Computer Architecture I: Digital Design

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Number Systems

Decimal Number System

- The decimal number system is employed in everyday arithmetic to represent numbers by strings of digits.
- Depending on its position in the string, each digit has an associated value of an integer raised to the power of 10.

For example,

the decimal 7392 number is interpreted as

$$7 \times 10^3 + 3 \times 10^2 + 9 \times 10^1 + 2 \times 10^0$$

Number Systems

Decimal Number System

- In general, a number with a decimal point is represented by a series of coefficients:

$$a_5 a_4 a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3}$$

- Where
  - the coefficients a_j are any of the 10 digits $(0, 1, 2 \ldots, 9)$
  - the subscript value j gives the place value and, hence, the power of 10 by which the coefficient must be multiplied
- Thus, the preceding decimal number can be expressed as

$$10^5 a_5 + 10^4 a_4 + 10^3 a_3 + 10^2 a_2 + 10^1 a_1 + 10^0 a_0 + 10^{-1} a_{-1} + 10^{-2} a_{-2} + 10^{-3} a_{-3}$$

- with $a_3 = 7$, $a_2 = 3$, $a_1 = 9$, $a_0 = 2$

Number Systems

Decimal Number System

The decimal number system is said to be of **base, or radix, 10** because it uses 10 digits (0,1,2,3,4,5,6,7,8,9) and the coefficients are multiplied by powers of 10.

Number Systems

Binary Number System

- The coefficients of the binary number system have only two possible values: 0 and 1
- Each coefficient a_j is multiplied by a power of the radix, e.g., 2^j, and the results are added to obtain the decimal equivalent of the number.

For example, (Binary to Decimal)

the decimal equivalent of the binary number 11010.11 is 26.75, as shown from the multiplication of the coefficients by powers of 2:

$1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} = 26.75$

Number Systems

base-r system

- In general, a number expressed in a base-r system has coefficients multiplied by powers of r:

$$a_n.r^n + a_{n-1}.r^{n-1} + \cdots + a_2.r^2 + a_1.r + a_0 + a_{-1}.r^{-1} + a_{-2}.r^{-2} + \cdots + a_{-m}.r^{-m}$$

- The coefficients a_j range in value from 0 to $r-1$.
- To distinguish between numbers of different bases, we enclose the coefficients in parentheses and write a subscript equal to the base used.

For example,

$(4021.2)_5, (127.4)_8, (B65F)_{16}, (110101)_2$

Number Systems

Commonly seen and used number systems:

- Decimal system (base-10 or radix-10): 0,1,2,3,4,5,6,7,8,9
- Binary system (base-2 or radix-2): 0,1
- Octal system (base-8 or radix-8): 0,1,2,3,4,5,6,7
- Hexadecimal system (base-16 or radix-16): 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

Number Systems

Can you think of any base-7, base-12, base-24, base-60 systems in everyday life?

- $(4021.2)_5 = 4 \times 5^3 + 0 \times 5^2 + 2 \times 5^1 + 1 \times 5^0 + 2 \times 5^{-1} = (511.4)_{10}$
- $(127.4)_8 = 1 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 + 4 \times 8^{-1} = (87.5)_{10}$
- $(B65F)_{16} = 11 \times 16^3 + 6 \times 16^2 + 5 \times 16^1 + 15 \times 16^0 = (46687)_{10}$
- $(110101)_2 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = (53)_{10}$

Number Systems: Practice Exercise

1. $(0011)_8 = (?)_{10}$
2. $(11010)_2 = (?)_{10}$
3. $(3A6)_{16} = (?)_{10}$

Number Systems

Powers of Two

Table 1.1 *Powers of Two*

n	2^n	*n*	2^n	*n*	2^n
0	1	8	256	16	65,536
1	2	9	512	17	131,072
2	4	10	1,024 (1K)	18	262,144
3	8	11	2,048	19	524,288
4	16	12	4,096 (4K)	20	1,048,576 (1M)
5	32	13	8,192	21	2,097,152
6	64	14	16,384	22	4,194,304
7	128	15	32,768	23	8,388,608

Number Systems

Arithmetic Operations

- Arithmetic operations with numbers in base r follow the same rules as for decimal numbers.
- When a base other than the familiar base 10 is used, one must be careful to use only the r-allowable digits.

Number Systems

Arithmetic Operations

Examples of addition, subtraction, and multiplication of two binary numbers are as follows:

augend:	101101	minuend:	101101	multiplicand:	1011
addend:	+100111	subtrahend:	−100111	multiplier:	× 101
sum:	1010100	difference:	000110		1011
					0000
				partial product:	1011
				product:	110111

Number-base Conversions

Number-base Conversions

- Representations of a number in a different radix are said to be equivalent if they have the same decimal representation.

For example,

$(0011)_8$ and $(1001)_2$ are equivalent—both have decimal value 9.

- The **conversion** of a number in base r to decimal is done by expanding the number in a power series and adding all the terms as shown previously.

Number-base Conversions

Decimal to base r

- We now present a general procedure for the reverse operation of converting a decimal number to a number in base r.
- If the number includes a **radix point**, it is necessary to separate the number into an **integer part** and a **fraction part**, since each part must be converted differently.
- The conversion of a decimal **integer** to a number in base r is done by dividing the number and all successive quotients by r and accumulating the remainders.
- The conversion of a decimal **fraction** to binary is accomplished by a method similar to that used for integers. However, multiplication is used instead of division, and integers instead of remainders are accumulated.

Number-base Conversions

Decimal to Binary

For example,

Convert decimal 41 to binary

	Integer Quotient		Remainder	Coefficient
$41/2 =$	20	$+$	$\frac{1}{2}$	$a_0 = 1$
$20/2 =$	10	$+$	0	$a_1 = 0$
$10/2 =$	5	$+$	0	$a_2 = 0$
$5/2 =$	2	$+$	$\frac{1}{2}$	$a_3 = 1$
$2/2 =$	1	$+$	0	$a_4 = 0$
$1/2 =$	0	$+$	$\frac{1}{2}$	$a_5 = 1$

Therefore, the answer is $(41)_{10} = (a_5a_4a_3a_2a_1a_0)_2 = (101001)_2$.

Number-base Conversions

Decimal to Binary

For example,

The arithmetic process can be manipulated more conveniently as follows:

Integer	Remainder
41	
20	1
10	0
5	0
2	1
1	0
0	1 $101001 = \text{answer}$

Number-base Conversions

Decimal to Octal

For example,

Convert decimal 153 to octal

153	
19	1
2	3
0	$2 = (231)_8$

Number-base Conversions

Decimal to Octal

For example,

Convert $(0.6875)_{10}$ to binary

	Integer		Fraction	Coefficient
$0.6875 \times 2 =$	1	$+$	0.3750	$a_{-1} = 1$
$0.3750 \times 2 =$	0	$+$	0.7500	$a_{-2} = 0$
$0.7500 \times 2 =$	1	$+$	0.5000	$a_{-3} = 1$
$0.5000 \times 2 =$	1	$+$	0.0000	$a_{-4} = 1$

Therefore, the answer is $(0.6875)_{10} = (0.a_{-1}\, a_{-2}\, a_{-3}\, a_{-4})_2 = (0.1011)_2$.

Number-base Conversions

Decimal to Octal

For example,

Convert $(0.513)_{10}$ to octal

$$0.513 \times 8 = 4.104$$
$$0.104 \times 8 = 0.832$$
$$0.832 \times 8 = 6.656$$
$$0.656 \times 8 = 5.248$$
$$0.248 \times 8 = 1.984$$
$$0.984 \times 8 = 7.872$$

The answer, to six significant figures, is obtained from the integer part of the products:

$$(0.513)_{10} = (0.406517\ldots)_8$$

Number-base Conversions

Number-base Conversions: Practice Exercise

1. $(27.315)_{10} = (?)_2$
2. $(369.3125)_{10} = (?)_8$
3. $(1938.257)_{10} = (?)_{16}$

Octal and Hexadecimal Numbers

Octal and Hexadecimal Numbers

- The conversion from and to binary, octal, and hexadecimal plays an important role in digital computers, because shorter patterns of hex characters are easier to recognize than long patterns of 1's and 0's.
- Since $2^3 = 8$ and $2^4 = 16$, each octal digit corresponds to three binary digits and each hexadecimal digit corresponds to four binary digits.
- The first 16 numbers in the decimal, binary, octal, and hexadecimal number systems are listed in Table 1.2.

Octal and Hexadecimal Numbers

Numbers with Different Bases

Table 1.2 *Numbers with Different Bases*

Decimal (base 10)	Binary (base 2)	Octal (base 8)	Hexadecimal (base 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Octal and Hexadecimal Numbers

Binary to Octal

- The conversion from binary to octal is easily accomplished by partitioning the binary number into groups of three digits each, starting from the binary point and proceeding to the left and to the right.
- The corresponding octal digit is then assigned to each group.
- The following example illustrates the procedure:

$$
\begin{array}{cccccccccc}
(10 & 110 & 001 & 101 & 011 & \cdot & 111 & 100 & 000 & 110)_2 = (26153.7406)_8 \\
2 & 6 & 1 & 5 & 3 & & 7 & 4 & 0 & 6
\end{array}
$$

Octal and Hexadecimal Numbers

Binary to Hexadecimal

- Conversion from binary to hexadecimal is similar, except that the binary number is divided into groups of four digits:

$$\begin{array}{cccccccl} (10 & 1100 & 0110 & 1011 & \cdot & 1111 & 0010)_2 & = (2C6B.F2)_{16} \\ 2 & C & 6 & B & & F & 2 & \end{array}$$

Octal and Hexadecimal Numbers

Octal/Hexadecimal to Binary

- Conversion from octal or hexadecimal to binary is done by reversing the preceding procedure.
- Each octal digit is converted to its three-digit binary equivalent.
- Similarly, each hexadecimal digit is converted to its four-digit binary equivalent.

$$\begin{array}{rcccccc} (673.124)_8 = & 110 & 111 & 011 & \cdot\ 001 & 010 & 100)_2 \\ & 6 & 7 & 3 & 1 & 2 & 4 \end{array}$$

and

$$\begin{array}{rcccc} (306.D)_{16} = & (0011 & 0000 & 0110 & \cdot\ 1101)_2 \\ & 3 & 0 & 6 & D \end{array}$$

Octal and Hexadecimal Numbers

Octal and Hexadecimal Numbers: Practice Exercise

1. $(326.5)_8 = (?)_2$
2. $(64CD)_{16} = (?)_2$