Continuity and Differentiation part 1

Calculus I

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Week 4

Outline for the Day

Definition of Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous** over its domain if we can sketch the graph in a single stroke of pencil (or pen for that matter). It means the graph has no holes nor asymptotes over the course of its domain.

Continuity Test at a Certain Point

A function f is continuous at $x = c$ if it satisfies these conditions

- $f(c)$ exists
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

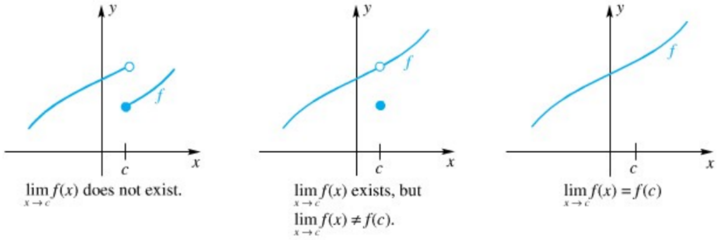


Figure: Illustration: Various cases of f at $x = c$

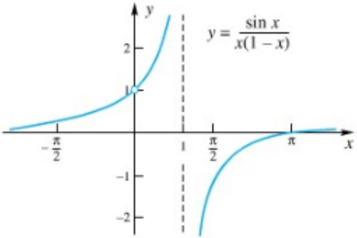
Interval Continuity

Let $f : \mathbb{R} \to \mathbb{R}$ and $I \subseteq \mathbb{R}$. Function f is said to be continuous over the interval I if it's continuous at $x = c$ for all $c \in I$.
Popular classes of continuous functions:

- A linear function $f(x) = ax + b$ is continuous in $\mathbb{R}$.
- A polynomial function in general is continuous in $\mathbb{R}$.
- A rational function $f(x) = \dfrac{p(x)}{q(x)}$ is continuous over $\mathbb{R}$ except at some c values such that $q(c) = 0$.
- A root function $f(x) = \sqrt{p(x)}$ is continuous over $\mathbb{R}$ except at some c values such that $q(c) < 0$.

Two Types of Discontinuity

Look at these following cases of discontinuity on function $f(x) = \dfrac{\sin x}{x(1-x)}$. From the previous slides we can easily verify that f discontinuous at both $x = 0$ and $x = 1$.



But there's a substantial difference with the behavior of f at those two points.

Two Types of Discontinuity

From the previous slides.

- The discontinuity at $x = 0$ is called a **hole** in f. This happens when $\lim_{x \to c} f(x)$ exists. This type of discontinuty is also called a **removable discontinuity**.
- The discontinuity at $x = 1$ is called an **asymptote** in f. In most of the cases, this happens when $\lim_{x \to c} f(x)$ doesn't exist in the first place. This type of discontinuity is also called a **non-removable discontinuity**.

Handling Removable Discontinuity

We can tinker with a removable discontinuity problem and making the function continuous over all real number by modifying the function a little. Let $f : \mathbb{R} \to \mathbb{R}$ and has a hole at $x = c$ but is continuous everywhere else. Then we can *plug the hole* at $x = c$ by redefining a new function:

$$f_1(x) = \begin{cases} f(x) & \text{when } x \neq c \\ \lim\limits_{x \to c} f(x) & \text{when } x = c. \end{cases}$$

Example of Function with Removable Discontinuity

- We're going to use the function that we used at the beginning of this chapter: $f(x) = \dfrac{x^3 - 1}{x - 1}$.
- As we learned last week, we know that $\lim_{x \to 1} f(x) = 3$ and from what we learned this week, f is **only** discontinuous at $x = 1$ hence creating a hole in $f(x)$ when $x = 1$.
- We can redefine the value of f at $x = 1$ that makes f continuous everywhere in $\mathbb{R}$:

$$f_1(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & \text{when } x \neq 1 \\ 3 & \text{when } x = 1 \end{cases}$$

The value 3 on the second case comes from the limit of f as x approaches 1.

Example of Function with Removable Discontinuity

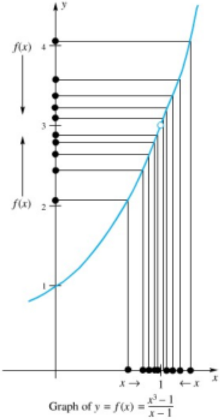


Figure: f has a hole at $x = 1$ that can be *plugged in* by redefining $f(1) = 3$

Case Study for Instantaneous Velocity

Consider this case:
A particle moves along a straight line with its position relative to origin follows the equation:

$$s(t) = 2t + t^2$$

Find the particle's *instantaneous* velocity at $t = 2$?

A Case of Average Velocity

Velocity of the particle at the instant of $t = 2$ is quite difficult to find. However, we can find an average velocity of the particle **around the time** $t = 2$. As we know,

$$v_{\text{average}} = \frac{\text{Particle's Displacement}}{\text{Time needed to travel}} = \frac{s_{t=c} - s_{t=2}}{c - 2}$$

For example, the average velocity between $t = 2$ and $t = 3$ is

$$v_{\text{average}} = \frac{s(3) - s(2)}{3 - 2} = \frac{15 - 8}{1} = 7 \text{ unit/sec}$$

This should give us an idea about the particle's instantaneous velocity at $t = 2$, *however...*

Getting the Gap as *Close* as Possible

- **A LOT** can happen in the one second between $t = 2$ and $t = 3$, so that the previous result (7 unit/sec) doesn't quite picture the case of *instantaneous* velocity at $t = 2$.
- To improve accuracy, we can close the gap a little, by getting the second position as close to $t = 2$ as possible.
- The following table gives you a various cases of c, position at $t = c$, and the average velocity between $t = 2$ and $t = c$.

Table of Average Velocity at Various Cases Close to $t = 2$

Position at $t = 2$ is 8 unit away from the origin.

c	s at $t = c$	$v_{\text{average}} = \dfrac{s(c) - s(2)}{c - 2}$
2.5	11.25	6.5
2.1	8.61	6.1
2.01	8.0601	6.01
2.001	8.006001	6.001
$2 + h$	$s(2 + h)$	$\dfrac{s(2 + h) - s(2)}{h}$

The closer the c value is to 2, the more accurate v_{average} is to its actual instantaneous velocity. Now what will happen if the h value from the last row is made to be **very small**?

Differentiation: Genesis

It makes perfect sense that instantaneous velocity of the particle at $t = 2$ is, in fact, **the average velocity between** $t = 2$ **and** $t = 2 + h$ **where** h **is very small**. Hence,

$$v(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$

with $v(2)$ being the **instantaneous velocity** at $t = 2$. Hence, by simple limit procedure, we can find that $v(2) = 6$.

Differentiation by First Principle

- That procedure from the past few slides is also called **differentiation** of the s function at $t = 2$. In differentiation notation, we can say that $s'(2) = v(2) = 6$.
- A lot of real world problems use similar concept like the ones we used on the instantaneous velocity problem.
- Mathematicians then generalize such process by defining a **first principle definition** of differentiation. Let $f : \mathbb{R} \to \mathbb{R}$, then
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$
- If the limit exists, then f is said to be **differentiable** at $x = c$ and the value $f'(c)$ is called **the (first) differentiation** of f at $x = c$

Differentiated Function

If $f : \mathbb{R} \to \mathbb{R}$ then the differentiation of (function) f, denoted by f', is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Other notation for differentiation of $y = f(x)$:

$$\frac{dy}{dx}, \frac{df(x)}{dx}, D_x y, D_x(f(x))$$

Example

Find $f'(x)$ if given that

1. $f(x) = 1 - 4x^2$
2. $f(x) = \dfrac{5}{x}$

The Power Rule

We can see from previous examples that differentiating a function by first principle takes a lot of brainpower. So here's a few general rules to speed up the differentiation process.

$$\textit{If } f(x) = x^n, \ \textit{then } f'(x) = nx^{n-1} \ \forall n \in \mathbb{Z}$$

Operation of Functions

If f, g are differentiable over $\mathbb{R}$, we have:

1. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g(x)$
2. $\dfrac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
3. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain Rule of Power Function

Let $f(x) = (g(x))^n$, then $f'(x) = n\,(g(x))^{n-1} \cdot g'(x)$.

On a more general form, let $f(x) = g(h(x))$, then $f'(x) = g'(h(x)) \times h'(x)$.

Example: Find $f'(x)$ when $f(x) = \left(x^3 - 2x^2 + x\right)^{15}$.

Implicit Differentiation

Not all relation can be expressed *explicitly* (in $y = f(x)$ form). Some other time the x's and y's gets tangled up in the middle of the expression making it pretty difficult to isolate the y to the form $y = f(x)$.

Here is one example, how are we going to find $\dfrac{dy}{dx}$ from this expression:

$$x^3y - 4y^2 = -2x^2 + 5$$

Keynote

- By (generalized) chain rule, we know that

$$\frac{d}{dx}g(y) = \frac{d}{dy}g(y) \times \frac{dy}{dx}$$

- To start the differentiation process, perform operator $\frac{d}{dx}$ in both sides of the equation. Then using the laws of differentiation, differentiate each term separately.
- Try to isolate $\frac{dy}{dx}$ by putting them on one side.

Example

Find $\dfrac{dy}{dx}$ if $x^3y - 4y^2 = -2x^2 + 5$.

$$
\begin{aligned}
x^3y - 4y^2 &= -2x^2 + 5 \\
\frac{d}{dx}\left(x^3y - 4y^2\right) &= \frac{d}{dx}\left(-2x^2 + 5\right) \\
\left(\frac{d}{dx}\left(x^3\right)\right)y + x^3\left(\frac{d}{dx}(y)\right) - \frac{d}{dx}\left(4y^2\right) &= \frac{d}{dx}\left(-2x^2\right) + \frac{d}{dx}(5) \\
3x^2y + x^3 \cdot 1 \cdot \frac{dy}{dx} - 8y\frac{dy}{dx} &= -4x + 0 \\
x^3 \cdot 1 \cdot \frac{dy}{dx} - 8y\frac{dy}{dx} &= -4x - 3x^2y \\
\frac{dy}{dx}\left(x^3 - 8y\right) &= -4x - 3x^2y \\
\frac{dy}{dx} &= \frac{-4x - 3x^2y}{x^3 - 8y}
\end{aligned}
$$

Exercise

- Exercise 1.6: Problems 18-23 and all odd numbers 1-15
- Exercise 2.2: All odd numbers 1-13
- Exercise 2.3: All odd numbers 9-45
- Exercise 2.5: All odd numbers 7-19
- Exercise 2.7: 5, 7, 13, 14, 17, 33.