# Computer Architecture I: Digital Design

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# Complements of Numbers

## Complements of Numbers

- Complements are used in digital computers to **simplify the subtraction operation** and **for logical manipulation**.
- Simplifying operations leads to simpler, less expensive circuits to implement the operations.
- There are two types of complements for each base-r system: the **radix complement ($r$'s complement)** and the **diminished radix complement ($(r-1)$'s complement)**.
- When the value of the base $r$ is substituted in the name, the two types are referred to as the 2's complement and 1's complement for binary numbers and the 10's complement and 9's complement for decimal numbers.

# Complements of Numbers

## Diminished Radix Complement (Find 9's complement)

- Given a number $N$ in base $r$ having $n$ digits, the $(r-1)$'s complement of $N$, i.e., its diminished radix complement, is defined as $(r^n - 1) - N$.
- **For decimal numbers,** $r = 10$ and $r - 1 = 9$, so the 9's complement of $N$ is $(10^n - 1) - N$.
  - In this case, $10^n$ represents a number that consists of a single 1 followed by n 0's.

For example,

The 9's complement of 546700 is
999999-546700=453299
The 9's complement of 012398 is
999999-012398=987601

# Complements of Numbers

## Diminished Radix Complement (Find 1's complement)

- **For binary numbers,** $r = 2$ and $r - 1 = 1$, so the 1's complement of $N$ is $(2^n - 1) - N$.
  - Again, $2^n$ is represented by a binary number that consists of a 1 followed by $n$ 0's.

For example,

The 1's complement of 1011000 is 0100111.

The 1's complement of 0101101 is 1010010.

# Complements of Numbers

## Radix Complement (Find 10's/2's complement)

- The $r$'s complement of an $n$-digit number $N$ in base $r$ is defined as $r^n - N$ for $N \neq 0$ and as 0 for $N = 0$.
- Comparing with the $(r-1)$'s complement, we note that the $r$'s complement is obtained by adding 1 to the $(r-1)$'s complement, since $r^n - N = [(r^n - 1) - N] + 1$.
- Thus, the 10's complement of decimal 2389 is $7610 + 1 = 7611$ and is obtained by adding 1 to the 9's complement value.
- The 2's complement of binary 101100 is $010011 + 1 = 010100$ and is obtained by adding 1 to the 1's-complement value.

# Complements of Numbers

## Radix Complement (Find 10's complement)

- The 10's complement of a decimal number is obtained by
  - leaving all least significant 0's unchanged,
  - subtracting the first nonzero least significant digit from 10, and
  - subtracting all higher significant digits from 9.

For example,

The 10's complement of 012398 is 987602

The 10's complement of 246700 is 753300

# Complements of Numbers

## Radix Complement (Find 2's complement)

- The 2's complement of a binary number is obtained by
  - leaving all least significant 0's, and
  - the first 1 unchanged, and
  - replacing 1's with 0's and 0's with 1's in all other higher significant digits.

For example,

The 2's complement of 1101100 is 0010100

The 2's complement of 0110111 is 1001001

# Complements of Numbers

## r's or (r-1)'s complement (Important notes)

- If the original number $N$ contains a radix point, the point should be removed temporarily in order to form the $r$'s or $(r-1)$'s complement. The radix point is then restored to the complemented number in the same relative position.
- It is also worth mentioning that the complement of the complement restores the number to its original value.
- To see this relationship, note that the $r$'s complement of $N$ is $r^n - N$, so that the complement of the complement is $r^n - (r^n - N) = N$ and is equal to the original number.

# Complements of Numbers

**Why we study complements? How Does Complements Help (Subtraction)?**

- With complements, you do not have to borrow anymore as you did in elementary school.

# Complements of Numbers

## Subtraction with $r$'s Complements

- The subtraction of two $n$-digit **unsigned numbers** $M - N$ in base $r$ can be done as follows:
  1. Add the minuend $M$ to the $r$'s complement of the subtrahend $N$. Mathematically, $M + (r^n - N) = M - N + r^n$.
  2. If $M \geq N$, the sum will produce an end carry $r^n$, which can be discarded; what is left is the result $M - N$.
  3. If $M < N$, the sum does not produce an end carry and is equal to $r^n - (N - M)$, which is the $r$'s complement of $(N - M)$. To obtain the answer in a familiar form, take the $r$'s complement of the sum and place a negative sign in front.

# Complements of Numbers

## Subtraction with $r$'s Complements: Example of Step-2: $M \geq N$

Using 10's complement, subtract $72532 - 3250$ .

$$\begin{aligned} M &= \quad 72532 \\ \text{10's complement of } N &= +\underline{96750} \\ \text{Sum} &= \quad 169282 \\ \text{Discard end carry } 10^5 &= -\underline{100000} \\ Answer &= \quad 69282 \end{aligned}$$

Note that *M* has five digits and *N* has only four digits. Both numbers must have the same number of digits, so we write *N* as 03250. Taking the 10's complement of *N* produces a 9 in the most significant position. The occurrence of the end carry signifies that $M \geq N$ and that the result is therefore positive.

# Complements of Numbers

## Subtraction with $r$'s Complements: Example of Step-3: $M < N$

Using 10's complement, subtract $3250 - 72532$ .

$$\begin{aligned} M &= \phantom{+}3250 \\ \text{10's complement of } N &= +27468 \\ \text{Sum} &= \phantom{+}30718 \end{aligned}$$

There is no end carry. Therefore, the answer is written with a minus sign as $-(\text{10's complement of } 30718) = -69282.$

Note that since $3250 < 72532,$ the result is negative. Because we are dealing with unsigned numbers, there is really no way to get an unsigned result for this case. When subtracting with complements, we recognize the negative answer from the absence of the end carry and the complemented result. When working with paper and pencil, we can change the answer to a signed negative number in order to put it in a familiar form.

# Complements of Numbers

## Subtraction with $r$'s Complements: Example-1.7 (Binary Numbers)

Given the two binary numbers $X = 1010100$ and $Y = 1000011$, perform the subtraction **(a)** $X - Y$ and **(b)** $Y - X$ by using 2's complements.

a.

$$
\begin{aligned}
X &= \phantom{+}1010100 \\
\text{2's complement of } Y &= +\underline{0111101} \\
\text{Sum} &= 10010001 \\
\text{Discard end carry } 2^7 &= -10000000 \\
\textit{Answer}: X - y &= \phantom{+}0010001
\end{aligned}
$$

b.

$$
\begin{aligned}
Y &= 1000011 \\
\text{2's complement of } X &= +\underline{0101100} \\
\text{Sum} &= 1101111
\end{aligned}
$$

There is no end carry. Therefore, the answer is
$Y - X = -(\text{2's complement of } 1101111) = -\ 0010001.$ ■

# Complements of Numbers

## Subtraction with $(r-1)$'s Complements

- Subtraction of unsigned numbers can also be done by means of the $(r-1)$'s complement.
- Remember that the $(r-1)$'s complement is one less than the $r$'s complement.
- Because of this, the result of adding the minuend to the complement of the subtrahend produces a sum that is one less than the correct difference when an end carry occurs.
- Removing the end carry and adding 1 to the sum is referred to as an **end-around carry**.

# Complements of Numbers

## Subtraction with $(r-1)$'s Complements: Example (Binary Numbers)

Repeat **Example 1.7**, but this time using 1's complement.

a. $X - Y = 1010100 - 1000011$

$$\begin{aligned} X &= \phantom{+}1010100 \\ \text{1's complement of } Y &= +\underline{0111100} \\ \text{Sum} &= 10010000 \\ \text{End-around carry} &= + \qquad\quad 1 \\ \textit{Answer}: X - Y &= \phantom{+}0010001 \end{aligned}$$

b. $Y - X = 1000011 - 1010100$

$$\begin{aligned} Y &= \phantom{+}1000011 \\ \text{1's complement of } X &= +0101011 \\ \text{Sum} &= \phantom{+}1101110 \end{aligned}$$

There is no end carry. Therefore, the answer is $Y - X = -(\text{1's complement of } 1101110) = -0010001.$ ■

# Signed Binary Numbers

## Signed Binary Numbers: Signed-magnitude System

- In previous discussions, are the binary numbers and/or complements positive or negative, in other words, signed or unsigned?
- Unsigned numbers are representing positive integers or zero.
- Then how to represent negative numbers?
- In ordinary arithmetic, a negative number is indicated by a minus sign and a positive number by a plus sign.
- This is referred to as the **signed-magnitude** convention.

# Signed Binary Numbers

## Signed Binary Numbers: Signed-complement System

- Because of hardware limitations, computers must represent everything with binary digits.
- When arithmetic operations are implemented in a computer, it is more convenient to use a different system, referred to as the **signed-complement** system, for representing negative numbers.
- In this system, a negative number is indicated by its complement.

# Signed Binary Numbers

## Signed Binary Numbers

As an example,

| | |
|---|---|
| signed-magnitude representation: | 10001001 |
| signed-1's-complement representation: | 11110110 |
| signed-2's-complement representation: | 11110111 |

# Signed Binary Numbers

## Signed Binary Numbers

Table 1.3 *Signed Binary Numbers*

| Decimal | Signed-2's Complement | Signed-1's Complement | Signed Magnitude |
|---|---|---|---|
| +7 | 0111 | 0111 | 0111 |
| +6 | 0110 | 0110 | 0110 |
| +5 | 0101 | 0101 | 0101 |
| +4 | 0100 | 0100 | 0100 |
| +3 | 0011 | 0011 | 0011 |
| +2 | 0010 | 0010 | 0010 |
| +1 | 0001 | 0001 | 0001 |
| +0 | 0000 | 0000 | 0000 |
| −0 | — | 1111 | 1000 |
| −1 | 1111 | 1110 | 1001 |
| −2 | 1110 | 1101 | 1010 |
| −3 | 1101 | 1100 | 1011 |
| −4 | 1100 | 1011 | 1100 |
| −5 | 1011 | 1010 | 1101 |
| −6 | 1010 | 1001 | 1110 |
| −7 | 1001 | 1000 | 1111 |
| −8 | 1000 | — | — |

# Signed Binary Numbers

## Signed Binary Numbers

- All negative numbers start with 1 in the left most position.
- **Singed-magnitude:** used in ordinal arithmetic, not in computer.
- **1's complement:** used for logical operation, not the best choice for arithmetic operation.
- **2's complement:** best choice so far for singed binary arithmetic.

# Signed Binary Numbers

## Arithmetic Addition: Signed-magnitude System

- The addition of two numbers in the signed-magnitude system follows the rules of ordinary arithmetic.
  - **If the signs are the same**, we add the two magnitudes and give the sum the common sign.
  - **If the signs are different**, we subtract the smaller magnitude from the larger and give the difference the sign of the larger magnitude.

    For example,

    (+25)+(-37)=-(37-25)=-12 is done by subtracting the smaller magnitude, 25, from the larger magnitude, 37, and appending the sign of 37 to the result.
  - This is a process that **requires a comparison of the signs and magnitudes** and then performing either addition or subtraction.

# Signed Binary Numbers

## Arithmetic Addition: With 2's complement system

- The same procedure applies to binary numbers in signed-magnitude representation.
- **In contrast**, the rule for adding numbers in the signed-complement system does not require a comparison or subtraction, but only addition.
- The addition of two signed binary numbers with negative numbers represented in signed- 2's-complement form is obtained from the addition of the two numbers, including their sign bits. A carry out of the sign-bit position is discarded.

# Signed Binary Numbers

## Arithmetic Addition: With 2's complement system

- The addition of two signed binary numbers with negative numbers represented in signed- 2's-complement form is obtained from the addition of the two numbers, including their sign bits. A carry out of the sign-bit position is discarded.

  For example,

  Numerical examples for addition follow:

| | | | |
|---|---|---|---|
| $+6$ | 00000110 | $-6$ | 11111010 |
| $+13$ | 00001101 | $+13$ | 00001101 |
| $+19$ | 00010011 | $+7$ | 00000111 |
| $+6$ | 00000110 | $-6$ | 11111010 |
| $-13$ | 11110011 | $-13$ | 11110011 |
| $-7$ | 11111001 | $-19$ | 11101101 |

- Note that negative numbers must be initially in 2's-complement form and that if the sum obtained after the addition is negative, it is in 2's-complement form. For example, -7 is represented as 11111001, which is the 2s complement of $+7$.

# Signed Binary Numbers

## Arithmetic Addition: With 2's complement system

For example,

1. Addition of two positive numbers, sum $< 2^{n-1}$

$$\begin{array}{rr l} +3 & 0011 & \\ \underline{+4} & \underline{0100} & \\ +7 & 0111 & \text{(correct answer)} \end{array}$$

2. Addition of two positive numbers, sum $\geq 2^{n-1}$

$$\begin{array}{rr l} +5 & 0101 & \\ \underline{+6} & \underline{0110} & \\ & 1011 & \longleftarrow \text{wrong answer because of overflow (+11 requires} \\ & & \quad \text{5 bits including sign)} \end{array}$$

3. Addition of positive and negative numbers (negative number has greater magnitude)

$$\begin{array}{rr l} +5 & 0101 & \\ \underline{-6} & \underline{1010} & \\ -1 & 1111 & \text{(correct answer)} \end{array}$$

# Signed Binary Numbers

## Arithmetic Addition: With 2's complement system

For example,

**4.** Same as case 3 except positive number has greater magnitude

$$\begin{array}{rr}
-5 & 1011 \\
\underline{+6} & \underline{0110} \\
+1 & (1)0001
\end{array}$$

$\longleftarrow$ correct answer when the carry from the sign bit is ignored (this is *not* an overflow)

**5.** Addition of two negative numbers, $|\text{sum}| \le 2^{n-1}$

$$\begin{array}{rr}
-3 & 1101 \\
\underline{-4} & \underline{1100} \\
-7 & (1)1001
\end{array}$$

$\longleftarrow$ correct answer when the last carry is ignored (this is *not* an overflow)

**6.** Addition of two negative numbers, $|\text{sum}| > 2^{n-1}$

$$\begin{array}{rr}
-5 & 1011 \\
\underline{-6} & \underline{1010} \\
 & (1)0101
\end{array}$$

$\longleftarrow$ wrong answer because of overflow ($-11$ requires 5 bits including sign)

# Signed Binary Numbers

## Arithmetic Addition: With 2's complement system

- Note that an overflow condition (case 2 and 6) is easy to detect because in case 2 the addition of two positive numbers yields a negative result, and in case 6 the addition of two negative numbers yields a positive answer (for four bits).
- The proof that throwing away the carry from the sign bit always gives the correct answer follows for cases 4 and 5:
  Case 4: $-A + B$(where $B > A$)
  $A^* + B = (2^n - A) + B = 2^n + (B - A) > 2^n$
- Throwing away the last carry is equivalent to subtracting $2^n$, so the result is $(B - A)$, which is correct. Case 5:
  $-A - B$(where $A + B \leq 2^{n-1}$)
  $A^* + B^* = (2^n - A) + (2^n - B) = 2^n + 2^n - (A + B)$
  Discarding the last carry yields $2^n - (A + B) = (A + B)^*$, which is the correct representation of $-(A + B)$.

# Signed Binary Numbers

## Arithmetic Subtraction: With 2's complement system

- Subtraction of two signed binary numbers when negative numbers are in 2's-complement form is simple and can be stated as follows:
- Take the 2's complement of the subtrahend (including the sign bit) and add it to the minuend (including the sign bit). A carry out of the sign-bit position is discarded.
- This procedure is adopted because a subtraction operation can be changed to an addition operation if the sign of the subtrahend is changed, as is demonstrated by the following relationship:
  $(\pm A) - (+B) = (\pm A) + (-B)$;
  $(\pm A) - (-B) = (\pm A) + (+B)$.

# Signed Binary Numbers

## Arithmetic Subtraction: With 2's complement system

To see this, consider the subtraction (-6)-(-13)=+7.

- In binary with eight bits, this operation is written as (11111010-11110011). The subtraction is changed to addition by taking the 2's complement of the subtrahend (-13), giving (+13).
- In binary, this is 11111010+00001101=100000111. Removing the end carry, we obtain the correct answer: 00000111 (+7).

# Signed Binary Numbers

## Signed Binary Numbers: Summary

- It is worth noting that binary numbers in the signed-complement system are added and subtracted by the same basic addition and subtraction rules as unsigned numbers.
- Therefore, computers need only one common hardware circuit to handle both types of arithmetic.
- This consideration has resulted in the signed-complement system being used in virtually all arithmetic units of computer systems.
- The user or programmer must interpret the results of such addition or subtraction differently, depending on whether it is assumed that the numbers are signed or unsigned.