# Advent of Code 2021 Day 7 Part 2

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December 7, 2021

Fuel usage for single ship from posistion $p_i$ to position $k$ :

$$\begin{aligned} f_s(p_i, k) &= 1 + 2 + \cdots + |k - p_i| \\ &= \sum_{n=0}^{|k-p+i|} n \\ &= \frac{(k-p_i)^2 + |k-p_i|}{2} \end{aligned} \tag{1}$$

Fuel usage for all $s$ ships to position $k$ :

$$\begin{aligned} f(k) &= \sum_{i=1}^{s} f_s(p_i, k) \\ &= \sum_{i=1}^{s} \frac{(k-p_i)^2 + |k-p_i|}{2} \end{aligned} \tag{2}$$

To find the solution we need to find the $k$ value that minimizes to function $f$. To find a function minimum we need to calculate it's derivative:

$$f'(k) = \sum_{i=1}^{s} \frac{2(k-p_i) + sgn(k-p_i)}{2} \tag{3}$$

If we comapre it to 0 and rearrange:

$$\sum_{i=1}^{s} \frac{2(k-p_i) + sgn(k-p_i)}{2} = 0 \tag{4}$$

$$\sum_{i=1}^{s} p_i = sk + \sum_{i=1}^{s} \frac{sgn(k-p_i)}{2} \tag{5}$$

This gives us and expression for k in almost closed form:

$$k = \frac{\sum_{i=1}^{s} p_i}{s} - \frac{1}{2} \frac{\sum_{i=1}^{s} sgn(k-p_i)}{s} \tag{6}$$

The second term depends on k, but we may find it's upper and lower bounds. $\sum_{i=1}^{s} sgn(k-p_i)$ is at most $s$, if $p_i > k$ for all $i$. At minimum it is $-s$ , if $p_i < k$ for all $i$. Taking it into account we can bound the second term to values between $-\frac{1}{2}$ and $\frac{1}{2}$. Additionaly if we notice that the first term $\frac{\sum_{i=1}^{s} p_i}{s}$ is the arithmetic mean of $p_i$, the value $k$ is bounded by:

$$\overline{p} - \frac{1}{2} \leq k \leq \overline{p} + \frac{1}{2} \tag{7}$$