Limit of Functions

Calculus I

Wisnu Aribowo

President University, Cikarang

Week 3

Outline for the Day

The Informal Definition of Limit

The expression

$$\lim_{x \to c} f(x)$$

is read as **the limit of** $f(x)$ **as** x **tends to (approaches)** c. If $f(x)$ approaches a particular $L \in \mathbb{R}$ value, then the expression

$$\lim_{x \to c} f(x) = L$$

roughly translates to $f(x)$ is *very close* to L whenever x is *very close* to c but not **exactly** equal to c.

Illustration

We can consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}.$$

From previous lesson we can straightaway figure out that the function is undefined at $x = 1$. However, we can calculate the values of $f(x)$ when x is *very close* to 1.

Illustration

x	$y = \frac{x^3-1}{x-1}$
1.25	3.813
1.1	3.310
1.01	3.030
1.001	3.003
$\downarrow$	$\downarrow$
1.000	?
$\uparrow$	$\uparrow$
0.999	2.997
0.99	2.970
0.9	2.710
0.75	2.313

Figure: Table of values for x close to 1

From the table, we can *intuitively* say that $f(x)$ is *very close* to 3 whenever x is *very close* to 1. Hence

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$$

.

One That Fools the Eyes

Find

$$\lim_{x \to 0} \left(x^2 - \frac{\cos x}{10000} \right)$$

Now from table of values:

x	$x^2 - \dfrac{\cos x}{10{,}000}$
± 1	0.99995
± 0.5	0.24991
± 0.1	0.00990
± 0.01	0.000000005
$\downarrow$	$\downarrow$
0	?

One That Fools the Eyes

From the previous table of values it seems like the $f(x)$ value is approaching 0 as x tends to 0. However, from direct substitution we can have:

$$f(0) = 0^2 - \frac{\cos 0}{100000} = 0 - \frac{1}{10000} = -0.0001$$

Which means:

$$\lim_{x \to 0} \left(x^2 - \frac{\cos x}{10000} \right) = -0.0001$$

Drawbacks

- The notion *very close* is so loosely defined—us mathematicians prefer a well-defined notion. How *close* is **very close**? 0.1? 0.01? 0.000001?! The previous slide is a perfect example of how it's not always possible to *intuitively* calculate a limit value.
- Doing table of values or graphing method for one limit problem is tedious. Takes a lot of time just to guess which values close enough to the c, let alone guessing the limit value.

Limit: The Formal Definition

The expression

$$\lim_{x \to c} f(x) = L$$

mathematically means: *For every* $\epsilon > 0$ *(no matter how small) there exists* $\delta > 0$ *such that whenever* $0 < |x - c| < \delta$ *we will have* $|f(x) - L| < \epsilon$*.*

This definition is often referred to as the $\epsilon - \delta$ definition.

Visualization of $\epsilon - \delta$ Definition

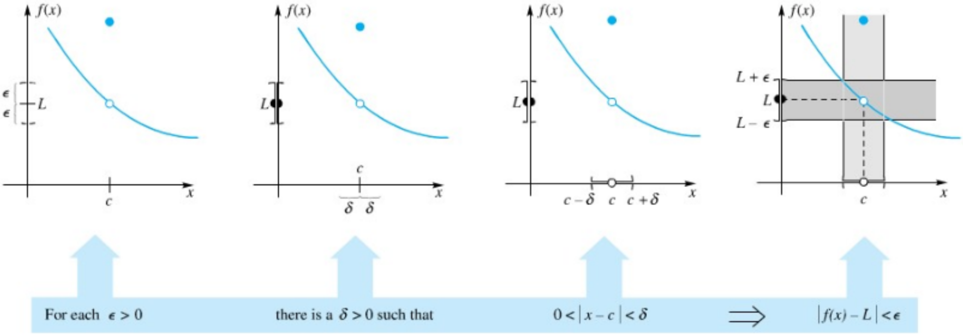


Figure: Illustration of the Formal Definition

Definition: One Sided Limit

- L is called **the left-side limit** of $f(x)$ as x tends to c, denoted by
$$\lim_{x \to c^-} f(x) = L;$$
if whenever x is *very close to but not equal to* c **from the left** $(x < c)$, $f(x)$ is also *very close* to L.
- L is called **the right-side limit** of $f(x)$ as x tends to c, denoted by
$$\lim_{x \to c^+} f(x) = L;$$
if whenever x is *very close to but not equal to* c **from the right** $(x > c)$, $f(x)$ is also *very close* to L.

Theorem: One Sided Limit

The limit

$$\lim_{x \to c} f(x)$$

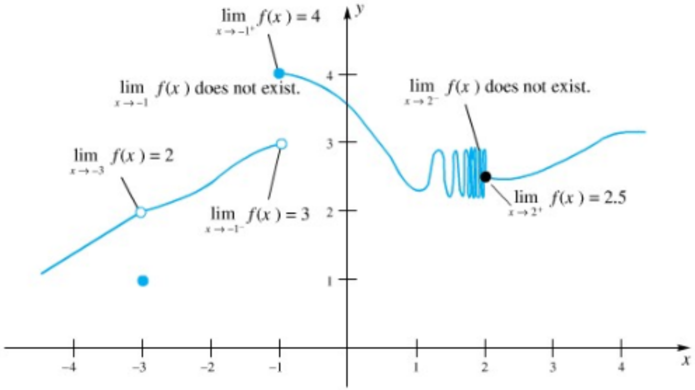
exists if and only if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x).$$

Please also note that *each* one-sided limit can exists without the existence of its *normal* limit.

Illusration: One Sided Limit

Look at the graph of a **very dynamic** $f(x)$ function to give you a better grasp of one-sided limit



Some Basic Results

Let $c, k \in \mathbb{R}$ be arbitrary constants.

1. $\lim\limits_{x \to c} 1 = 1$
2. $\lim\limits_{x \to c} k = k$
3. $\lim\limits_{x \to c} x = c$

Algebraic Properties of Limit

Let $f(x), g(x)$ be functions of real values and $k \in \mathbb{R}$ is an arbitrary constant.

1. $\lim\limits_{x \to c} (f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$
2. $\lim\limits_{x \to c} (f(x) - g(x)) = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$
3. $\lim\limits_{x \to c} (kf(x)) = k \left(\lim\limits_{x \to c} f(x) \right)$
4. $\lim\limits_{x \to c} (f(x) \times g(x)) = \lim\limits_{x \to c} f(x) \times \lim\limits_{x \to c} g(x)$
5. $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ provided that $g(x) \neq 0$ whenever x is close to c and $\lim\limits_{x \to c} g(x) \neq 0$

Polynomial Functions

If a function $p(x)$ takes the form of

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

with $n \in \mathbb{N}$, $a_0, a_1, \cdots, a_n \in \mathbb{R}$, and $a_n \neq 0$, then $p(x)$ is called a **polynomial function**.
Since the domain of any polynomial function is all real number, then its limit at any $c \in \mathbb{R}$ value is equal to its function value, or

$$\lim_{x \to c} p(x) = p(c)$$

Rational and Square Root Function

Let $p(x), q(x)$ be polynomials. We have:

1. $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$ provided that $q(c) \neq 0$ and $q(x) \neq 0$ whenever x is close to c.
2. $\lim_{x \to c} \sqrt{p(x)} = \sqrt{p(c)}$ provided that $p(c) \geq 0$ and $p(x) \geq 0$ whenever x is close to c.

The Squeeze Theorem

Let f, g, h be real valued functions that satisfy

$$f(x) \leq g(x) \leq h(x)$$

for x values near c. If we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L$$

Illustration for Squeeze Theorem

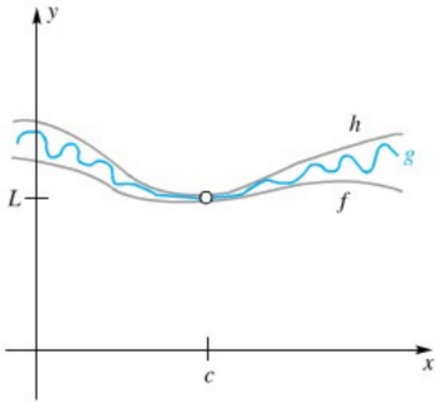


Figure: Visualization of the Squeeze Theorem

Evaluating Limit Values for Algebraic Functions

Here are the steps of determining the value of $\lim_{x \to c} f(x)$.

1. Try plugging in $x = c$ to the function $f(x)$
   - If $f(c) \in \mathbb{R}$, then $\lim_{x \to c} f(x) = f(c)$
   - If $f(c)$ involves $p/0$ form with $p \neq 0$, then the limit doesn't exist
   - If $f(c)$ involves $0/0$, then go to step 2
2. Try to factorize $(x - c)$ from the numerator and denominator, then cancel them out.
3. Return to step 1.

Example

Calculate

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

Plugging in $x = 1$ will lead to $0/0$. Hence, we should try to factorize $(x - 1)$ from both the numerator and denominator.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x - 1)}(x^2 + x + 1)}{\cancel{x - 1}} = \lim_{x \to 1} \left(x^2 + x + 1\right) = 3$$

Trig Identities

Identities = Equations that hold through for *all* x values. Example of some of the more popular trigonometric identities:

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Special Trigonometric Limit

Here are a couple of trigonometric limits as $x \to 0$ that's going to be important in evaluating trigonometric limit in general:

1. $\lim_{x \to 0} \cos x = 1$
2. $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1$
3. $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{x}{\tan x} = 1$

Examples

Calculate

$$\lim_{x \to 0} \frac{5x^2}{4 - 4\cos^2 x}$$

Direct substitution should easily gives us $0/0$ expression so we have to tinker the limit a little with identities.

$$\lim_{x \to 0} \frac{5x^2}{4 - 4\cos^2 x} = \lim_{x \to 0} \frac{5x^2}{4(1 - \cos^2 x)}$$

$$= \lim_{x \to 0} \frac{5x^2}{4\sin^2 x} = \frac{5}{4}\left(\lim_{x \to 0} \frac{x}{\sin x}\right)\left(\lim_{x \to 0} \frac{x}{\sin x}\right)$$

$$= \frac{5}{4} \times 1 \times 1 = \frac{5}{4}$$

Definitions for Limit Involving Infinity

- The expression
$$\lim_{x\to\infty} f(x) = L$$
means $f(x)$ is close to L whenever x is very large.
- The expression
$$\lim_{x\to c} f(x) = +\infty$$
means $f(x)$ is very large whenever x is close to c.
- The expression
$$\lim_{x\to\infty} f(x) = +\infty$$
means $f(x)$ is very large whenever x is very large.

Evaluating Infinity Limits: Rational Form

Let $a_m, b_n > 0$ and

$$L := \lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}.$$

We will have

1. $L = +\infty$ only if $m > n$
2. $L = 0$ only if $m < n$
3. $L = a_m/b_n$ only if $m = n$

Examples

Calculate

$$\lim_{x\to\infty} \frac{(-2x+1)^5}{(x^2-4x+1)(16x^3-10x^2+7x-99)} = \cdots$$

Answer:

$$\lim_{x\to\infty} \frac{(-2x+1)^5}{(4x^2-4x+1)(16x^3-10x^2+7x-99)}$$

$$= \lim_{x\to\infty} \frac{-32x^5+\cdots}{64x^5+\cdots} = \frac{-32}{64} = -0.5$$

Some Exercises

- Exercise 1.1: All odd numbers from 1-17
- Exercise 1.3: All odd numbers from 13-30
- Exercise 1.4: 8-14
- Exercise 1.5: All odd numbers from 1-23