Basic notions

Definition 0.1 We consider a chance experiment. The set of all outcomes that may appear in a realization of this experiment is called **sample space**, denoted by Ω. We call $\omega \in \Omega$ **sample points** and the associated sets $\{\omega\}$ **elementary events** for this experiment. A given subset $A \subseteq \Omega$ is called **event**.

Definition 0.2 Let Ω be a sample space and $A, B \subseteq \Omega$ events.

- A and B are **equivalent**, if $A = B$.
- The **union** of A and B is $A \cup B$.
- The **intersection** of A and B is $A \cap B$.
- A and B are **disjoint** or **mutually exclusive** if $A \cap B = \emptyset$.
- The complement of A is $\bar{A} := \Omega \setminus A$.

Definition 0.3 Let Ω be a sample space and $P : \mathcal{P}(\Omega) \to \mathbb{R}$ a real-valued function on events on Ω. P is called **probability on** Ω, if the following axioms hold:

1. $P(A) \geq 0$ for all events A.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ is a pairwise disjoint sequence of events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)\,.$$

The real number $P(A)$ is the **probability of the event** A.

Toss of two coins

$$\Omega_2 = \{HH, HT, TH, TT\}$$

Event "at least one head":

$$A = \{HH, HT, TH\}$$

$$P(\{HH\}) = P(\{HT\}) = \_ = \frac{1}{4}$$

Definition 0.4 Let Ω be a sample space, $B \subseteq \Omega$ be an event with $P(B) > 0$ and $A \subseteq \Omega$ be an arbitrary event. We define the **conditional probability of** A **given** B as

$$P(A|B) := \frac{P(A \cap B)}{P(B)} = \frac{P(A, B)}{P(B)}$$

Definition 0.5 Let $A, B \subseteq \Omega$ be events. They are called **independent** if

$$P(A \cap B) = P(A)P(B)\,.$$

Otherwise, they are called **dependent**.

Discrete random variables

Definition 0.6 Let Ω be a sample space. We call a function

$$X : \Omega \to \mathbb{R}$$

random variable, if for any interval $I \subset \mathbb{R}$, the set $\{\omega \in \Omega | X(\omega) \in I\}$ is an event on Ω. By $P(X \in I)$, we denote the probability of X to take values on I.

Definition 0.7 Let $X : \Omega \to \mathbb{R}$ be a random variable. The function $F : \mathbb{R} \to \mathbb{R}$ with

$$F(t) = P(X \leq t) = P(\omega \in \Omega | X(\omega) \leq t)$$

is called **(cumulative) distribution function** (CDF) of X.

Definition 0.8 Let Ω be a sample space and X a random variable on Ω. We call X a **discrete random variable** if X can take only a finite or at most infinite but countable number of values. In this case, the CDF F of X is called **discrete distribution**.

Definition 0.9 Let $X : \Omega \to \mathbb{R}$ be a discrete random variable. We call the function $p : \mathbb{R} \to \mathbb{R}$ with

$$p(x) := P(X = x)$$

probability (mass) function (PMF) of X. It describes the probability of the event $X = x$, i.e. that X takes the value x.

Definition 0.10 Let X be a discrete random variable with PMF p and range R_X. The **expected value / expectation / mean** of X is denoted by $E(X)$ and is given by

$$\mathrm{E}(X) = \sum_{x \in R_X} x\, p(x) = \sum_{x \in R_X} x\, P(X = x),$$

under the assumption that this series converges absolutely. The expectation *does not exist*, if this assumption is not fulfilled.

Throw of dice

$$X : \Omega \to \mathbb{R}$$

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

$$p(x) = \begin{cases} \frac{1}{6} & x \in R_X \\ 0 & \text{else}. \end{cases}$$

$$\mathrm{E}(X) = \sum_{x \in R_X} x\, p(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3.5$$

1 Discrete random variables

We condsider the chance experiment of rolling two fair dices.

1. Give the sample space Ω.
2. Formally give the event *the sum of the thrown values is four*.
3. Introduce the random variable X of the sum of the thrown values.
4. Give the PMF p of X.
5. Compute the mean of X.

1) $\Omega = \{1, \dots, 6\} \times \{1, \dots, 6\} = \{ (1,1), (1,2), \dots (2,1), (2,2), \dots \}$

2) $A = \{(1,3), (2,2), (3,1)\}$

3) $X : \Omega \to \{2, 3, \dots, 12\}$ $\quad X(\omega) = X((a,b)) := a+b$
$\forall a, b \in \{1, \dots, 6\}$

4) $p(x)$, $R_x = \{2, \dots, 12\}$

$p(2) = P(X=2) = P(\{(1,1)\}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$p(3) = P(X=3) = P(\{(1,2),(2,1)\}) = 2 \cdot \frac{1}{36} = \frac{1}{18}$

$\vdots$

5) $E(X) = \sum_{x \in R_X} x \, p(x) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + \dots + 12 \cdot \frac{1}{36} = 7$

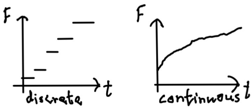
alternative: $X_1 : \Omega_1 \to \{1, \dots, 6\}$
$X_2 : \Omega_1 \to \{1, \dots, 6\}$

$E(X_1 + X_2) = E(X_1) + E(X_2) = 3.5 + 3.5 = 7$

Continuous random variables

Definition 0.12 Let Ω be a sample space and X a random variable on Ω. We call its CDF $F(t) = P(X \leq t)$ a **continuous distribution**, if F is continuous everywhere. A random variable with a continuous distribution is called **continuous random variable**. It has an uncountable range.

$$X : \Omega \to [0, 4]$$



Definition 0.13 Let X be a continuous RV. Let us assume that there exists a function $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that it holds

$$P(X \in A) = \int_A \rho(x) dx$$

for all subsets $A \subseteq \mathbb{R}$ that can be written as the union of a finite / infinite number of intervals. If ρ and the above integral exist, we call X **absolutely continuous** and ρ the **(probability) density function** (PDF) or **density** of X.

$$\rho(x) = \frac{3}{32}(4x - x^2)$$

Definition 0.14 Let X be a continuous RV with density ρ. We define the **expected value / expectation / mean** of X by

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \, \rho(x) dx$$

under the assumption that the integral exists and converges absolutely, hence

$$\int_{-\infty}^{\infty} |x| \, \rho(x) dx < \infty \, .$$

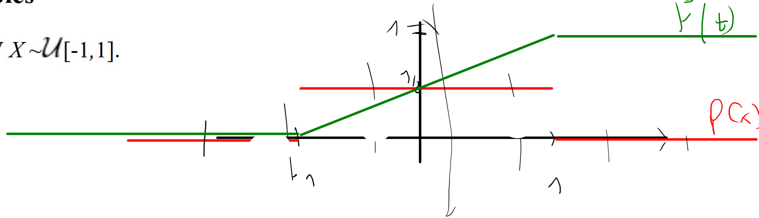
If the assumption is not fulfilled, the expectation of X does not exist.

$$\mathrm{E}(X) = \int_0^4 x(4x - x^2) dx = \left[4 \cdot \frac{1}{3} x^3 - \frac{1}{4} x^4\right]_0^4 = \ldots$$

2 Continuous random variables

Let X be a uniformly distributed RV $X \sim \mathcal{U}[-1,1]$.

1. Give its density.
2. Give its CDF and plot it.
3. Compute its mean.



1) $p(x) = \begin{cases} \frac{1}{1-(1)} = \frac{1}{2} & \text{if } x \in [-1,1] \\ 0 & \text{else} \end{cases}$

2) $F(t) = P(X \leq t) = \int_{-\infty}^{t} p(x)\,dx$

$= \begin{cases} 0 & t < -1 \\ \frac{1}{2}x + \frac{1}{2} & -1 \leq t \leq 1 \\ 1 & t > 1 \end{cases}$

$t \in [-1,1]$

$\int_{-\infty}^{t} \frac{1}{2}\,dx = \left[\frac{1}{2}x\right]_{-1}^{t}$

$= \frac{1}{2}t - \left(-\frac{1}{2}\right)$

$= \left(\frac{t+1}{2}\right)$

3) $E[X] = \int_{-\infty}^{\infty} x\,p(x)\,dx = \int_{-1}^{1} x \frac{1}{2}\,dx = \left[\frac{1}{2}\frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$