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FA20-BCS-049

LINEAR ALGEBRA

ASSIGNMENT 2

QUESTION 1

Q. What is matrix determinant?

Determinant is a scalar value. It is function of the elements of a square matrix. It identifies the nature of a matrix.

e.g.:

A matrix is invertible if its determinant is non-zero.

Determinant can be determined by:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(2×2)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

$$= aei - ahf - bdi - bgf + cdh - cge$$

(3×3)

Co-factor of matrix A

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ a_{31} & \cdots & a_{3m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

Area of parallelogram:

Determinant can also determine area of a parallelogram.

$$\text{Area of parallelogram} = |\det(A)|$$

Volume of parallelpiped

$$= |\det(A)|$$

Example:

Q1 $A = \begin{vmatrix} 1 & 3 & 6 \\ 5 & 2 & 9 \\ 4 & 7 & 8 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & 9 \\ 7 & 8 \end{vmatrix} - 3\begin{vmatrix} 5 & 9 \\ 4 & 8 \end{vmatrix} + 6\begin{vmatrix} 5 & 2 \\ 4 & 7 \end{vmatrix}$$

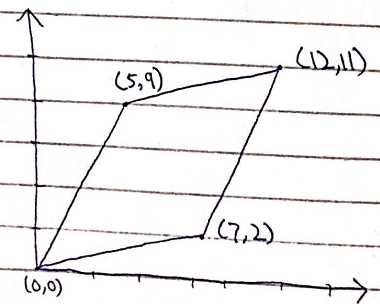
$$= 1(16-63) - 3(40-36) + 6(35-8)$$

$$= -47 - 12 + 162$$

$$= 103$$

Q2. Calculate area of a parallelogram using determinants.

let (0,0) (7,2) (5,9) (12,11) be a parallelogram.



$$V_1 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix} \Rightarrow \text{Area of parallelogram} = |\det A|$$

$$= \begin{vmatrix} 5 & 7 \\ 9 & 2 \end{vmatrix} \Rightarrow 10 - 63 = -53 \Rightarrow |-53| = \boxed{53}$$

Q3 Calculate volume of parallelpiped using determinants.

let (0,0,0), (2,2,-1), (1,3,0), (-1,1,4) make a parallelpiped.

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{bmatrix}$$

$$= \left\| \begin{matrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{matrix} \right\| \Rightarrow 2\begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}$$

$$= |12| = \boxed{12}$$

QUESTION 2-3

Properties of determinants:

1) Matrix is only invertible when determinant is non-zero i.e matrix A is only invertible i.e. A^{-1} when $\det A \neq 0$

$$A^{-1} = \frac{\text{Adj} A}{\det A}, \det A \neq 0$$

(2) Matrix's determinant has the same value as the determinant of its transpose.

i.e $\det A = \det A^t$.

Example

if $A = \begin{bmatrix} 6 & 8 \\ 7 & 11 \end{bmatrix} \Rightarrow \det A = 66 - 56 = \boxed{10}$

$A^t = \begin{bmatrix} 6 & 7 \\ 8 & 11 \end{bmatrix} \Rightarrow \det A^t = 66 - 56 = \boxed{10}$

Proved

(3) When rows or columns of a matrix are interchanged the determinant is negative.

Example

$A = \begin{bmatrix} 11 & 10 \\ 2 & 3 \end{bmatrix} \Rightarrow \det A = 33 - 20 = \boxed{13}$

$A = \begin{bmatrix} 10 & 11 \\ 3 & 2 \end{bmatrix} \Rightarrow \det A = 20 - 33 = \boxed{-13}$

(4) Determinant of a matrix is 0. when:

i) Matrix A has 2 identical rows or columns.

ii) Matrix A has all elements zero.

(i)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \det A = 2-2 = 0$$

(ii)

A null matrix A of order $n \times m$ has $\det A = 0$ (always)

(5) Determinant of identity matrix is always 1.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = \boxed{1}$$

(6) If each entry of a row or a column consists of two terms as a sum then its determinant can be expressed as.

$$A = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

(7) When each entry of a column or a row is added to a non-zero multiple of the corresponding entries of another row or column then determinant of the matrix ~~is~~ remains same.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

if $|B| = \begin{vmatrix} a_{11} & a_{12} + ka_{11} \\ a_{21} & a_{22} + ka_{21} \end{vmatrix}$

$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & ka_{11} \\ a_{21} & ka_{21} \end{vmatrix}$

$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + k \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{vmatrix}$

$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + 0$

$\boxed{|B| = |A|}$

(8) If matrix is triangular then product of entries in its diagonal gives the determinant of that matrix.

$A = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 21 \times 6 \times 2 = 252$