.Y] \neq \mathbb{E}[X]\,\mathbb{E}[Y]$. not always true.

Suppose n letters are placed in n envelopes. Let X denote the number of letters placed in the right envelope. Find $\mathbb{E}[X]$.

Ex. $n=3$

envelops 1, 2, 3

$123 \xrightarrow{\times} 3$

$132 \longrightarrow 1$

$231 \longrightarrow 0$

$213 \longrightarrow 1$

$312 \longrightarrow 0$

$321 \longrightarrow 1$

X	0	1	3
P	1/3	1/2	1/6

$$\mathbb{E}[X] = 0\cdot\frac{1}{3} + 1\cdot\frac{1}{2} + 3\cdot\frac{1}{6} = \frac{1}{2} + \frac{1}{2} = 1$$

general case: $n!$ scenarios.

values of X: $0, 1, \dots, n$

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \dots \pm \frac{1}{n!} \sim \mathbb{P}(X=0)$ $\quad \mathbb{P}(X=1)$ $\quad \frac{1}{n!}$

Trick

$$\underset{\substack{\uparrow\\\text{"complicated"}\\\text{RV}}}{\mathbb{E}[X]} \qquad X = X_1 + X_2 + \cdots + X_n$$

"Simple" RV

linearity $\quad \mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

In the above example

$X = \#$ letters placed into the right envelope.

goal: $X = X_1 + X_2 + \cdots + X_n$. each X_i is a Bernoulli.

$$X_1 = \begin{cases} 1 & \text{when letter \#1 is placed into envelope \#1} \\ 0 & \text{otherwise.} \end{cases}$$

$$X_i = \begin{cases} 1 & \text{when letter } i \text{ goes into envelope \#} i. \\ 0 & \text{otherwise.} \end{cases}$$

- $X = X_1 + X_2 + \cdots + X_n$
- each X_i is a Bernoulli random variable with $p = \frac{1}{n}$.

$$\mathbb{E}[X_i] = \frac{1}{n} \cdot 1 + \left(1 - \frac{1}{n}\right) \cdot 0 = \frac{1}{n}.$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$$

$$= \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n} = 1.$$

check $\underline{X = X_1 + X_2 + \cdots + X_n}$

1 or 0 1 or 0

This expression is a sum of 1s and 0s

\# 1s = \# letters that have gone into the right envelope. $= X$

Reason

each X_i is a Bernoulli random variable with $p = \frac{1}{n}$.

$$\mathbb{P}[X_i = 1] = \mathbb{P}\left[\begin{matrix}\text{letter } i \text{ goes to} \\ \text{envelope } i\end{matrix}\right] = \frac{1}{n}$$

— — — . . . —

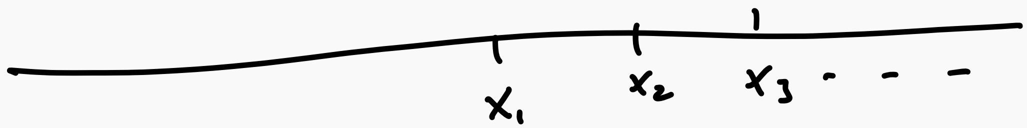
i th envelope.

Elevator stops

There are 5 people in an elevators. An elevator goes up a building with 10 floors and stops at each floor where at least one person wants to get off. If X denote the number of stops, find $\mathbb{E}[X]$.

Measuring the spread of a random variable

expected value



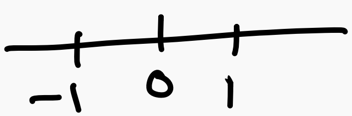
X takes value 1 or −1 each with prob ½

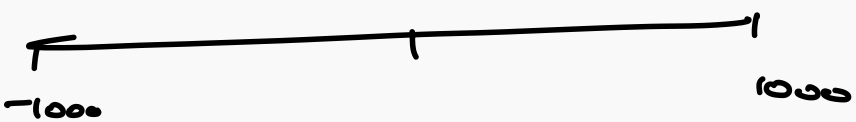
X	−1	1
	½	½

$$E[X] = \frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot 1 = 0.$$

Y	−1000	1000
	½	½

$$E[Y] = \frac{1}{2}(-1000) + \frac{1}{2}\cdot 1000 = 0.$$





Z	−1	1	10^2
	½	0.499	↑ 0.001

max possible value of X − minimum possible value of X

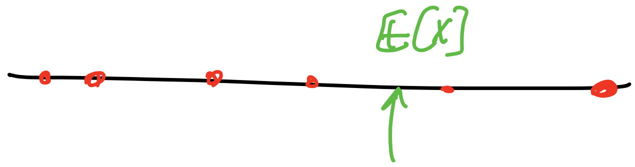
$$100 - (-1) = \boxed{101}$$

(not ideal)

One measure

X random variable

$\mathbb{E}[X]$



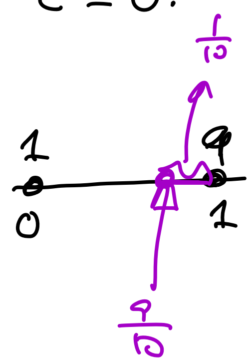
$X' = X - \underbrace{\mathbb{E}[X]}_{c}$. Centered version of X

$$\mathbb{E}[X'] = \mathbb{E}[X-c] = \underbrace{\mathbb{E}[X]}_{c} - \mathbb{E}[c] = c - c = 0.$$

$|X'| = |X - \mathbb{E}[X]|$ $\qquad$ $\mathbb{E}[|X - \mathbb{E}[X]|]$

	1	−1
	1/2	1/2

	+1000	−1000
	1/2	1/2



$\mathbb{E}[X] = 0$ both centered

look at $|X| = X'$

X'	1
$P(X'=)$	1

Y'	1000
	1

	c
	1

$\mathbb{E}[X'] = 1 \cdot 1 = 1$ $\qquad$ $\mathbb{E}[Y'] = 1 \cdot 1000 = 1000.$

downside $\quad$ it's not easy to work with

substitute

$$Y = (X - \mathbb{E}[X])^2 \geq 0.$$

$$\mathbb{E}[Y] = \mathbb{E}\left((X - \mathbb{E}[X])^2\right)$$

Variance of a random variable

Variance of a random variable

Definition
The variance of a random variable X is defined by

$$\mathrm{Var}\left[X\right] = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right].$$

Example

Let X be the outcome of a fair die. Find $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

X	1	2	3	4	5	6
	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\mathbb{E}[X] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \cdots + \frac{1}{6}\cdot 6 = 3.5 = \frac{7}{2}$$

$\mathrm{Var}[X]$

$Y = (X-\mathbb{E}[X])^2$	$\left(\frac{-5}{2}\right)^2$	$\left(\frac{-3}{2}\right)^2$	$\left(\frac{-1}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{3}{2}\right)^2$	$\left(\frac{5}{2}\right)^2$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

2. —

$$\mathrm{Var}[X] = \frac{1}{6}\left[\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}\right] = \frac{70}{24} \approx$$