Differentiation: Applications

Calculus I

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Week 6

Outline for the Day

Trends of Functions

Let $f : \mathbb{R} \to \mathbb{R}$ and $I \subseteq \mathbb{R}$ being an interval. The function f is said to be:

1. **increasing** within the interval I if $f(a) < f(b)$ whenever $a < b$.
2. **decreasing** within the interval I if $f(a) > f(b)$ whenever $a < b$.

Illustration of Function Trends

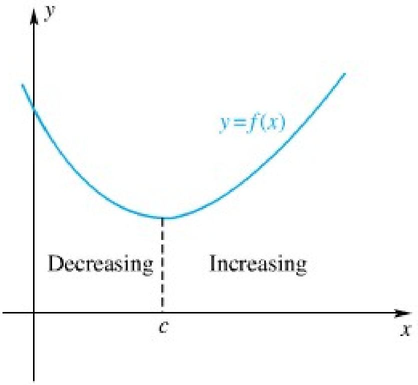

Figure: A function that is increasing and decreasing on a certain interval

The Tangent Slope

To find a gradient of a tangent at a certain point $x = c$, we do:

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{c+h-c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

Hence we can use differentiation to determine a trend of a function.

The Slope-Derivative Connection

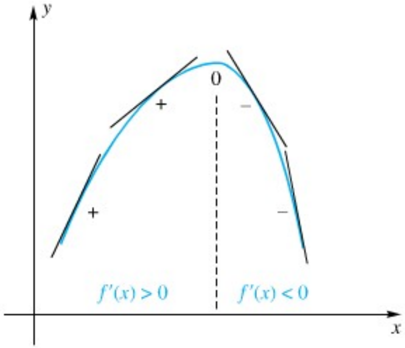

Figure: Connection between tangent slopes and trends

From the figure above, we can see that increasing function on a certain interval will have positive slope over the interval—while decreasing function will have negative slope.

The Slope-Derivative Connection

Let $f : \mathbb{R} \to \mathbb{R}$ and $I \subseteq \mathbb{R}$, we have:

1. f is increasing over interval I only if $f'(x) > 0$ for all $x \in I$
2. f is decreasing over interval I only if $f'(x) < 0$ for all $x \in I$

Illustration of Concavity

A differentiable function f is said to be:

1. concave up over interval I if f' is increasing over I.
2. concave down over interval I if f' is decreasing over I.

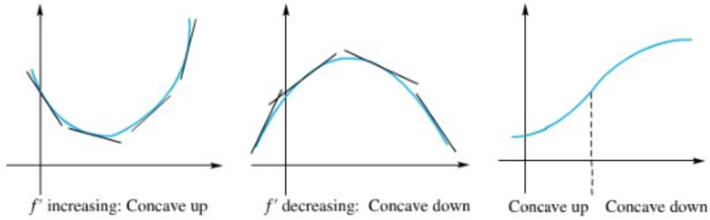

Figure: Illustration of concavity

Concavity Conditions

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable twice over the interval $I \subseteq \mathbb{R}$, we have:

1. f is concave up over interval I only if $f''(x) > 0$ for all $x \in I$
2. f is concave down over interval I only if $f''(x) < 0$ for all $x \in I$

Local Extrema

Let $f : \mathbb{R} \to \mathbb{R}$ be a function,

1. The value $f(a)$ is said to be a **local maximum** if $f(a) \geq f(x)$ for all $x \in I$, with I being some interval that includes a.
2. The value $f(a)$ is said to be a **local minimum** if $f(a) \leq f(x)$ for all $x \in I$, with I being some interval that includes a.

Conditions for Extremes of Function

Let f be a real valued continuous function defined over some interval $I \subseteq R$. Let $a \in I$ and f is differentiable twice at $x = a$.

1. f reach its local maximum at $x = a$ (with $f(a)$ as its local maximum value) if:
   - $f'(a) = 0$
   - $f''(a) < 0$
2. f reach its local minimum at $x = a$ (with $f(a)$ as its local minimum value) if:
   - $f'(a) = 0$
   - $f''(a) > 0$

Steps for the Problem Solving

1. Translate the Problems into Math Equations
   - Assign variable(s) to various value(s) mentioned. (pro tip: it helps if you don't assign that many)
   - Generate the function that you want to optimize. (key: make it in one variable!)
2. Work the Math out
   - Solve the equation $f' = 0$
   - Check the "x" values you find in the previous part for second derivative test
     - $f''(x) > 0$ then it's a minimum value
     - $f''(x) < 0$ then it's a maximum value
3. Clearly State the Solution
   - Try to "*word*" the math result(s) you now have into sentence(s).

Example

A box with a square base has no top. If 64 cm^2 of material is used, what is the maximum possible volume for the box?

Worked Example

Step 1

- **Assigning variables:** A topless box with a square base has two key features, its height and base's edge length. Let h be the height and x be the base's edge of said box.
- **Stating the soon-to-be optimized function:** We know that they ask for a maximum volume, hence the function f should signify the volume of said box. Using the rectangular box formula, we know that

$$f = \text{Box Volume} = x \times x \times h = hx^2$$

Only problem with this is that we still need to make f as a one-variable function.

Worked Example

Step 1 (continued) We need to find the relationship between x and h so we can substitute one with the other. The problem states that its area is 64 cm^2. Using this, we have

$$
\begin{aligned}
x^2 + 4xh &= 64 \\
4xh &= 64 - x^2 \\
h = \frac{64 - x^2}{4x} &= \frac{16}{x} - \frac{x}{4}
\end{aligned}
$$

Hence, substituting this to the f from previous slide:

$$f(x) = hx^2 = \left(\frac{16}{x} - \frac{x}{4}\right)x^2 = 16x - \frac{x^3}{4}$$

Worked Example

Step 2

- **Solving** $f'(x) = 0$

$$f' = \frac{d}{dx}\left(16x - \frac{x^3}{4}\right) = 0$$
$$16 - \frac{3x^2}{4} = 0$$
$$x^2 = 64/3$$
$$x = \frac{8\sqrt{3}}{3}$$
$$(\text{ditch } x = -\frac{8\sqrt{3}}{3})$$

Worked Example

Step 2 (continued)

- **Confirming the types of Extremes**: We're expecting f to be maximized, so checking the second derivative for x-value we found previously $\left(x = \dfrac{8\sqrt{3}}{3}\right)$

$$f''(x) = -3x/2 \Rightarrow f''\left(\frac{8\sqrt{3}}{3}\right) = -4\sqrt{3} < 0$$

Hence f will reach a maximum value when $x = \dfrac{8\sqrt{3}}{3} \approxeq 4.619$ with

$$f\left(\frac{8\sqrt{3}}{3}\right) \approxeq 49.267$$

as the maximum value.

Worked Example

Step 3

- This is equally (if not more) important than the first two steps. We cannot leave the answer in *mathematical* form. We need to somehow *word* it out.
- **We need to give meaning to the result that we have:** So the maximum value of the box will be 49.267 cm^3 which will be obtained when the base's edge is 4.619 cm.

Exercises

1. Exercise 3.2: 1-18 all odd numbers, 31, 33.
2. Exercise 3.3: 1-20 all odd numbers
3. Exercise 3.4: 12-25 all odd numbers (except 19 and 21)