Trying out GroupMath features

In[92]:=
```
(* ::Package::*)(*SU(2) Tensor Products—Minimal Script with GroupMath`*)
(*Load package*)Get["GroupMath`"];

Print["—— SU(2) quick demo (GroupMath`) ——"];

(*THEORY CHEATSHEET-CG rule:j1⊗j2=
 ⊕_{J= |j1-j2|}^{j1+j2} J (step 1)-Dimension check:(2 j1+1)(2 j2+1)=
  Σ_J (2 J+1)-GroupMath reps use Dynkin n=2 j so {1}≡doublet (dim 2),
{2}≡triplet (dim 3),{3}≡quartet (dim 4),…*)

(*Friendly header helper*)
section[s_] := Print["\n--- ", s, " ---\n"];

section["Basic reps (Dynkin → spin → dimension)"];
Print["{1} = spin 1/2 = dim 2"];
Print["{2} = spin 1   = dim 3"];
Print["{3} = spin 3/2 = dim 4"];

section["Decompositions (Clebsch-Gordan)"];
(*2⊗2=3⊕1*)
Print["2⊗2 → ", ReduceRepProduct[SU2, {{1}, {1}}, UseName → True]];

(*3⊗2=4⊕2*)
Print["3⊗2 → ", ReduceRepProduct[SU2, {{2}, {1}}, UseName → True]];

(*3⊗3=5⊕3⊕1*)
Print["3⊗3 → ", ReduceRepProduct[SU2, {{2}, {2}}, UseName → True]];

(*4⊗4=7⊕5⊕3⊕1*)
Print["4⊗4 → ", ReduceRepProduct[SU2, {{3}, {3}}, UseName → True]];

(*2⊗2⊗2*)
Print["2⊗2⊗2 → ", ReduceRepProduct[SU2, {{1}, {1}, {1}}, UseName → True]];

section["Singlet invariants (tensor contractions)"];
(*Note:we predefine symbols so results don’t show as Removed[...]*)
ClearAll[a, b, c];
SetAttributes[{a, b, c}, {NHoldAll}];

Print["Invariants for 2⊗2: ", Invariants[SU2, {2, 2}]];

Print["Invariants for 2⊗2⊗2⊗2: ", Invariants[SU2, {2, 2, 2, 2}]];

section["Permutation symmetry of invariant subspace"];
```

```
Print["Sym(2⊗2): ", PermutationSymmetry[SU2, {2, 2}, UseName → True]];

Print["Sym(2⊗2⊗2⊗2): ", PermutationSymmetry[SU2, {2, 2, 2, 2}, UseName → True]];

section["Generators in fundamental ({1}) and adjoint (3) reps"];
(*Fundamental (doublet)*)
Print["Generators for {1} (2×2):"];
Scan[Print@*MatrixForm, RepMatrices[SU2, {1}]];

(*Adjoint (triplet)*)
Print["\nGenerators for 3 (3×3):"];
Scan[Print@*MatrixForm, RepMatrices[SU2, 3]];

Print["\n-- Done --"];
```

```
XXXXXXXXXXXXXXXXXXXXXXXXXXX GroupMath XXXXXXXXXXXXXXXXXXXXXXXXXXX
Version: 1.1.2 (6/May/2020)
Author: Renato Fonseca
Reference: 2011.01764 [hep-th]
Website: renatofonseca.net/groupmath
Built-in documentation: here
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX-
   XXX
-- SU(2) quick demo (GroupMath`) --

--- Basic reps (Dynkin → spin → dimension) ---

{1} = spin 1/2 = dim 2
{2} = spin 1   = dim 3
{3} = spin 3/2 = dim 4

--- Decompositions (Clebsch-Gordan) ---

2⊗2 → {{3, 1}, {1, 1}}
3⊗2 → {{4, 1}, {2, 1}}
3⊗3 → {{5, 1}, {1, 1}, {3, 1}}
4⊗4 → {{7, 1}, {1, 1}, {5, 1}, {3, 1}}
2⊗2⊗2 → {{4, 1}, {2, 2}}

--- Singlet invariants (tensor contractions) ---

Invariants for 2⊗2: {Removed[a][2] Removed[b][1] - Removed[a][1] Removed[b][2]}
```

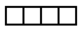
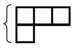
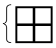
Invariants for 2⊗2⊗2⊗2:

SparseArray[ArrayRules[SparseArray[{}].SparseArray[Specified elements: 2 Dimensions: {2, 2}]], {2}].

{{Removed[a][2] Removed[b][1] Removed[c][1] - Removed[a][1] Removed[b][2]
Removed[c][1]} /. {MapThread[Rule, {{b[1]}, {0, 0, 0, 0, 0}}]}},
{$\sqrt{2}$ Removed[a][1] Removed[b][1] Removed[c][1] + Removed[a][2] Removed[b][1]
Removed[c][2] + Removed[a][1] Removed[b][2] Removed[c][2] +
$\sqrt{2}$ Removed[a][2] Removed[b][2] Removed[c][3]} /.
{MapThread[Rule, {{b[1], b[2], b[3]}, {0, 0, 0, 0, 0, 0, 0}}]}}

--- Permutation symmetry of invariant subspace ---

Sym(2⊗2): {{{1, 2}}, {{{**3**, {}}, 1}, {{**1**, {}}, 1}}}

Sym(2⊗2⊗2⊗2): {{{1, 2, 3, 4}}, {{{**5**, {}}, 1}, {{**3**, {}}, 1}, {{**1**, {}}, 1}}}

--- Generators in fundamental ({1}) and adjoint (3) reps ---

Generators for {1} (2×2):

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Generators for 3 (3×3):

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

–– Done ––

In[78]:=
```
(*Dimensions of fully symmetric/antisymmetric tensor powers*)
(*Sym^k(V) where dim V=d*)
SymDim[d_Integer?NonNegative, k_Integer?NonNegative] /; d ≥ 0 && k ≥ 0 :=
 Binomial[d + k - 1, k]

(*Λ^k(V) where dim V=d*)
AltDim[d_Integer?NonNegative, k_Integer?NonNegative] /; d ≥ 0 && k ≥ 0 :=
 Binomial[d, k] (*this is 0 automatically when k>d*)
```

In[88]:=
```
SymDim[2, 3] (*SU(2) doublet⊗^3,fully symmetric→4 (spin-3/2)*)
AltDim[3, 3] (*ε_{ijk} singlet in SU(3)→1*)
AltDim[3, 4] (*vanishes,cannot antisymmetrize 4 indices in dim 3→0*)
SymDim[5, 0] (*Sym^0 is the scalar→1*)
```

Out[88]=
```
4
```

Out[89]=
```
1
```

Out[90]=
```
0
```

Out[91]=
```
1
```

In[128]:=
```
Symmetric * vs. Antisymmetric * Subspaces – Theory (Mathematica * Text)
```

Out[128]=
```
Mathematica * Subspaces * Symmetric * Text * Theory – vs.Antisymmetric
```

Symmetric vs. Antisymmetric Subspaces — Theory (Mathematica Text)

Setup

We consider a d-dimensional complex vector space ℂ^d (e.g. the defining representation of SU(d)). For k identical copies, the tensor power V⊗…⊗V (k factors) decomposes under S_k into sectors with different permutation symmetry. Two extreme cases are the completely symmetric and completely antisymmetric subspaces.

Dimensions

The dimensions of these extremal subspaces are:

dim $\mathrm{Sym}^k(\mathbb{C}^d)$ == Binomial[$d + k - 1$, k]

dim $\Lambda^k(\mathbb{C}^d)$ == Binomial[d, k]

Here Sym^k denotes the k-fold symmetric power and Λ^k the k-fold exterior (antisymmetric) power. The antisymmetric dimension vanishes for k > d.

In[135]:=

```
(*Dimensions*) SymDim[d_, k_] := Binomial[d + k - 1, k];
AltDim[d_, k_] := Binomial[d, k]; (*fermions, for comparison*)

(*All d-mode occupancy vectors summing to k (stars-and-bars)*)
Occupations[d_Integer?Positive, k_Integer?NonNegative] :=
  Select[Tuples[Range[0, k], d], Total[#] == k &];

(*Quick examples*)
SymDim[3, 2] (* =6 states for 2 photons in 3 modes*)
AltDim[3, 2] (* =3 antisymmetric 2-fermion states in 3 modes*)
Occupations[3, 2]
```

Out[138]=

6

Out[139]=

3

Out[140]=

{{0, 0, 2}, {0, 1, 1}, {0, 2, 0}, {1, 0, 1}, {1, 1, 0}, {2, 0, 0}}

Connection to SU(n) Representations

For SU(n):

• The completely symmetric rank-k irrep corresponds to Dynkin label {k,0,…,0} and has the above dimension Binomial[n + k - 1, k].

• The completely antisymmetric rank-k irrep corresponds to a single box in the k-th Dynkin position (Young diagram (1^k)) and has the above dimension Binomial[n, k].

Linear Optics Interpretation

In linear optics with d spatial/polarization modes and k indistinguishable photons:

• The k-photon Fock subspace across d modes is the completely symmetric subspace of (ℂ^d)^{⊗k }, hence its size is Binomial[d + k - 1, k].

• For fermions, the accessible subspace is antisymmetric with size Binomial[d, k].

A lossless interferometer is a unitary U ∈ U(d) acting on mode creation operators a_j^† → \!\(* UnderoverscriptBox[\(∑\), \(m = 1\), \(d\)]\) U_{m j} b_m^†. For indistinguishable bosons, transition amplitudes between Fock states are given by matrix permanents; for indistinguishable fermions, by determinants.

In[147]:=

```
(*Build a repeated-index list given an occupation vector,
e.g.{2,0,1}→{1,1,3}*)IndexMultiset[occ_List] :=
  Flatten[MapIndexed[ConstantArray[First@#2, #1] &, occ]];

(*Submatrix with rows/cols repeated by occ vectors for output/input*)
RepeatedSubmatrix[U_, outOcc_, inOcc_] := Module[
   {rows = IndexMultiset[outOcc], cols = IndexMultiset[inOcc]}, U[[rows, cols]]];

(*Bosonic transition amplitude:Permanent normalization*)
BosonAmplitude[U_, inOcc_, outOcc_] :=
  Module[{M = RepeatedSubmatrix[U, outOcc, inOcc], nin = inOcc, nout = outOcc},
   Permanent[M] / Sqrt[Times @@ (Factorial /@ nin) * Times @@ (Factorial /@ nout)]];

(*Fermionic transition
 amplitude:Determinant (Pauli exclusion already enforced by occ∈{0,1})*)
FermionAmplitude[U_, inOcc_, outOcc_] :=
  Module[{M = RepeatedSubmatrix[U, outOcc, inOcc]}, Det[M]];

(*Probability*)
BosonProb[U_, inOcc_, outOcc_] := Abs[BosonAmplitude[U, inOcc, outOcc]]^2;
FermionProb[U_, inOcc_, outOcc_] := Abs[FermionAmplitude[U, inOcc, outOcc]]^2;
```

Hong–Ou–Mandel (HOM) Example

Two photons, two modes, balanced beamsplitter:

$$U == 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Input $|1,1\rangle$ transforms such that the coincidence $|1,1\rangle$ amplitude vanishes ("HOM dip"), while photons bunch into $|2,0\rangle$ and $|0,2\rangle$ with equal probability 1/2. For fermions, Pauli exclusion forbids $|2,0\rangle$ and $|0,2\rangle$, leaving perfect anti-bunching into $|1,1\rangle$.

In[153]:=

```
(*50:50 beamsplitter (real convention)*)UBS = (1 / Sqrt[2]) {{1, 1}, {1, -1}};

in = {1, 1};
outs = {{2, 0}, {0, 2}, {1, 1}};

(*Bosons (photons)*)
Table[out → BosonProb[UBS, in, out], {out, outs}]
(*Expected:{2,0}→1/2,{0,2}→1/2,{1,1}→0 (HOM dip)*)

(*Fermions (for contrast;only occupancies≤1 are physical)*)
fermOuts = {{1, 1}}; (*{2,0} and {0,2} are excluded by Pauli*)
Table[out → FermionProb[UBS, in, out], {out, fermOuts}]
(*Expected:{1,1}→1*)
```

Out[156]=

$$\left\{\{2, 0\} \to \frac{1}{2}, \{0, 2\} \to \frac{1}{2}, \{1, 1\} \to 0\right\}$$

Out[158]=

$$\{\{1, 1\} \to 1\}$$

Three-mode interferometer with two photons

In[159]:=

```
SeedRandom[1];
d = 3; k = 2;
basis = Occupations[d, k]; (*six states*)

(*Random unitary via QR of random complex matrix*)
rand = RandomComplex[NormalDistribution[0, 1], {d, d}] +
   I RandomComplex[NormalDistribution[0, 1], {d, d}];
{q, r} = QRDecomposition[rand];
phase = DiagonalMatrix[Exp[-I Arg[Diagonal[r]]]];
U3 = q.phase; (*Haar-ish unitary*)

(*Choose input|2,0,0>*)
in = {2, 0, 0};

(*All output probabilities*)
assoc = Association@Table[out → BosonProb[U3, in, out], {out, basis}];
N@Normal@assoc
Total[assoc] (*sanity:sums to 1*)
```

Out[168]=

```
{{0., 0., 2.} → 0.195296, {0., 1., 1.} → 0.409754, {0., 2., 0.} → 0.214928,
 {1., 0., 1.} → 0.0835001, {1., 1., 0.} → 0.0875965, {2., 0., 0.} → 0.00892525}
```

Out[169]=

```
1.
```