Functions

Calculus I

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Week 2

Outline for the Day

Definition of Functions

Function is a rule of *correspondence* that assigns each object x in one set, called the **domain**, to a single value of $f(x)$ from a second set. The set of all values so obtained is called the *range*.

Illustration

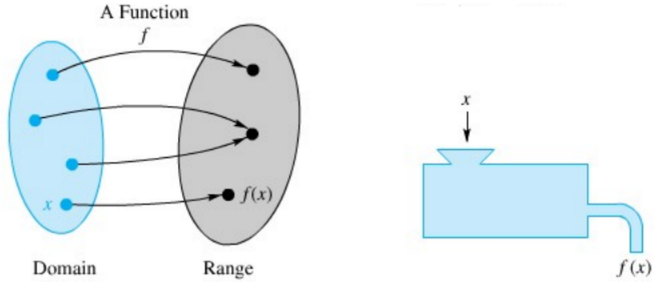


Figure: Visualization of Function as a Machine

Some Notes

- Not all "materials" can fit into any type of machine. So are functions, the collection of materials that can be put in some particular function, are called the **domain** of such functions.
- The output of a function *machine* is called the **range** of said function.
- Consistency: If two **identical** materials were put into two **identical** machines, the results will always be **identical**.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ (is read as "f is a function from a real number to a real number" or "f is a real-to-real function" for short) with $f : x \mapsto 3x - 1$ (read as "f sends/maps/assigns x to $3x - 1$", some other times it's going to be written as $f(x) = 3x - 1$).

- $f(1) = 3 \times 1 - 1 = 2$
- $f(-2) = 3 \times (-2) - 1 = -7$
- $f(a) = 3a - 1$

On Finding the (natural) Domains of a Function

Key point: **Exclude** the *problematic* numbers—if there aren't any, then the domain is **all Real numbers.**

Example: Find the natural domains of these functions:

1. $f(x) = \dfrac{x+1}{x-1}$
2. $g(x) = 4x - \sqrt{x^2 - 4}$
3. $h(x) = \log(3x+4)$

Steps of sketching a simple function:

1. Plug in *some* values from the domain into the function.
2. Plot the resulting points you have from step 1 into the cartesian coordinate as an ordered pair $(x, f(x))$. The horizontal axis (x-axis) and vertical axis (y-axis) will correspond to the domain (x value) and range (f(x) value) respectively.
3. Connect the dots to approximate the in-between values to make a **smooth curve**.

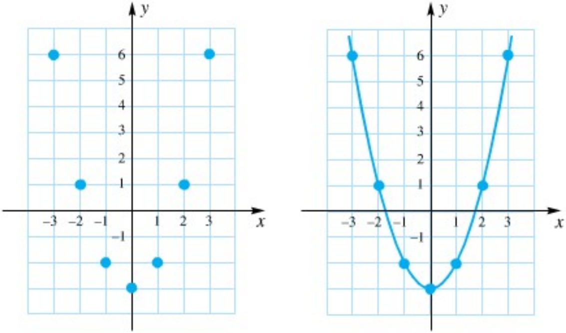
Example

Sketch the function $f(x) = x^2 - 3$.

$y = x^2 - 3$

x	y
−3	6
−2	1
−1	−2
0	−3
1	−2
2	1
3	6

Step 1
Make a table of values.



Step 2
Plot those points.

Step 3
Connect those points with a smooth curve.

Figure: Steps of Sketching Simple Functions

Special Function Graphs

Special Function 1: **The Absolute Value Function**

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

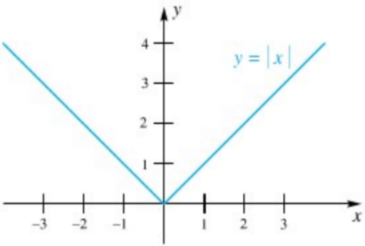


Figure: The Graph of Absolute Function $f(x) = |x|$

Special Function Graphs

Special Function 2: **The Floor Function**
It's to state the biggest integer less than or equal to x.

$$f(x) = [x] = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ \dots \end{cases}$$

Special Function Graphs

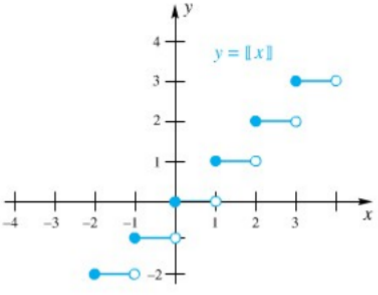


Figure: The Graph of a floor function $f(x) = [x]$

On Odd and Even Functions

We can divide a function into two specific classes in regards of their **symmetry**.

- A function f is called an **even function** if it satisfies $f(-x) = f(x)$ for all x in its domain.
- A function f is called an **odd function** if it satisfies $f(-x) = -f(x)$ for all x in its domain.

These two types of functions can be distinguished easily by their graphs, for even functions are symmetrical to the **y-axis** while odd functions are symmetrical to **the origin point (0,0)**.

Example

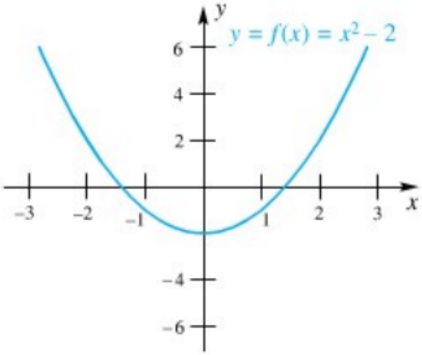


Figure: Example of an Even Function

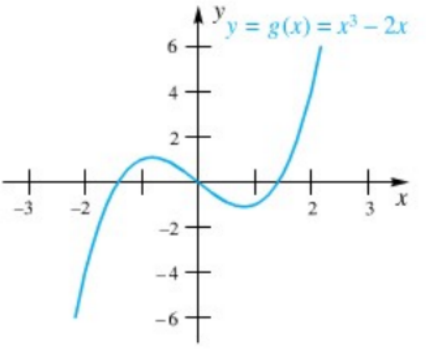


Figure: Example of an Odd Function

Keynote

Important note: A function can be **neither odd nor even**, so if the function is not even, it doesn't necessarily mean that the function is odd.

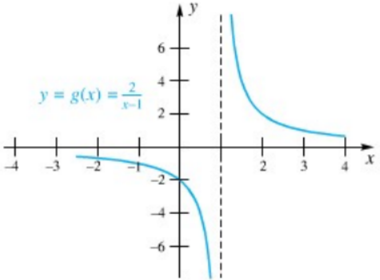


Figure: The function $g(x) = \dfrac{2}{x-1}$ is neither odd nor even

Algebraic Operations of Functions

Just like numbers, we can also perform some basic algebraic operations on functions. Let $f, g : \mathbb{R} \to \mathbb{R}$, with D_f and D_g being the domains of f and g respectively

- $(f+g)(x) = f(x) + g(x)$ with domain $= D_f \cap D_g$
- $(f-g)(x) = f(x) - g(x)$ with domain $= D_f \cap D_g$
- $(fg)(x) = f(x) \times g(x)$ with domain $= D_f \cap D_g$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ with domain $= (D_f \cap D_g) - \{x : g(x) \neq 0\}$

Composition of Functions

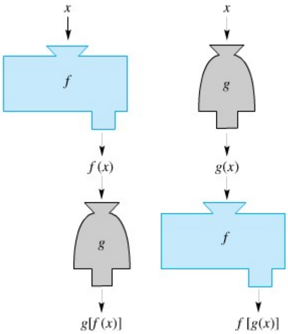


Figure: Illustration of Composite Functions

Composition of Functions

Let $f, g : \mathbb{R} \to \mathbb{R}$ and Range of $f \subseteq$ Domain of g, then:

$$g \circ f(x) = g\left(f(x)\right)$$

Example: If $f(x) = x - 3$ and $g(x) = x^2 + 2$, then we can formulate the composite function $g \circ f(x)$ as

$$g \circ f(x) = g\left(f(x)\right) = g(x-3) = (x-3)^2 + 2 = x^2 - 6x + 11$$

Basic $\sin x$ and $\cos x$ function

The values of $\sin x$ and $\cos x$ on some special angles

t	$\sin t$	$\cos t$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$
π	0	-1

Figure: Trig values for special angles

Graph of $y = \sin x$ and $y = \cos x$

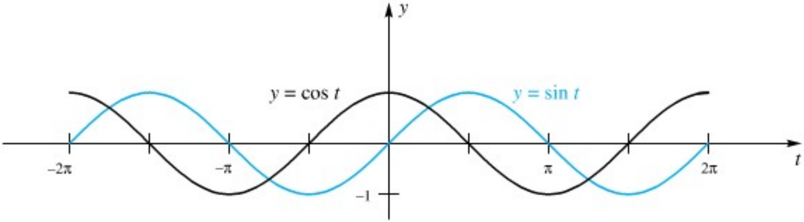


Figure: The basic graph of $\sin x$ and $\cos x$

Some Key Takeaways

1. The domain for $\sin x$ and $\cos x$ functions are all real numbers ($\mathbb{R}$).
2. The range for $\sin x$ and $\cos x$ functions are $-1 \leq y \leq 1$.
3. Both graph repeat itself after every 2π, this value is called the **period** of each graph.
4. The graph $\sin x$ is symmetrical by the origin while $\cos x$ is symmetrical by the $y-$axis. Hence, we know that $\sin x$ is odd while $\cos x$ is even.
5. The two graphs are practically similar, you can obtain $\cos x$ by translating $\sin x$ by $\pi/2$ units to the **left**.

The graph $y = \tan x$

Since we know that

$$\tan x = \frac{\sin x}{\cos x},$$

you can obtain the values for $\tan x$ in special angles by straight division. The graph will look like this:

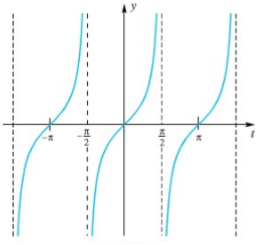


Figure: The graph of $y = \tan x$

On Amplitudes and Periods

- **Amplitudes** of a trigonometric functions indicates the maximum **deviation** of such function from the baseline (or x-axis). The amplitude of a normal $\sin x$ and $\cos x$ function is 1.
- **Periods** of a trigonometric functions indicates **how long it takes** for such function **to repeat itself**. The period of a normal $\sin x$ and $\cos x$ function is 2π.

The functions $y = A \sin Bx$ and $y = A \cos Bx$

1. The functions $y = A \sin Bx$ and $y = A \cos Bx$ will be similar to its *parent function* of $y = \sin x$ and $y = \cos x$ respectively.
2. The functions $y = A \sin Bx$ and $y = A \cos Bx$ have an amplitude of A and a period of $\dfrac{2\pi}{B}$.
3. The **key component** of each graph (axes intercept(s), maximum/minimum value, etc) change slightly with regards to their **amplitude** and **period**.

Some Examples

For the function $y = \sin 2\pi t$, the amplitude is 1 while the period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{2\pi} = 1$. Hence,

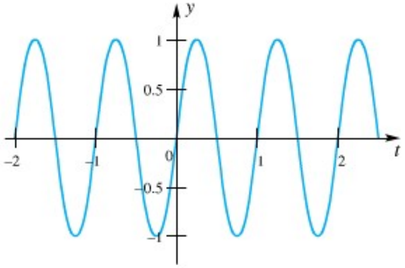


Figure: The graph of $y = \sin 2\pi t$

Some Examples

For the function $y = 2\cos 4x$, the amplitude is 2 while the period is $\frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$. Hence,

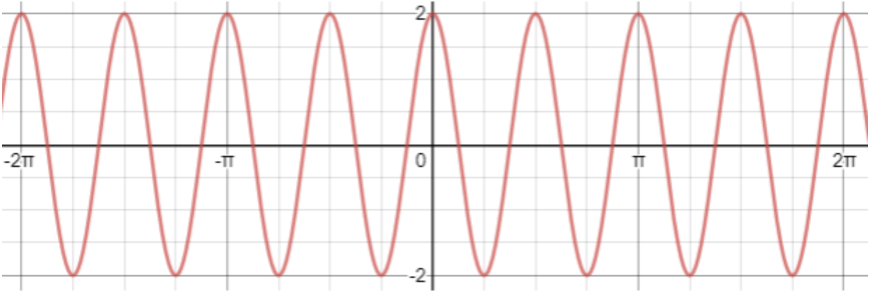


Figure: The graph of $y = 2\cos 4t$

Some Exercise

- Exercise 0.5: Problem 1, 4, 11, 13, 15, 19, 21, 25, 27, 31, 33, 35, 37
- Exercise 0.6: Problem 1, 4, 5, 11-14, 33
- Exercise 0.7: Problem 9, 10, 14ab, 16, 17