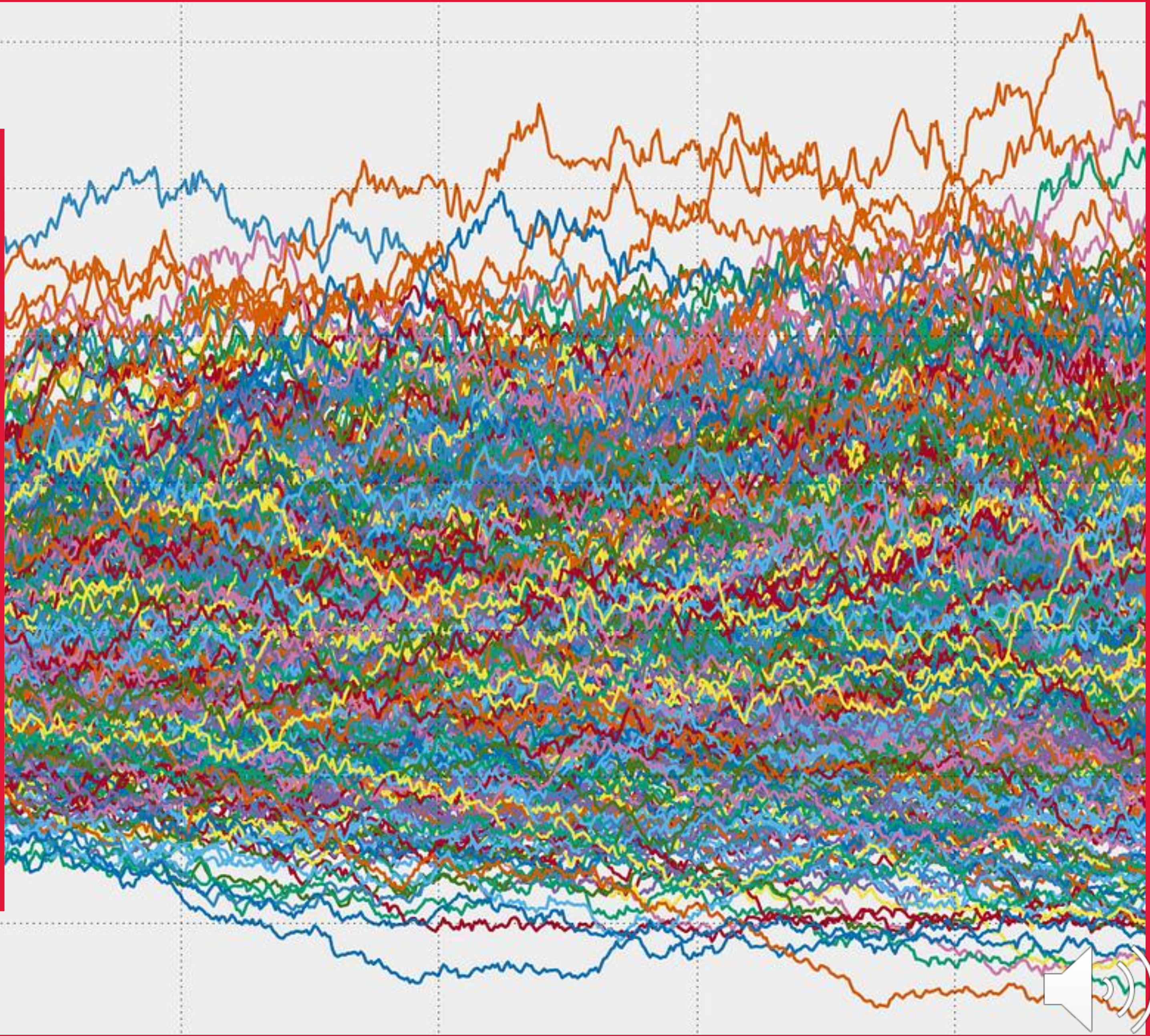
Interest Rate Forecasting Using the Vasicek Model and Monte Carlo

FARES MANSI, 218305854



Introduction

1. Objective
2. Motivation
3. Overview of Methods

Objective: What Do We Aim to Accomplish?

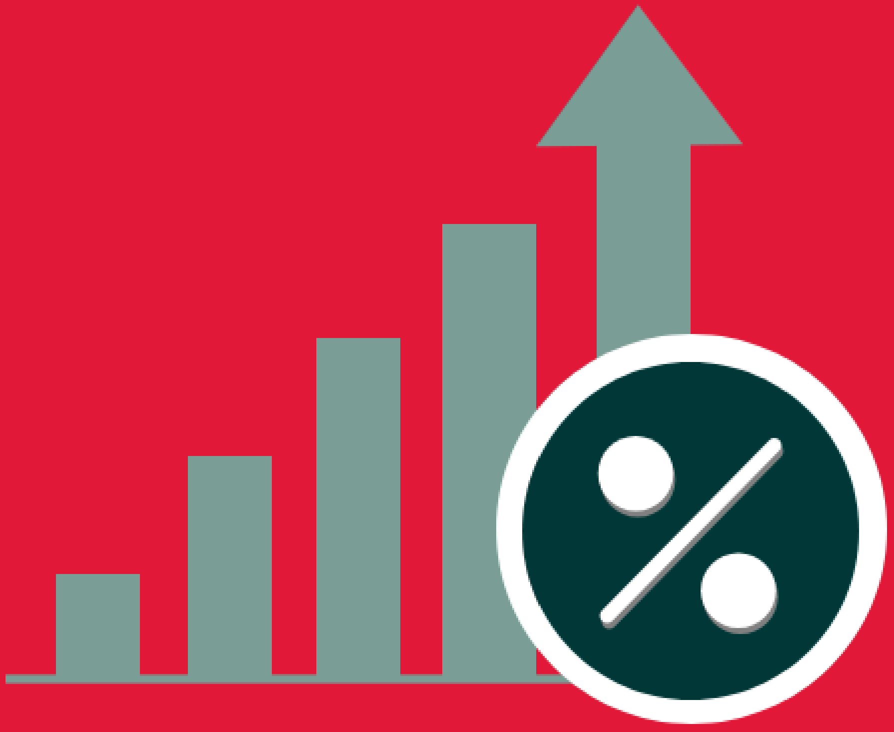
- **Explain Interest Rate Forecasting:**
  - Discuss why predicting interest rates is crucial for financial markets.
  - Highlight challenges in forecasting.
- **Introduce the Vasicek Model:**
  - Outline key parameters
  - Discuss analytical solution, strengths & limitations
- **Apply Monte Carlo Simulations**
- **Highlight Practical Applications:**
  - Discuss real world implications for investors, central banks, and policy makers
  - Compute predicted bond yields and prices
- **Summarize Key Insights:**
  - Evaluate the models accuracy & limitations

"All models are wrong, but some are useful." – George Box

Motivation: What Are Interest Rates? Why Are They Important?

Interest Rates Are:

- The cost of borrowing money or the return on investment for lending money.



Why Are They so Important?

1. **Borrowing & Spending:**
   - High Rates → Expensive Loans → Less Borrowing & Spending.
   - Low Rates → Cheaper Loans → More Borrowing & Economic Growth.
2. **Monetary Policy & Inflation Control:**
   - Central banks adjust rates to control inflation & stabilize the economy.
3. **Impact on Investments:**
   - Bond Yields & Stock Market, Real Estate Market.
4. **Influence on Exchange Rates & Global Currency:**
   - Higher Rates → Attracts Foreign Investors → Strengthen Currency
   - Low Rates → Boosts Exports → May Weaken Currency

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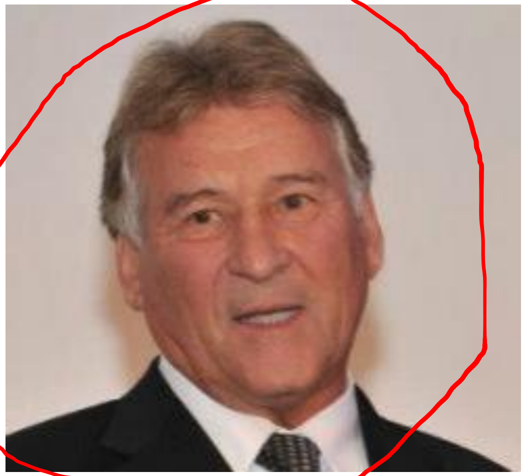
Overview of Methods: Vasicek Model and Monte Carlo



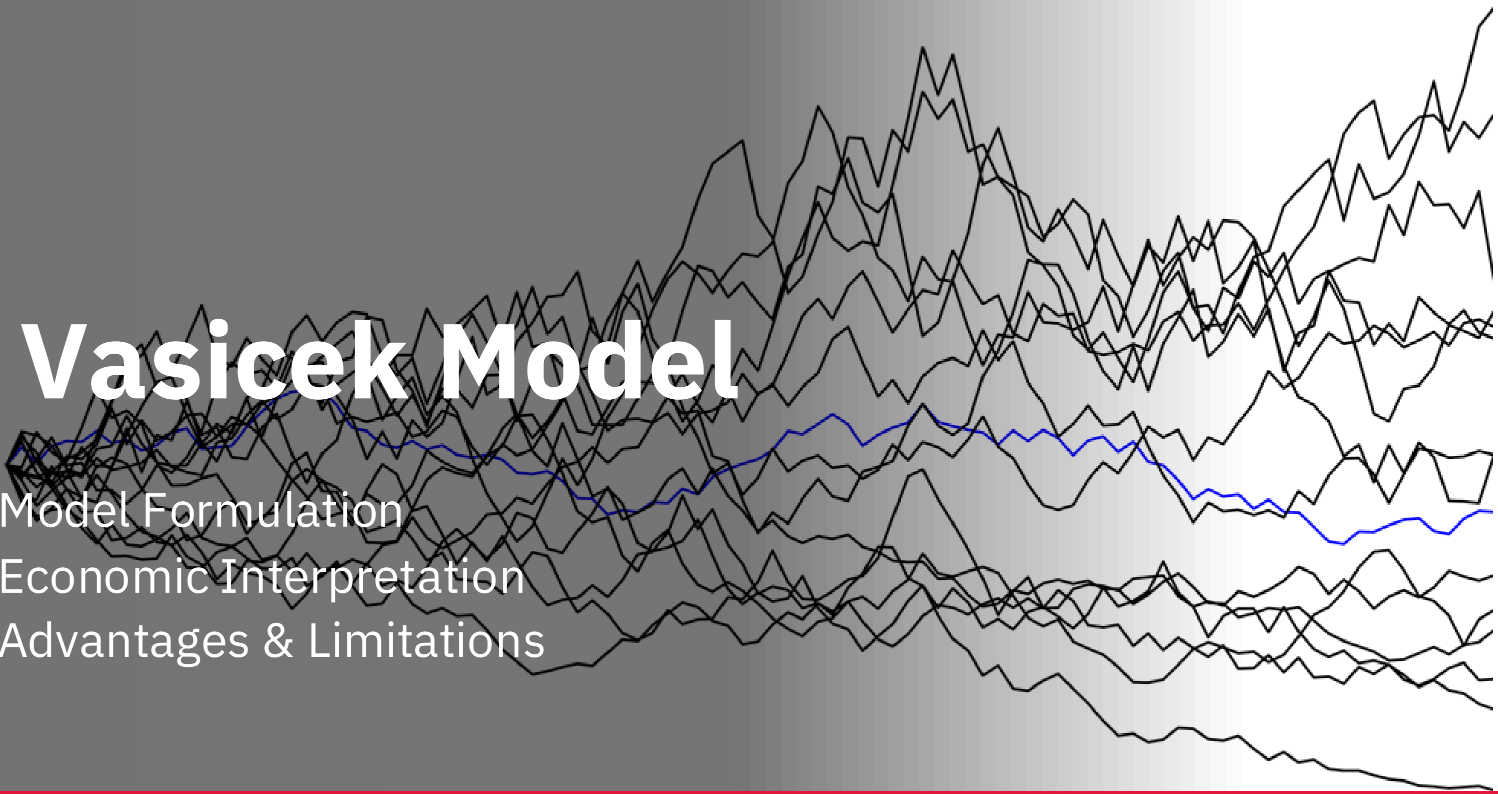
1. The Vasicek Model

- Introduced by Oldrich Vasicek in 1977
- Formulated as a type of SDE, captures random fluctuations in interest rates
- Mean-reverting model, ensures rates do not drift infinitely
- **Key Assumptions:**
  - Constant volatility
  - Rates can become negative

2. Monte Carlo Simulation

- Primarily developed by Stainslaw Ulam, who was inspired by his uncle's gambling habits.
- Broad class of computer algorithms, relies on repeated random sampling
- Simulate random paths, generate distribution of possibilities
- Help account for uncertainty in financial markets

The Vasicek Model



1. Model Formulation
2. Economic Interpretation
3. Advantages & Limitations

Mathematical Formulation

Type of **Stochastic Differential Equation (SDE)** that describes the evolution of short-term interest rate r_t over time

$$dr_t = \theta(\mu - r_t)dt + \sigma dW_t$$

Variables Breakdown:

- $\boldsymbol{r_t}$ – Instantaneous short-term interest rate at time t
- $\boldsymbol{\Theta}$ – Magnitude of mean reversion
- $\boldsymbol{\mu}$ – Long-term mean interest rate
- $\boldsymbol{\sigma}$ – Volatility of the changes in interest rates
- $\boldsymbol{W_t}$ – Standard Brownian motion process

Deterministic Term: $\boldsymbol{\theta(\mu - r_t)}$

- If $r_t > \mu \longrightarrow (\mu - r_t) < 0 \longrightarrow$ r_t starts to decrease
- If $r_t < \mu \longrightarrow (\mu - r_t) \geq 0 \longrightarrow$ r_t starts to increase
    - *Then, $\boldsymbol{\theta}$ dictates the strength of the mean reversion*

Random Term: $\boldsymbol{\sigma dW_t}$

- Represents random fluctuations in interest rate
- Useful when modeling complex systems with random elements

Economic Interpretation

1. Analytical Solution

$$r_t = r_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma \int_0^t e^{-\theta(t-s)}\, dW_s$$

Where:

- r_0 – Initial interest rate
- μ – Long-term mean interest rate
- $e^{-\theta t}$ – Decay factor
- $\sigma \int_0^t e^{-\theta(t-s)}\, dW_s$ – Random fluctuations driven by market shocks
- $E[r_t] = r_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right)$
- $Var[r_t] = \frac{\sigma^2}{2\theta}\left(1 - e^{-\theta t}\right)$
  - As $t \to \infty$:
    - $r_0 e^{-\theta t}$ – Decay of the initial interest rate over time
    - $\mu\left(1 - e^{-\theta t}\right)$ – Drift toward the long term mean

Understanding Mean Reversion in Interest Rates

1. The Vasicek Model assumes that interest rates **revert to a long-term mean** over time
2. This reflect **central bank policies**, where rates are adjusted to stabilize inflation and economic growth
3. Θ controls how quickly rates **return to equilibrium**

Key Economic Implications

1. If θ is high → Rates adjust quickly to economic changes (strong central bank control)
2. If θ is low → Rates take longer to revert, indicating prolonged economic fluctuations
3. Rates converge towards μ, the long term mean
4. Past interest rate influence fades over time

Advantages and Limitations

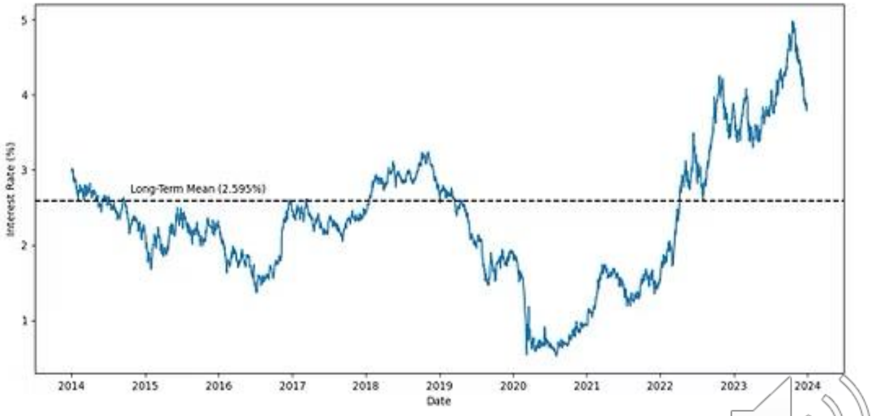
Advantages

1. Mean Reversion Property
   - Reflects real-world central bank policies that stabilize rates over time
2. Closed-Form Analytical Solution
   - Allows for explicit calculation of expected future interest rates
3. Computational Efficiency
   - Works well for pricing interest rate derivatives (bonds, swaps)

Limitations

1. Allows Negative Interest Rates
   - Normally distributed shocks (Brownian motion) can lead to negative interest rates (unrealistic in some markets)
2. Constant Volatility Assumption
   - σ varies over time
3. Over-Simplicity in Market Dynamics
   - Does not account for factors like liquidity constraints, macroeconomic shocks or policy shifts

Foundational tool for interest rate forecasting, however, its assumptions limit its real-world accuracy, making it less suitable for complex financial environments



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Monte Carlo Simulation

1. Simulation Process
2. Building Blocks
3. Graphs

Simulation Process

- **Discretization of the Model (Euler Method)**

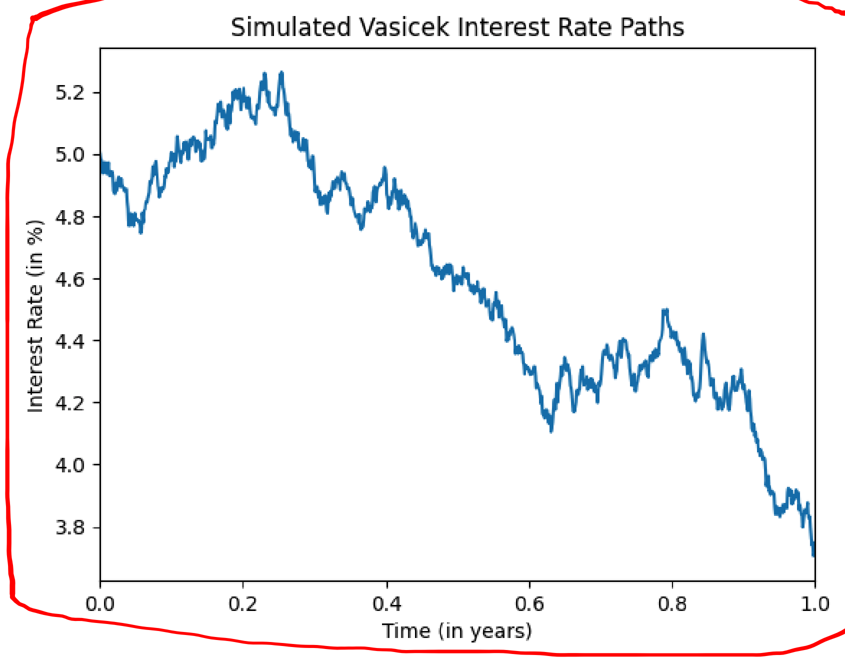
1. Choose a large T and large N
   - Consider the SDE on the interval $\boxed{t \in [0, T]}$
   - Define $\Delta t = \frac{T}{N}$
2. Generate (iid) Random Variables
   - $Z_1, Z_2, \dots, Z_N$
   - Define $\boxed{\Delta B_i = Z_{i+1}\sqrt{\Delta t}}$
3. Discretize the SDE

$$\boxed{r_{i+1} = r_i + \theta(\mu - r_i)\Delta t + \sigma \Delta B_i}$$

   - For each time step $i = 0, 1, 2, \dots, N-1$, we update the short rate using the iterative approach
   - Store each r_i for analysis

- **Simulating Multiple Paths**

1. Start with initial interest rate r_0
   - This value is chosen according to the market conditions that we want to model
2. Generate random shocks Z_t for each time step Δt



Variable	Value
θ	24.75%
μ	3.25%
σ	0.64%
r_0	5%

Model parameters estimated using MLE

Building Blocks: Python Functions

```python
def vasicek_euler(T, N, r_0, theta, mu, sigma):
  dt = T/N
  Z = np.random.normal(0, 1)
  r_next = r_0 + theta*(mu - r_0)*dt + sigma*np.sqrt(dt)*Z
  return r_next
```

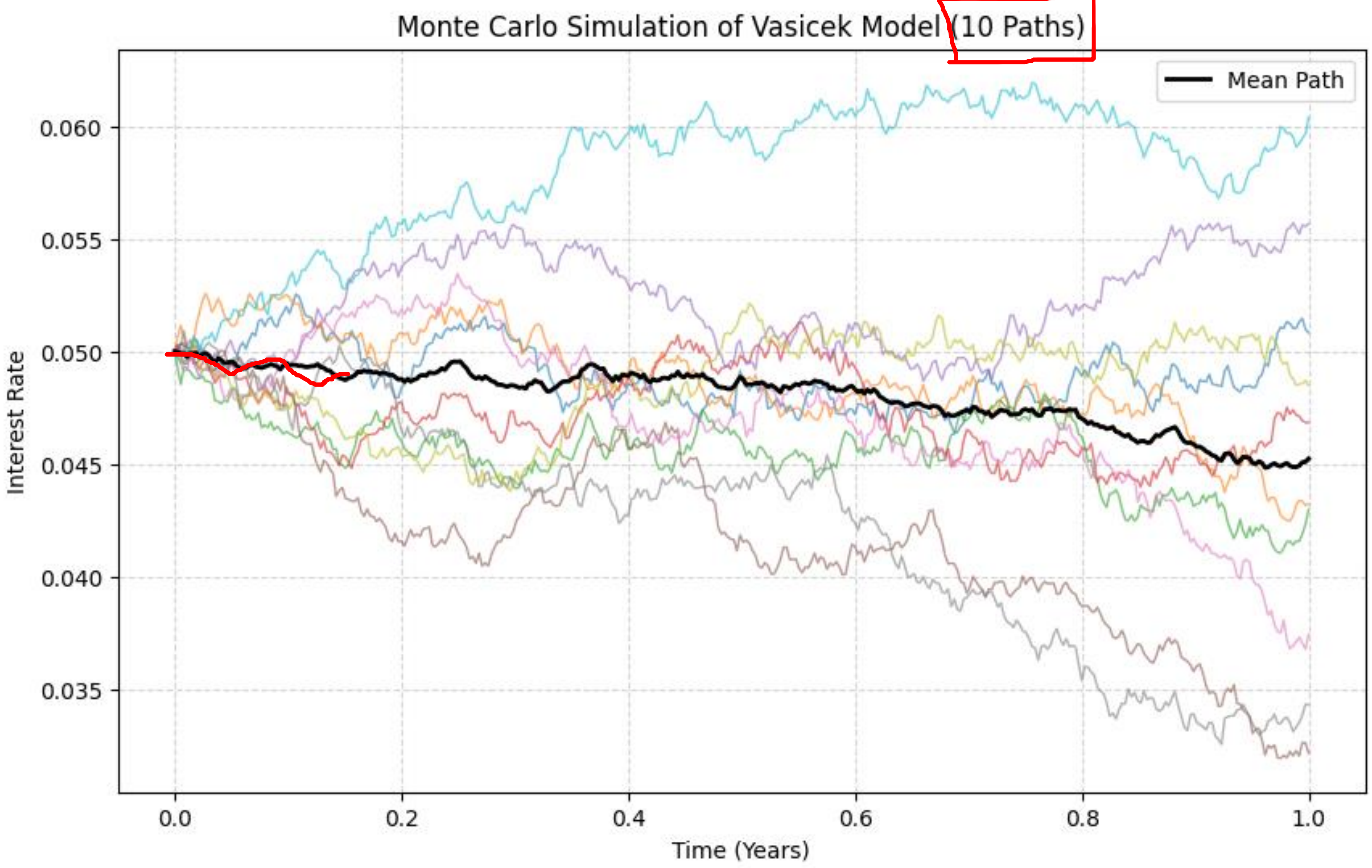
```python
def vasicek_path(T, N, r_0, theta, mu, sigma):
  interest_rates = [r_0]
  for i in range(N):
    interest_rates.append(vasicek_euler(T, N, interest_rates[-1], theta, mu, sigma))
  return interest_rates
```

- Iterate forward in time by Δt, find the next interest rate

```python
def MC_vasicek(T, N, r_0, theta, mu, sigma, num_simulations):
  all_simulations = np.zeros((num_simulations, N+1))
  for i in range(num_simulations):
    all_simulations[i, :] = vasicek_path(T, N, r_0, theta, mu, sigma)
  return all_simulations
```

- Create an array to store data, iterate through the number of simulations and store each interest rate path in the array

Graphs



Key Metrics at t=T

Metric	Value
N	365
Mean	4.63%
Variance	1.84×10^{-5}
Standard Deviation	0.0043

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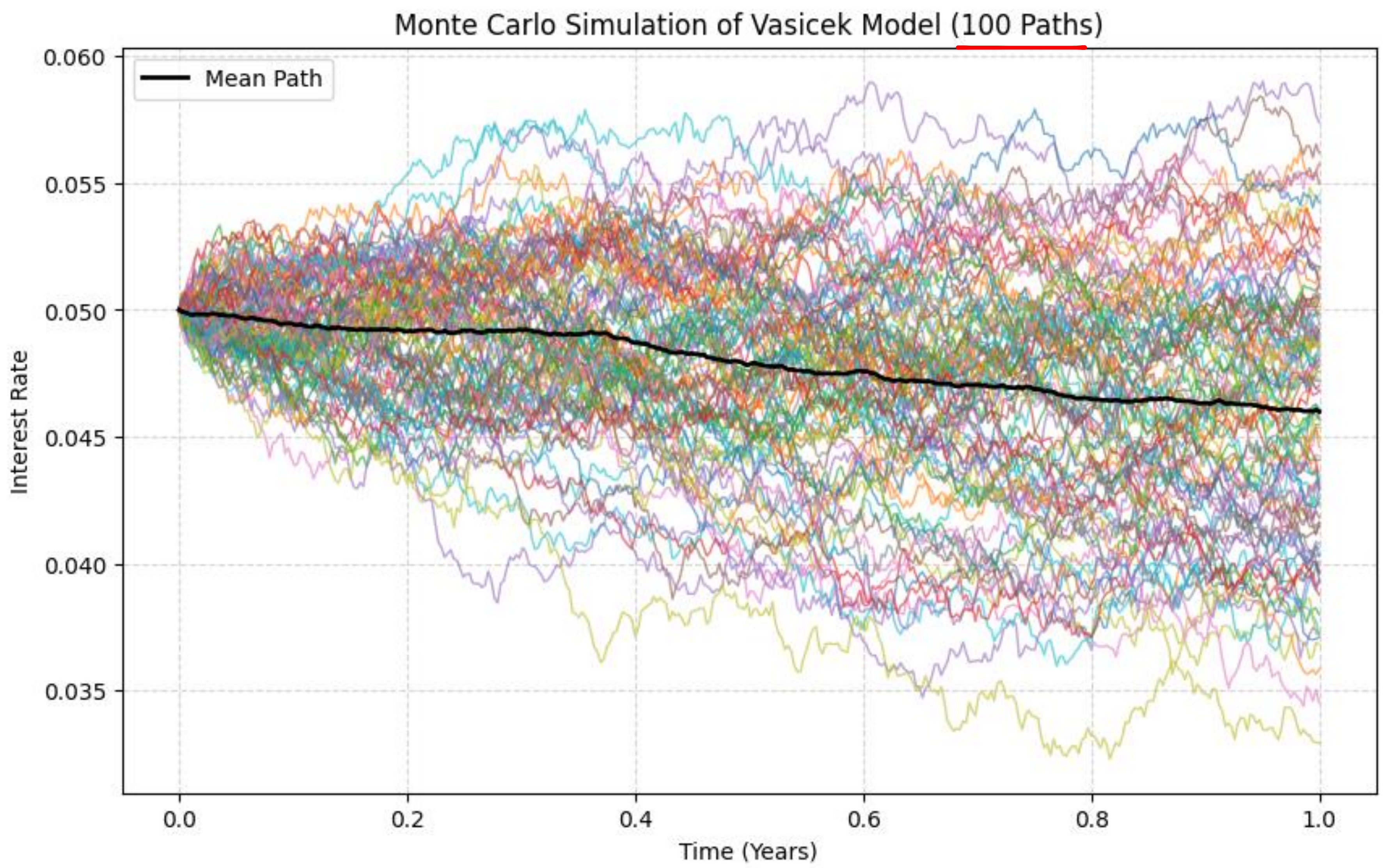

Key Metrics at t=T

Metric	Value
N	365
Mean	4.62%
Variance	$3.07\text{x}10^{-5}$
Standard Deviation	0.0055

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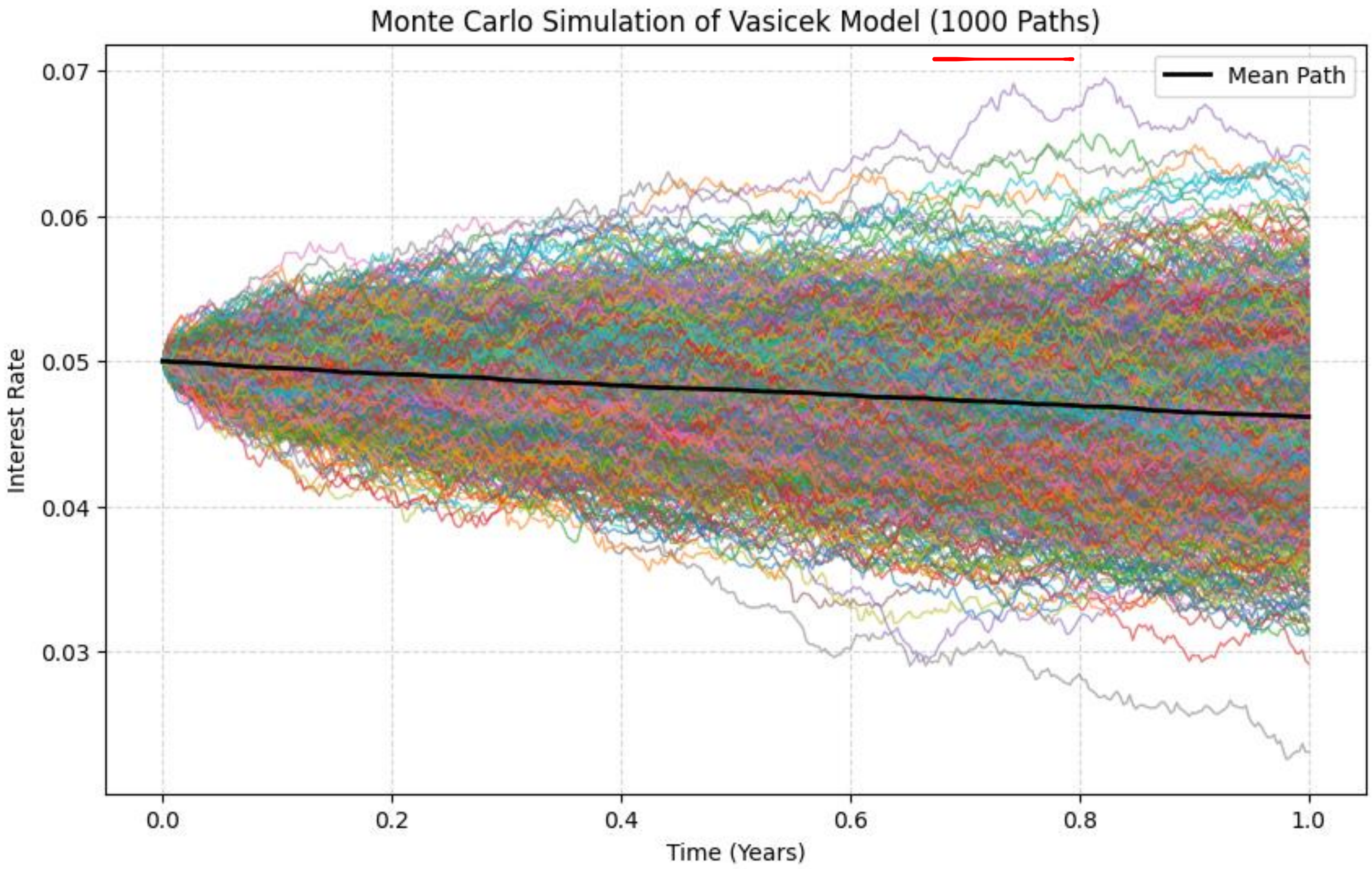
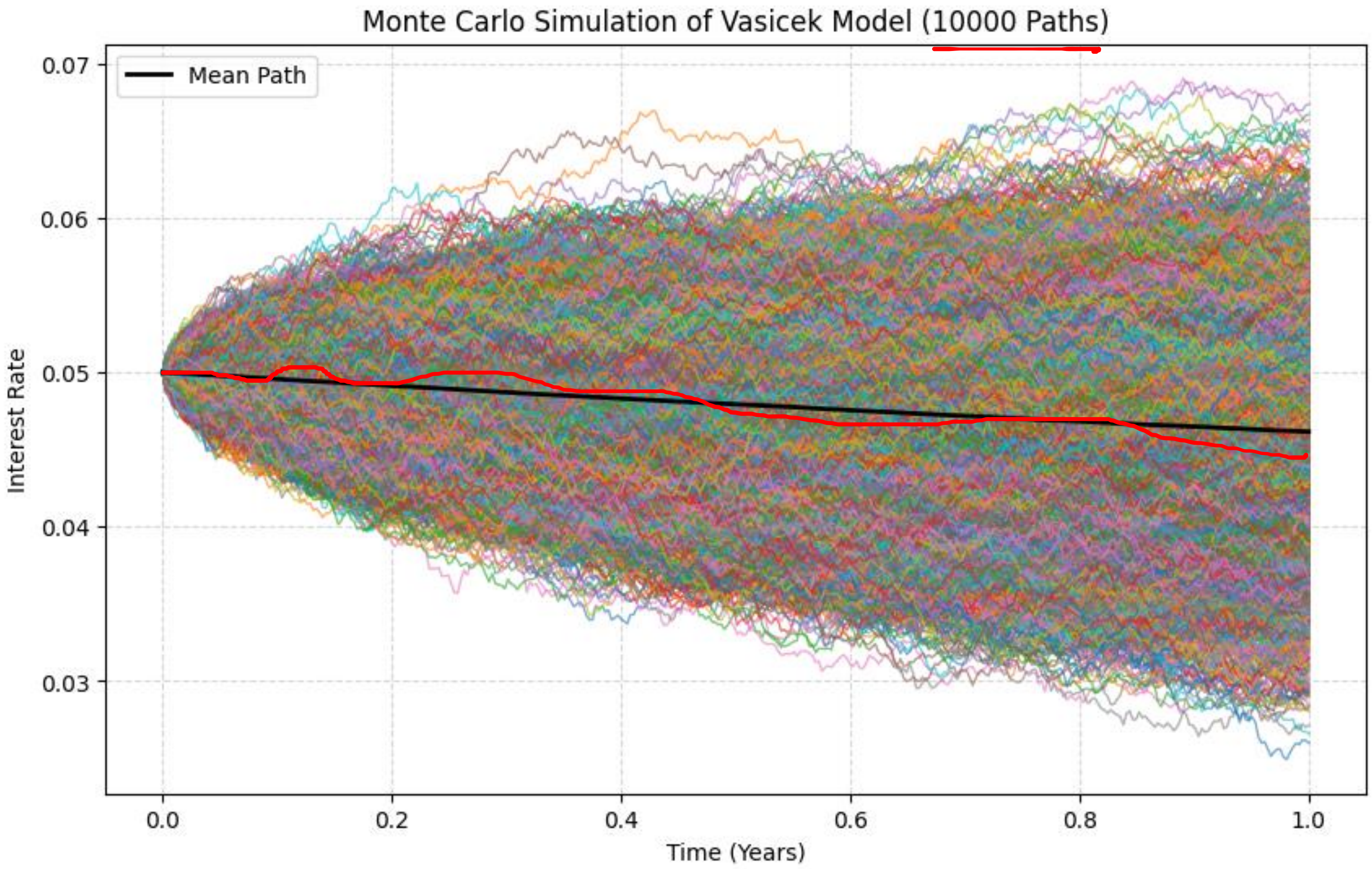


Key Metrics at t=T

Metric	Value
N	365
Mean	4.64%
Variance	3.03×10^{-5}
Standard Deviation	0.0055



Key Metrics at t=T

Metric	Value
N	365
Mean	4.61%
Variance	3.28×10^{-5}
Standard Deviation	0.0057

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Analysis

1. Key Observations
2. Comparison with Theoretical Expectation
3. Implications for Finance & Closing Remarks

Key Observations

1. CONVERGENCE AND MEAN REVERSION

- Simulated paths exhibit mean reversion
- Long-term trend demonstrates tendency to revert towards the long term mean μ
- Aligns with theoretical expectation
  - $E[r_1] = r_0 e^{-\theta} + \mu(1 - e^{-2\theta}) \approx 0.0517$
  - $\sigma = \sqrt{Var[r_1]} = \sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta})} \approx 0.00568$

2. IMPACT OF THE NUMBER OF SIMULATIONS

- **100 paths:** High variability, less smooth mean path due to statistical noise
- **1000 paths:** Lower variability, more stable mean path, clearer convergence towards μ
- **10000 paths:** Smoothest mean path, strong law of large numbers

Then,

- Higher (M) $\longrightarrow$ Reduces statistical noise
- Higher (N) $\longrightarrow$ More accurate discretization of continuous time process

3. STATISTICAL INSIGHTS

- Mean paths in the graphs represent mean reversion and models convergence properties
- Distribution shape suggests that interest rates remain within a controlled range, rather than exploding to extreme values
- Spread of simulated paths widens over time, but dispersion is contained within realistic bounds

4. DISPERSION AND VOLATILITY TRENDS

- Spread of simulated paths increases over time, shows uncertainty in interest rates as we progress further in time
- Model accounts for stochastic fluctuations, but retains bounded volatility

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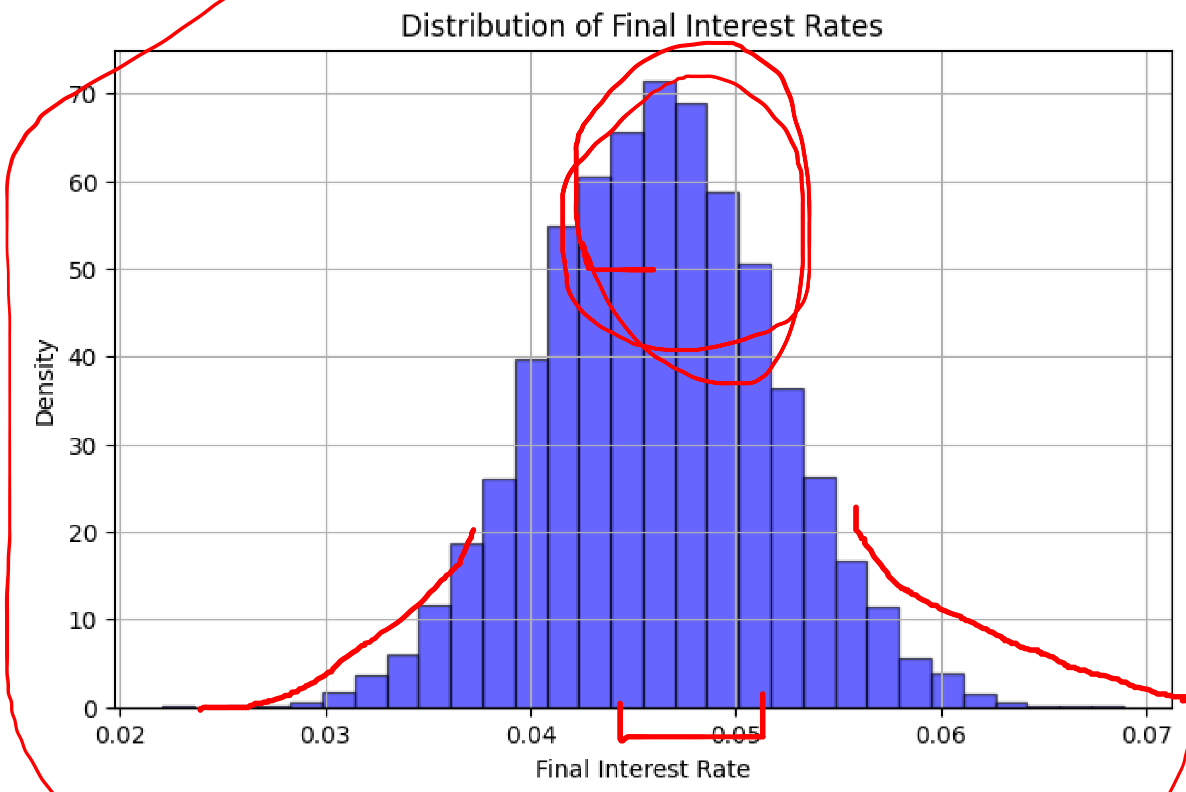
Comparison with Theoretical Expectations

1. **Shape of Distribution**
   - **Theoretical Expectation:**
     - Follows a normal distribution
     - Centered around long term mean
   - **Observation:**
     - Roughly shows a normal distribution
     - Highest density around 0.05
     - Decreasing frequency of extreme rates
2. **Mean and Spread of Interest Rates**
   - **Theoretical Expectation:**
     - Expected mean should align with theoretical mean
     - Extreme deviations should be rare
   - **Observation:**
     - Mean is around 0.05, aligns with expected value from analytical solution
     - Spread ranges from 0.02 to 0.07, suggests a moderate level of volatility
     - Frequency of very low or very high interest rates decrease



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Implications for Finance & Closing Remarks

QUANTITATIVE RISK METRICS

Using M = 10,000:

Metric	Value
μ	4.61%
σ	0.57%
σ^2	$3.27\text{x}10^{-5}$
r_0	5%
N	365

ZERO-COUPON BOND PRICING

The price of a zero-coupon bond at time t is given by:

$$\boxed{P(t,T) = A(T) \times e^{-B(T)\times r_t}}$$

Where:

- $B(T) = \frac{1-e^{-\theta T}}{\theta}$
- $A(T) = \exp\left(\left(\mu - \frac{\sigma^2}{2\theta^2}\right) \times (B(T) - T) - \frac{\sigma^2 B(T)^2}{4\theta}\right)$
- $P(0,1) = A(1) \times e^{-0.884*0.0461} \approx 0.9564$
- For a $1 face value bond:

	Simulated	Market (March 2025)
Yield	4.61%	4.07%
Price	$0.9564	$0.9610

PRACTICAL IMPLICATIONS & MODEL LIMITATIONS

1. Model Accuracy
   - Bond price is slightly lower than the market price ⟶ Higher yield under Vasicek model
     - We start at $r_0 = 5\%$, which reflects an inflationary market
2. Risk Management Applications
   - Output of simulations provides quantifiable measures of interest rate risk
   - Can inform:
     - **VAR** for fixed-income portfolios
     - **Stress testing** under adverse interest rate scenarios
3. Model Limitations
   - Assumes constant volatility
   - Allows for negative interest rate
   - Needs periodic recalibrations to be used effectively in practice