Probability and Random Processes

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Agenda

1. . The notion of expected value
2. Simple computations

The notion of average

The notion of average

The average of a set of numbers $x_1, \ldots, x_n$ is

$$\frac{x_1 + \cdots + x_n}{n}$$

$$x_1 \longrightarrow 1 = \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ terms}}$$

The average of a set of numbers $x_1, \ldots, x_n$ is

$$\frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}x_1 + \frac{1}{n}x_2 + \cdots + \frac{1}{n}x_n$$

This is useful when numbers are equally important, and hence must receive the same weight which is $1/n$.

GPA

- averaging with different weights

2.5	1
5	2
7.5	3

$$\frac{1+2+3}{3} = 2$$

$$\frac{2.5 \times 1 + 5 \cdot 2 + 3 \cdot 7.5}{2.5 + 5 + 7.5}$$

$$= \frac{2.5}{2.5+5+7.5} \cdot 1 + \frac{5}{2.5+5+7.5} \cdot 2 + \frac{7.5}{2.5+5+7.5} \cdot 3$$

$\frac{2.5}{15}$ $1/6$ $\frac{5}{15}$ $1/3$ $1/2$

The notion of average

The average of a set of numbers $x_1, \dots, x_n$ is

$$\frac{x_1 + \cdots + x_n}{n}$$

This is useful when numbers are equally important, and hence must receive the same weight which is $1/n$.

What is the points are not equally important?

$$x_1, x_2, \dots, x_n$$

<u>weights</u> $w_1 \quad w_2 \quad \dots \quad w_n$

$w_i \geq 0 \quad 1 \leq i \leq n$

$w_1 + w_2 + \dots + w_n = 1$

$w_1 x_1 + w_2 x_2 + \dots + w_n x_n$ is a weighted average of $x_1, \dots, x_n$ with weights $w_1, \dots, w_n$.

Expectation of a random variable

Expected value or Expectation random value.

X discrete random variable, taking values

$x_1, x_2, \dots, x_n$ with prob $p_1, \dots, p_n$

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one-number summary of the values of X ?

X	x_1	x_2	$\dots$	x_n
$P(X=x_i)$	p_1	p_2	$\dots$	p_n

Bernoulli

X	0	1
	0.01	0.99

naive suggestion $\dfrac{x_1 + x_2 + \dots + x_n}{n}$

better suggestion $p_1 x_1 + p_2 x_2 + \dots + p_n x_n$

naive suggestion : $\dfrac{0+1}{2} = \frac{1}{2}$

better suggestion : $0.01 \cdot 0 + 0.99 \times 1 = 0.99$

Expectation of a random variable

Definition
For a discrete random variable X with values $x_1, x_2, \ldots, x_n$ obtained with probabilities $p_1, \ldots, p_n$. Then the *expected value* of X is defined by

$$\mathbb{E}[X] = p_1 x_1 + \cdots + p_n x_n.$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i \cdot \mathbb{P}(X = x_i) = \sum_{i=1}^{n} x_i \cdot p_i$$

X	x_1	x_2	...	x_n
$\mathbb{P}(X=x_i)$	p_1	p_2	...	p_n

$$\mathbb{E}[X] = p_1 x_1 + p_2 x_2 \quad + p_n x_n$$

Example

A fair die is rolled. Suppose that X is the number shown. Compute $\mathbb{E}[X]$.

Clearly X takes value $1 \leq n \leq 6$, each with probability $1/6$. From here we have:

$$\mathbb{E}[X] = \frac{1}{6}\sum_{n=1}^{6} n = \frac{21}{6} = 3.5$$

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}$$

$$= \frac{1}{6}\left[1 + 2 + 3 + \cdots + 6\right]$$

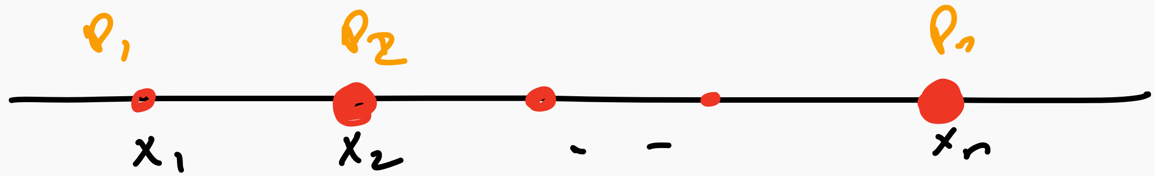
$$= \frac{21}{6} = 3.5$$

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

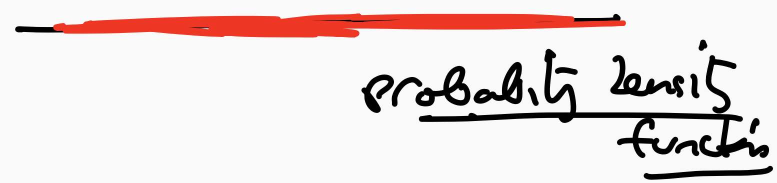
real-valued random variable

$\updownarrow$

distribution of mass on the real axis



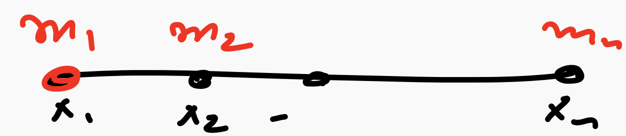
$x_1 < x_2 < \cdots < x_n$			
p_1	p_2	$\cdots$	p_n



Does the notion of expected value have a physical interpretation?

Center of gravity

Center of mass



$$\text{Center of mass} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$

$$M = m_1 + m_2 + \cdots + m_n$$

$$= \frac{m_1}{M} x_1 + \frac{m_2}{M} x_2 + \cdots + \frac{m_n}{M} x_n$$

Physical interpretation of the expected value

The center of mass of a finite set of points:

Physical interpretation of the expected value

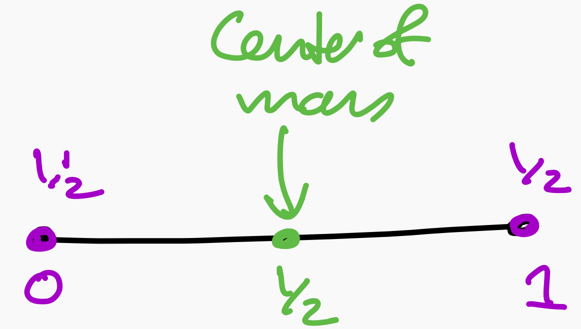
The center of mass of a finite set of points:

Put weights $m_1, \ldots, m_n$ at locations $x_1, \ldots, x_n$. Then the center of mass is at

$$\frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n}.$$

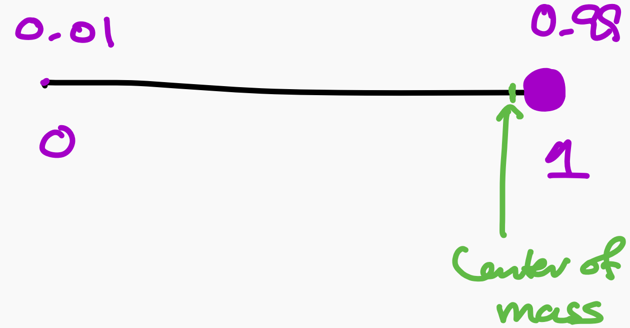
0	1
$1/2$	$1/2$

$$1/2 = 1/2 \cdot 0 + 1/2 \cdot 1$$



0	1
0.01	0.99

$$0 \cdot 0.01 + 0.99 \times 1 = 0.99$$



Expected value of Bernoulli random variables

X Bernoulli
with parameter p

x	0	1
	$1-p$	p

$$\mathbb{E}[X] = \sum_{i=1}^{n} p_i x_i = (1-p)\times 0 + p \times 1$$

$$= p$$

Expected value of Bernoulli random variables

Example
Let X be a Bernoulli random variable with parameter p. Then

$$\mathbb{E}[X] = p \cdot 1 + (1 - p) \cdot 0 = p.$$

X binomial RV with parameters (n, p)

X counts the number of successes in n Bernoulli trials

X takes values $0, 1, \dots, n$. $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

Compute $\mathbb{E}[X]$.

X	0	1	2	·	k	–	–	n
	$(1-p)^n$	–	–	–	$\binom{n}{k} p^k (1-p)^{n-k}$	·	–	p^n

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X=k)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!\,(n-k)!} \cdot p^k (1-p)^{n-k}$$

$1 \times 2 \times \dots \times k$

$(n-1)!$

$1 \times 2 \times \cdots \times (n-1) \times n$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$n-k+k-1 = n-1$

$(n-1)-(k-1) = n-k$

$$= \sum_{k=1}^{n} \frac{(n-1)! \times n}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$\boxed{(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}}$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j+1} \cdot (1-p)^{(n-1)-j}$$

$k-1 = j$

$\underline{k = j+1}$

$n-k = n-j-1$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}$$

$$= np \cdot (\underbrace{p + 1-p}_{1})^{n-1} = np.$$

X binomial random variable with parameters (n, p)

<u>Then</u> <u>$E[X] = np.$</u>

<u>Ex.</u> Coin

H $1/4$ success 100 times

T $3/4$

$p = 1/4$ $n = 100$

Example
Let X be a Binomial random variable with the parameters (n, p). Find $\mathbb{E}[X]$ in terms of n and p.

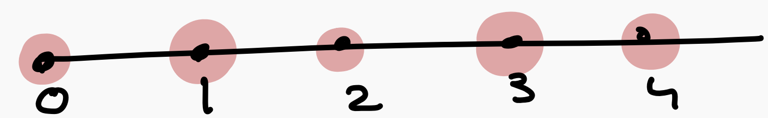
Formula for specific random variables

Theorem

Suppose X is a random variable which only takes values $0, 1, 2, \dots$. Then the expected value of X is given by

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i].$$

$$\mathbb{E}[X] = \sum_{j=0}^{\infty} j \cdot P_j$$



0	1	2	3	…
P_0	P_1	P_2	P_3	…

$$= \cancel{0 \cdot P_0} + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 + \dots$$

$$\begin{aligned}
= \;& P_1 + P_2 + P_3 + P_4 + \dots = \mathbb{P}(X \geq 1) \\
+ \;& P_2 + P_3 + P_4 + \dots = \mathbb{P}(X \geq 2) \\
+ \;& P_3 + P_4 + \dots = \mathbb{P}(X \geq 3) \\
+ \;& P_4 + \dots = \mathbb{P}(X \geq 4)
\end{aligned}$$

$P_0 + P_1 + P_2 + \dots = 1$

Example

Let X have the geometric distribution with parameter p, i.e.

$$\mathbb{P}[X = j] = p(1-p)^{j-1},$$

for $j \geq 1$. Compute $\underline{\mathbb{E}[X]}$.

j	1	2	3	4	...
	p	$(1-p)p$	$(1-p)^2 p$	$(1-p)^3 p$	...

$$\mathbb{E}[X] = 1 \cdot p + 2(1-p)p + 3(1-p)^2 p + 4(1-p)^3 p + \dots$$

Can we do better?

Vaules of $X = \{1, 2, 3, \dots\} \subseteq \{0, 1, 2, 3, \dots\}$

$1 - p = q$

$$\mathbb{E}[X] = \sum_{j=1}^{\infty} \underset{(1-p)^{j-1}}{\underset{\parallel}{\mathbb{P}[X \geq j]}} = \sum_{j=1}^{\infty} (1-p)^{j-1} = \sum_{k=0}^{\infty} (1-p)^k = 1 + q + q^2 + \dots$$

$j - 1 = k$

$$= \frac{1}{1-q} = \frac{1}{p}$$

Summary

X random variable with a geometric distribution with parameter p, Then

$$\mathbb{E}[X] = 1/p.$$

X : first success in a Bernoulli trials.
prob of success $= p$.

p close to 1. $\quad p = 0.8 \quad \mathbb{E}[X] = \frac{1}{p} = \frac{1}{8/10} = \frac{10}{8} = 1.25$

1	2	3 ...
0.8	$\frac{16}{100}$	

p close to 0 $\quad p = 0.1$

$$\mathbb{E}[X] = \frac{1}{1/10} = 10$$

1	2	3
$\frac{1}{10}$	$\frac{9}{10} \cdot \frac{1}{10} = \frac{9}{100} \approx \frac{1}{10}$	

$$X = Y_1 + Y_2$$

$$\mathbb{E}[X] = \mathbb{E}[Y_1 + Y_2]$$