Exercise 2 - Linear regression by least squares

$$T = \left\{ \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix}, 3 \right), \left( \begin{pmatrix} 2 \\ 3 \end{pmatrix}, 1 \right), \left( \begin{pmatrix} 3 \\ 3 \end{pmatrix}, 6 \right), \left( \begin{pmatrix} 4 \\ 3 \end{pmatrix}, 5 \right) \right\}$$

a) The least squares estimator $\hat{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$ (that minimizes the sum of the squared residuals) is given by:

$$\hat{\beta} = (X^T X)^{-1} X^T y \iff (X^T X)\hat{\beta} = X^T y$$

where

$$X = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 3 \\ 1 \\ 6 \\ 5 \end{pmatrix}$$

compute $X^T X$ and $X^T y$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 11 & 10 \\ 11 & 33 & 29 \\ 10 & 29 & 28 \end{pmatrix}$$

$$X^T y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 15 \\ 46 \\ 39 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 11 & 10 \\ 11 & 33 & 29 \\ 10 & 29 & 28 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 46 \\ 39 \end{pmatrix}$$

Solve the system

Using Gaussian Elimination

$$\left(\begin{array}{ccc|c} 4 & 11 & 10 & 15 \\ 11 & 33 & 29 & 46 \\ 10 & 29 & 28 & 39 \end{array}\right) \xrightarrow[R_3 - \frac{5}{2}R_1]{R_2 - \frac{11}{4}R_1} \left(\begin{array}{ccc|c} 4 & 11 & 10 & 15 \\ 0 & 11/4 & 3/2 & 19/4 \\ 0 & 3/2 & 3 & 3/2 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{6}{11}R_2} \left(\begin{array}{ccc|c} 4 & 11 & 10 & 15 \\ 0 & 11/4 & 3/2 & 19/4 \\ 0 & 0 & 24/11 & -24/22 \end{array}\right) \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

From (3)

$$\frac{24}{11} \cdot \beta_2 = -\frac{24}{22} \Rightarrow \boxed{\beta_2 = -\frac{1}{2}}$$

From (2)

$$\frac{11}{4}\beta_1 + \frac{3}{2}\left(-\frac{1}{2}\right) = \frac{19}{4}$$

$$\frac{11}{4}\beta_1 - \frac{3}{4} = \frac{19}{4}$$

$$11\beta_1 - 3 = 19$$

$$\boxed{\beta_1 = \frac{22}{11} = 2}$$

From (1)

$$4\beta_0 + 11 \cdot 2 + 10 \cdot \left(-\frac{1}{2}\right) = 15$$

$$4\beta_0 = 15 - 22 + 5$$

$$\boxed{\beta_0 = -\frac{2}{4} = -\frac{1}{2}}$$

The predictor function $f_{\hat{\beta}}(\vec{x})$ is given by:

$$f_{\hat{\beta}}(\vec{x}) = -\frac{1}{2} + 2x_1 - \frac{1}{2}x_2$$

b) Predict the output for $\vec{x} = \begin{pmatrix} 2.5 \\ 5 \end{pmatrix}$

$$f_{\hat{\beta}}\left(\begin{pmatrix} 2.5 \\ 5 \end{pmatrix}\right) = -\frac{1}{2} + 2 \cdot \frac{5}{2} - \frac{1}{2} \cdot 5 = -3 + 5 = 2$$