Interest Rate Forecasting Using the Vasicek Model and Monte Carlo Simulations

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1 Introduction

Interest rate forecasting is a crucial aspect of financial risk management. It plays a role in bond pricing, portfolio management, and monetary policy decision making. To be able to accurately model interest rates is essential for financial institutions, central banks and investors to make informed decisions [4].

The Vasicek model, introduced by Oldřich Vasicek in 1977, is widely recognized as a mean-reverting stochastic process used to describe the evolution of interest rates over short periods of time [14]. It belongs to a class of "one-factor short-rate models", where the short-term interest rate is modeled as a single stochastic process [12]. The model assumes that interest rates are driven by economic forces that push them towards a long-term mean level with incorporating random fluctuations. The Vasicek model is known to produce analytically tractable solutions for bond prices and has a tendency to ensure that interest rates revert to realistic levels over time [14].

Alongside the Vasicek model, this project will use Monte Carlo simulations (MC) which is a powerful computational technique widely used in finance for modeling uncertainty and predicting outcomes. MC simulations involve generating a large number of random scenarios to approximate the behavior of complex systems [6]. By simulating a variety of different potential outcomes, this method allows for a comprehensive understanding of possible future interest rate paths. MC simulations are particularly useful in financial modeling due to their flexibility in handling non-linear systems and their ability to incorporate randomness, which is a very demanding and computationally difficult task when using other methods [11]. In the context of interest rate forecasting, these simulations will provide a range of possible future scenarios, allowing for better assessment and decision making.

2 Literature Review

The Vasicek model has been studied in finance for many years.. Vasicek (1977) initially proposed the model as a solution to term structure dynamics with mean-reverting properties [14]. His model demonstrated the analytical ability to price bonds, making it widely adopted in financial risk management [8]. Subsequent research by Hull and White (1990) incorporated time varying parameters to the Vasicek model, which improved its ability to capture term structure volatility [7]. This enhancement addressed some of the models limitations, specifically how it allows for negative interest rates in extreme conditions.

Monte Carlo simulation's (MC) have also gained popularity in stochastic modeling. Boyle (1977) used MC simulations in option pricing, demonstrating its ability to estimate financial derivatives under uncertainty [2]. Since then, MC methods have become essential in risk management portfolio optimization and forecasting [10]. Research by Glasserman (2003) provides a framework for implementing MC methods in financial applications, including interest rate modeling [6]. These studies have shown the power of combining the Vasicek model with MC in order to enhance predictive accuracy by capturing a broader range of potential market conditions and creating a distribution of interest-rate paths.

This project builds upon these foundational works by integrating the Vasicek model with Monte Carlo simulations to forecast interest rate movements. By doing so, it aims to provide a comprehensive approach to risk assessment in dynamic financial markets.

3 Model

The Vasicek model is a type of stochastic differential equation (SDE) that describes the evolution of the short-term interest rate r_t over time. The SDE is defined below:

$$dr_t = \theta(\mu - r_t)dt + \sigma dW_t \tag{1}$$

Where:

- r_t: The instantaneous short-term interest rate at time t. (r_0 represents the initial rate at the beginning of the observation period).
- θ: The magnitude of the mean reversion, indicating how quickly the rate reverts to the mean. A higher value of θ means that the interest rate reverts to the mean faster, while a lower value indicates a slower adjustment.
- μ: The long-term mean of the interest rate. The is a central value around which the interest rate will fluctuate. It represents the average level to

which the interest rate is expected to revert over time.

- σ: The volatility (standard deviation) of the changes in interest rates. This parameter controls the level of randomness or uncertainty in the movement of the interest rate. Higher volatility leads to higher fluctuations in interest rates.
- W_t: This is a standard Brownian motion process that is used to introduce randomness (also known as the Wiener process).

The model essentially has two components. The first is the deterministic term $\theta(\mu - r_t)$ which drives the mean reverting behavior. The second is dW_t, which is a standard Brownian motion process that introduces randomness to the system [3]. The mean reverting behavior captured by the equation (1) has the following properties:

- If $r_t > \mu$, the term $(\mu - r_t) < 0$, causing the interest rate (r_t) to decrease.
- If $r_t < \mu$, the term $(\mu - r_t) \geq 0$, causing r_t to increase.

Hence, the term θ dictates how strongly the interest rate reverts back to μ.

To capture randomness, the term dW_t which is the standard Brownian motion is used to represent random fluctuations in the interest rate, with the following properties:

- $E[dW_t] = 0$
- $Var(dW_t) = dt$

Additionally, choosing model parameters is crucial in modeling specific market conditions. This project aims to predict interest rate behavior as accurately as possible, so the following model parameters are chosen, which reflect stable market conditions:

Variable	Value	Ref
μ	0.0325 (3.25%)	[15]
θ	0.2475 (24.75%)	[15]
σ	0.0064 (0.64%)	[15]
r_0	0.05 (5%)	

Table 1: Model Variables and Parameters

4 Analytical Solution

Using Ito's Lemma, we can integrate the SDE (Stochastic Differential Equation). The goal is to isolate r_t. We define $X_t = e^{\theta t}$, and multiply both sides of (1) by X_t to obtain:

$$X_t dr_t = \theta\mu X_t d_t - \theta X_t r_t d_t + \sigma X_t dW_t$$

Undoing the product rule, we can obtain a simplified expression:

$$d(X_t r_t) = \theta\mu X_t d_t + \sigma X_t dW_t$$

Simplifying further to obtain:

$$\frac{d(X_t r_t)}{d_t} = \theta\mu X_t + \sigma X_t dW_t$$

Integrating both sides over the interval $[0, t]$:

$$X_t r_t = r_0 + \int_0^t \theta\mu X_s \, ds + \int_0^t \sigma X_s \, dW_s$$

Now we must evaluate the integrals. The first integral is the deterministic term of the model, and it can be solved simply by substituting the expression for X_t.

$$\int_0^t \theta\mu e^{\theta s} \, ds = \mu(e^{\theta t} - 1)$$

For the second integral, we must use properties of stochastic integrals to obtain:

$$\int_0^t \sigma e^{\theta s} \, dW_s = \sigma e^{\theta t} \int_0^t e^{-\theta(t-s)} \, dW_s$$

Reducing the equation by $e^{\theta t}$ gives:

$$r_t = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} \, dW_s \tag{2}$$

Now, we can compute both the expectation (mean) and the variance of the given SDE:

- $E[r_t] = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$
- $Var(r_t) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$

And thus, the analytical solution is found with both the mean and variance of our SDE given a specific set of parameters. Before conducting any simulations, we must first analyze the given terms in (2).

- $r_0 e^{-\theta t}$: This term represents the decay of the initial interest rate r_0 over time. As time progresses, $(t \to \infty)$ since $e^{-\theta t} \to 0$. A key observation is that the speed of the decay depends on the size of θ, so a higher θ means the interest rate will revert back to the mean faster and vice versa.
- $\mu(1 - e^{-\theta t})$: Represents the drift toward the long term mean μ. As $(t \to \infty)$, the term approaches μ since $e^{-\theta t} \to 0$ making $1 - e^{-\theta t} \to 1$. This term ensures that over time, the model's expected value aligns with the long-term mean μ.
- $\sigma \int_0^t e^{-\theta(t-s)}\, dW_s$: Introduces random fluctuations driven by Brownian motion. The volatility σ controls the magnitudes of the random shocks. A key observation is that this term models unpredictable market dynamics, ensuring randomness is realistically incorporated into our model [12].

5 Euler-Maruyama Method

The Euler-Maruyama method is a method for approximating Stochastic Differential Equations [1]. First, we start with discretizing the time variable by considering our model on the interval $t \in [0, T]$ and choosing a large $N \in Z$. We then define:

$$\triangle t = \frac{T}{N}$$

This term represents each individual time increment. We then define the time points $t_0, t_1, ..., t_N$ where $t_i = i\triangle t$ for $i = 0, 1, 2, ..., N$. These represent our increment's forward in time.

Next, we generate N independent random variables $Z_1, Z_2, ..., Z_N$. These represent the increments of Brownian motion $\triangle B_i = Z_{i+1}\sqrt{\triangle t}$.

Finally, for each time step $i = 0, 1, 2, ..., N-1$, we update the short rate using Euler's discretization of the Vasicek SDE:

$$r_{i+1} = r_i + \theta(\mu - r_i)\triangle t + \sigma \triangle B_i \tag{3}$$

Given initial conditions $r_0 = 5\%$, we can iterate forward in time to approximate the short-rate over period of time $\triangle t$. This setup can be implemented through Python. We store each short-rate r_i for analysis.

6 Python Simulation

To demonstrate the implementation of the Vasicek model using Monte Carlo and Euler-Maruyama discretization, we utilize Python for its clarity, computational efficiency, and powerful numerical tools. The Python code developed for this project is structured into clearly defined functions, which handle model simulation, parameter specification, and result visualization.

6.1 Python Implementation

First, we implement a function to increment a single step forward in time. Specifically, the function takes the parameters of the model as inputs as well as the previous spot rate, computes $\triangle t$ and the value of Brownian Motion, then uses the Euler discretization of the Vasicek model in order to increment the interest rate forward one step in time.

```
#iterates forward a single step in time, computes interest rate
def vasicek_euler(T, N, r_0, theta, mu, sigma):
  dt = T/N
  Z = np.random.normal(0, 1)
  r_next = r_0 + theta*(mu - r_0)*dt + sigma*np.sqrt(dt)*Z
  return r_next
```

Figure 1: Single Step Forward

Our next function computes an entire interest rate path over a time interval T. The function does this by calling our previous function N times and storing the outputted spot rate, then uses that spot rate to continue incrementing forward in time.

```
#creates a single path of interest rates for t between 0 and T
def vasicek_path(T, N, r_0, theta, mu, sigma):
  interest_rates = [r_0]
  for i in range(N):
    interest_rates.append(vasicek_euler(T, N, interest_rates[-1], theta, mu, sigma))
  return interest_rates
```

Figure 2: Vasicek Interest Rate Path

Finally, we create our final function which is the Monte Carlo simulation. This function will call the previous function M times, creating a distribution of possible paths, and outputs the data of those paths for our analysis.

```
#runs vasicek_path functions M times
def MC_vasicek(T, N, r_0, theta, mu, sigma, num_simulations):
  all_simulations = np.zeros((num_simulations, N+1))
  for i in range(num_simulations):
    all_simulations[i, :] = vasicek_path(T, N, r_0, theta, mu, sigma)
  return all_simulations
```

Figure 3: Monte Carlo Function

6.2 Visualization of Simulations

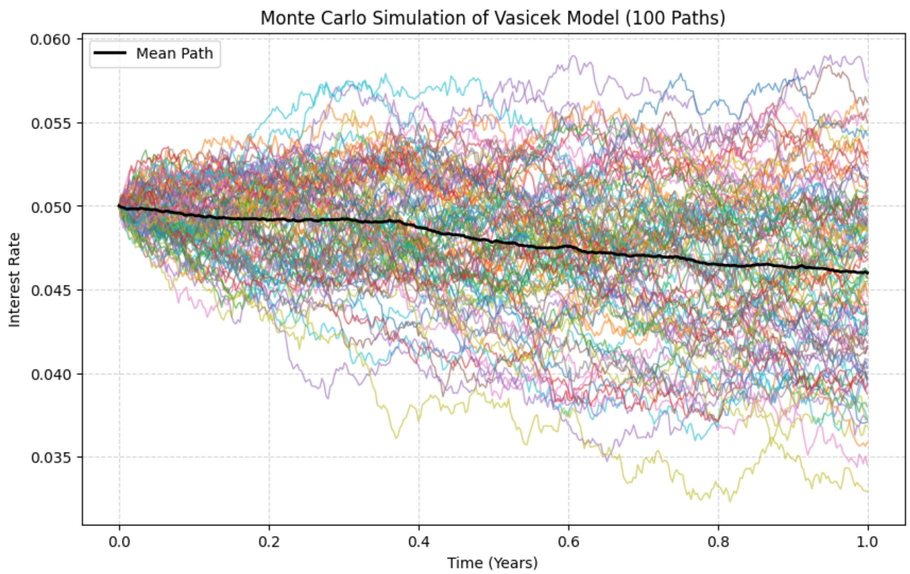

Figure 4: Monte Carlo Simulation (100 Interest Rate Paths, 1-year horizon)

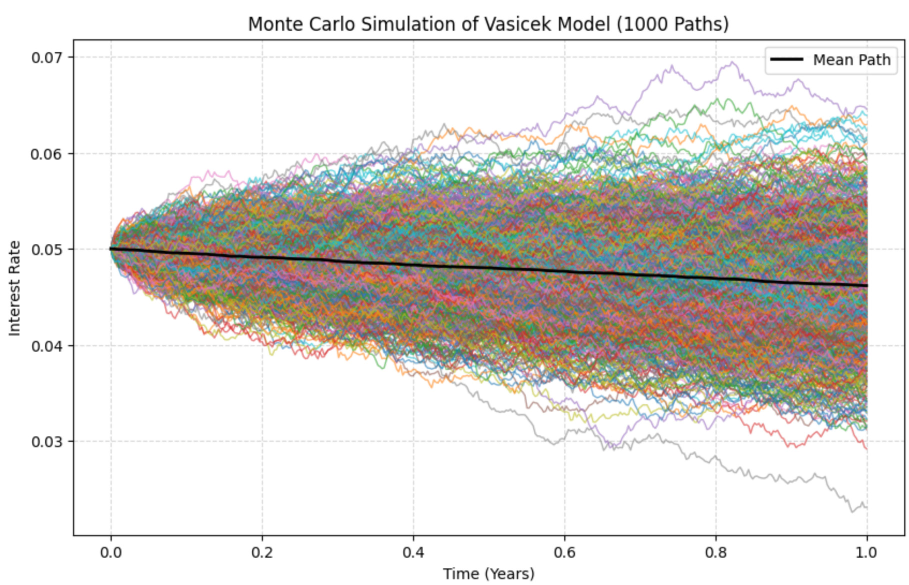

Figure 5: Monte Carlo Simulation (1000 Interest Rate Paths, 1-year horizon)

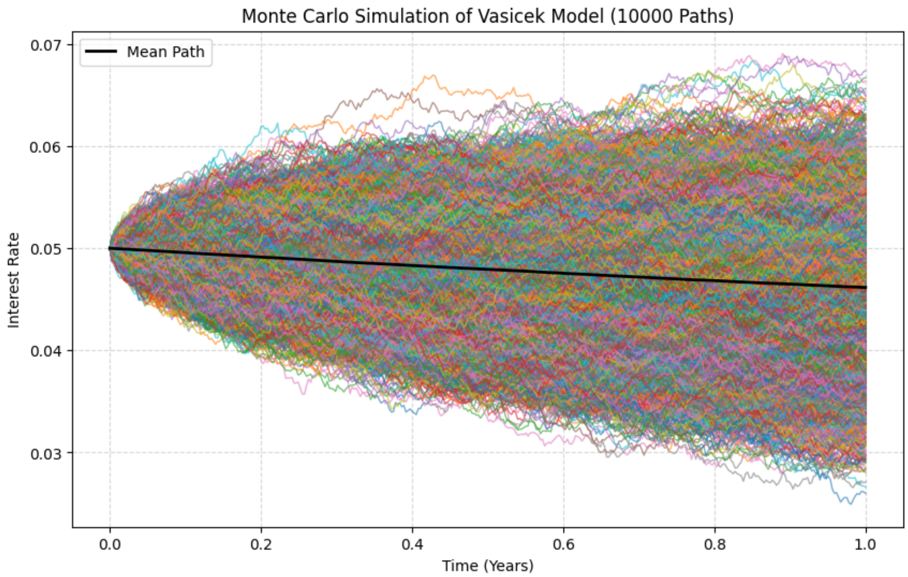

Figure 6: Monte Carlo Simulation (10,000 Interest Rate Paths, 1-year horizon)

7 Outcome Analysis

This section analyzes the results obtained from conducting 10,000 simulations, visualized in Figure 5. The graph illustrates multiple simulated paths of short-term interest rates over a one-year horizon. This simulation provides valuable insight into the stochastic behavior of interest rates under realistic financial conditions [9], as well as key metrics that we can use for asset pricing or risk management.

7.1 Analysis of Simulation Results

Figure 5 clearly demonstrates the stochastic and mean-reverting behavior of the Vasicek model. Each colored line represents a possible evolution of the short-term interest rate over the simulation period. Despite the initial interest rate being higher than our mean (starting at $r_0 = 5\%$), simulated paths quickly diverge due to random fluctuations driven by Brownian Motion and the volatility parameter ($\sigma = 0.0064$).

Variable	Value
θ	24.75%
μ	4.61%
σ	0.57%
r_0	5%

Table 2: Simulated Key Metrics at $T = 1$ Year

Key descriptive statistics derived from the simulated data are summarized in Table 2.

It is clear to see that the mean simulated interest rate (4.61%) starts to converge with the theoretical long-term mean ($\mu = 3.25\%$). Furthermore, the observed standard deviation (0.57%) closely matches the input volatility parameter ($\sigma = 0.64\%$), confirming the accuracy and consistency of the simulation methodology.

7.2 Comparison with Analytical Expectations

To validate the accuracy of the numerical implementation, the simulated outcomes are compared with the analytical solution provided earlier. The analytical mean and variance of the interest rate at time $T = 1$ year are given by:

- $E[r_t] = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) \approx 3.27\%$
- $Var(r_t) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}) \approx 0.64\%$

The most notable difference between the simulated mean interest rate (approximately 4.61%) and the theoretical long-term mean ($\mu = 3.25\%$) can be attributed to the relatively short time horizon ($T = 1$ year). The Vasicek model's mean-reverting characteristic implies that the short-term interest rate moves progressively, not instantaneously, towards the long-term mean [14]. A one-year period is insufficient for the interest rate, initially set at $r_0 = 5\%$, to fully revert to its theoretical mean of 3.25%. A longer time frame (greater T) would result in the simulated mean interest rate converging closer to the theoretical long-term mean, as the impact of initial conditions diminishes with increased maturity. This explains the observed difference and is consistent with the model's fundamental dynamics.

7.3 Distribution Analysis of Final Rates

Figure 7 presents a histogram depicting the distribution of final interest rates from the 10,000 simulated Monte Carlo paths at the one-year mark. The histogram provides further insights into the statistical properties of the Vasicek model simulations.

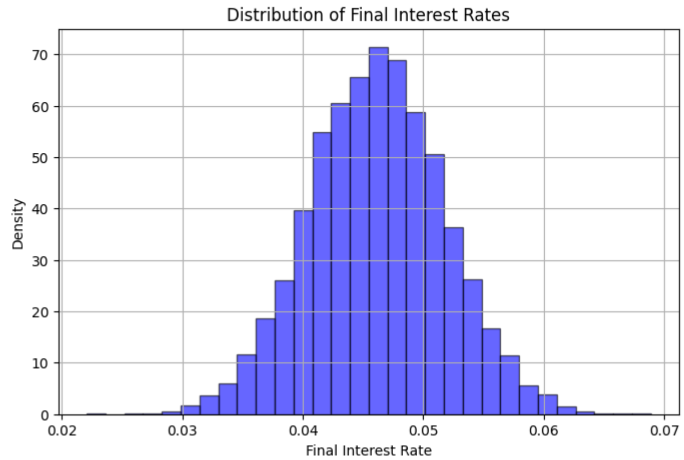

Figure 7: Final Rates Histogram

The histogram visually demonstrates a clear bell-shaped distribution, closely resembling a normal distribution. This observation aligns with theoretical expectations, as the Vasicek model predicts a normally distributed interest rate at any fixed time point due to its construction around a Gaussian stochastic process [8].

The histogram shows that the distribution is centered around approximately 4% to 5%. Additionally, the distribution's spread provides a visualization of the volatility in interest rate forecasting. Most outcomes cluster tightly around the central mean, yet the presence of significant tails emphasizes the realistic potential for considerable deviations from the expected interest rate.

Interestingly, the histogram does have minor skewness towards higher values, indicating slightly more deviations on the upper side. This skewness remains small and practically negligible, which reinforces the assumption that the Vasicek model follows a normal distribution [14]. The tails indicate rare but possible extreme rates, an important consideration for stress testing and risk assessment purposes. These tails imply that even though unlikely, financial institutions must account for scenarios involving unusually low or high interest rates.

7.4 Practical Finance Implications

A zero-coupon bond provides a single payoff at maturity without coupon payments. Its price is directly influenced by the interest rate, calculated using the following discounting formula [7]:

$$P(0,T) = e^{-rt}$$

Where:

- $P(0,T)$ is the bond price at time 0 for maturity T
- T is the time to maturity
- r is the interest rate

Using the simulated mean interest rate (approximately 4.61%) at $T = 1$ year, we calculate the expected bond price. Using the Empirical Rule, approximately 68% of simulated interest rates fall within one standard deviation of the mean [13], providing a confidence interval to assess bond price uncertainty.

Specifically, we calculate bond prices at interest rates one standard deviation above and below the mean:

$$P_{mean}(0,1) = e^{-0.0461*1} \approx \$0.9549$$

$$P_{upper}(0,1) = e^{-(0.0461+0.0057)*1} \approx \$0.9495$$

$$P_{lower}(0,1) = e^{-(0.0461-0.0057)*1} \approx \$0.9604$$

Consequently, we obtain a 68% confidence interval for bond prices ranging from \$0.9495 to \$0.9604. This interval implies that there is a 68% probability that the bond's price will fall within this range, given the modeled interest rate variability.

Bonds and interest rates have an inverse relationship, meaning that we expect the price of this current market bond to be higher than our predicted bond price 1 year from today [5]. Today, the current yield for a 1 year bond is 4.07%, which is lower than our predicted interest rate in 1 year. The price of these bonds is approximately \$0.961.

8 Conclusion

Throughout this project, we explored the application of the Vasicek model in tandem with Monte Carlo and the Euler Maruyama Discretization, to forecast short-term interest rates. The Vasicek model's mean reverting behavior provides a mathematically elegant and analytically sound framework for modeling interest rate dynamics. When paired with Monte Carlo methods, it allows for the simulation of thousands of possible paths, offering valuable insight into the distribution and volatility of future rates.

Through simulation, it was observed that the mean of the interest rate distribution at the one-year horizon (4.61%) did not fully converge to the theoretical long-term mean ($\mu = 3.25\%$). This difference is attributed to the short time horizon, which demonstrates a gradual reversion to the mean over time. The analysis further showed the empirical distribution of simulated rates closely resembled a normal distribution, as theoretically expected. Our analysis also provided realistic scenarios for bond pricing, reflecting real world economic implications of our findings.

While the results were insightful, there are clear limitations such as the short forecast horizon. Further research could involve extending the simulation period, incorporating varying parameters, or comparing alternative interest rate models such as the Cox-Ingersoll-Ross (CIR) or Hull-White model. Ultimately, this project emphasizes the utility of combining analytical stochastic models with computational techniques to enhance forecasting methods and accuracy, which is extremely helpful for financial risk management. These results are particularly valuable for risk managers and fixed-income analysts, providing a practical framework for forecasting interest rate behavior and assessing bond price volatility under uncertain (stochastic) dynamics.

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