Differentiation part 2

Calculus I

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Week 5

Outline for the Day

Some Trig Identities

1. $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
2. $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
3. $\lim_{h\to 0}\frac{\sin h}{h} = 1$

Finding $d/dx\,(\sin x)$ and $d/dx\,(\cos x)$

Questions posed: How do we find $\dfrac{d}{dx}(\sin x)$ and $\dfrac{d}{dx}(\cos x)$?

A reminder on first principle differentiation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation of $\sin x$

Given $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\dfrac{x+h+x}{2}\right)\sin\left(\dfrac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{2x+h}{2}\right) \lim_{h\to 0} \frac{\sin\left(\dfrac{h}{2}\right)}{h}$$

$$= 2\cos x \times \frac{1}{2}$$

$$= \cos x$$

Differentiation of $\cos x$

Given $f(x) = \cos x$, then

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h\to 0} -2\sin\left(\frac{2x+h}{2}\right) \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{h}$$

$$= -2\sin x \times \frac{1}{2}$$

$$= -\sin x$$

For Other Trig Functions

To find $\frac{d}{dx}(\tan x)$, we use the fact that

$$\tan x = \frac{\sin x}{\cos x}$$

then

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

For Other Trig Functions

The case $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$ can be handled similarly:

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{(1)'\cos x - 1(\cos x)'}{(\cos x)^2}$$

$$= \frac{0-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

You can differentiate the remaining trig functions (i.e. $\csc x$, $\cot x$) by first principle using similar method as an exercise.

Recap Table

Here's a recap table of derivatives for basic trigonometric function.

y	y'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\tan x \sec x$
$\csc x$	$-\cot x \csc x$
$\cot x$	$-\csc^2 x$

Chain Rule for Trig Functions

Rule of thumb:

$$\frac{d}{dx}\left(\mathrm{trig}(u(x))\right) = \mathrm{trig}'(u(x)) \times u'(x)$$

Example: Find dy/dx if

1. $y = \tan(3x^2 + x - 11)$
2. $-4\csc\left((3x+2)^{10}\right)$

The Natural Exponential Function

1. The number $e = 2.78...$, like π, is one of the most important irrational numbers in maths.
2. It is used in the study of compound interests and infinite series among other important applications.
3. In differentiation, the natural exponential function $f(x) = e^x$ also holds some significance, since:

$$\frac{d}{dx}\left(e^x\right) = e^x$$

4. Chain rule version of the differentiation:

$$\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)} \cdot u'(x)$$

Natural Logarithm Function

1. The natural logarithm function, $\ln x$, is basically a logarithmic function with e as the base.

$$\ln x = \log_e x$$

2. We use implicit method to differentiate $\ln x$, that is:

$$y = \ln x \Rightarrow e^y = x \Rightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\Rightarrow e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Hence,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

and

$$\frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)}$$

Differentiating $\log_a x$

You may recall one particular property of logs:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Hence,

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{(\ln a)x}$$

Or in chain rule form,

$$\frac{d}{dx}(\log_a(u(x))) = \frac{u'(x)}{(\ln a)u(x)}$$

Differentiating a^x

We can once again use an implicit method

$$y = a^x \Rightarrow \log_a y = x \Rightarrow \frac{d}{dx}(\log_a y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{(\ln a)y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = (\ln a)y = (\ln a)a^x$$

Or in the chain rule form

$$\frac{d}{dx}\left(a^{u(x)}\right) = a^u(x) \cdot (\ln a) \cdot u'(x)$$

Recap Table

Here's a recap table of derivatives for basic exponential and logarithmic function.

y	y'
e^x	e^x
a^x	$(\ln a)a^x$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{(\ln a)x}$

Example: Chain Rule

Find y' if:

1. $y = \ln(3x^2 - 5)$
2. $y = e^{-\sin(4x^2 - x + 5)}$

Advanced Implicit Differentiation involving Trigs and Logs

Find $\dfrac{dy}{dx}$ if:

1. $\sin(xy) = 4x^2 - 9y^2$
2. $e^{x^2} + \tan(xy) = \dfrac{4x}{y} - y^3$

Exercises

1. Exercise 2.4: 5-12 all odd numbers
2. Exercise 2.5: 9, 11, 15, 27, 35, 38
3. Exercise 2.7: 11, 12, 29, 30
4. Exercise 6.1: 3, 5, 7, 9, 14
5. Exercise 6.3: 11-22 all odd numbers