Modern Control

Project: Ball & Beam System (Phase 1)

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Questions

1.

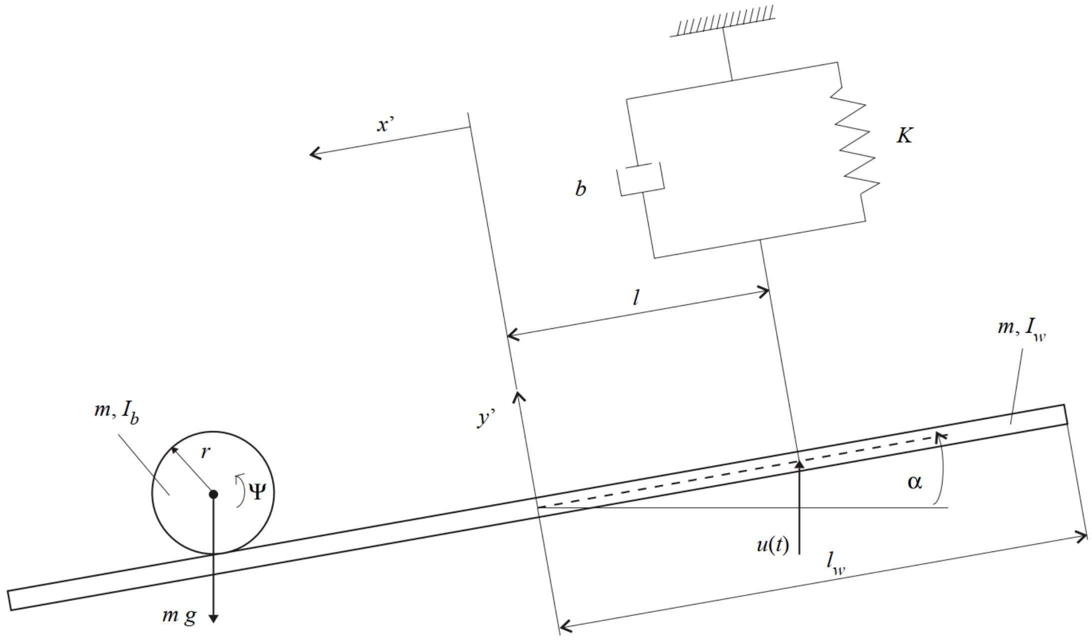
1.1.

$$x = x_1 \rightarrow \dot{x} = \dot{x_1}$$

$$\dot{x_1} = x_2 \rightarrow x_2 = \ddot{x}$$

$$a = x_3 \rightarrow \dot{a} = \dot{x_3}$$

$$\dot{x_3} = x_4 \rightarrow x_4 = \ddot{a}$$



$$\left(\frac{J_b}{r^2} + m\right)\ddot{x} + \frac{mr^2 + J_b}{r}\ddot{a} - mx\dot{a}^2 = mgsin(a)$$

$$(mx^2 + J_b + J_w)\ddot{a} + (2mx\dot{x} + bl^2)\dot{a} + Kl^2a + \frac{mr^2 + J_b}{r}\ddot{x} - mgcos(a) = lu(t)\,cos(a)$$

For making the simulation easier we can create new paramiters:

$$A_{11}\ddot{x} + A_{12}\ddot{a} = B_{11} + B_{22}$$

$$A_{22}\ddot{a} + A_{21}\ddot{x} = B_{21} + B_{22} + B_{23} + B_{24}$$

Which we have:

$$A_{11} = \frac{J_b}{r^2} + m\,, A_{12} = \frac{(m * r^2 + J_b)}{r}\,, A_{21} = A_{12}\,, A_{22} = m * x_1^2 + J_w + J_b$$

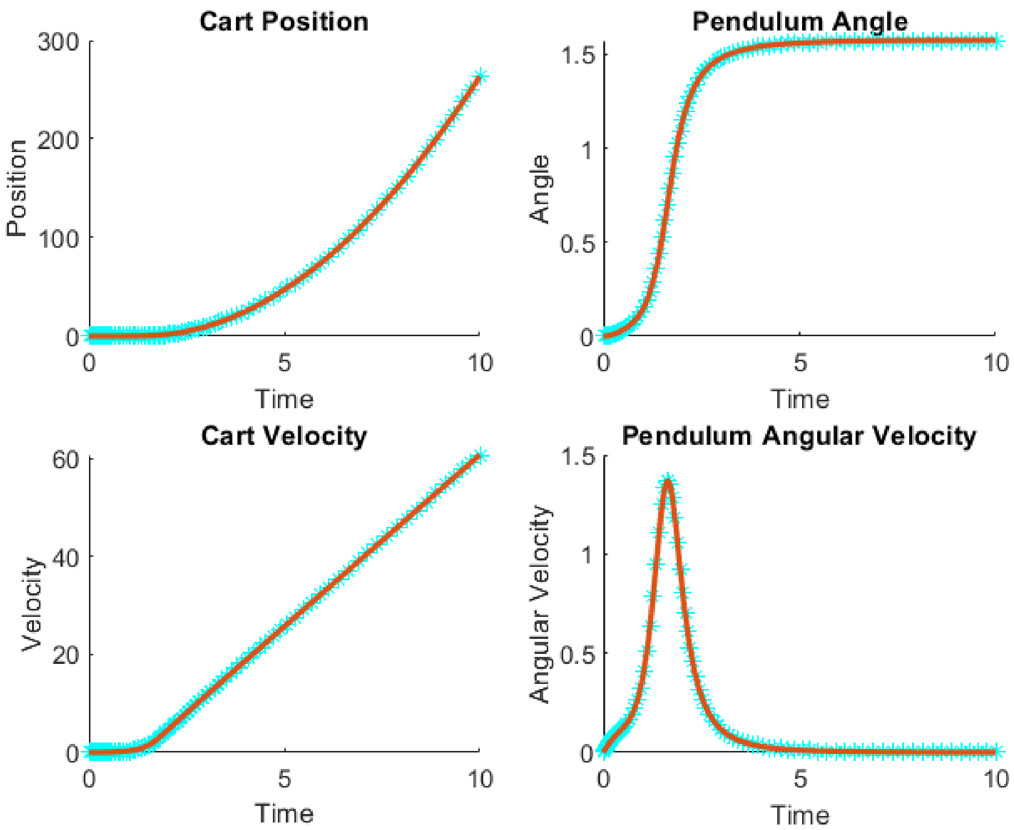
$$B_{11} = m\ g\ sin(x_3)\,, B_{12} = m\, x_1\, x_4^2$$

$$B_{21} = l\ cos(x_3)\ u\ , \qquad B_{22} = m\ g\ \ x_1\, cos(x_3)$$
$$B_{23} = -\,k\ *\ l^2 * x_3\ , \qquad B_{24} = -(2\,m\ \ x_1 x_2 +\ b\, l^2)\ x_4$$
$$C_1 = B_{11} + B_{12}\ , \qquad C_2 = B_{21} + B_{22} + B_{23} + B_{24}$$
$$detA = A_{11} * A_{22} - A_{12} * A_{21}$$

State Equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{(A_{22}\ *\ C_1\ -\ A_{12}\ *\ C_2)}{detA}$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{(-A_{21}\ *\ C_1\ +\ A_{11}\ *\ C_2)}{detA}$$

1.2.



The simulation and the equations we derived match, as depicted in the picture.

2.

The set of differential equations is linearized around the $X_{eq} = (0,0,0,0)^T$ in the following manner:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.378 & 0 & 7.0147 & 0.0343 \\ 0 & 0 & 0 & 1 \\ 18.9001 & 0 & -0.3797 & -1.7133 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -0.0699 \\ 0 \\ 3.4965 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3.

$$\lambda_1 = 2.9896\,, \lambda_2 = -3.8454$$

$$\lambda_3 = -0.4288 + 3.367i\,, \lambda_4 = -0.4288 - 3.367i$$

Since λ_1 is positive, system is not stable unless this mode is controlable.

4.

For controlable matrix:

$$Co = [B, AB, A^2B, A^3B]$$

$$Co = \begin{bmatrix} 0 & -0.0699 & 0.1198 & 24.3479 \\ -0.0699 & 0.1198 & 24.3479 & -41.8051 \\ 0 & 3.4965 & -5.9903 & 7.6137 \\ 3.4965 & -5.9903 & 7.6137 & -8.5053 \end{bmatrix}$$

The row rank of Co is 4 so the system is controlable.

For *observable* matrix:

$$Ob = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

```
Ob = 8×4
    1.0000         0         0         0
         0         0    1.0000         0
         0    1.0000         0         0
         0         0         0    1.0000
   -0.3780         0    7.0147    0.0343
   18.9001         0   -0.3797   -1.7133
    0.6476   -0.3780   -0.0130    6.9560
  -32.3809   18.9001    0.6506    2.5556
```

The column rank of Ob is 4 so the system is *observable*.

The transform function is written as follows: $G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-0.07s^2+24.5}{s^4+1.71s^3+0.75s^2-132.43} \\ \frac{3.5s^2}{s^4+1.71s^3+0.75s^2-132.43} \end{bmatrix}$

Now we see Dim(A) = 4 and the order of the transform function is 4, so the system is minimal.

5.

State transition matrix is written as follows:

$e^{At} =$

$$\begin{bmatrix} 0.274e^{3t} + 0.21e^{-3.85t} + e^{-0.43t}(0.516\cos(3.37t) + 0.0616\sin(3.37t)) & 0.092e^{3t} - 0.054e^{-3.85t} + e^{-0.43t}(-0.037\cos(3.37t) + 0.149\sin(3.37t)) & 0.21e^{3t} + 0.095e^{-3.85t} + e^{-0.43t}(-0.3\cos(3.37t) - 0.116\sin(3.37t)) & 0.04e^{3t} - 0.44e^{-3.85t} + 0.09e^{-0.43t}\sin(3.37t) \\ 0.82e^{3t} - 0.807e^{-3.85t} + e^{-0.43t}(-0.014\cos(3.37t) - 1.764\sin(3.37t)) & 0.274e^{3t} + 0.21e^{-3.85t} + e^{-0.43t}(0.516\cos(3.37t) + 0.0616\sin(3.37t)) & 0.63e^{3t} - 0.37e^{-3.85t} + e^{-0.43t}(-0.26\cos(3.37t) + 1.07\sin(3.37t)) & 0.13e^{3t} + 0.17e^{-3.85t} + e^{-0.43t}(-0.3\cos(3.37t) + 0.043\sin(3.37t)) \\ 0.36e^{3t} + 0.46e^{-3.85t} + e^{-0.43t}(-0.82\cos(3.37t) + 0.1\sin(3.37t)) & 0.12e^{3t} - 0.12e^{-3.85t} - 0.25e^{-0.43t}\sin(3.37t) & 0.27e^{3t} + 0.21e^{-3.85t} + e^{-0.43t}(0.52\cos(3.37t) + 0.06\sin(3.37t)) & 0.06e^{3t} - 0.97e^{-3.85t} + e^{-0.43t}(0.037\cos(3.37t) + 0.14\sin(3.37t)) \\ 1.07e^{3t} - 1.78e^{-3.85t} + e^{-0.43}(0.7\cos(3.37t) + 2.72\sin(3.37t)) & 0.36e^{3t} + 0.46e^{-3.85t} + e^{-0.43t}(-0.82\cos(3.37t) + 0.1\sin(3.37t)) & 0.82e^{3t} - 0.8e^{-3.85t} + e^{-0.43t}(-0.14\cos(3.37t) - 1.76\sin(3.37t)) & 0.177e^{3t} + 0.371e^{-3.85t} + e^{-0.43t}(0.45\cos(3.37t) - 0.18\sin(3.37t)) \end{bmatrix}$$

6.

The transform function was already computed in part 4 as:

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-0.07s^2 + 24.5}{s^4 + 1.71s^3 + 0.75s^2 - 132.43} \\ \frac{3.5s^2}{s^4 + 1.71s^3 + 0.75s^2 - 132.43} \end{bmatrix}$$

$$Zeros\ of\ G_1(s) = \pm 18.7$$

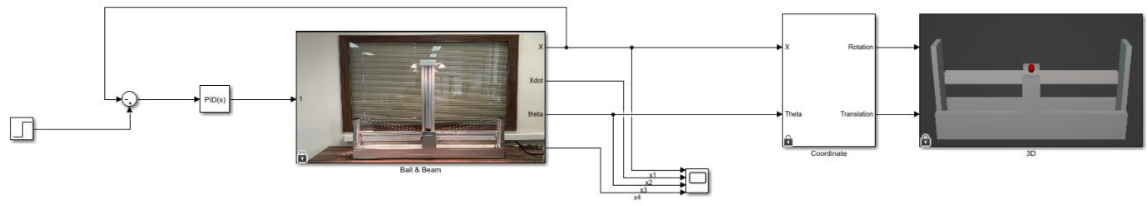
$$Zeros\ of\ G_2(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Poles\ of\ G(s) = \begin{bmatrix} 2.989614354234491 \\ -0.428760361976171 \pm\ 3.366927012696459\text{i} \\ -3.845355892844408 \end{bmatrix}$$

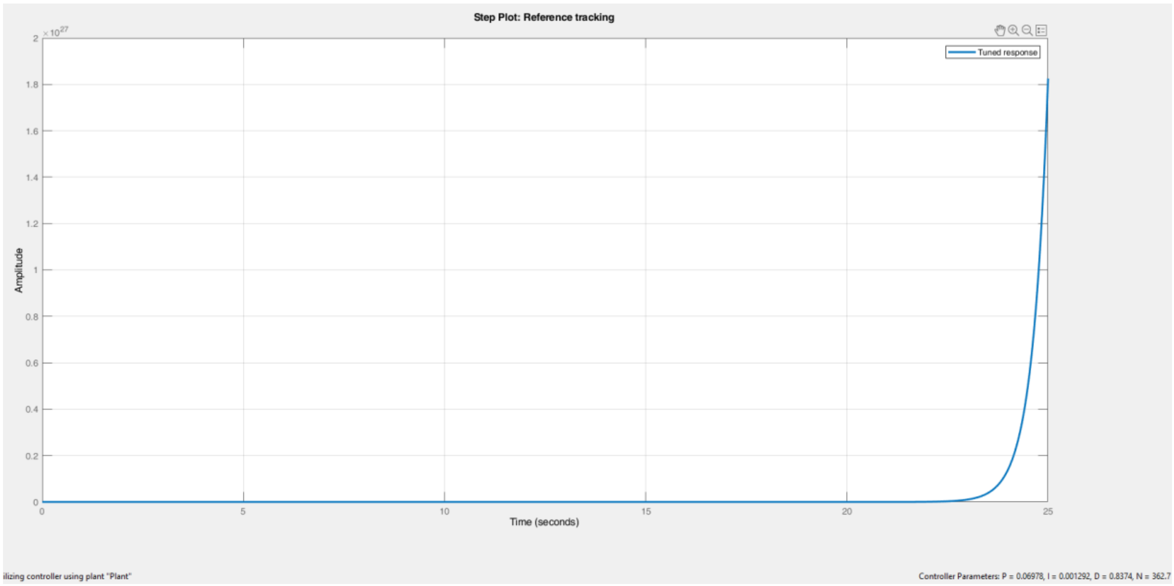
7.

The step response for non-linear and linearized systems using the PID controller is depicted below:

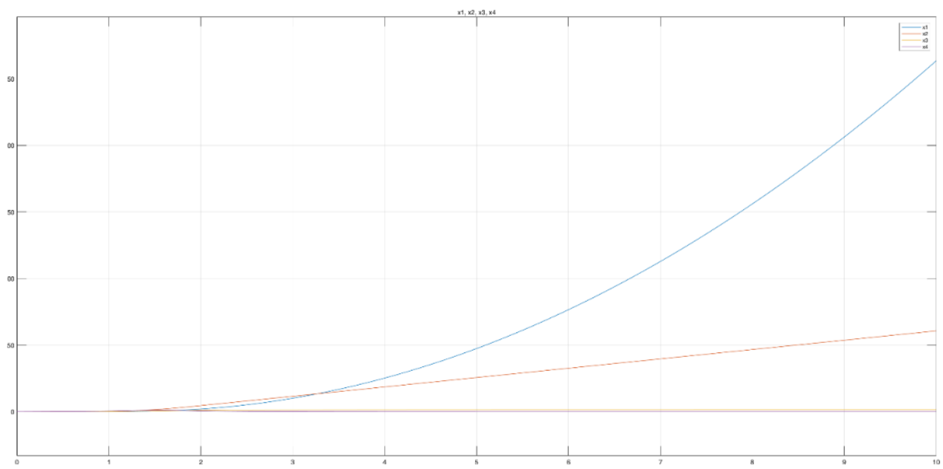
Non-linear system response:



PID controller is tuned with the following controller properties:

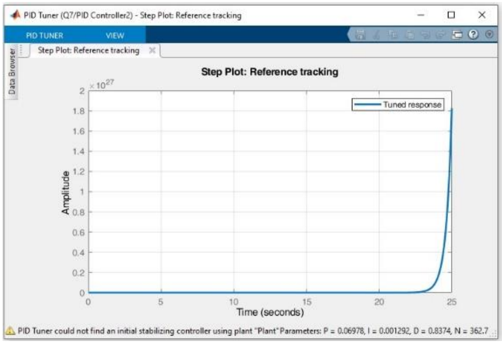


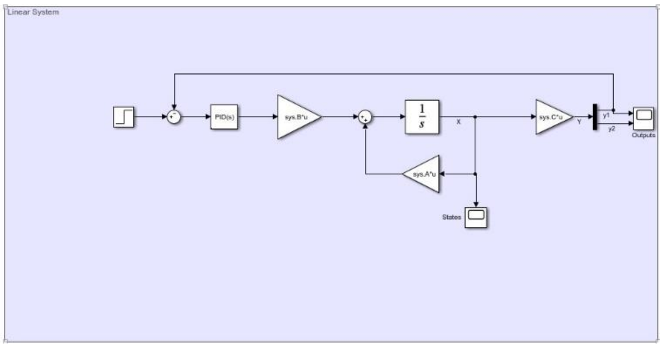
Step response:



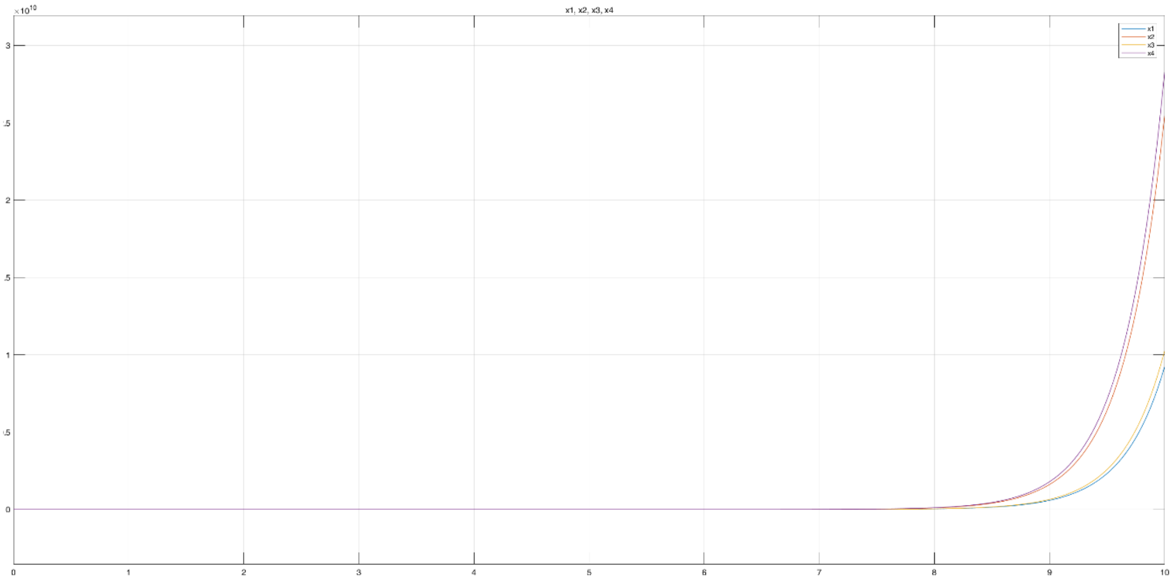
Linear system response:

PID controller is tuned with the following controller properties:



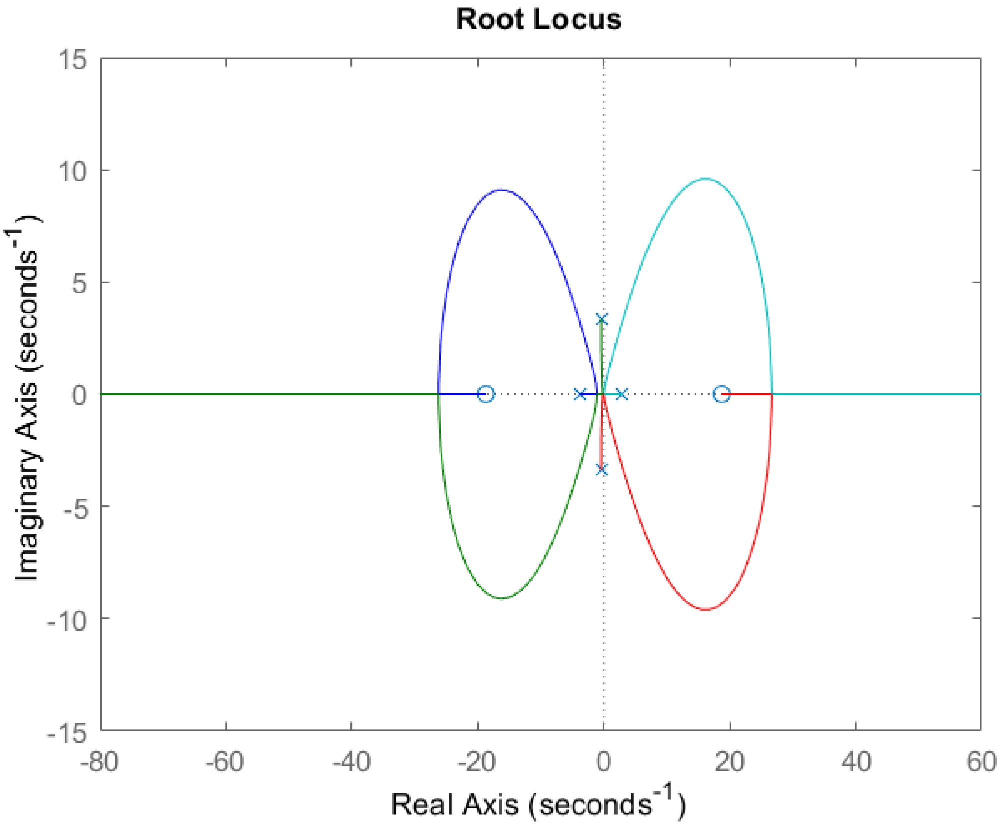


Step response:

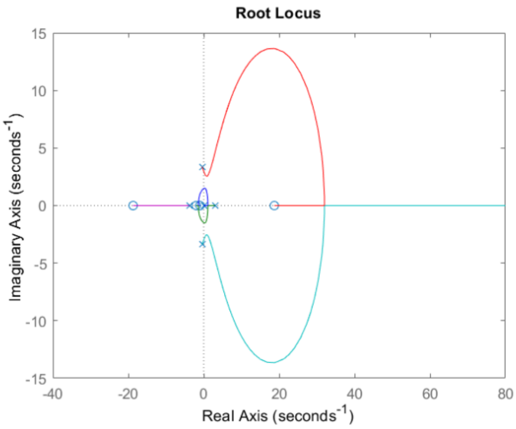


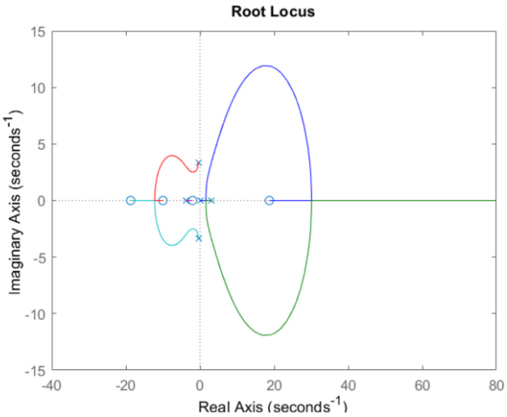
This system cannot be stabilized using a compensator.

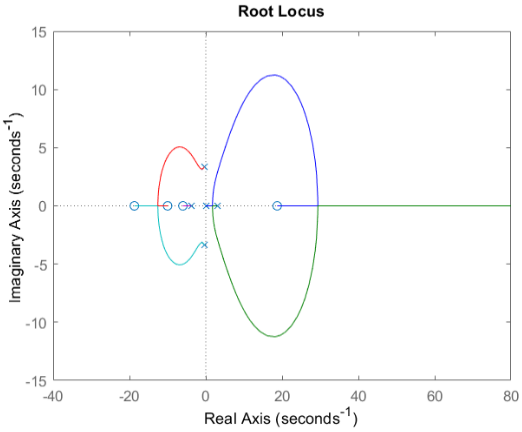
But why can't we use a PID controller? First, we must examine the root locus of the transfer function.

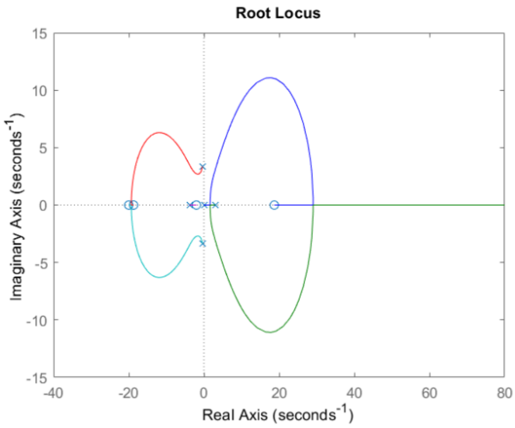


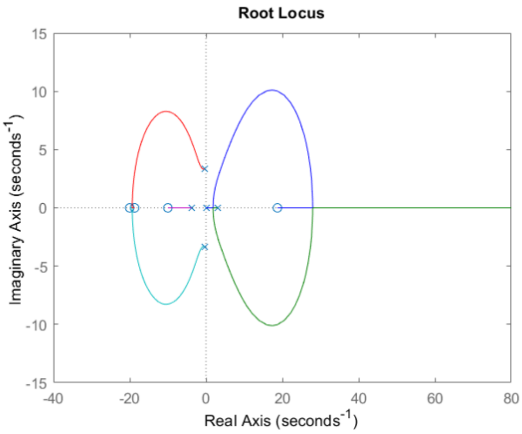
We know that PID systems add two zeros and a pole. We can only determine the location of two zeros with K_p, K_i, K_D, so we must check different states to see if we can truly stabilize the system or not.

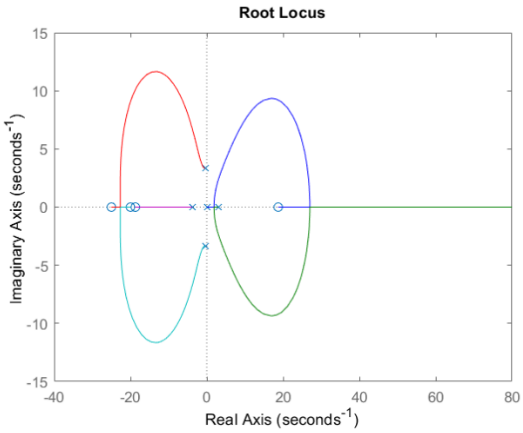












As you can see, when we place both zeros of the PID controller in different locations by defining K_p, K_i, K_D, our system couldn't stabilize with PID control. Another approach is to calculate the closed-loop transfer function and demonstrate that we cannot control our system by changing the PID controller parameters.

8.

A new system is obtained by removing unstable pole of the linearized system:

$$G_{mod}(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{-0.07s^2 + 24.5}{s^3 + 4.7s^2 + 14.82s + 44.3} \\ \dfrac{3.5s^2}{s^3 + 4.7s^2 + 14.82s + 44.3} \end{bmatrix}$$

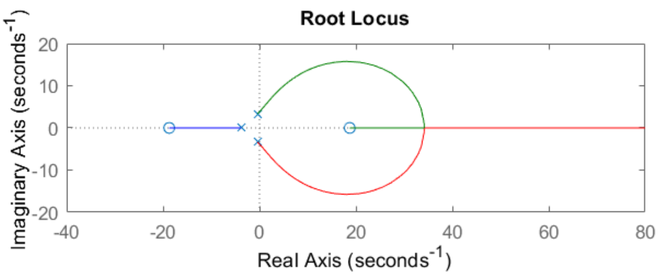
$$A_{mod} = \begin{bmatrix} -4.7 & -14.81 & -44.3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_{mod} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

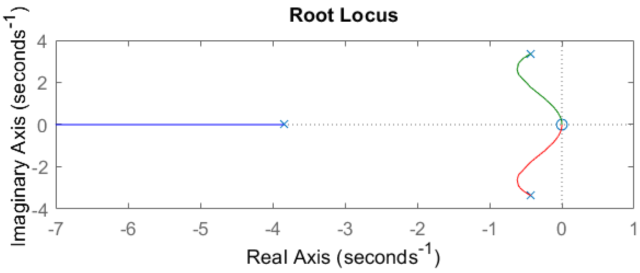
$$C_{mod} = \begin{bmatrix} -0.07 & 0 & 24.5 \\ 3.5 & 0 & 0 \end{bmatrix}, D_{mod} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Co_{mod} = \begin{bmatrix} 1 & -4.7 & 7.3 \\ 0 & 1 & -4.7 \\ 0 & 0 & 1 \end{bmatrix}$$

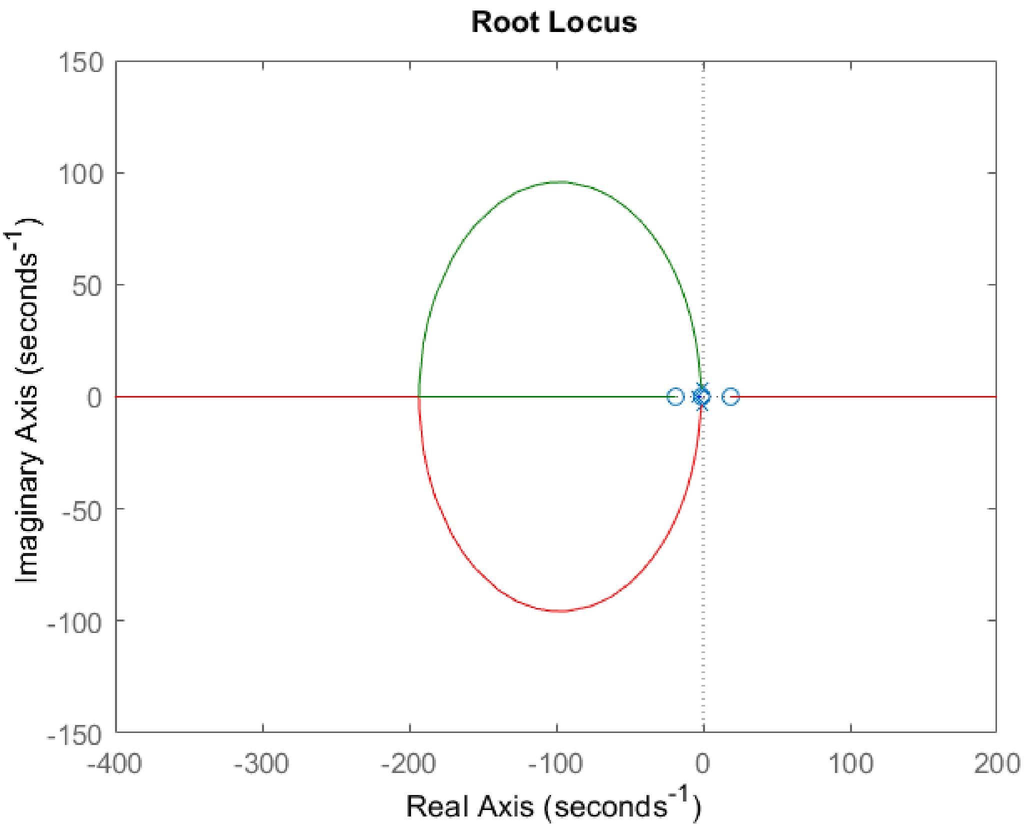
```
Ob_mod = 6×3
   -0.0699         0   24.5001
    3.4965   -0.0000    0.0000
    0.3289   25.5363    3.0978
  -16.4434  -51.8087 -154.8881
   23.9897   -1.7752  -14.5684
   25.5225   88.7619  728.4196
```

$$Rank(Co_{mod}) = Rank(Ob_{mod}) = Dim(G_{mod}) = 3 \rightarrow System\ represantion\ is\ minimal$$

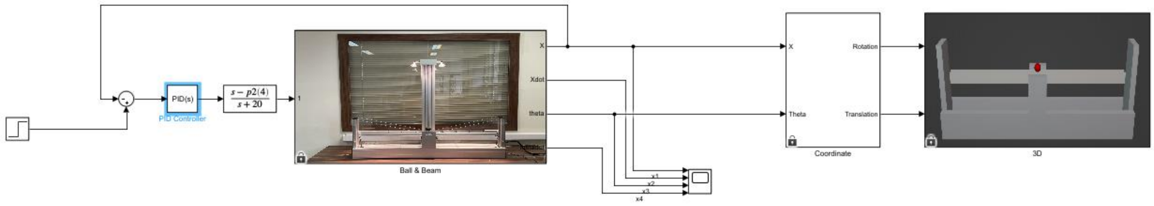




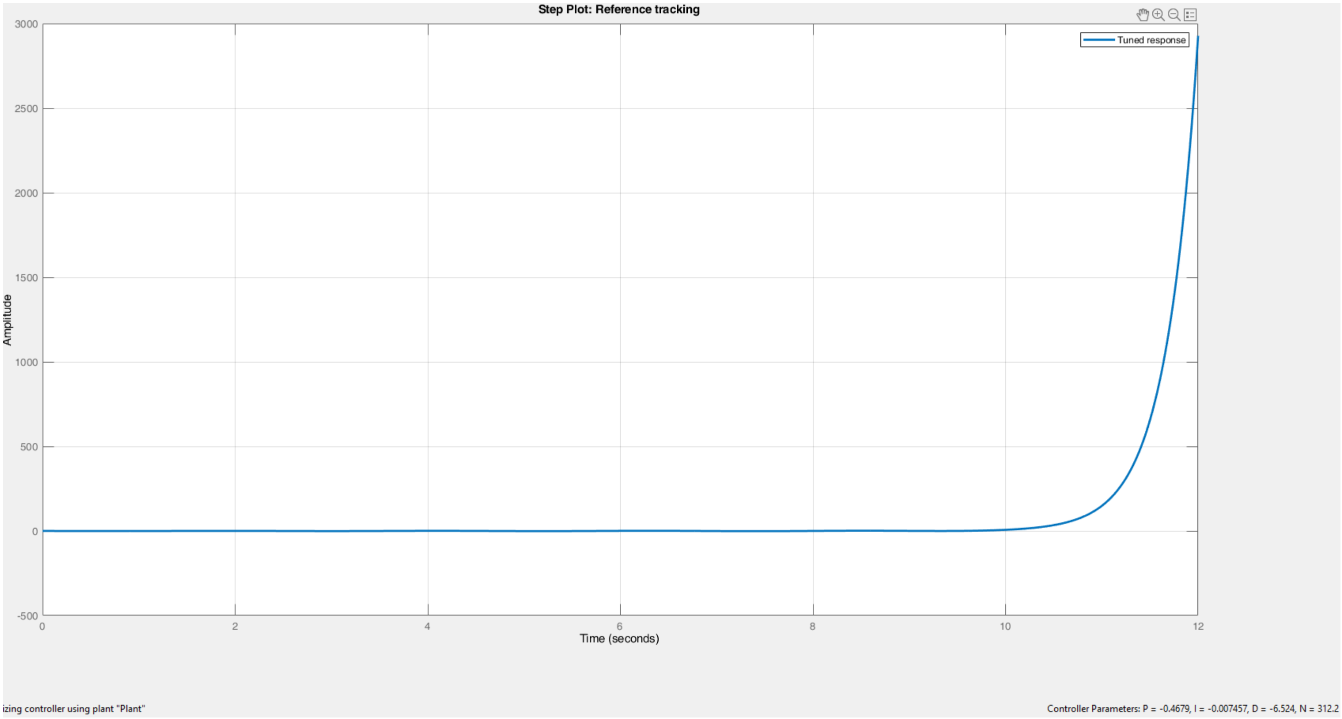
Now root locus with PID controller without unstable pole:



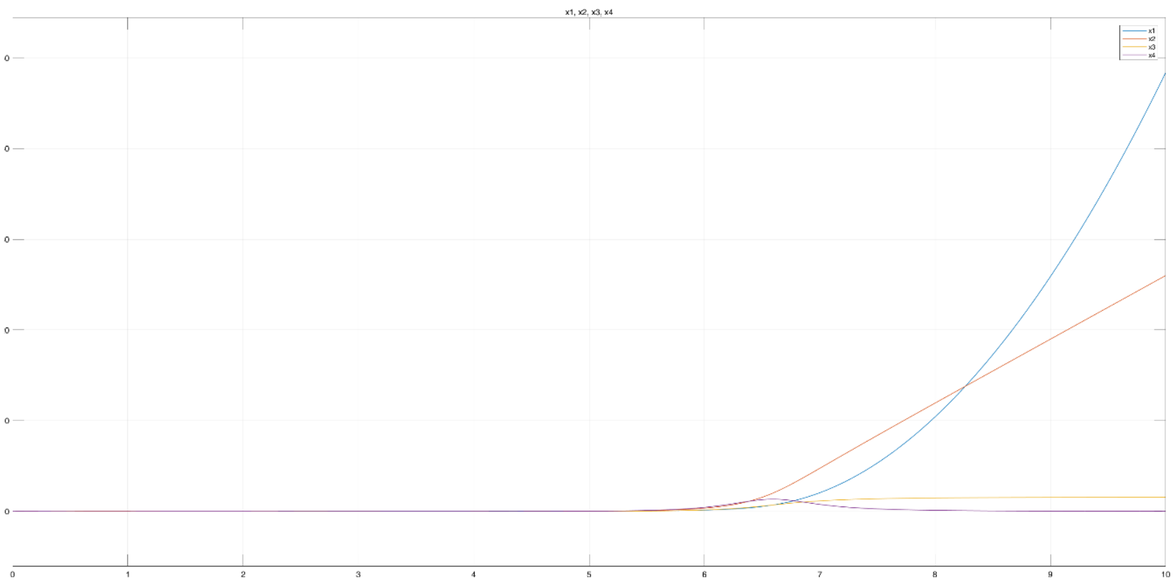
Non-linear system without unstable pole:



PID controller is tuned with the following controller properties:

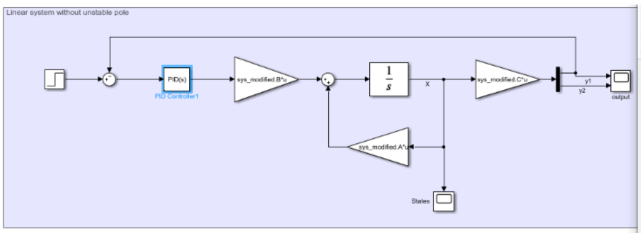


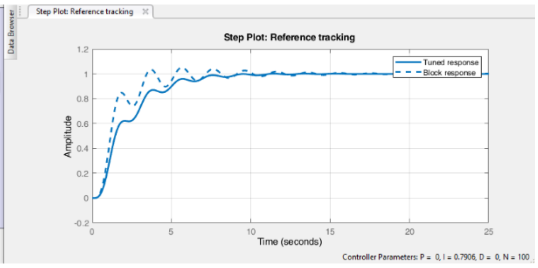
Step response:



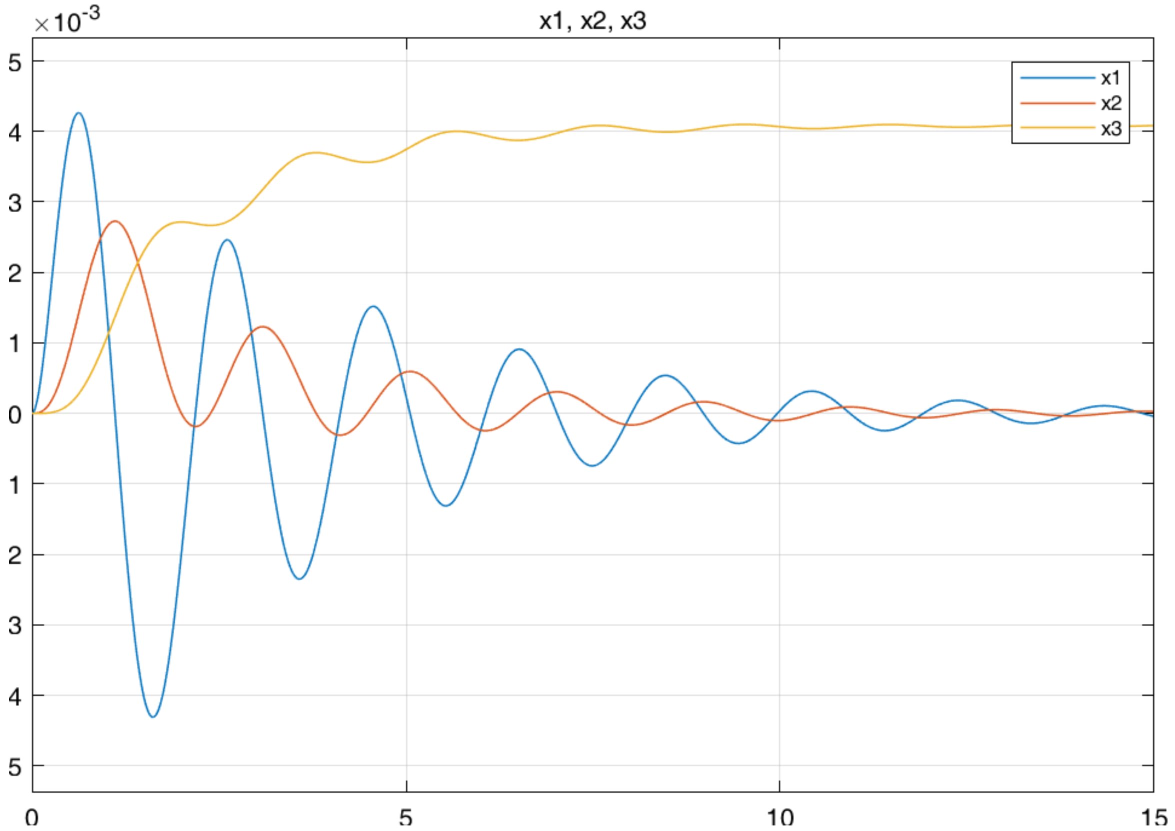
The Linear system without unstable pole:

PID controller is tuned with the following controller properties:

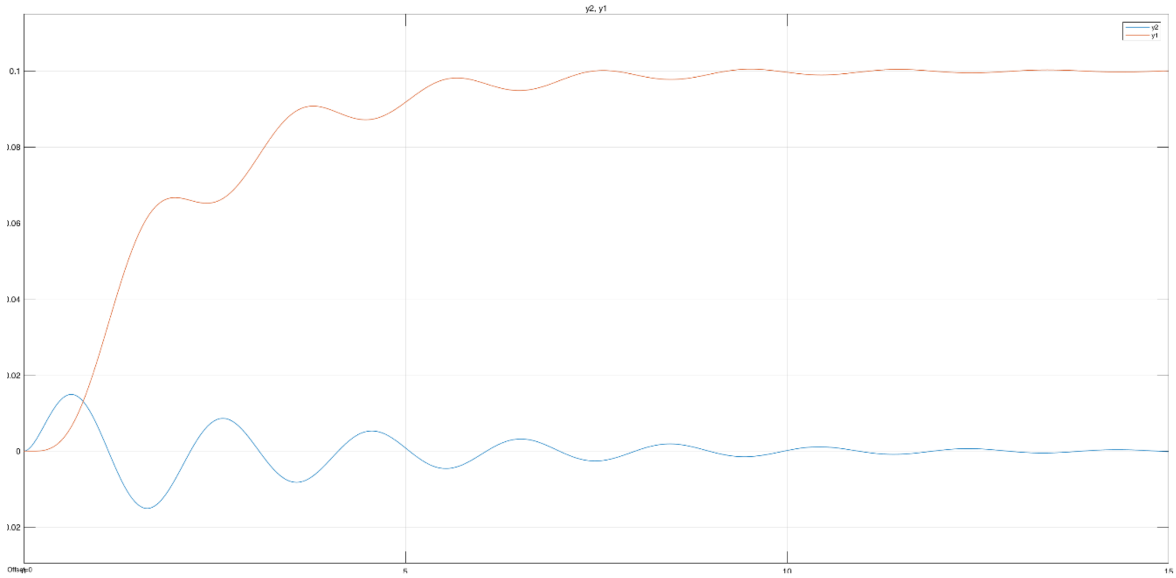




Step response:



As demonstrated, the x_1 state stabilizes in the linear system. However, the nonlinear system remains unsteady even after the removal of the unstable pole with the PID controller.



y_1 $output\ has\ converged\ to\ step\ input\ with\ amplitude = 0.1$